Chains in the noncrossing partition lattice of a reflection group

Ketten im Kreuzungsfreipartitionsverband einer Spiegelungsgruppe (?)

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Arbeitsgemeinschaft Diskrete Mathematik
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Introduction

- V: real vector space of finite dimension.
- $W \leq \mathrm{GL}(V)$: a finite reflection group, *i.e.* finite subgroup of $\mathrm{GL}(V)$ generated by reflections
($\rightsquigarrow$ structure of a finite Coxeter group).

Note: results remain valid for a more general class of groups
(**well-generated complex reflection groups**).

Combinatorics of the noncrossing partition lattice of W (*via* factorisations of a Coxeter element)	$\leftrightarrow$	Invariant theory of W (*via* geometry of the discriminant of W)

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1. Combinatorics of the noncrossing partition lattice
   - The noncrossing partition lattice of a reflection group
   - Chains and Fuß-Catalan numbers
   - Factorisations of a Coxeter element
2. Geometry of the hyperplane arrangement and of the discriminant
   - Discriminant and braid group
   - Geometric factorisations
   - Stratification and parabolic Coxeter elements
3. Lyashko-Looijenga covering and geometric factorisations
   - The Lyashko-Looijenga covering
   - Enumeration of maximal factorisations
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The noncrossing partition lattice of type W

- Define $R := \{\text{all reflections of } W\}$.
- $\rightsquigarrow$ reflection length (or absolute length) ℓ_R. (forget about the usual Coxeter length ℓ_S !)
- Absolute order $\preccurlyeq_R$:

$$u \preccurlyeq_R v \text{ if and only if } \ell_R(u) + \ell_R(u^{-1}v) = \ell_R(v) .$$

- Fix c : a Coxeter element in W (particular conjugacy class of elements of length $n = \mathrm{rk}(W)$).

Definition (Noncrossing partition lattice of type W)

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Prototype: noncrossing partitions of an n-gon

- $W := \mathfrak{S}_n$, with generating set $R := \{\text{all transpositions}\}$
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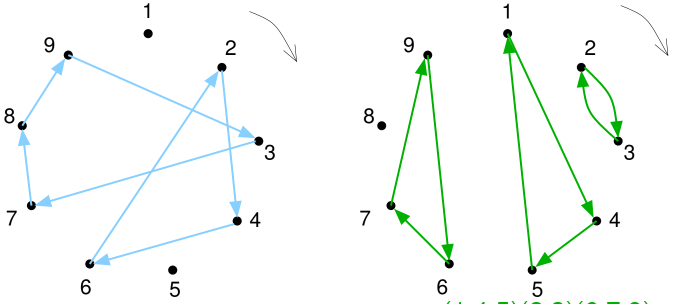

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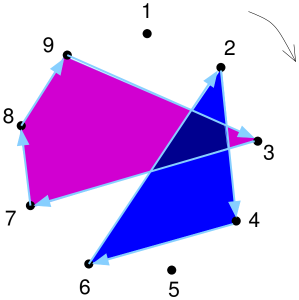

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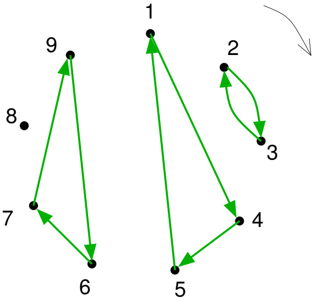

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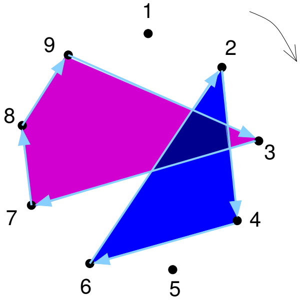

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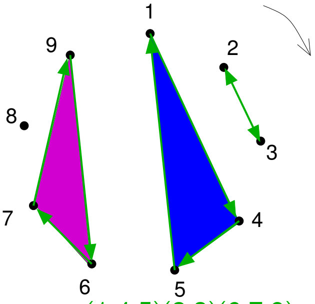

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Fuß-Catalan numbers

Kreweras's formula for multichains of noncrossing partitions

- $W := \mathfrak{S}_n$;
- c : an n-cycle.

The number of multichains $w_1 \preccurlyeq w_2 \preccurlyeq \ldots \preccurlyeq w_p \preccurlyeq c$ in $\mathrm{NC}(W, c)$ is the **Fuß-Catalan number**

$$\mathrm{Cat}^{(p)}(n) = \prod_{i=2}^{n} \frac{i + pn}{i} = \frac{1}{pn+1} \binom{(p+1)n}{n} .$$

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Proof: [Athanasiadis, Reiner, Bessis...] case-by-case!
Remark: $\mathrm{Cat}^{(1)}(W)$ (and $\mathrm{Cat}^{(p)}(W)$) appear in other contexts: Fomin-Zelevinsky cluster algebras, nonnesting partitions...

Factorisations of a Coxeter element

Definition (Block factorisations of c)

$(w_1, \ldots, w_p) \in (W - \{1\})^p$ is a block factorisation of c if

- $w_1 \ldots w_p = c$.
- $\ell_R(w_1) + \cdots + \ell_R(w_p) = \ell_R(c) = n$.

$\mathrm{FACT}_p(c) := \{$block factorisations of c in p factors$\}$.

- "Factorisations $\leftrightarrow$ chains".
- **Problem** : $\preccurlyeq$ *vs* $\prec$? Use conversion formulas:

$$\#\{w_1 \preccurlyeq \ldots \preccurlyeq w_p \preccurlyeq c\} = \sum_{k=1}^{p+1} \binom{p+1}{k} \# \mathrm{FACT}_k(c)$$

- **Bad news** : we obtain much more complicated formulas.
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Hyperplane arrangement

$\mathcal{A} := \{\text{reflecting hyperplanes of } W\}$ (Coxeter arrangement).

It's too simple, now make the ambient space V complex!
(replace V with $V \otimes \mathbb{C}$)

$$\begin{array}{ccc} V & \supset & \bigcup_{H \in \mathcal{A}} H \\ \downarrow & & \downarrow \\ V/W & \supset & \left(\bigcup_{H \in \mathcal{A}} H \right) \Big/ W \end{array}$$

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The quotient-space V/W

W acts on the polynomial algebra $\mathbb{C}[V]$.

Chevalley-Shephard-Todd's theorem

There exist invariant polynomials $f_1, \ldots, f_n$, homogeneous and algebraically independent, s.t. $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.

The degrees $d_1 \leq \cdots \leq d_n = h$ of $f_1, \ldots, f_n$ (called invariant degrees) do not depend on the choices of the fundamental invariants.

$\rightsquigarrow$ isomorphism: $$\begin{array}{ccc} V/W & \xrightarrow{\sim} & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \ldots, f_n(v)). \end{array}$$

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Discriminant of W

For H in $\mathcal{A}$, denote by α_H a linear form of kernel H.

$$\prod_{H \in \mathcal{A}} \alpha_H \quad \in \quad \mathbb{C}[V]$$

equation of $\bigcup_{H \in \mathcal{A}} H$.

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equation of $p(\bigcup_{H \in \mathcal{A}} H) = \mathcal{H}$, where $p : V \twoheadrightarrow V/W$.

Example $W = A_3$: discriminant ("swallowtail")

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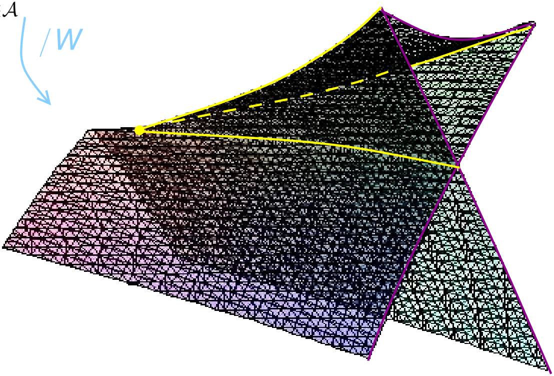

hypersurface $\mathcal{H}$ (discriminant) $\subseteq W \backslash V \simeq \mathbb{C}^3$

Braid group

- $V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H$
- W acts on V^{reg} (freely)
- Braid group of W:

$$B(W) := \pi_1(V^{\mathrm{reg}}/W) = \pi_1(\mathbb{C}^n - \mathcal{H})$$

Unramified covering $V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/W$

$\rightsquigarrow$ fibration exact sequence

$$1 \to \pi_1(V^{\mathrm{reg}}) \hookrightarrow \pi_1(V^{\mathrm{reg}}/W) \twoheadrightarrow W \to 1$$

$\pi : B(W) \twoheadrightarrow W$ "canonical" surjection.

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- $V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H$
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An element of $B(W) =$ a loop around $\mathcal{H}$, up to homotopy.

Consider “vertical loops”, i.e. for which $f_1, \dots, f_{n-1}$ remain constant. They are just loops in a punctured complex plane.

Call $\delta \in B(W)$ the simplest (clock-wise) vertical loop around $(0, 0, \dots, 0)$.

Facts:

- up to homotopy, this is also the simple vertical loop “around all $\mathcal{H}$”.
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Bifurcation locus

Theorem (Orlik-Solomon)

If W is a real (or complex well-generated) reflection group, then the discriminant Δ_W is monic of degree n in the variable f_n.

So if we fix $f_1, \ldots, f_{n-1}$, the polynomial Δ_W, viewed as a polynomial in f_n, has generically n distinct roots...
... it has multiple roots whenever $(f_1, \ldots, f_{n-1})$ is a zero of

$$D_W := \mathrm{Disc}(\Delta_W(f_1, \ldots, f_n)\ ;\ f_n) \in \mathbb{C}[f_1, \ldots, f_{n-1}].$$

Definition

The bifurcation locus of Δ_W (w.r.t. f_n) is the hypersurface of $\mathbb{C}^{n-1}$:

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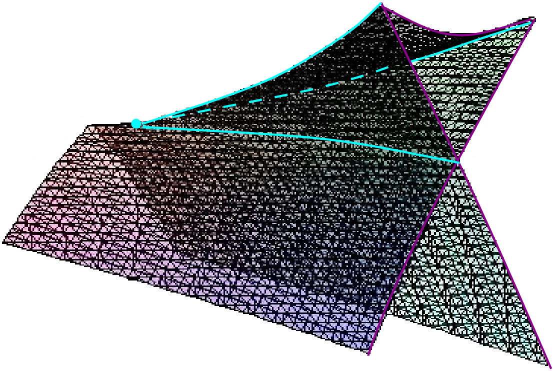

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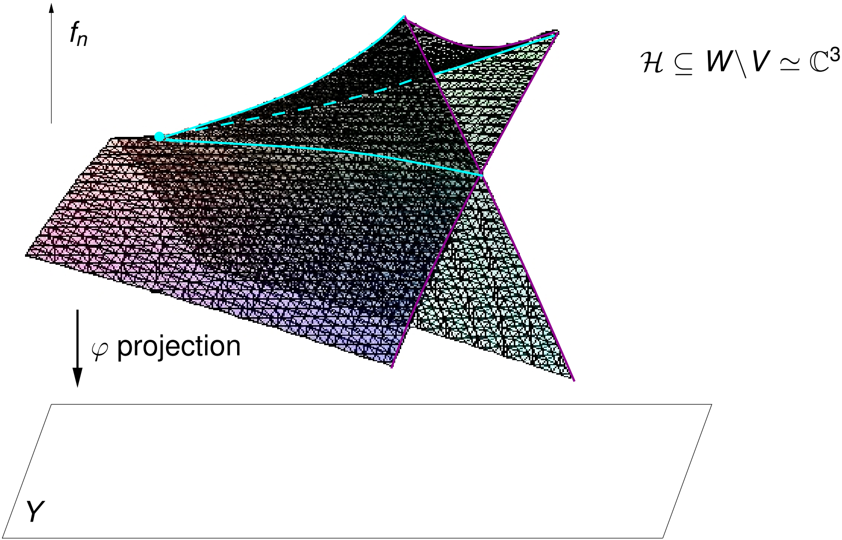

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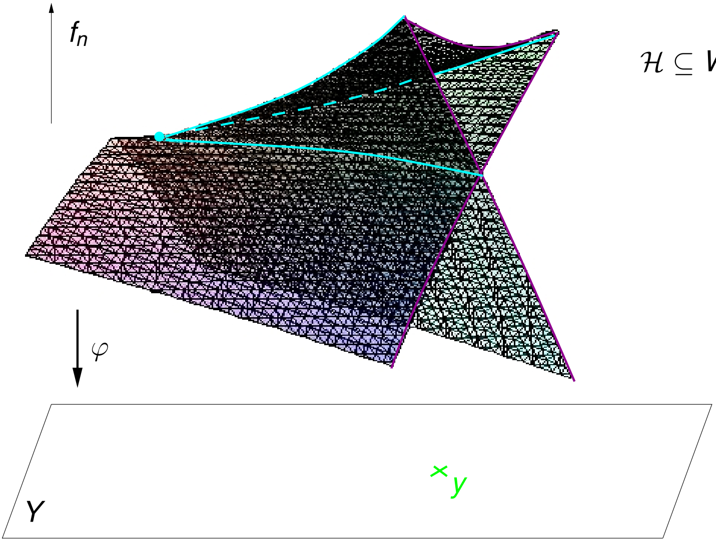

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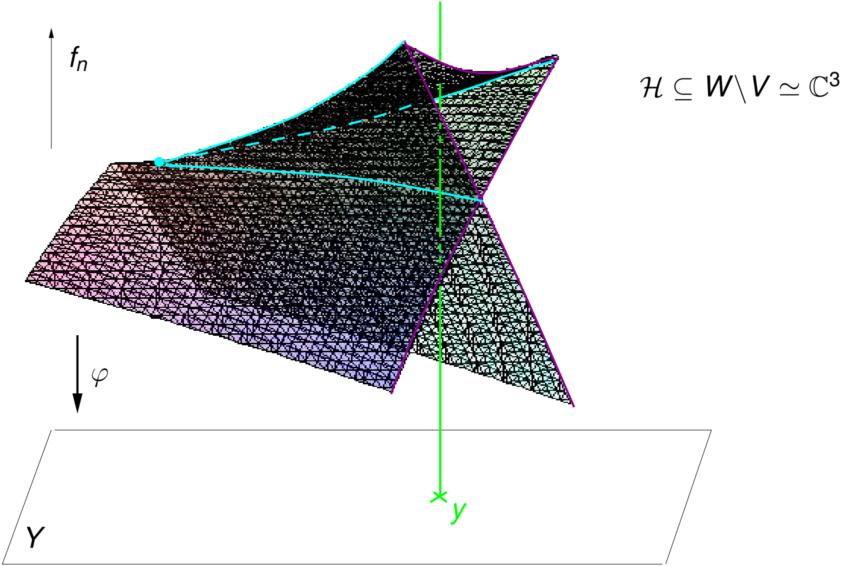

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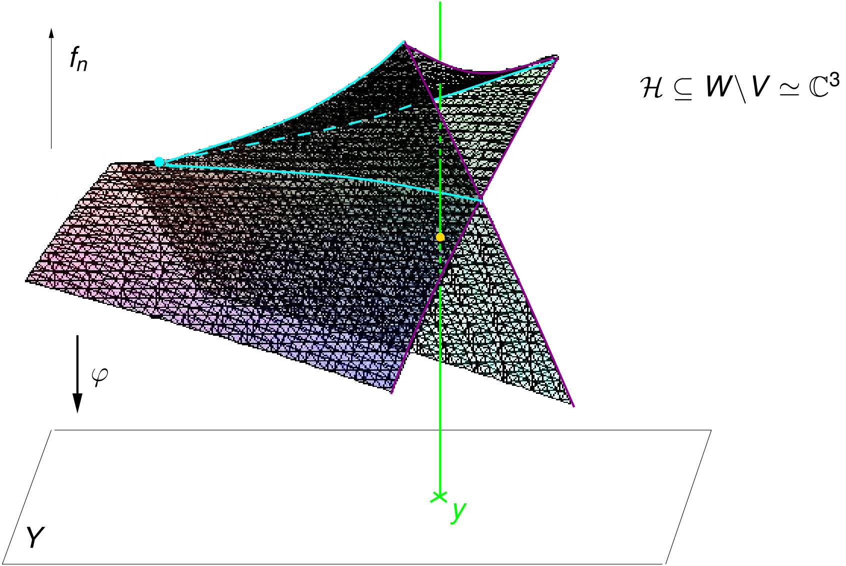

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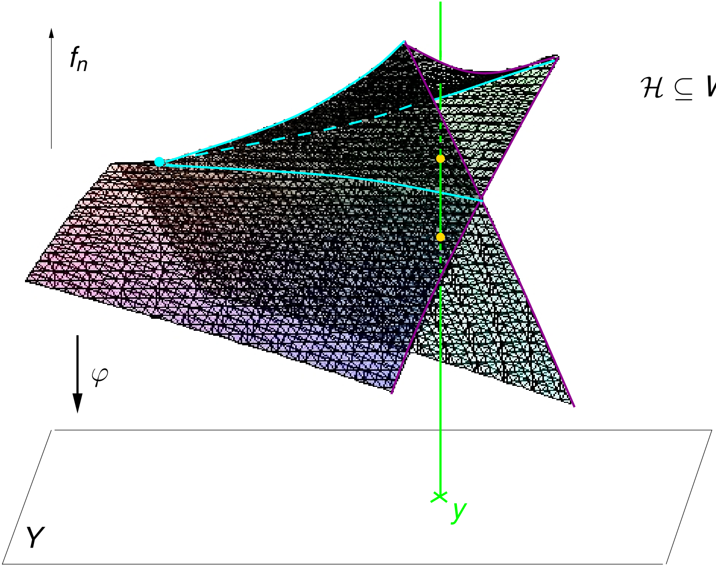

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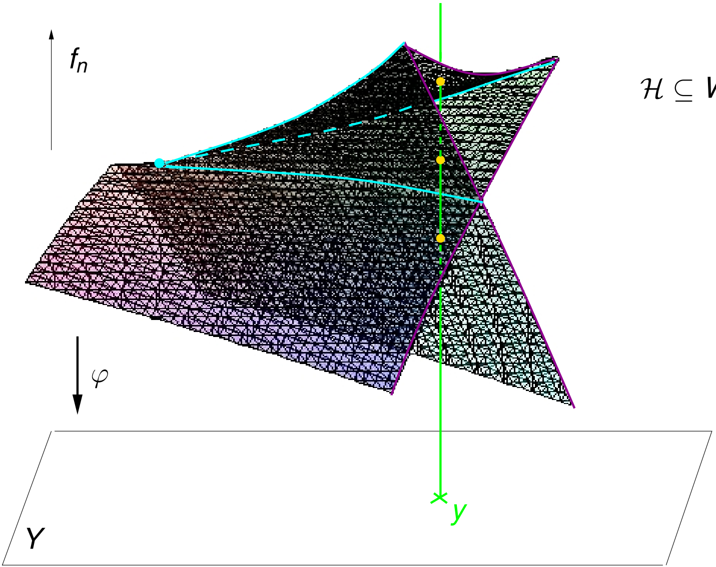

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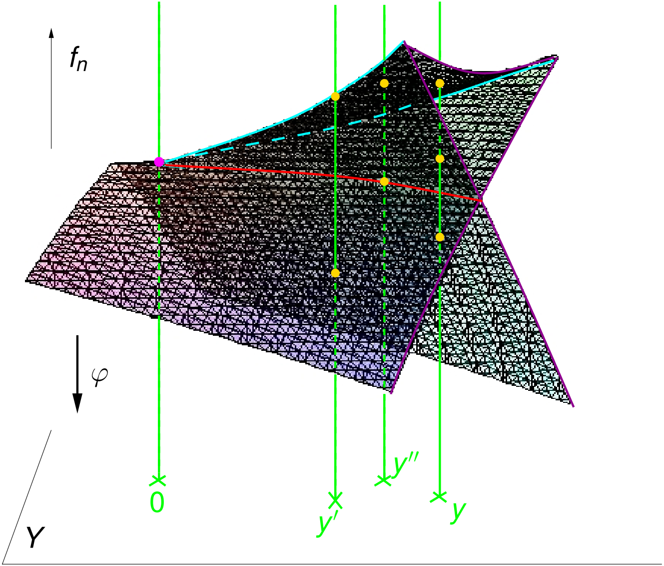

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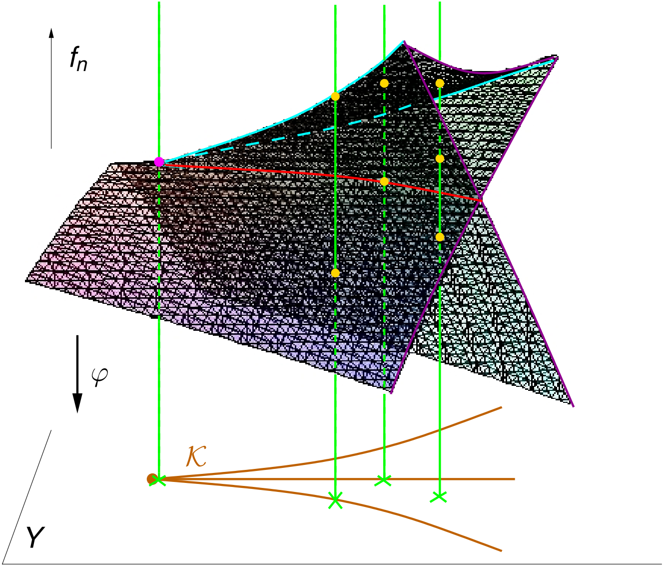

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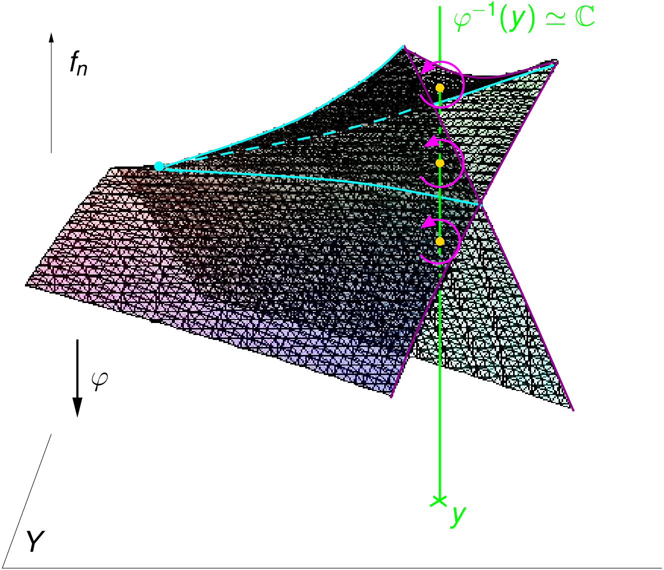

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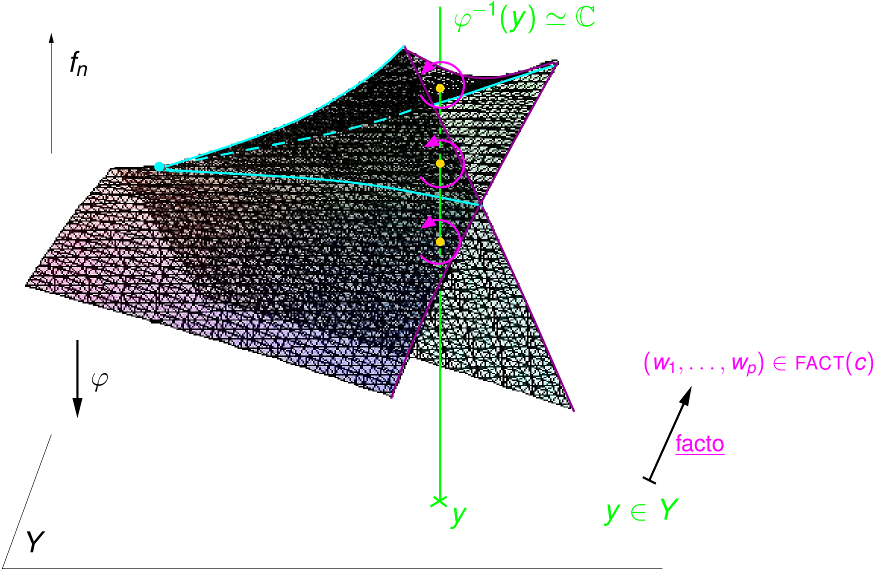

Geometric factorisations

$Y \simeq \mathbb{C}^{n-1}$ (with coordinates $f_1, \ldots, f_{n-1}$).

$\underline{\text{facto}} : y \in Y \mapsto (\gamma_1, \ldots, \gamma_p) \in B(W)^p \mapsto (w_1, \ldots, w_p) \in W^p$

(where $w_i = \pi(\gamma_i)$)

Facts:

- $\gamma_1 \ldots \gamma_p = \delta$ and $w_1 \ldots w_p = \pi(\delta) = c$.
- If $y \in Y - \mathcal{K}$, then $p = n$ and $w_1, \ldots, w_n$ are reflections.
- In general, $\ell_R(w_i)$ equals the multiplicity of the correponding point (y, x_i) in the discriminant.
- $\sum_i \ell_R(w_i) = n$, i.e., $\underline{\text{facto}}(y)$ is always a block factorisation of c.
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Stratification of V with the “flats” (**intersection lattice**):

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- A parabolic subgroup is a reflection group [Steinberg].
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$$\begin{array}{ccccc} L_0 \in \mathcal{L} & \leftrightarrow & W_0 \in \mathrm{PSG}(W) & \leftarrow & c_0 \text{ parabolic Coxeter elt} \\ \mathrm{codim}(L_0) & = & \mathrm{rk}(W_0) & = & \ell_R(c_0) \end{array}$$

Strata in $\mathcal{H}$

Construct a stratification of V/W, image of the stratification $\mathcal{L}$:
$\bar{\mathcal{L}} = \mathcal{L}/W = (\mathrm{p}(L))_{L\in\mathcal{L}} = (W\cdot L)_{L\in\mathcal{L}}$.

$\mathcal{L}$	$\leftrightarrow$	{parabolic subgroups of W}	$\leftrightarrow$	{parab. Coxete
$\mathrm{codim}(\Lambda)$	$=$	$\mathrm{rank}(W_\Lambda)$	$=$	$\ell_R(w$

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The set $\bar{\mathcal{L}}$ is in canonical bijection with:

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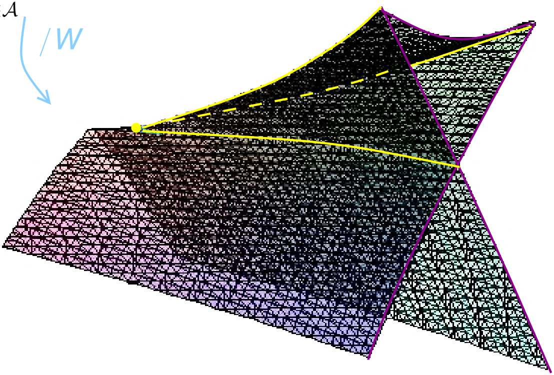
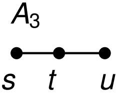
Example of $W = A_3$: stratification of the discriminant

$$\bigcup_{H\in\mathcal{A}} H \subseteq V$$

$\big\downarrow /W$





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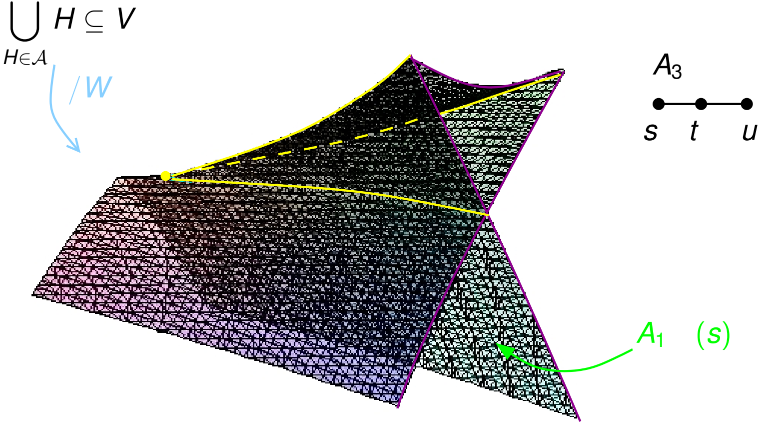

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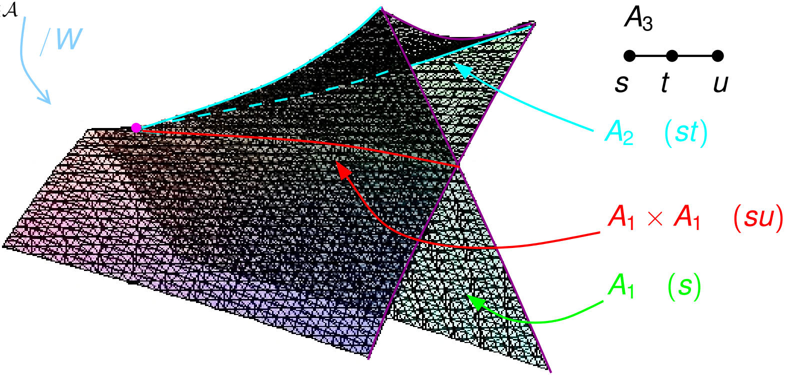

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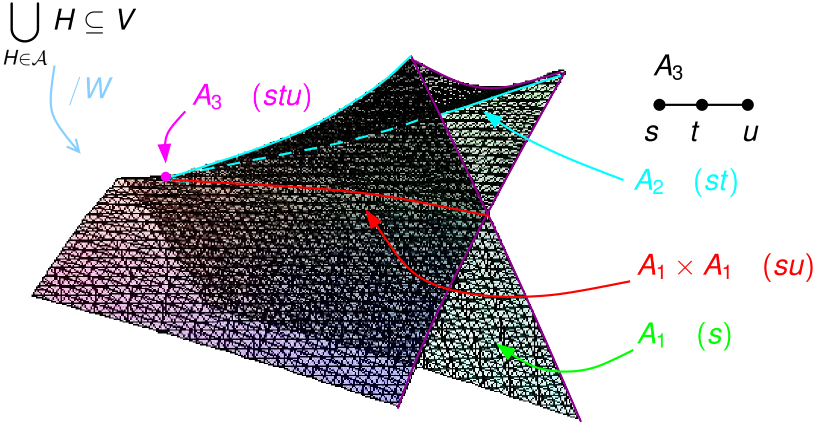

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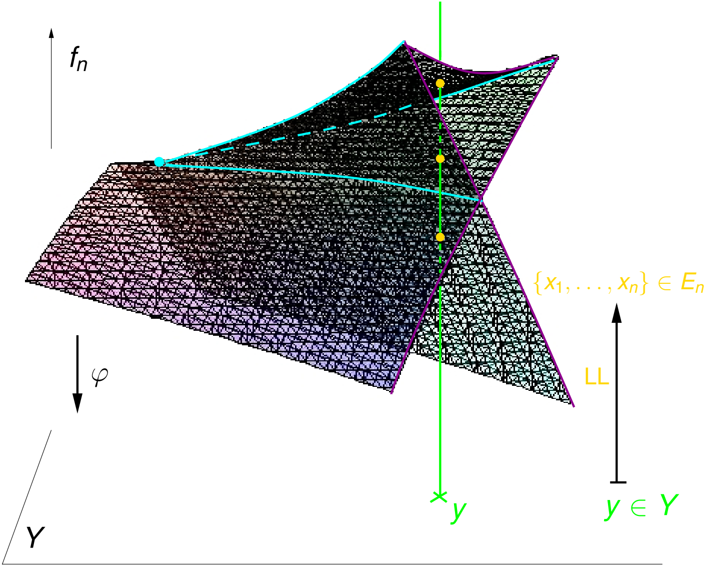
Conjugacy classes of factors

For any factor w_i in some $\underline{\text{facto}}(y)$:

- w_i is a parabolic Coxeter element;
- its conjugacy class corresponds (via bijection above) to the minimal stratum of $\bar{\mathcal{L}}$ in which lies the corresponding point (y, x_i).

Outline

Lyashko-Looijenga morphism



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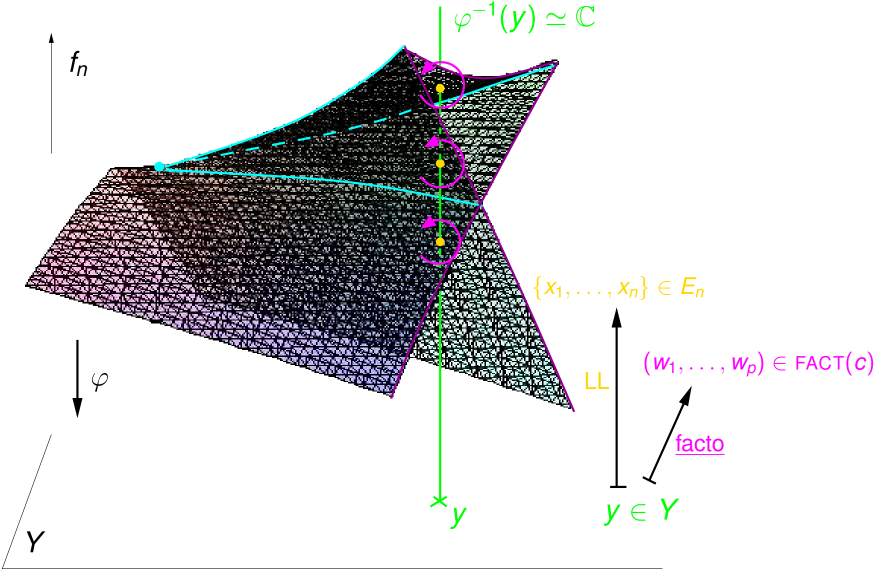

Properties of the Lyashko-Looijenga morphism

Definition

$$\begin{array}{rccl} \mathrm{LL}: & Y & \to & E_n := \{\text{multisets of } n \text{ points in } \mathbb{C}\} \\ & y & \mapsto & \{\text{roots, with multiplicities, of } \Delta_W(y, f_n) \text{ in } f_n\} \end{array}$$

$$\Delta_W = f_n^n + a_2 f_n^{n-2} + a_3 f_n^{n-3} + \cdots + a_{n-1} f_n + a_n.$$

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$\mathrm{LL} : Y - \mathcal{K} \twoheadrightarrow E_n^{\mathrm{reg}}$ *is a topological covering, of degree*

$$\frac{2h \cdot 3h \ldots nh}{d_1 \ldots d_{n-1}} = \frac{n!\, h^n}{|W|}$$

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Fibers of LL and block factorisations of c

Note: for $y \in Y$, $\mathrm{LL}(y)$ and $\underline{\mathrm{facto}}(y)$ have necessarily the same associated *composition of n*.

Theorem (Bessis '07)

The product map:

$$Y \xrightarrow{\mathrm{LL} \times \underline{\mathrm{facto}}} E_n \times \mathrm{FACT}(c)$$

is injective, and is a bijection onto the set of "compatible" pairs.

Equivalently, for $\omega \in E_n$, the map $\underline{\mathrm{facto}}$ induces a bijection between the fiber $\mathrm{LL}^{-1}(\omega)$ and the set of block factorisations of same "composition" as ω.

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Maximal factorisations of a Coxeter element

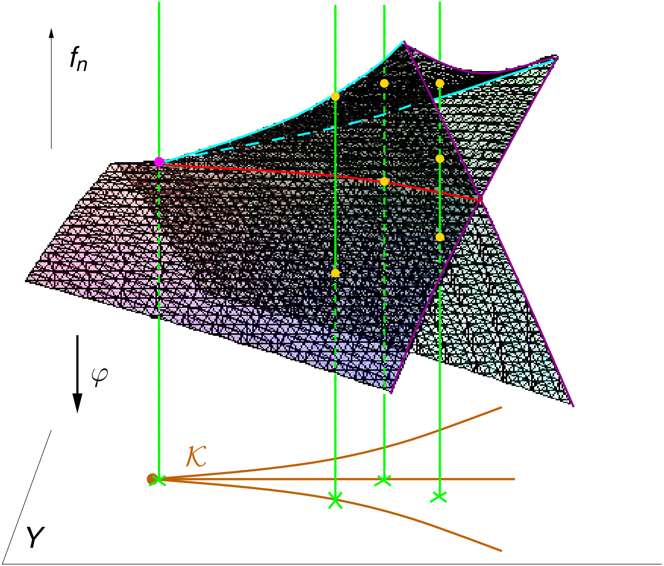
(a.k.a *reduced decompositions of c*)

Corollary

$|\,\mathrm{FACT}_n(c)|$ *equals the cardinality of a generic (regular) fiber of* LL, *i.e.*,

$$|\,\mathrm{FACT}_n(c)| = \frac{n!\,h^n}{|W|}$$

Strata of codimension 2



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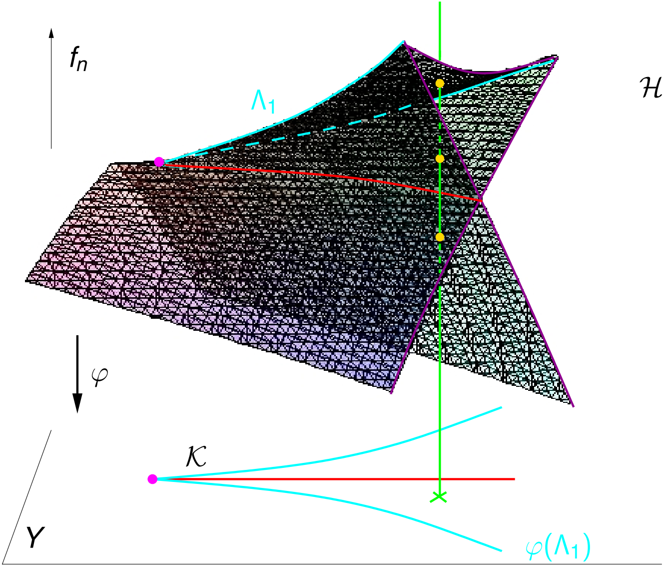

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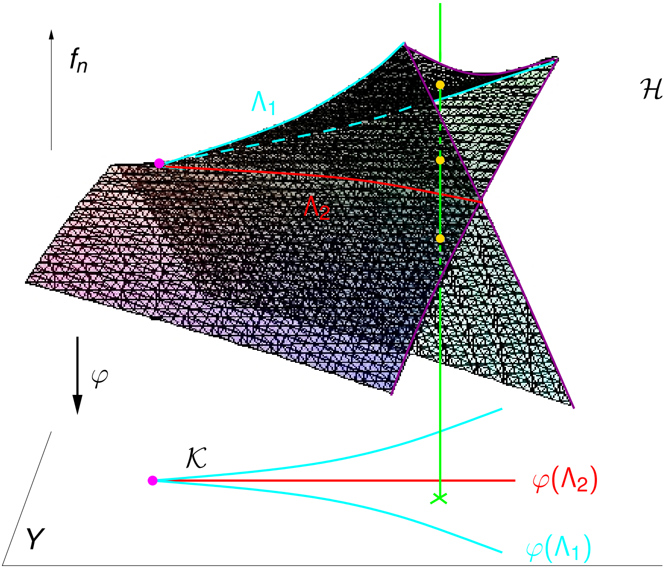

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Submaximal factorisations of type Λ

$$\bar{\mathcal{L}}_2 := \{\text{strata of } \bar{\mathcal{L}} \text{ of codimension } 2\}$$
$$\leftrightarrow \{\text{conjugacy classes of elements of } \mathrm{NC}(W) \text{ of length } 2\}$$

Proposition

The $\varphi(\Lambda)$, for $\Lambda \in \bar{\mathcal{L}}_2$, are the irreducible components of $\mathcal{K}$.

$\rightsquigarrow$ we can write $D_W = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$, where $r_\Lambda \geq 1$ and the D_Λ are polynomials in $f_1, \ldots, f_{n-1}$.

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For $\Lambda \in \bar{\mathcal{L}}_2$, the number of submaximal factorisations of c of type Λ (i.e. , whose unique length 2 element lies in the conjugacy class Λ) is:

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- Recall that $D_W = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{f_\Lambda}$.
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Danke schön!

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