# Limit points of root systems of infinite Coxeter groups

Vivien RIPOLL

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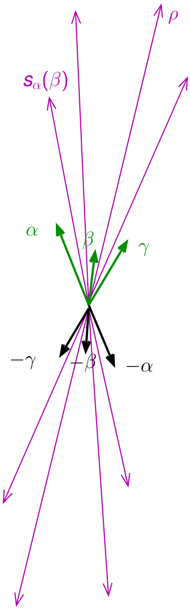
From joint works with

- **Matthew Dyer** (University of Notre Dame)
- **Christophe Hohlweg** (UQÀM)
- **Jean-Philippe Labbé** (FU Berlin)

# Overview




- root system $\Phi$ : set of vectors encoding the reflections of a Coxeter group
- General property : $\Phi = \Phi^+ \sqcup (-\Phi^+)$, where $\Phi^+ \subseteq \text{cone}(\Delta)$, $\Delta$ simple roots.
- Get a projective version of $\Phi$ by constructing normalized roots in a cutting hyperplane $H$.
- draw examples, get amazing pictures, try to understand

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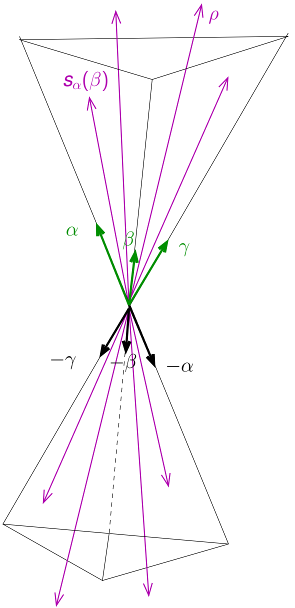


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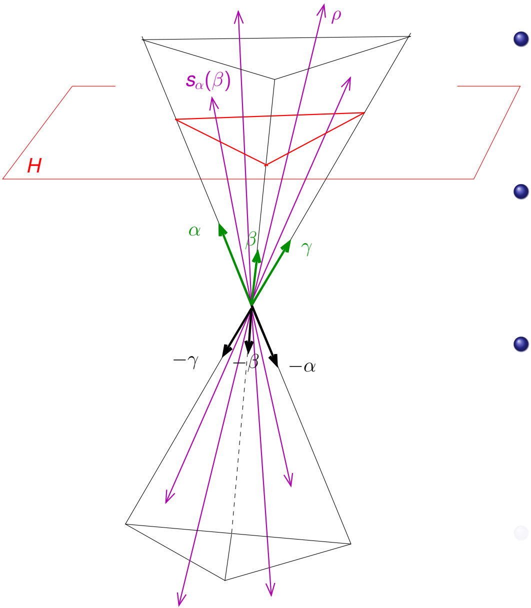


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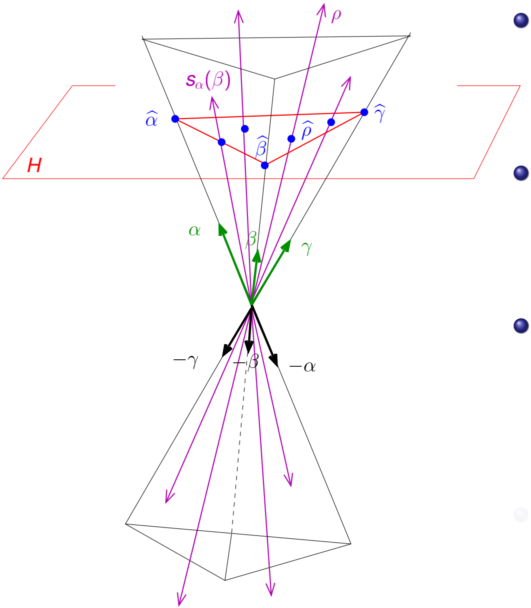


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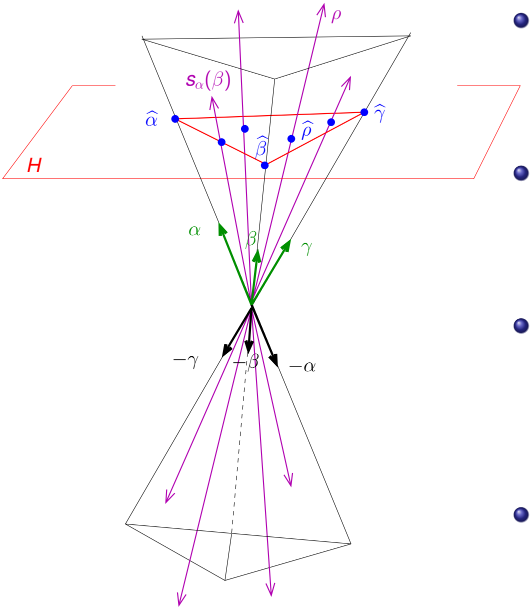


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1. Root system, limit roots and isotropic cone
2. Action of $W$ on the limit roots : faithfulness, density of the orbits
3. Fractal description of the limit roots, and the hyperbolic case

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# A “dynamical” construction of a root system

- $V$: a real vector space, of finite dimension $n$
- $B$: a symmetric bilinear form on $V$

Construction of a root system in $(V, B)$:

1. Start with a simple system $\Delta$:
   - $\Delta$ is a basis for $V$;
   - $\forall \alpha \in \Delta, B(\alpha, \alpha) = 1$;
   - $\forall \alpha \neq \beta \in \Delta$:
     - either $B(\alpha, \beta) = -\cos\left(\frac{\pi}{m}\right)$ for some $m \in \mathbb{Z}_{\geq 2}$,
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# A “dynamical” construction of a root system

2. For each $\alpha \in \Delta$, define the *B*-reflection $s_\alpha$:

$$s_\alpha : \begin{array}{ccc} V & \to & V \\ v & \mapsto & v - 2B(\alpha, v)\,\alpha. \end{array}$$

Check: $s_\alpha(\alpha) = -\alpha$, and $s_\alpha$ fixes pointwise $\alpha^\perp$.

Notation: $S = \{s_\alpha,\ \alpha \in \Delta\}$.

3. Construct the *B*-reflection group $W := \langle S \rangle$.

4. Act by $W$ on $\Delta$ to construct the based root system

$$\Phi := W(\Delta)\ .$$

**Note:** if $\rho = w(\alpha)$ (with $\alpha \in \Delta$), $ws_\alpha w^{-1}$ is the *B*-reflection associated to the root $\rho$.

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# Coxeter group and root system

## Proposition

- $(W, S)$ *is a* *Coxeter system*, *with Coxeter presentation:*

$$W = \left\langle S \;\middle|\; s^2 = 1 \ (\forall s \in S);\ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \right\rangle,$$

*where* $m_{s_\alpha, s_\beta} = \begin{cases} m & \text{if } B(\alpha,\beta) = -\cos(\pi/m), \\ \infty & \text{if } B(\alpha,\beta) \leq -1. \end{cases}$

- *Let* $\Phi^+ := \Phi \cap \mathrm{cone}(\Delta)$. *Then:* $\Phi = \Phi^+ \sqcup (-\Phi^+)$.

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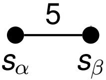
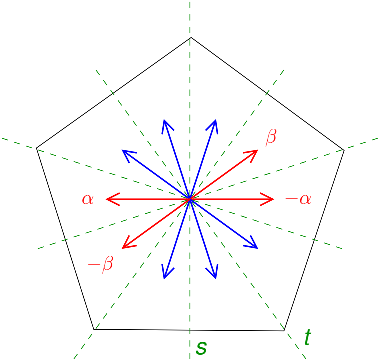
# Infinite root systems

For finite root systems:
$\Phi$ is finite $\Leftrightarrow$ $W$ is finite
($\Leftrightarrow$ $B$ is positive definite).

**Example:** $W = I_2(5)$,

$|\Phi| = 10$, 


What does an infinite root system look like?

Simplest example, in rank 2:




Matrix of $B$ in the basis $(\alpha, \beta)$: $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

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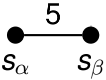
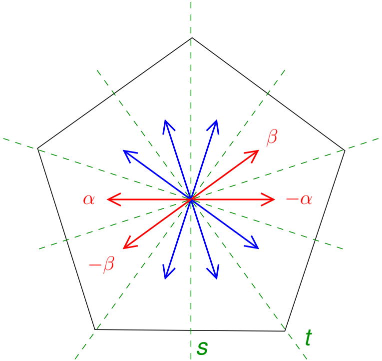
$|\Phi| = 10$, 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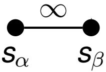
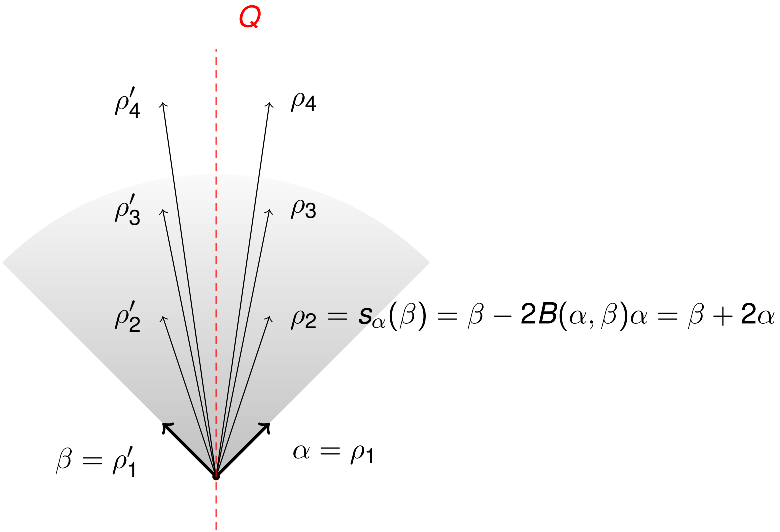



Matrix of $B$ in the basis $(\alpha, \beta)$: $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

# Infinite dihedral group, case $B(\alpha, \beta) = -1$




$$\rho'_n = n\alpha + (n+1)\beta \quad ; \quad \rho_n = (n+1)\alpha + n\beta$$

# Observations

- The **norms** of the roots tend to $\infty$;
- The **directions** of the roots tend to the direction of the isotropic cone $Q$ of $B$:

$$Q := \{v \in V,\ B(v,v) = 0\}.$$

(in the example the equation is $v_\alpha^2 + v_\beta^2 - 2v_\alpha v_\beta = 0$, and $Q = \text{span}(\alpha + \beta)$.)

# What if $B(\alpha, \beta) < -1$?

- Matrix of $B$: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write $s_\alpha \overset{\infty(\kappa)}{\text{———}} s_\beta$
- Then $Q$ is the union of 2 lines.

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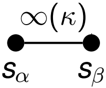
- Matrix of $B$: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 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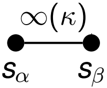
- Matrix of $B$: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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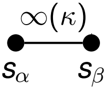
- Matrix of $B$: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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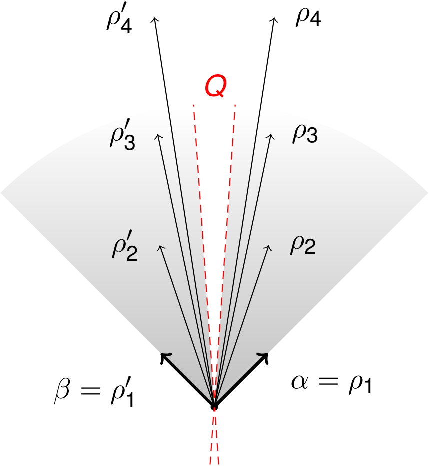
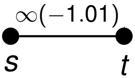
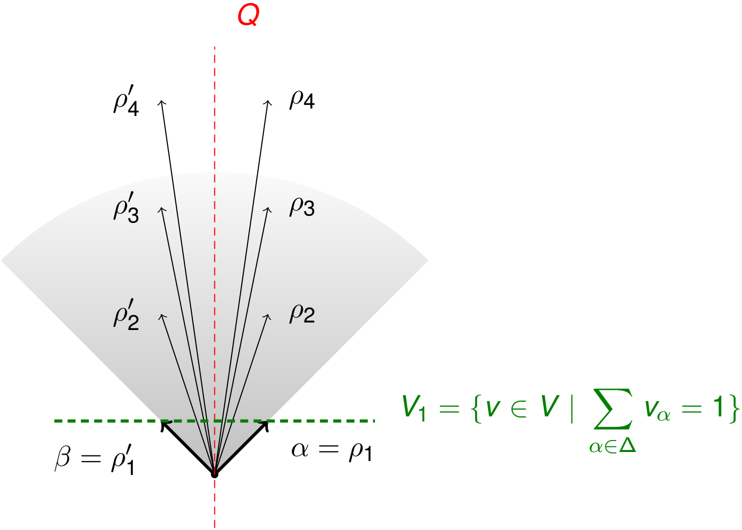


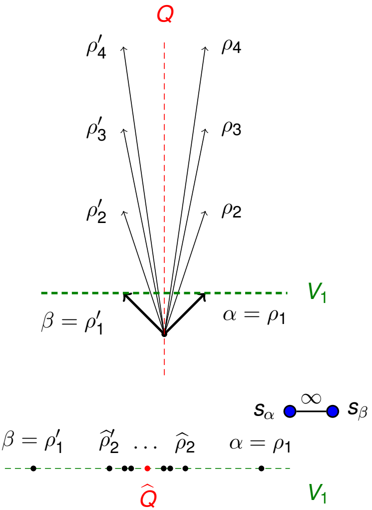



# “Normalization” of the roots

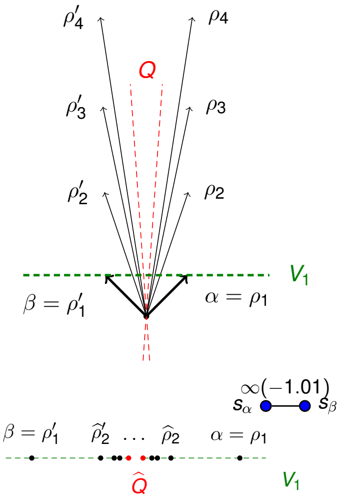
Cut the directions of the roots with an affine hyperplane
$\rightsquigarrow$ get a picture for the projective version of $\Phi$.



# Normalized roots in rank 2




case $B(\alpha, \beta) = -1$




case $B(\alpha, \beta) = -1.01 < -1$

# Limit roots and isotropic cone

**Theorem (Hohlweg-Labbé-R. '11)**

*Let $\Phi$ be an infinite root system, $Q$ its isotropic cone, and $(\rho_n)_{n\in\mathbb{N}}$ an injective sequence in $\Phi$. Then:*

- $||\rho_n||$ *tends to* $\infty$ *(for any norm on* $V$*);*
- *if the sequence of normalized root* $(\widehat{\rho_n})_{n\in\mathbb{N}}$ *has a limit* $\ell$*, then*

$$\ell \in \widehat{Q} \cap \operatorname{conv}(\Delta).$$

See also:

- [Kac 90] for Weyl groups of Kac-Moody algebras,
- generalized by [Dyer '12] (work on the imaginary cone of a Coxeter group).

$\rightsquigarrow$ **Problem:** understand the set of possible limits, i.e., the accumulation points of $\widehat{\Phi}$:

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# Zoology of root systems and limit roots

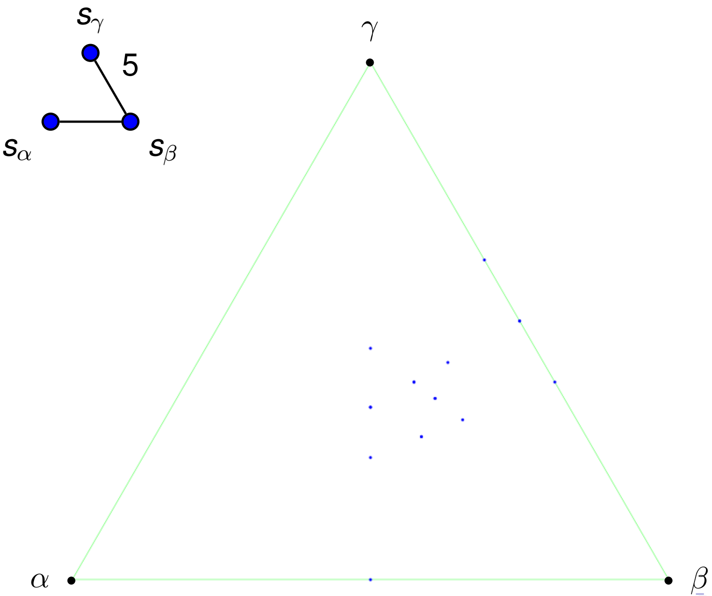
- $\Phi$ **finite** ($W$ finite Coxeter group) :
  $B$ positive definite, $\widehat{Q} = \varnothing$, $E = \varnothing$.

- $\Phi$ **of affine type** :
  $B$ positive. Actually $\operatorname{sgn} B = (n-1, 0)$ if $\Phi$ irreducible.
  $\widehat{Q}$ is a singleton, $E = \widehat{Q}$.

- **otherwise: $\Phi$ of indefinite type**
  - particular case : **weakly hyperbolic** type,
    $\operatorname{sgn} B = (n-1, 1)$.
    $\widehat{Q}$ is a sphere (if we choose well the cutting hyperplane).
    $E$ is pretty and well understood.

  - other cases : still work to do!

# Examples in rank 3: finite group, sgn $B = (3, 0)$. $(H_3)$



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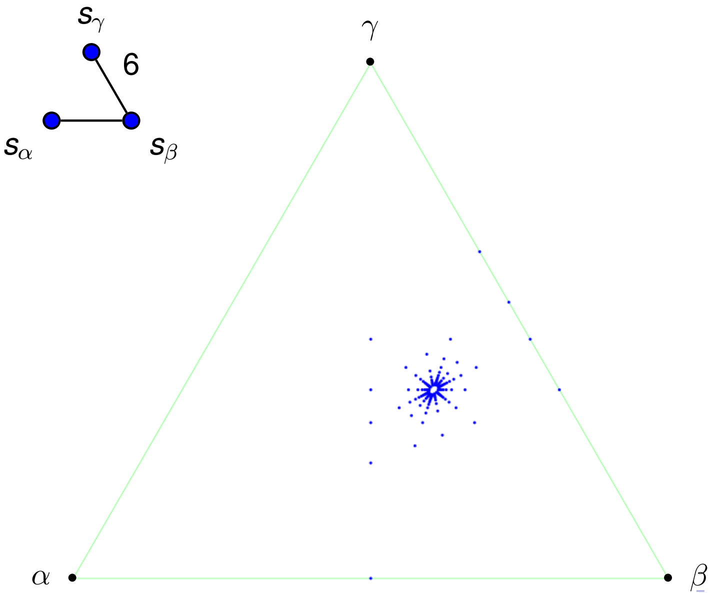

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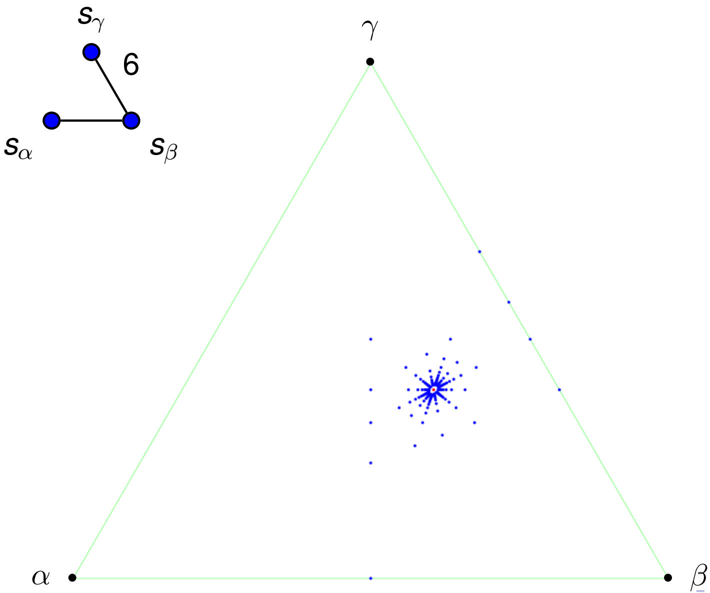

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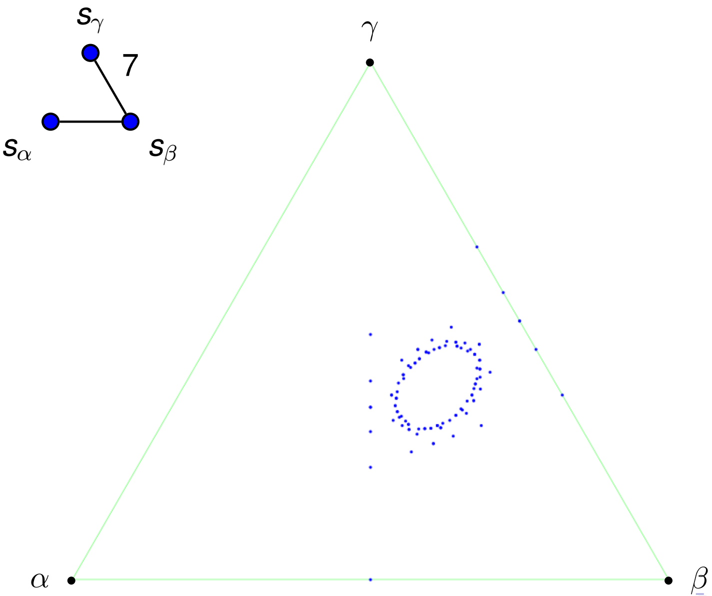

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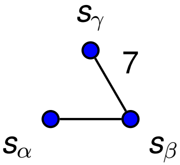


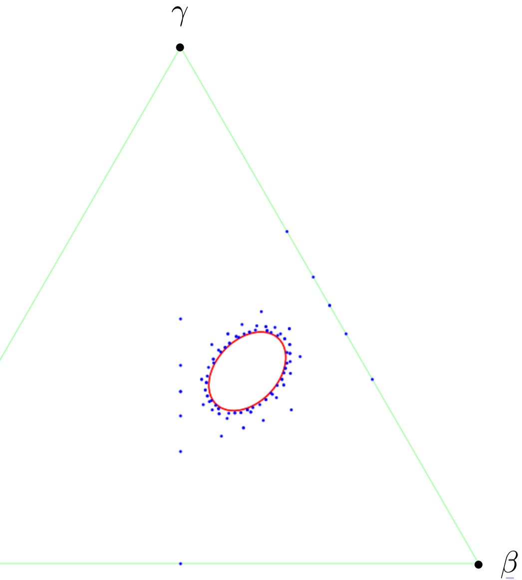

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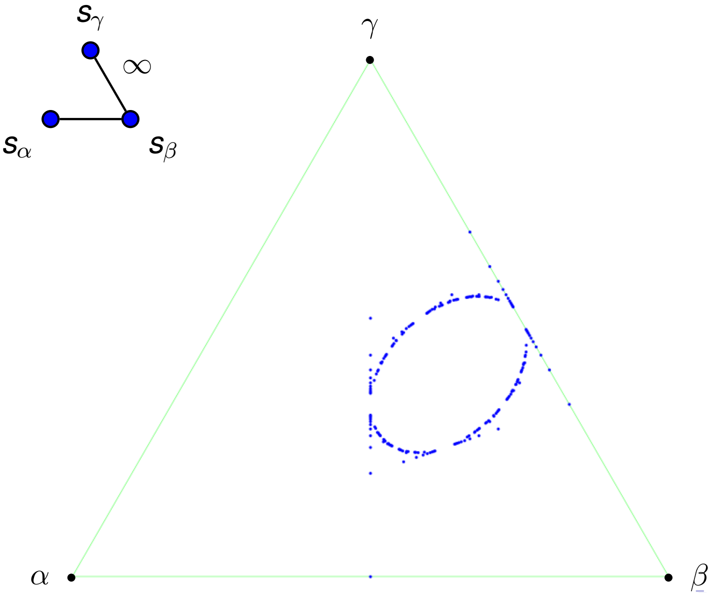

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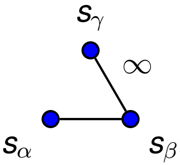


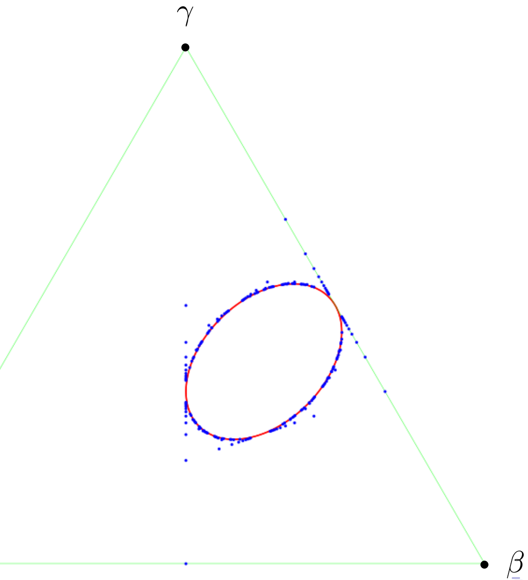

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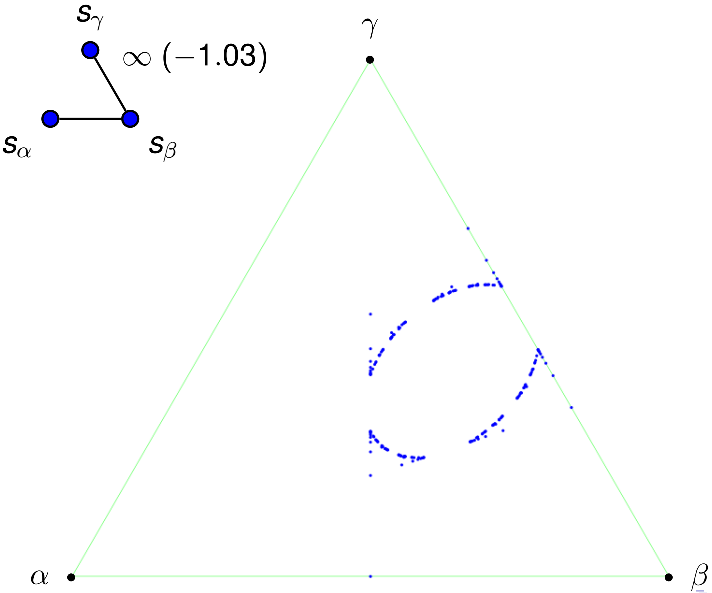

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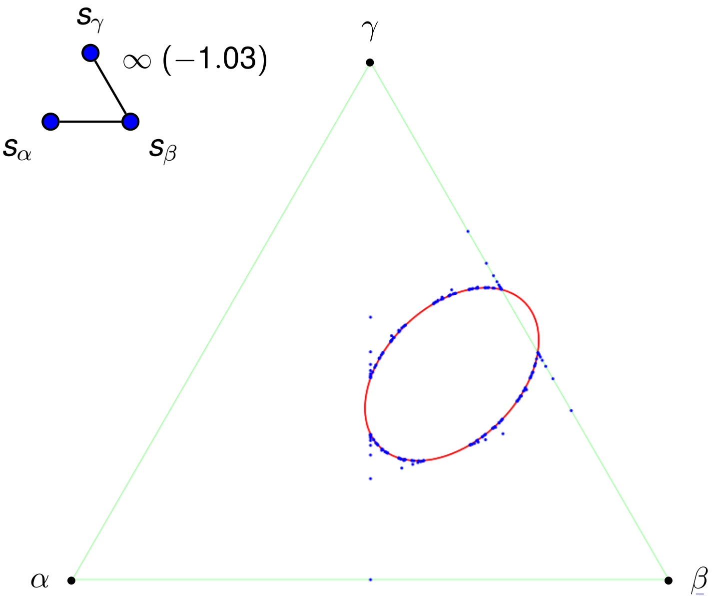

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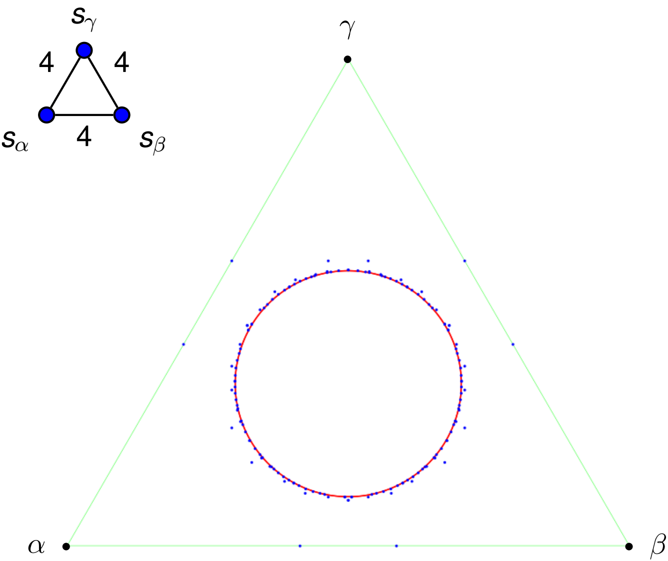

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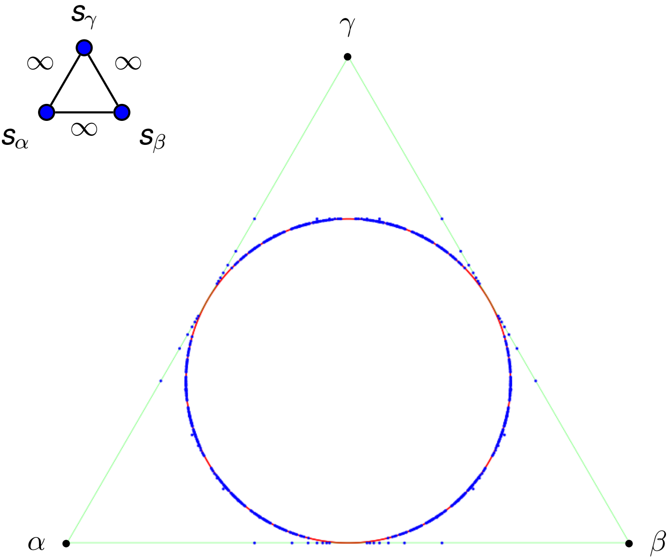

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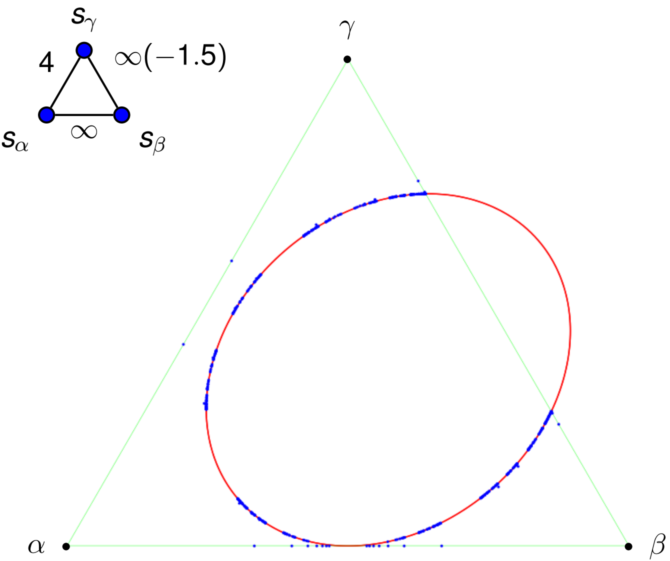
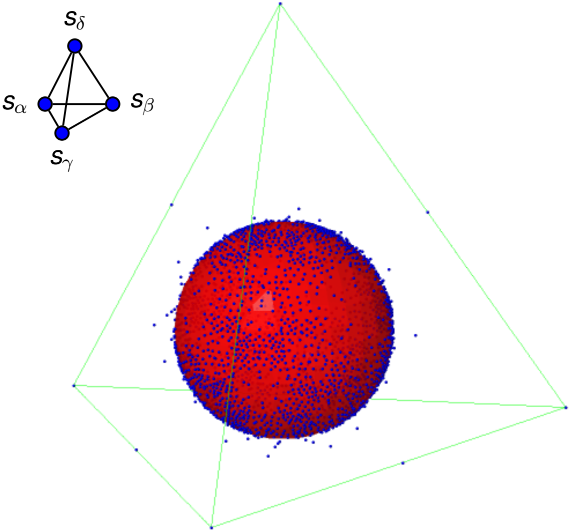


# Examples in rank 4



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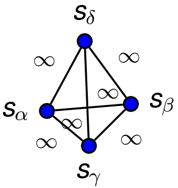
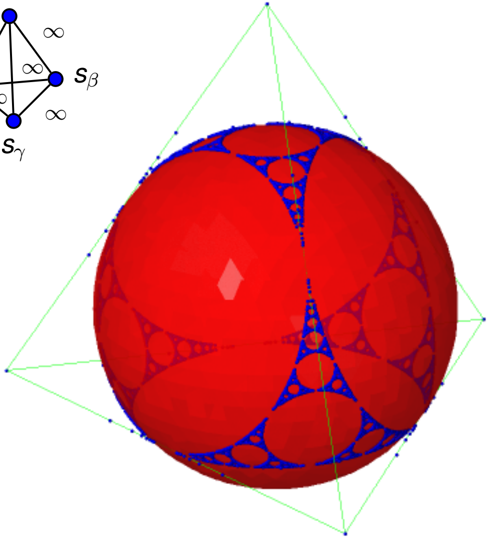
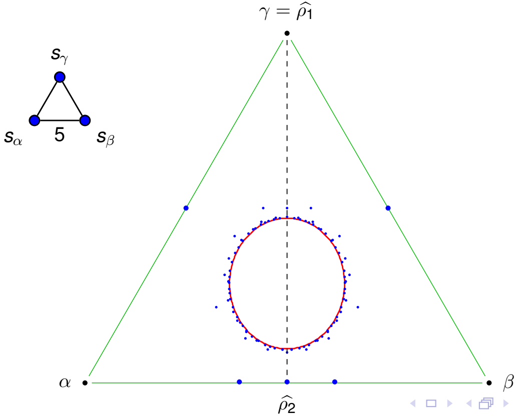


# Dihedral limit roots

Fix 2 roots $\rho_1, \rho_2$ in $\Phi^+$ $\rightsquigarrow$ get a reflection subgroup of rank 2 of $W$, and a root subsystem $\Phi'$.

- $\widehat{\Phi}'$ lives in the line $L(\widehat{\rho_1}, \widehat{\rho_2})$ ;
- the isotropic cone of $\Phi'$ is $Q \cap \mathrm{Vect}(\rho_1, \rho_2)$ ;
- $\rightsquigarrow$ we can construct limit roots of $\Phi'$ : $E(\Phi') = Q \cap L(\widehat{\rho_1}, \widehat{\rho_2})$ (0,1 or 2 points).



# Outline

1. Root system, limit roots and isotropic cone
2. Action of $W$ on the limit roots : faithfulness, density of the orbits
3. Fractal description of the limit roots, and the hyperbolic case

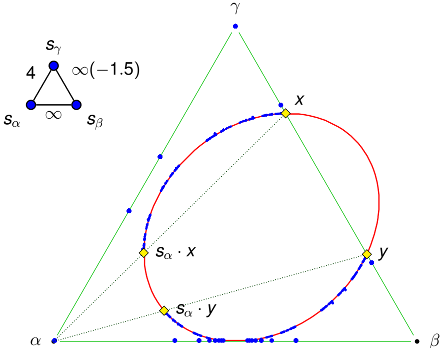
# A natural group action of $W$ on $E$

Geometric action of $W$ on a part of $V_1$: $w \cdot v := \widehat{w(v)}$.

Defined on $D = V_1 \cap \bigcap_{w \in W} w(V \setminus V_0)$, where $V_0 = \overrightarrow{V_1}$.

## Proposition

- $E(\Phi) \subseteq D$ *and* $E(\Phi)$ *is stable under the action of* $W$.
- *For* $\alpha \in \Phi$ *and* $x \in E$, $\widehat{Q} \cap L(\widehat{\alpha}, x) = \{x, s_\alpha \cdot x\}$.



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*If $W$ is infinite, non-affine and irreducible, then the action of $W$ on $E$ is faithful.*

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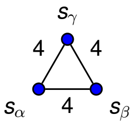
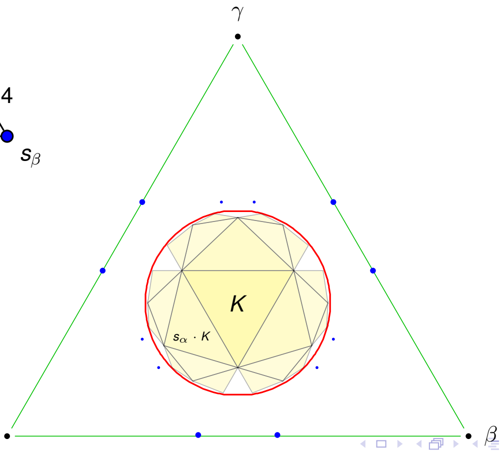
## Definition (Kac, Hée, Dyer...)

- $\mathcal{K} := \{v \in \text{cone}(\Delta) \mid \forall \alpha \in \Delta, B(\alpha, v) \leq 0\}$
- the imaginary cone $\mathcal{Z}$ of $\Phi$ is the $W$-orbit of $\mathcal{K}$ : $\mathcal{Z} := W(\mathcal{K})$.



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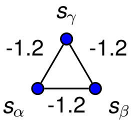


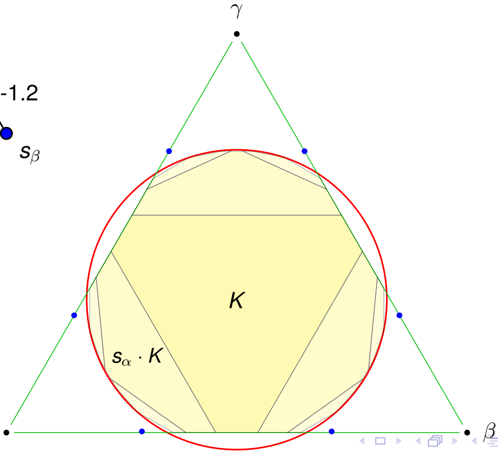

# Minimality of the action

## Relation limit roots/imaginary cone, [Dyer]

Let $Z$ be the normalized isotropic cone $\mathcal{Z} \cap V_1$.
Then : $\overline{Z} = \mathrm{conv}(E)$.

## Theorem (Dyer-Hohlweg-R. '12)

*If $W$ is irreducible infinite, then the action of $W$ on $E$ is minimal, i.e., for all $x \in E$, the orbit of $x$ under the action of $W$ is dense in $E$:*

$$\overline{W \cdot x} = E.$$

The proof uses:

- the properties of the action on $\overline{Z} = \mathrm{conv}(E)$ [Dyer]: if $W$ is irreducible infinite, then

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# The hyperbolic case

$\Phi$ is hyperbolic if:

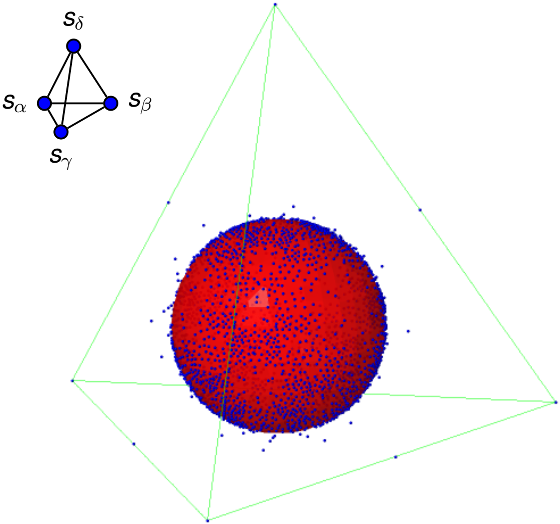
- $\operatorname{sgn} B = (n-1, 1)$ **and**
- every proper parabolic subgroup of $W$ is finite or affine

**Theorem (Dyer-Hohlweg-R.)**

*Let $\Phi$ be irreducible of indefinite type. Then:*

$$\Phi \textit{ is } \textbf{\textit{hyperbolic}} \iff \widehat{Q} \subseteq \operatorname{conv}(\widehat{\Delta}) \iff E(\Phi) = \widehat{Q}.$$

# A hyperbolic example



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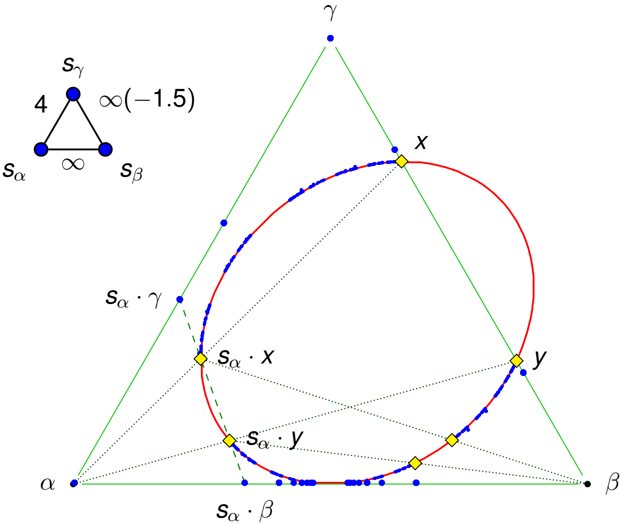

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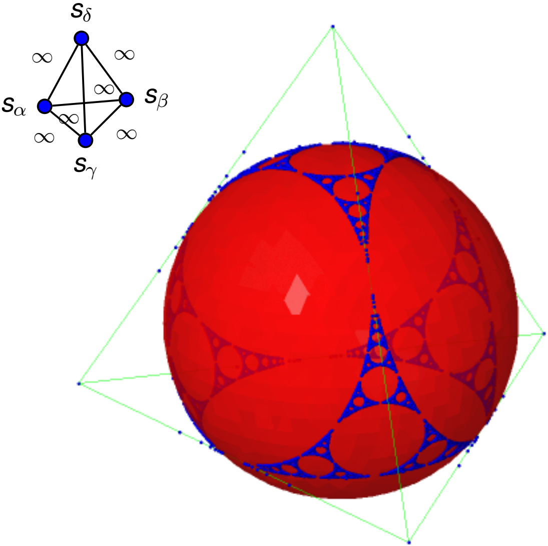

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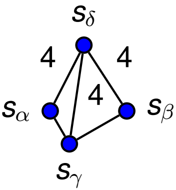


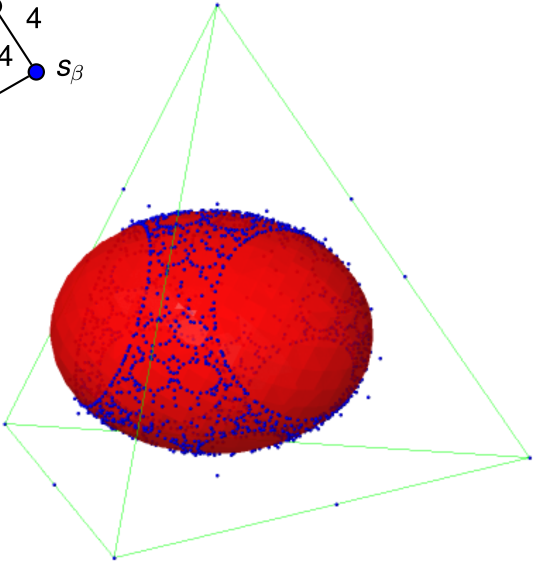

# Describe $E$ directly?

## Conjecture

*If $W$ is irreducible, then $E(\Phi) = \widehat{Q} \setminus$ all the images by $W$ of the parts of $\widehat{Q}$ which are outside* $\operatorname{conv}(\Delta)$, *i.e. :*

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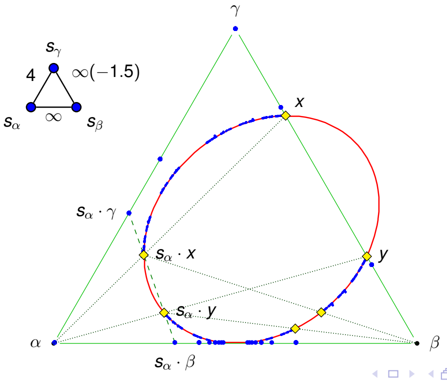

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$$\text{conjecture} \iff E = \mathrm{conv}(E) \cap \widehat{Q}.$$

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# Other questions

- How does $E$ behave in regards to restriction to parabolic subgroups? Take $I \subseteq \Delta$, $W_I$ its associated parabolic subgroup, $\Phi_I = W_I(\Delta_I)$, and $V_I = \mathrm{Vect}(I) \cap V_1$. Then $E(\Phi_I) \neq E(\Phi) \cap V_I$ in general! (counterexample in rank 5). But this type of property of good restriction works for other "natural" subsets of $E$...
- Explicit construction of converging sequences, links with the dominance order on $\Phi$.
- What can be said about the dynamics of the projective action of $W$ on the whole space $V$ (not only $\Phi$, $E$ and $Z$) ?

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The normalized imaginary cone conv($E$) (an artist's impression)