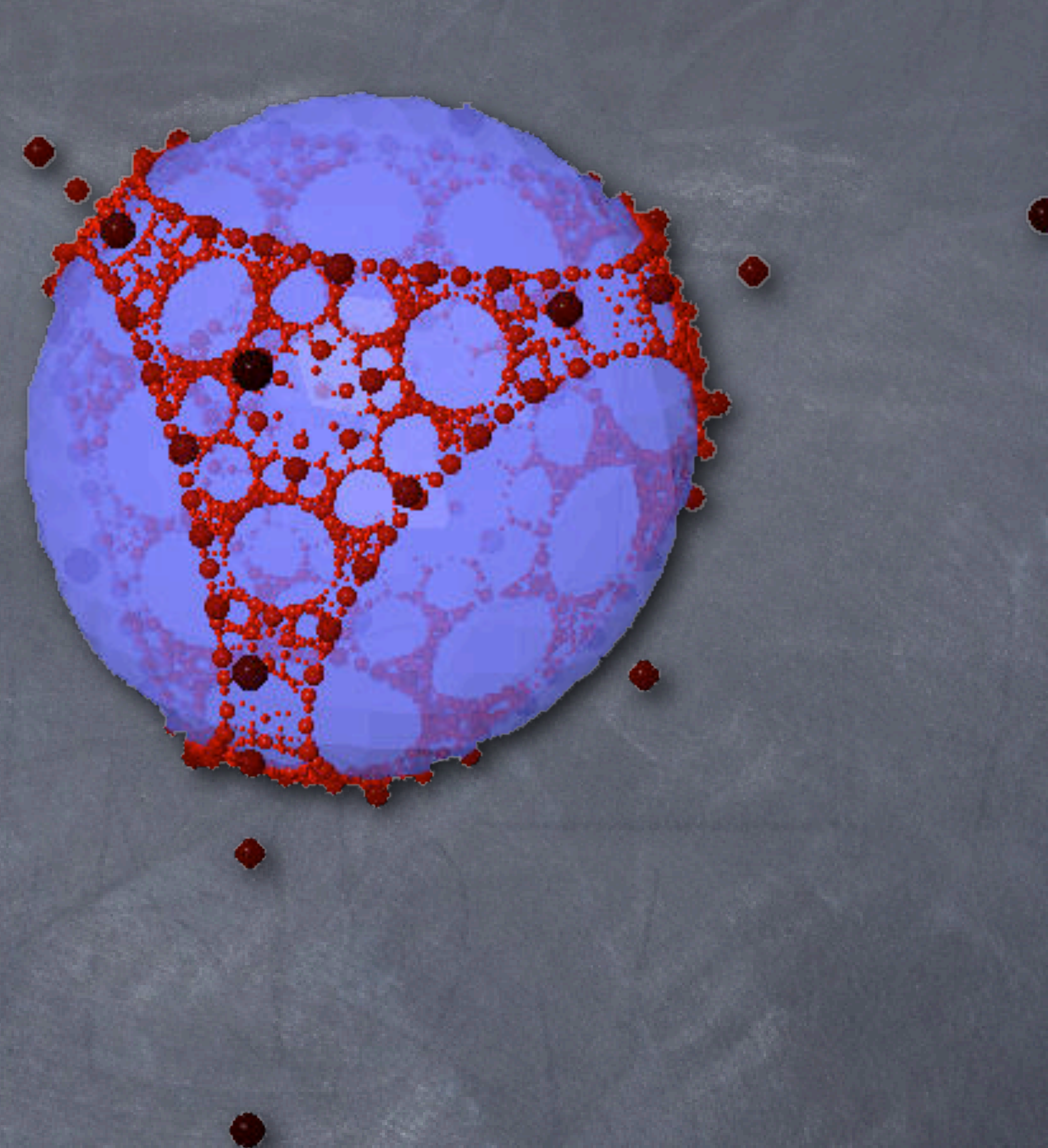
Ottawa-Carleton
Algebra Seminar
March 21st 2012

Asymptotical behaviour of roots in infinite Coxeter groups



joint works with:
Matthew **Dyer** (Notre-Dame),
Christophe **Hohlweg** (LaCIM, UQÀM),
Jean-Philippe **Labbé** (FU Berlin).

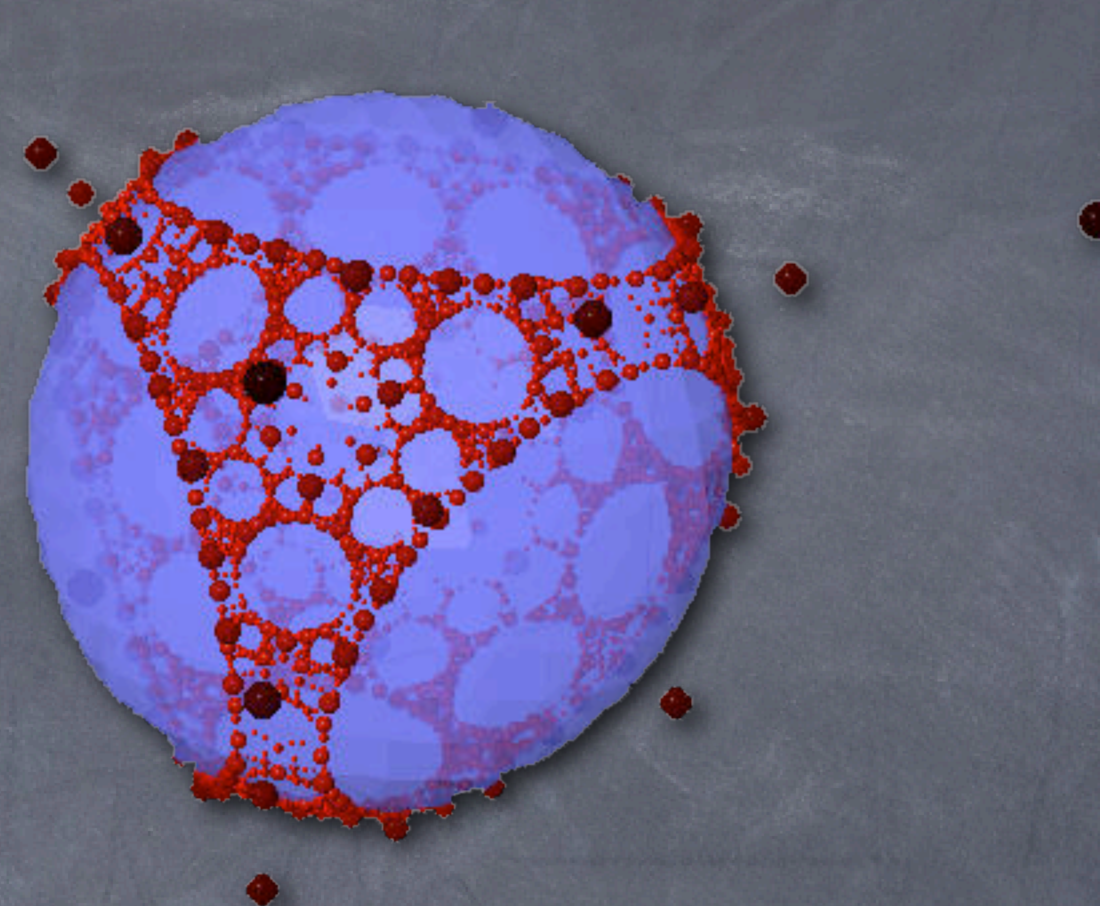
Vivien **Ripoll**
LaCIM, UQÀM
(Montréal, Canada)

What do we see?

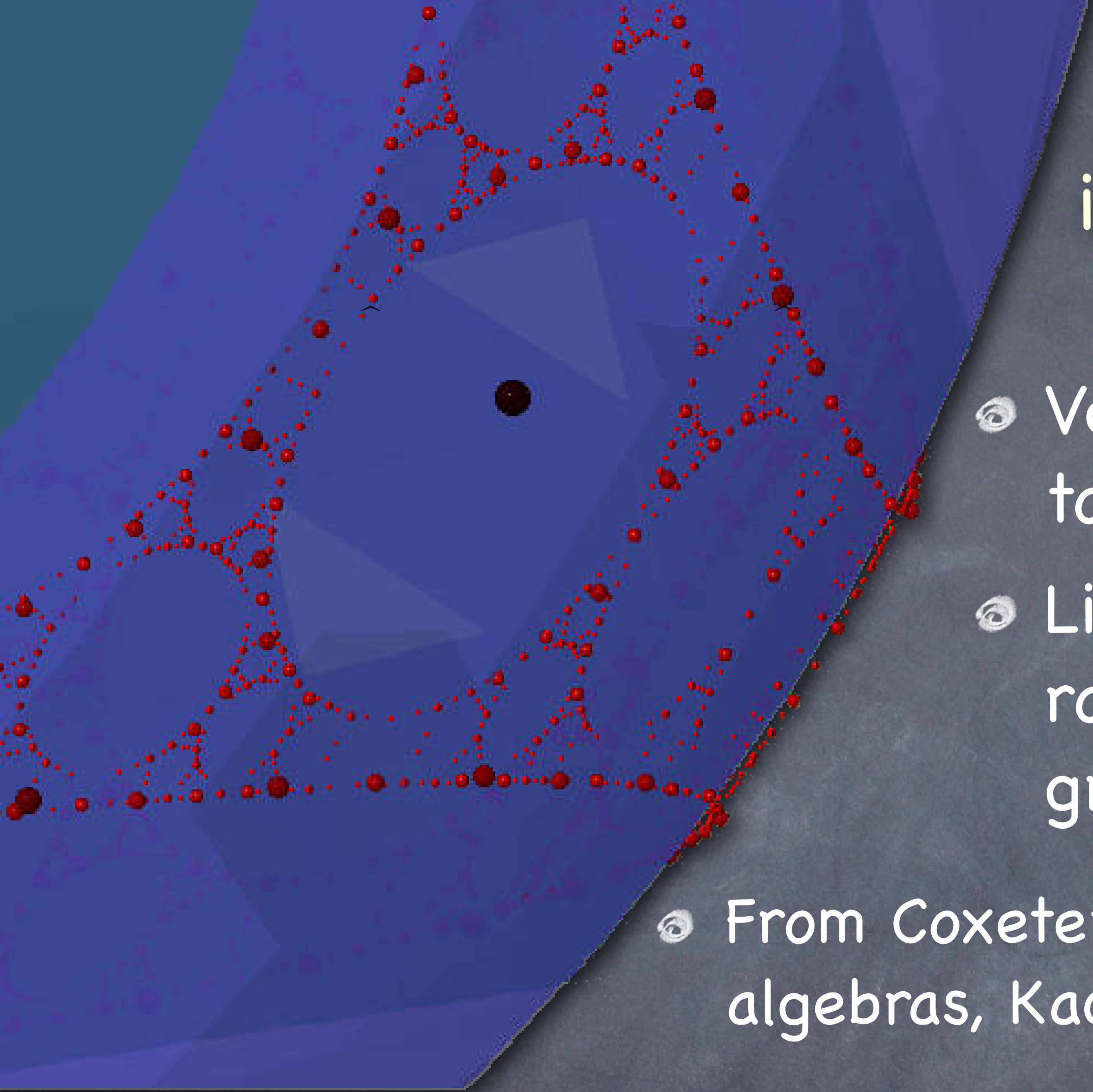
An affine picture built with

of the following:

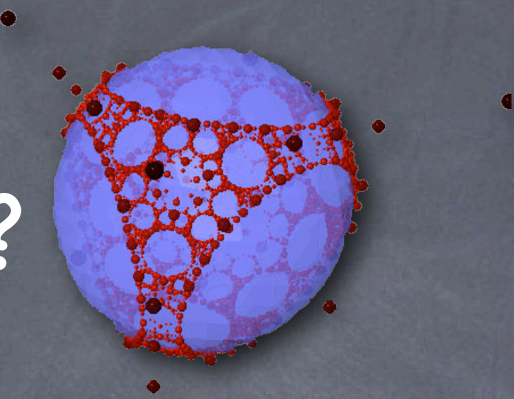
- (in blue) the isotropic cone of a symmetric bilinear form B on a vector space V;
- (in red) the first few thousands roots of an infinite root system related to B.



Motivation: to understand how roots are distributed over the space



Why study infinite root systems?



- Very useful and powerful tool to study Coxeter groups;
- Little is known for non affine root systems of infinite Coxeter groups (see Brink-Howlett, Dyer);
- From Coxeter groups to other structures (e.g. Lie algebras, Kac-Moody algebras, cluster algebras).

- Original motivation of this work: weak order and convexity of subsets of roots, to extend Reading's Cambrian fan.
- And because the pictures we obtain are nice and intriguing ...

What is a root system ?

Basic construction:

- V f.d. vector space, B a symmetric bilinear form
- The simple roots: start with a set Δ of vectors in V (usually a basis), such that $\forall\alpha\in\Delta, B(\alpha,\alpha)=1.$
- For each $\alpha\in\Delta$, define a B-reflection s_α

$$s_\alpha(v)=v-2B(v,\alpha)\alpha$$

- Construct the B-reflection group

$$W=\langle s_\alpha\,|\,\alpha\in\Delta\rangle\subseteq \mathrm{O}_B(V)$$

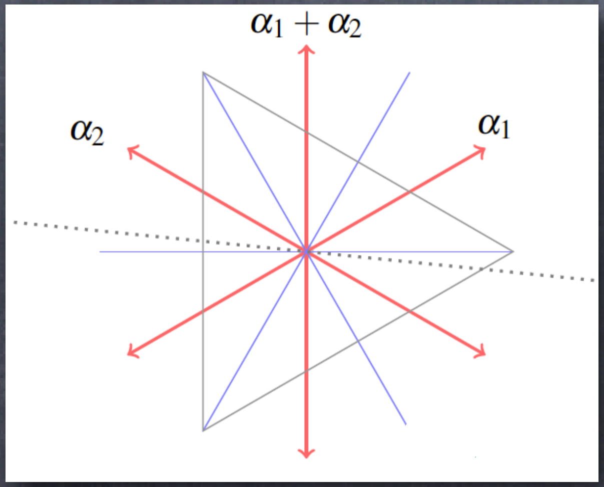
- Act by W on Δ, construct the root system

$$\Phi=W(\Delta)$$

What is a root system ?

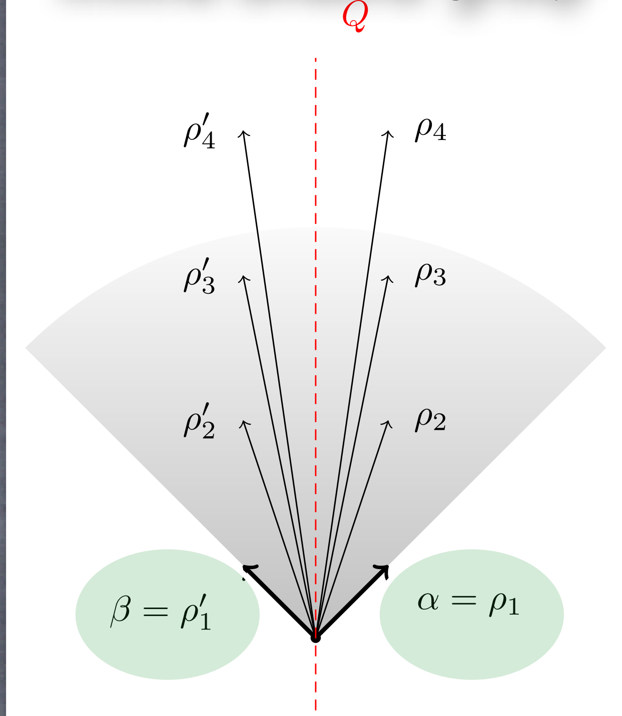
Examples

Finite (Euclidean) reflection groups (when B is positive definite)



Affine root systems (when B is positive semidefinite)

infinite dihedral group

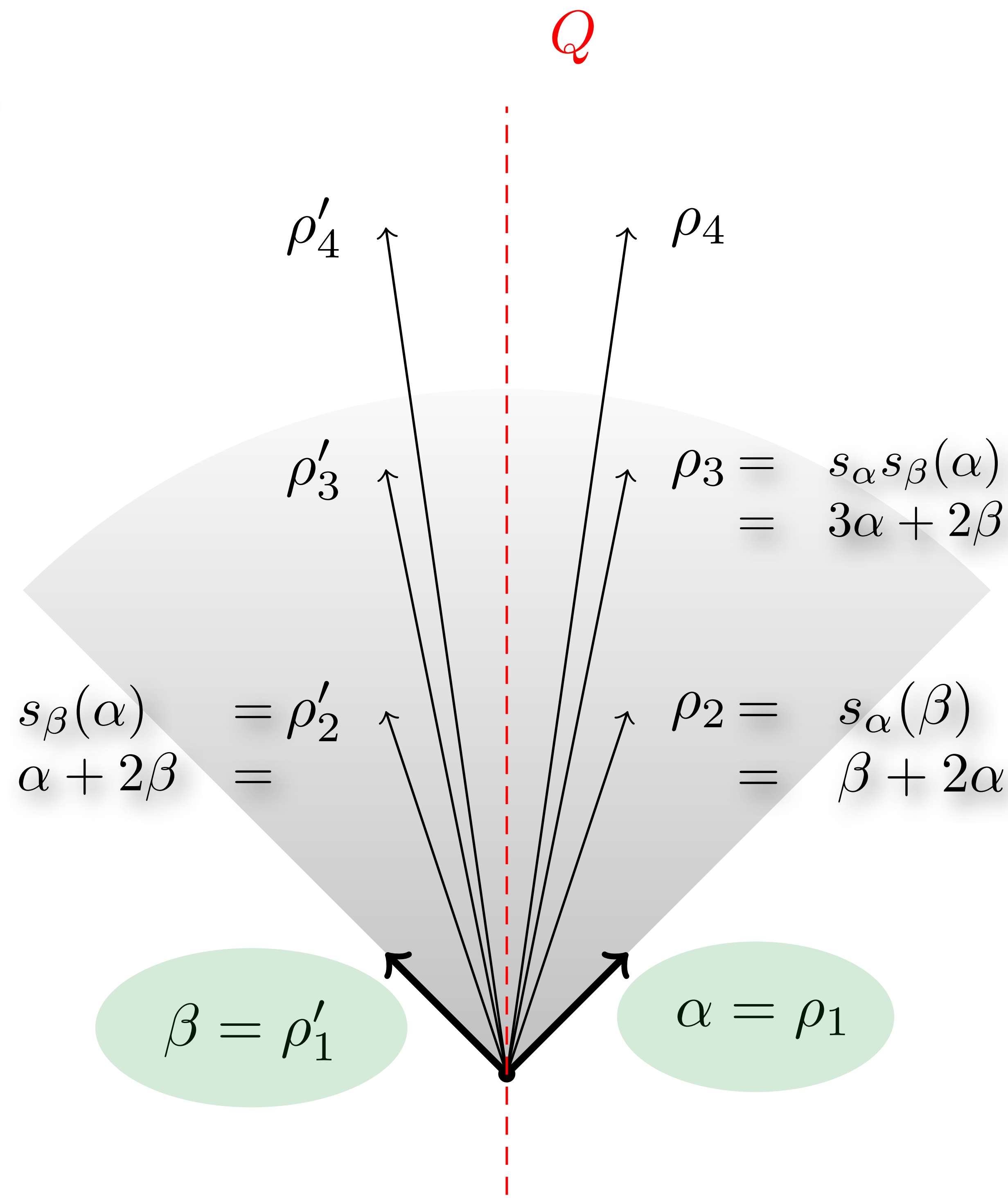


(a) $B(\alpha,\beta)=-1$

Isotropic cone of B: $Q=\{v\in V \mid B(v,v)=0\}$

What is a root system ?

$$\rho'_n = n\alpha + (n+1)\beta \qquad\qquad \rho_n = (n+1)\alpha + n\beta$$

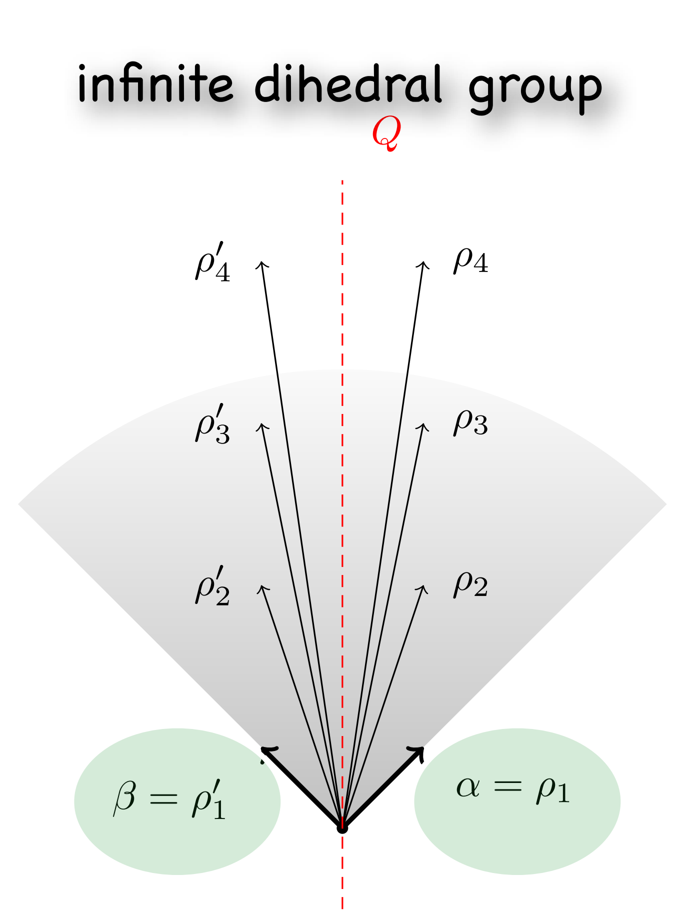


(a) $B(\alpha, \beta) = -1$

$$s_\alpha(v) = v - 2B(v, \alpha)\alpha.$$

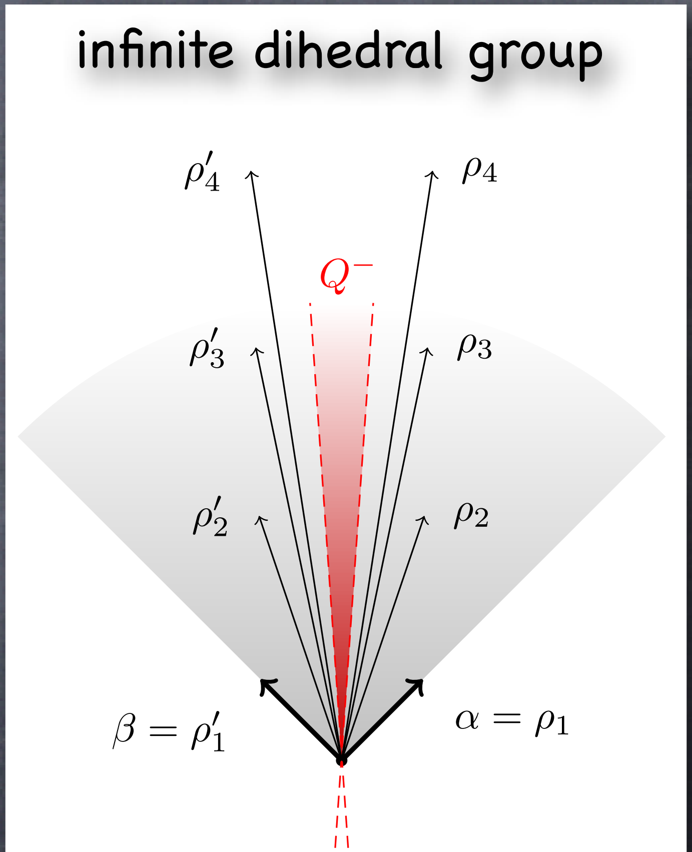
What is a root system ?
More examples

Affine root systems (when B is positive semidefinite)



(a) $B(\alpha, \beta) = -1$

Non-affine root sytems



(b) $B(\alpha, \beta) = -1.01 < -1$

$$Q^- = \{v \in V \mid B(v, v) \leq 0\}$$

What is a root system ? (in this talk)

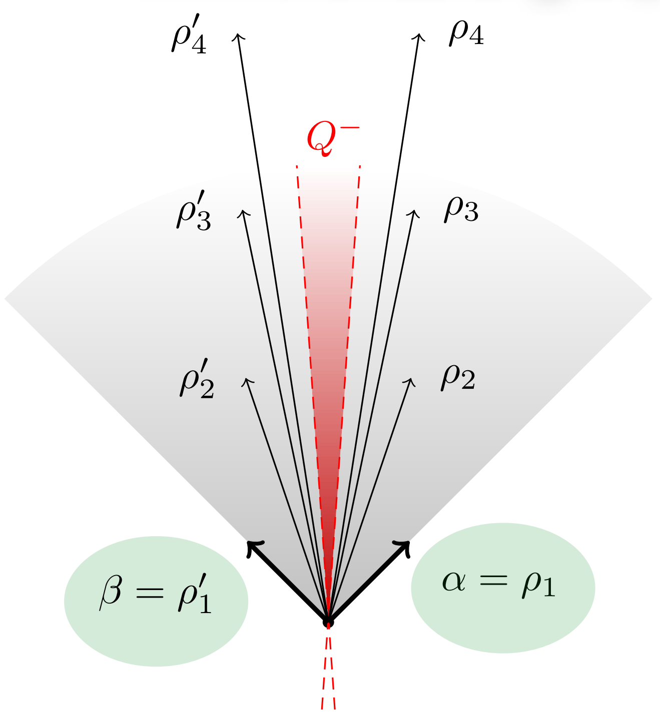
A simple system Δ, i.e.,

- Δ is a basis for V;
- $B(\alpha, \alpha) = 1$ for all $\alpha \in \Delta$;
- $B(\alpha, \beta) \in \,]-\infty, -1] \cup \{-\cos\left(\frac{\pi}{k}\right), k \in \mathbb{Z}_{\geq 2}\}$ for $\alpha \neq \beta \in \Delta$.

A B-reflection group W generated by $S := \{s_\alpha \mid \alpha \in \Delta\}$.

Root system: $\Phi = W(\Delta)$

W infinite dihedral group



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A B-reflection group W generated by $S := \{s_\alpha \,|\, \alpha \in \Delta\}$.

Root system: $\Phi = W(\Delta)$

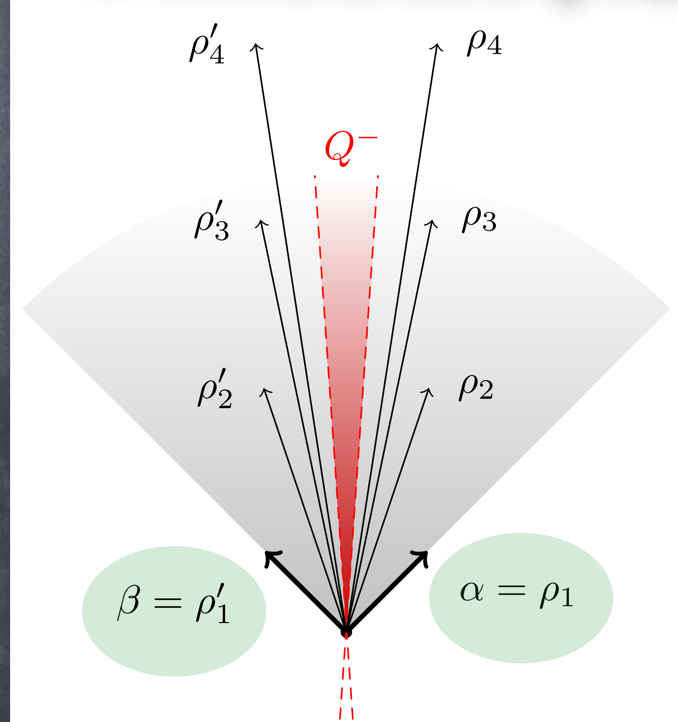
Proposition (see Krammer)

(a) (W,S) is a Coxeter system;

(b) the order of $s_\alpha s_\beta$ is k (or ∞) if $B(\alpha,\beta) = -\cos(\frac{\pi}{k})$ (or $B(\alpha,\beta) \leq -1$)

(c) $\Phi^+ := \mathrm{cone}(\Delta) \cap \Phi$ is a positive root system: $\Phi = \Phi^+ \sqcup -\Phi^+$.

W infinite dihedral group

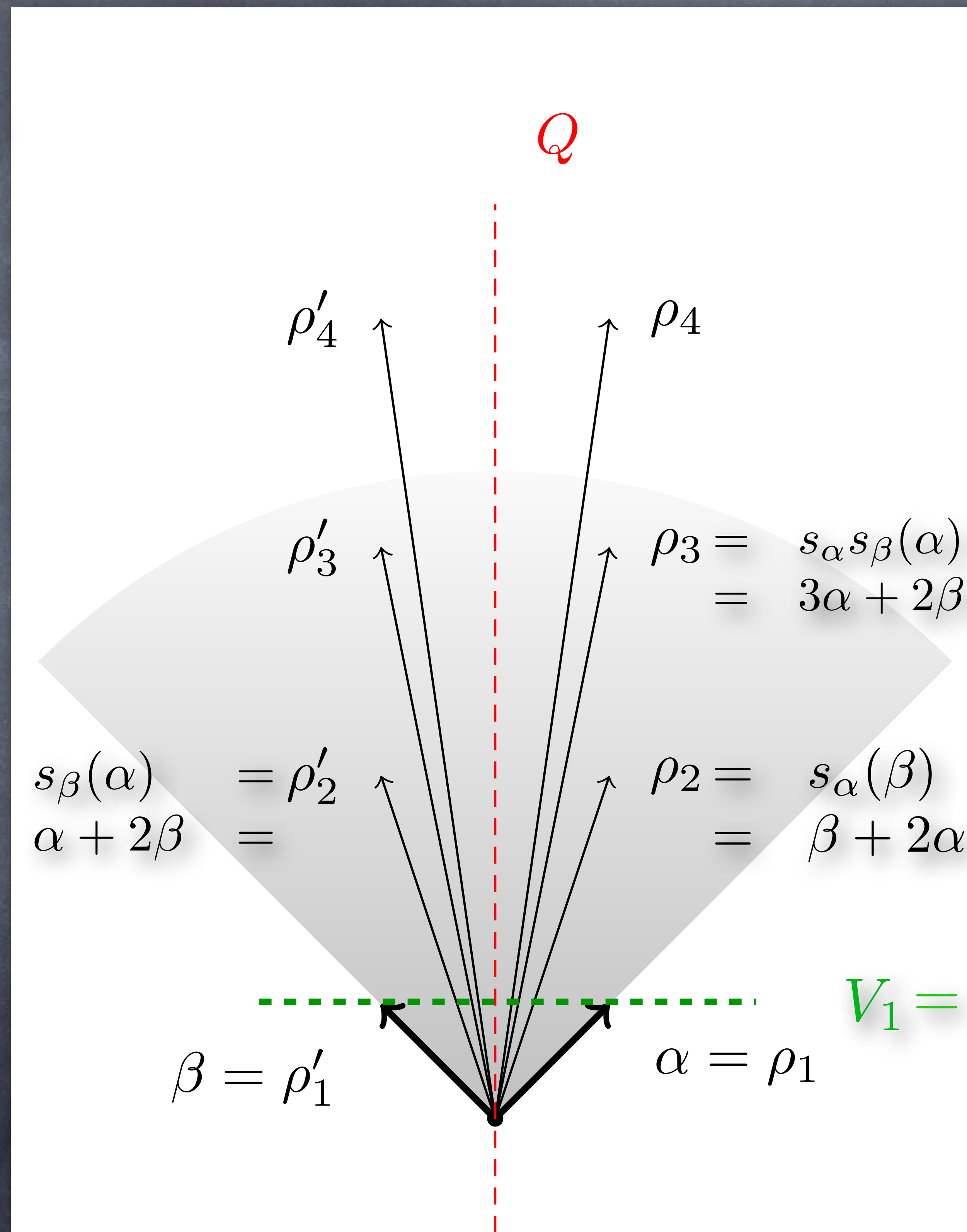


(b) $B(\alpha,\beta) = -1.01 < -1$

How to see examples of higher rank?

$$\rho'_n = n\alpha + (n+1)\beta$$

$$\rho_n = (n+1)\alpha + n\beta$$

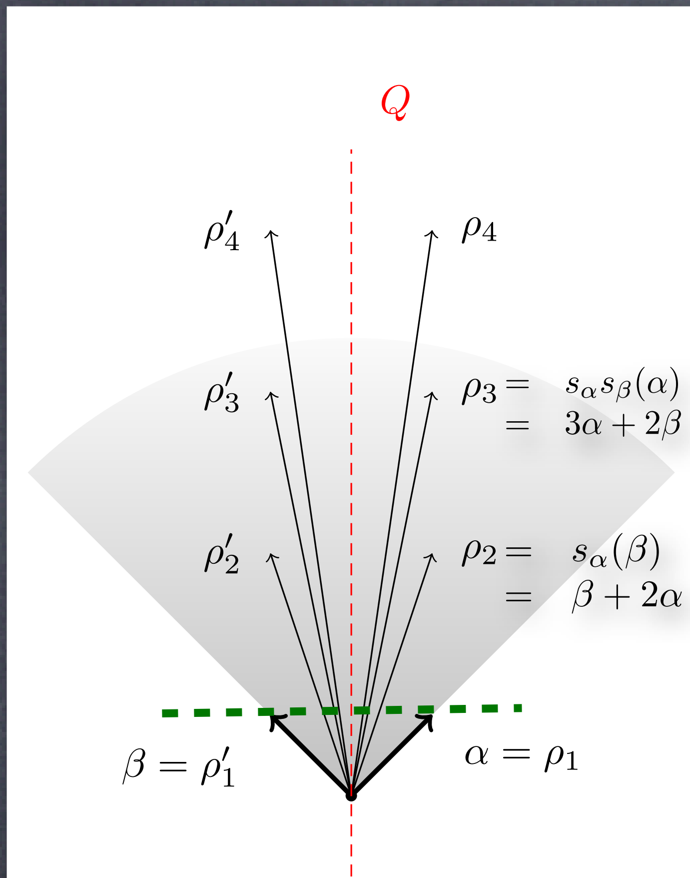


(a) $B(\alpha, \beta) = -1$

'Cut' the rays of Φ^+ by an affine hyperplane

$$V_1 = \{v \in V \mid \sum_{\alpha \in \Delta} v_\alpha = 1\}$$

How to see examples of higher rank?



(a) $B(\alpha,\beta) = -1$

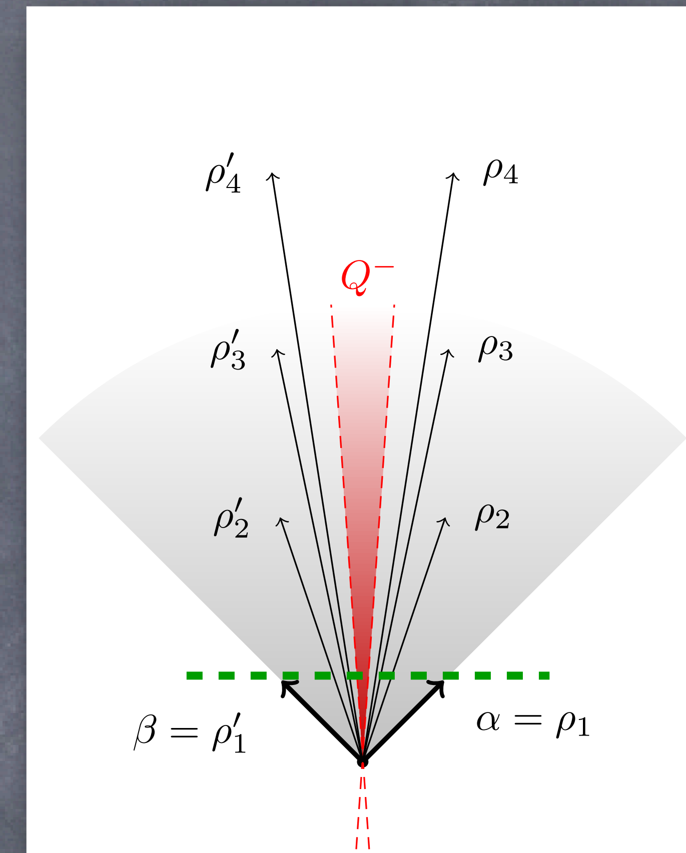
Affine hyperplane

$$V_1 = \{v \in V \mid \sum_{\alpha\in\Delta} v_\alpha = 1\}$$

Normalized isotropic cone: $\widehat{Q} := Q \cap V_1$

Normalized roots

$$\widehat{\rho} := \rho / \sum_{\alpha\in\Delta} \rho_\alpha$$



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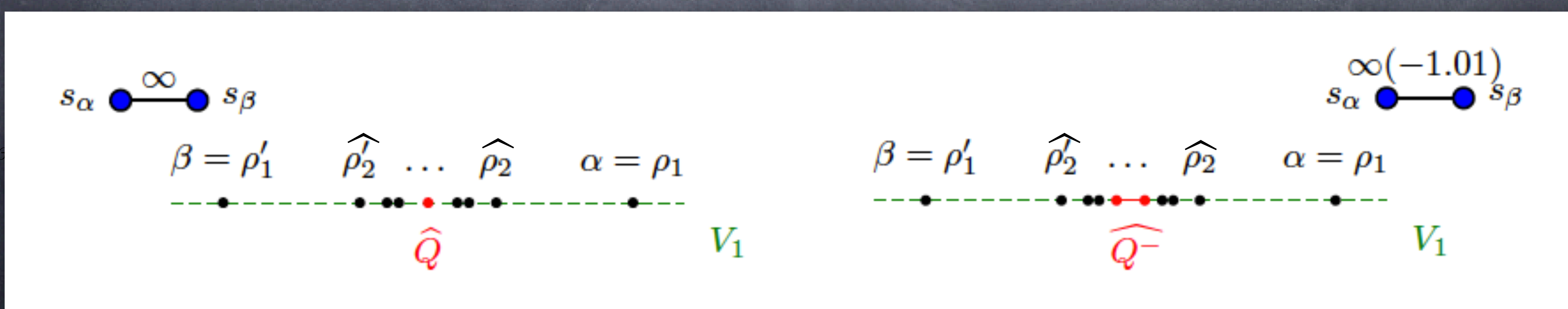
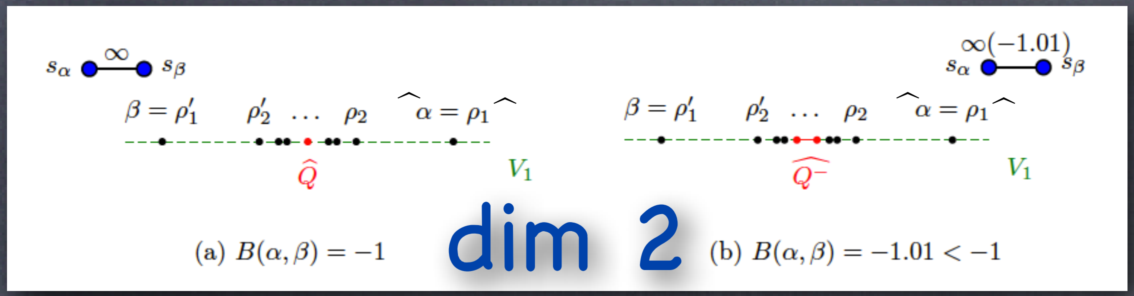
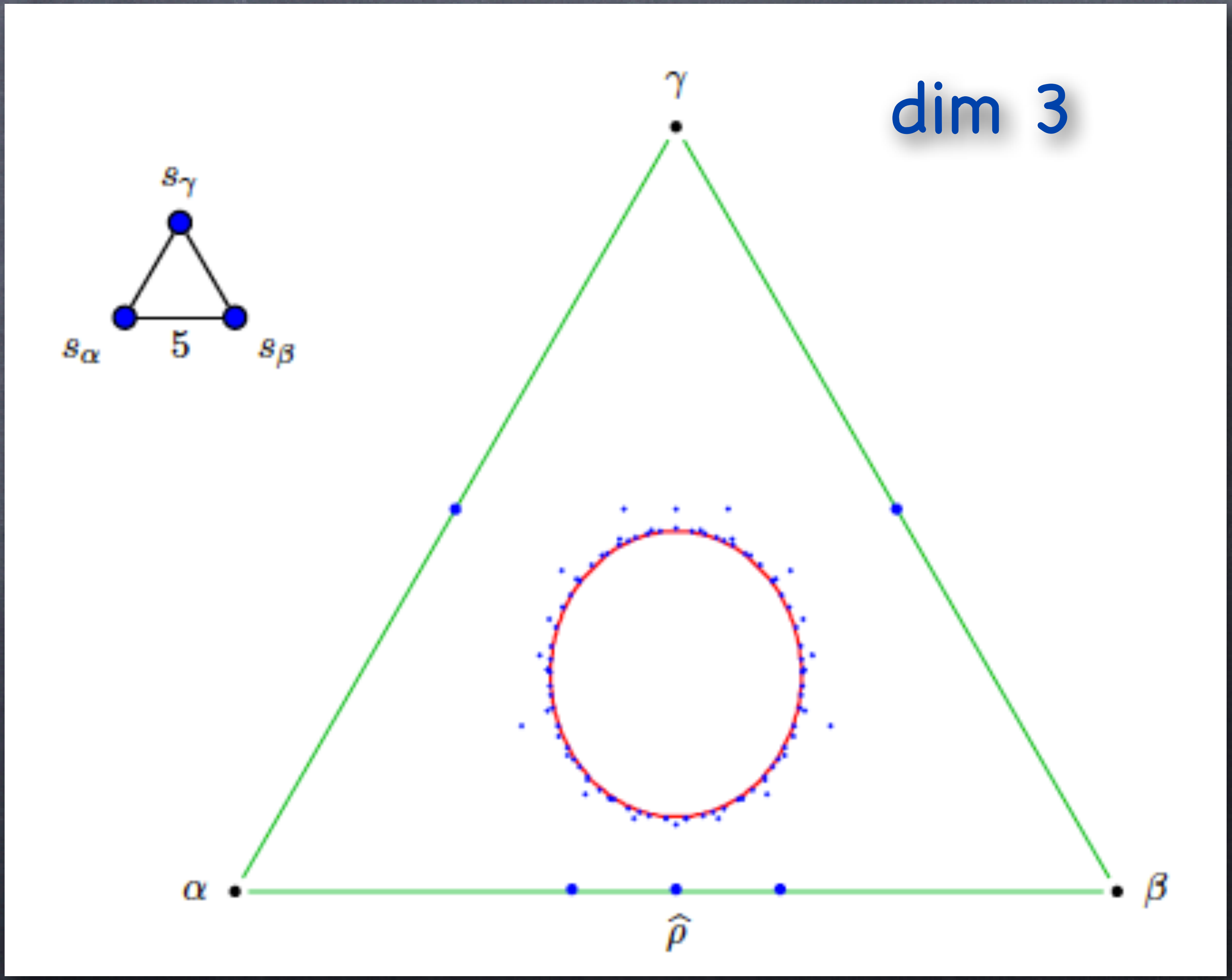


(a) $B(\alpha,\beta) = -1$ (b) $B(\alpha,\beta) = -1.01 < -1$

Other examples of infinite root systems in rank 3 and 4





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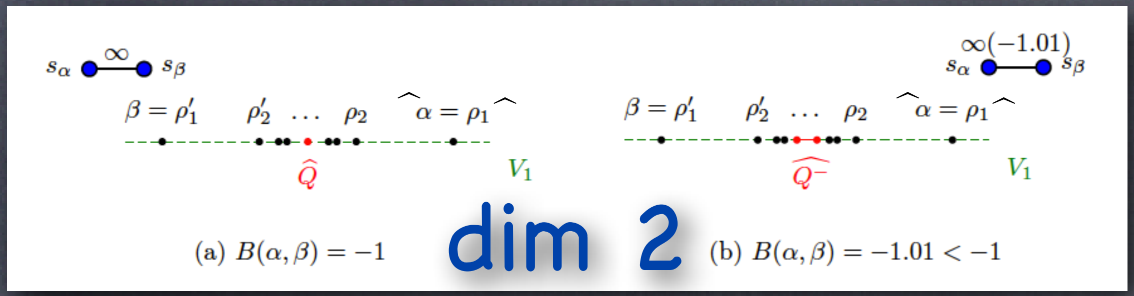

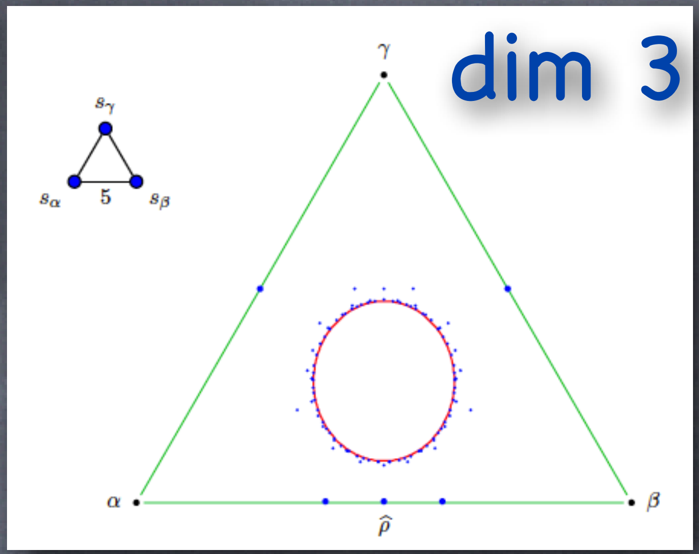

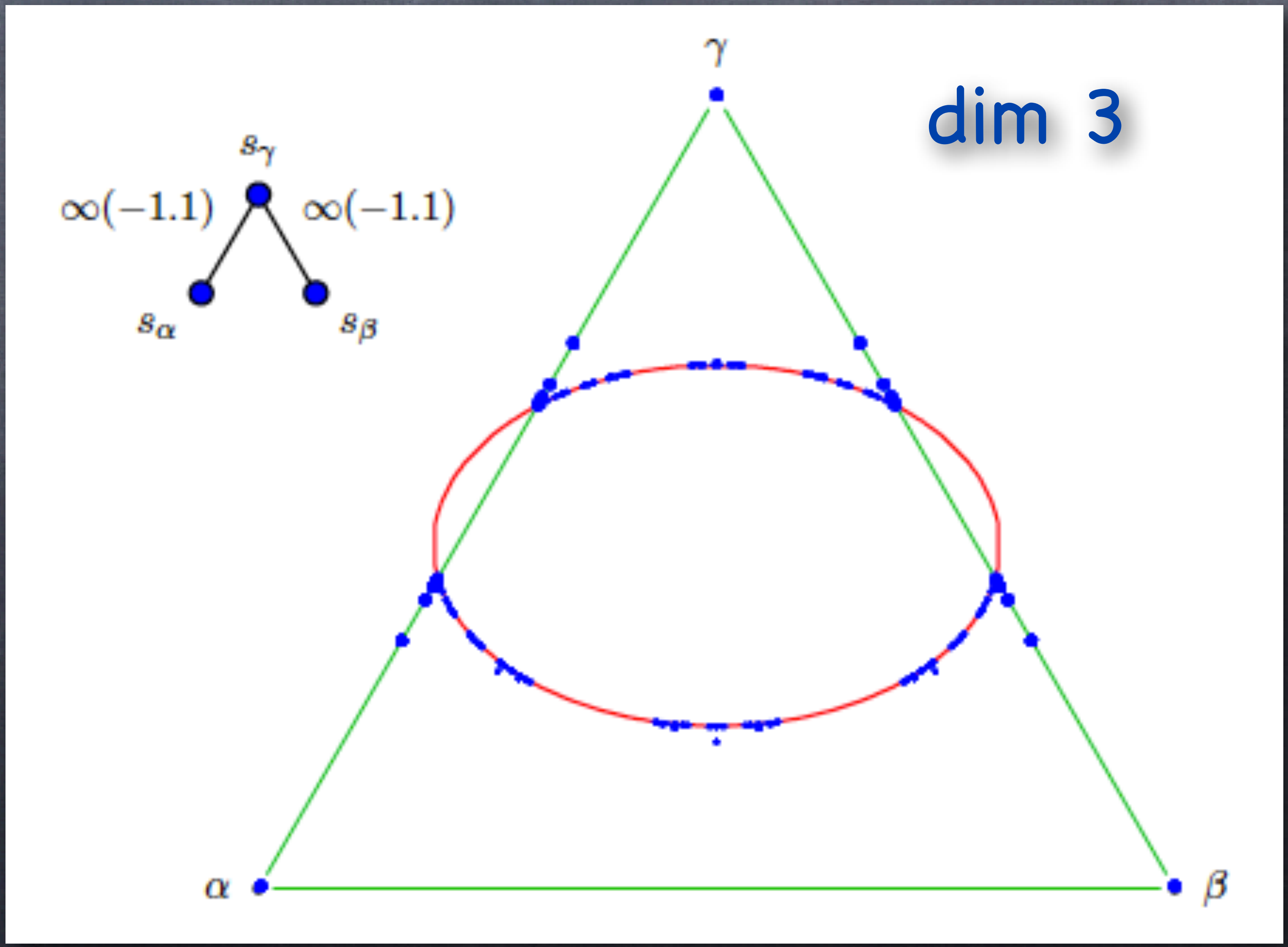

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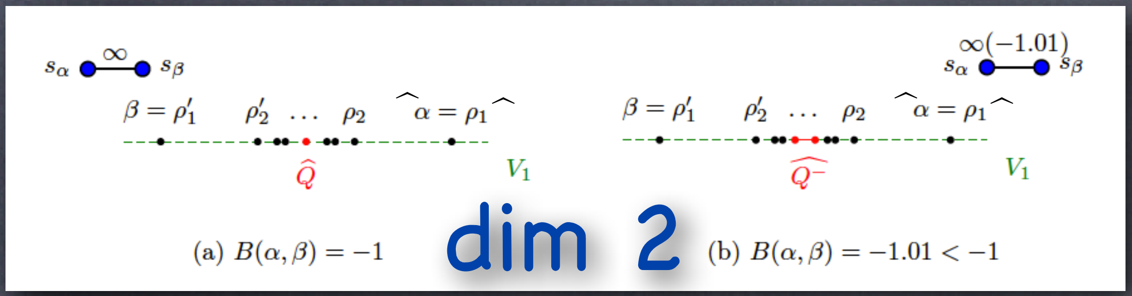

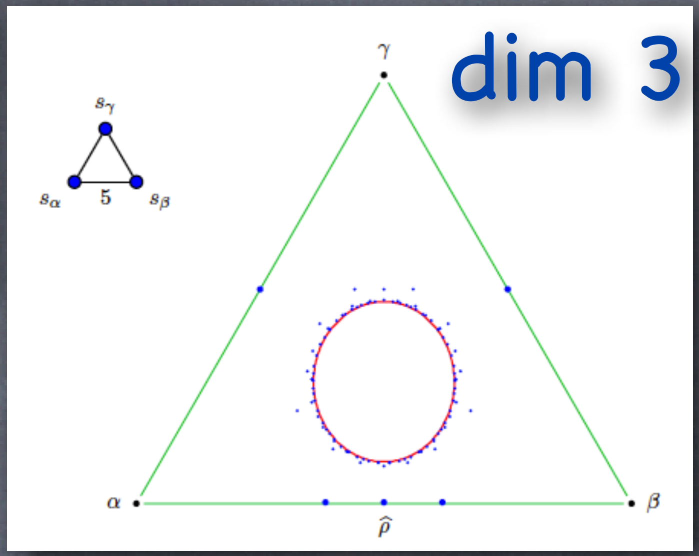

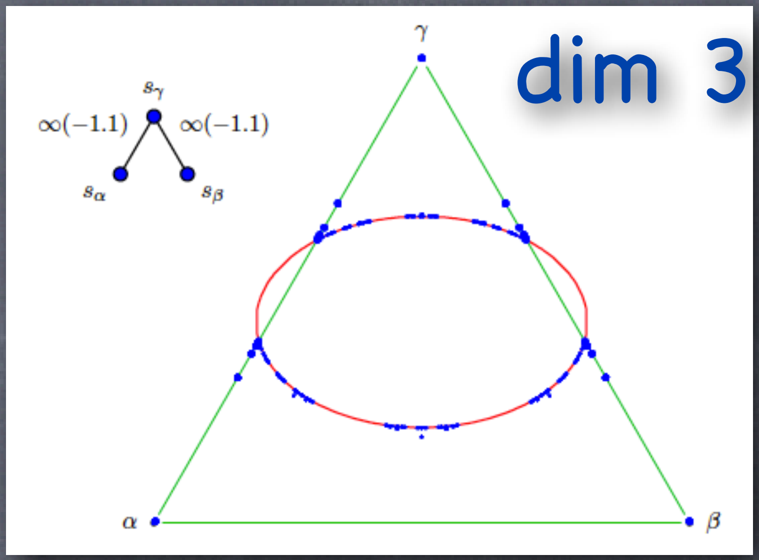

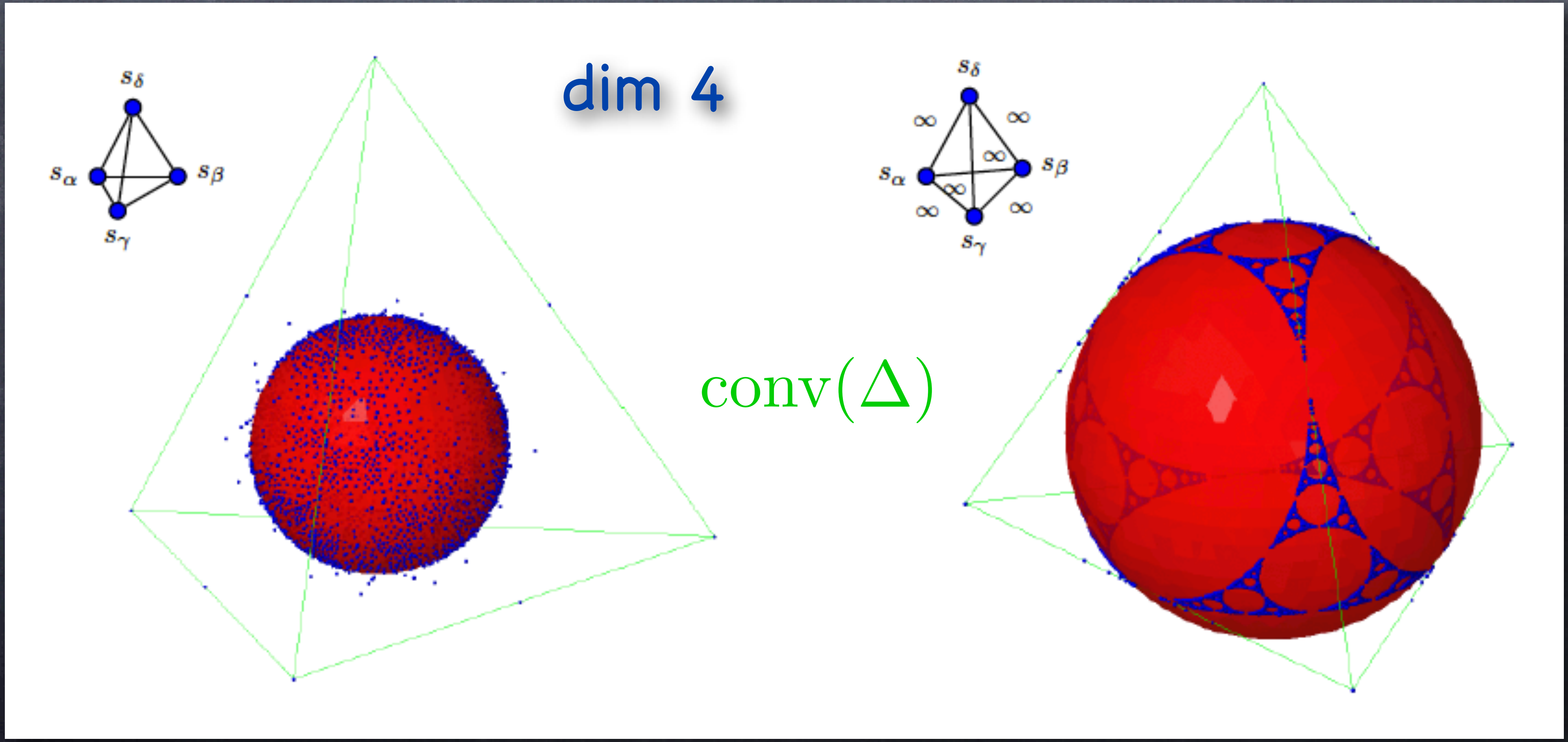

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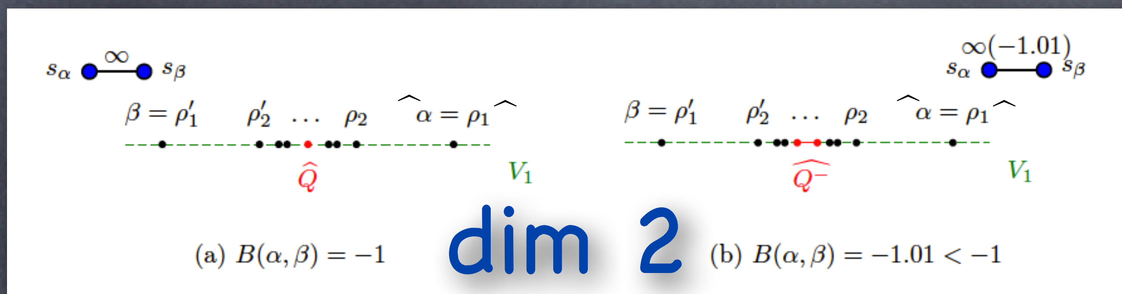
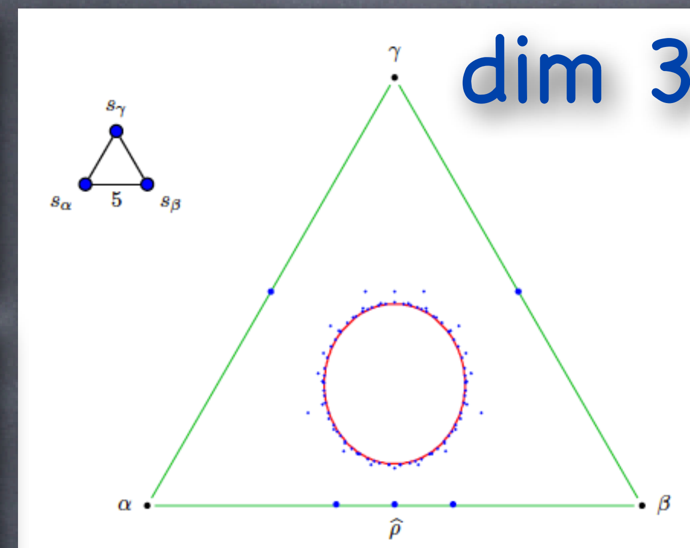
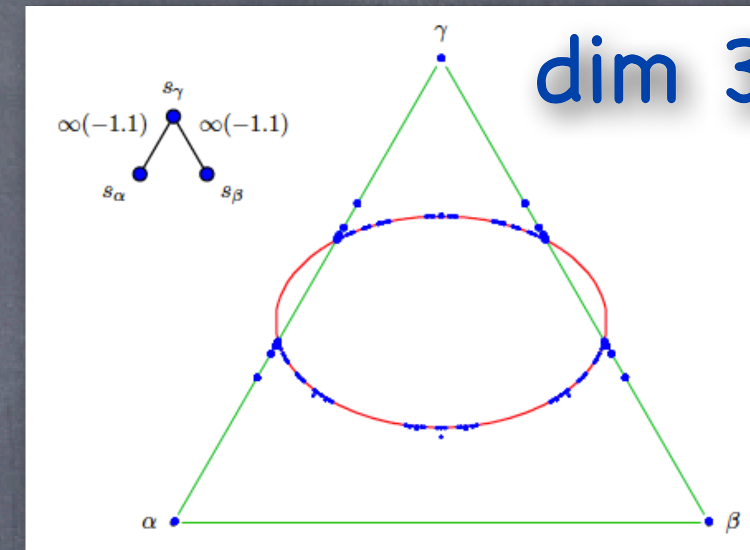
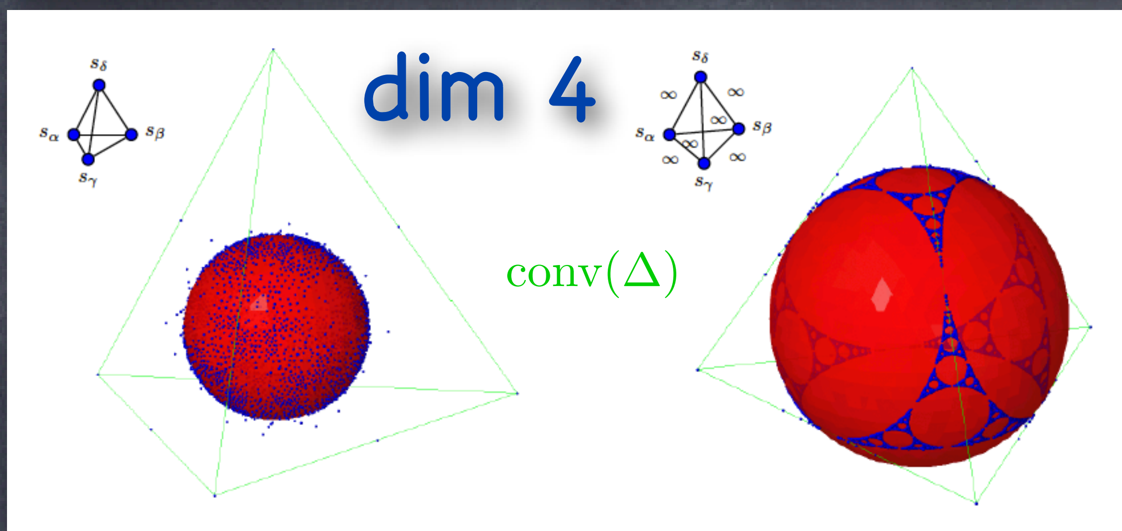
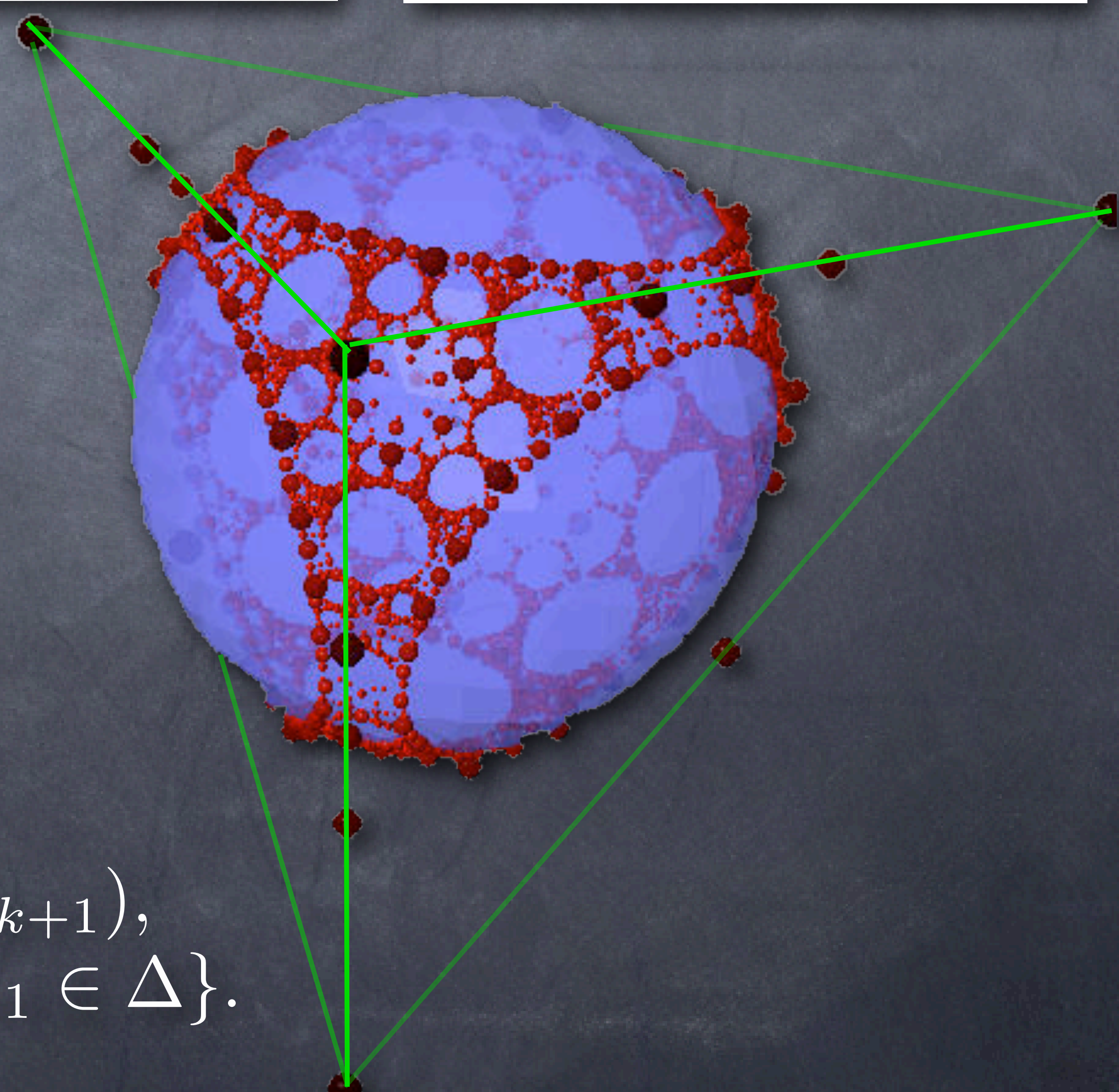






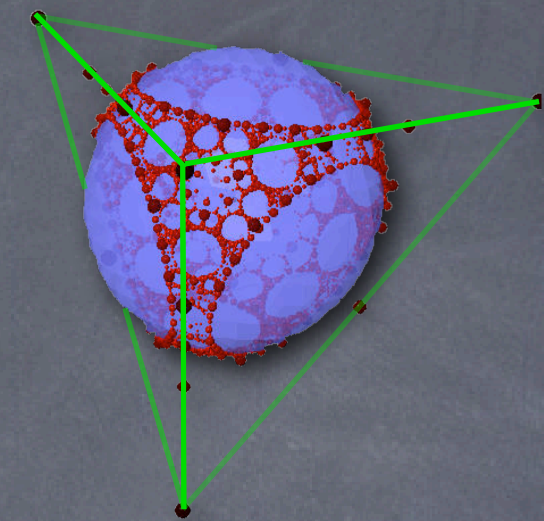




The displayed size of a normalized root (in red in this last picture) is decreasing as the depth of the root is increasing.

$$\mathrm{dp}(\rho) = 1 + \min\{k \mid \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}),\ \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}.$$

Limits of roots



Root system: $\Phi = W(\Delta)$

Depth of a root:

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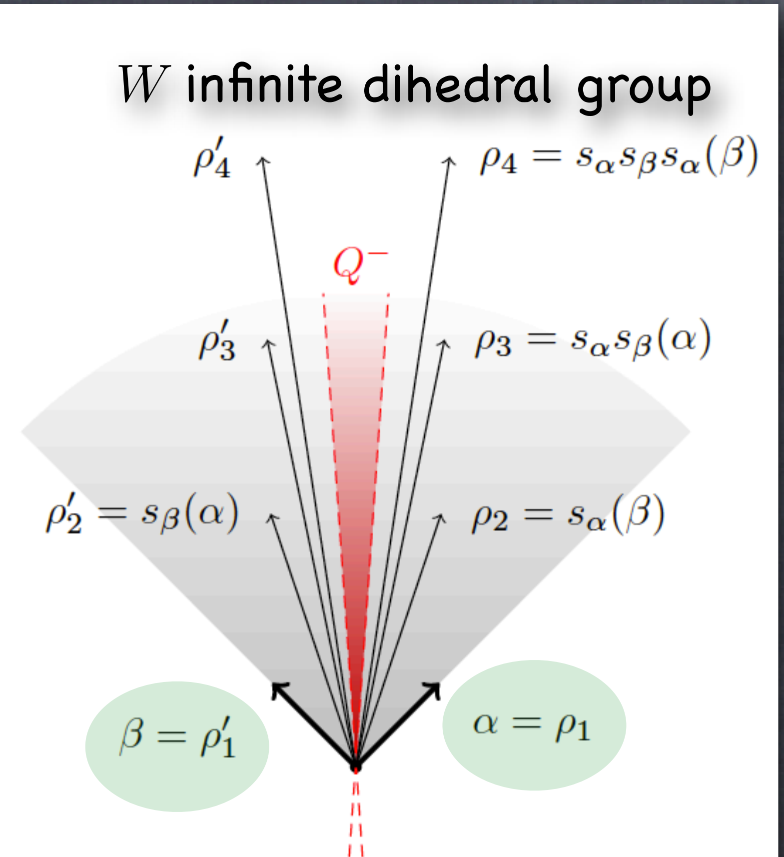

(b) $B(\alpha,\beta) = -1.01 < -1$

Euclidean norm for Δ orthonormal basis

Lemma

$$\exists\lambda>0,\ \forall\rho\in\Phi^+,\ ||\rho||^2 \geq 1+\lambda(\mathrm{dp}(\rho)-1).$$

Theorem 1 (Hohlweg-Labbé-R. 2011 ?)

Consider an injective sequence of positive roots $(\rho_n)_{n\in\mathbb{N}}$.
Then the norm $||\rho_n||$ tends to $+\infty$ (for any norm on V).

Limits of normalized roots

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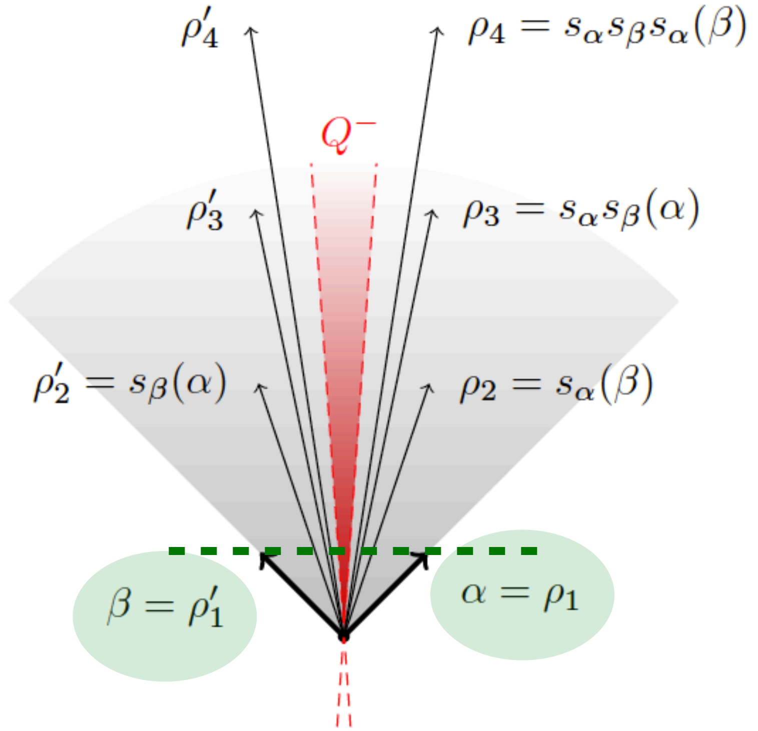
$\widehat{\Phi}$ is contained in the compact set $\mathrm{conv}(\Delta)$.

W infinite dihedral group



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Cut by the affine hyperplane

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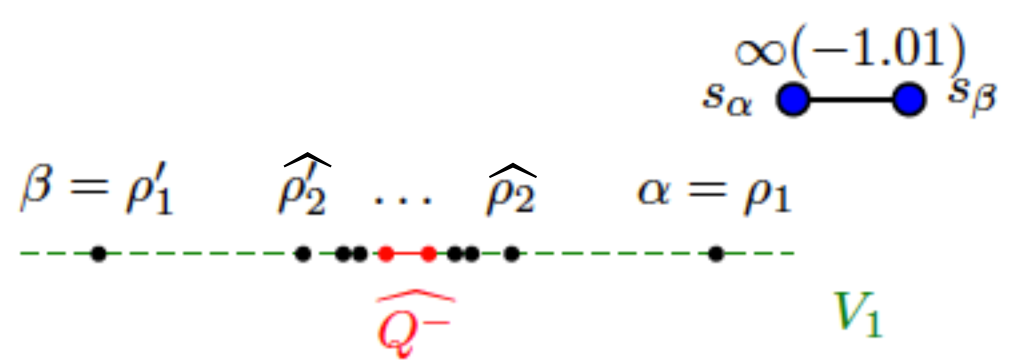

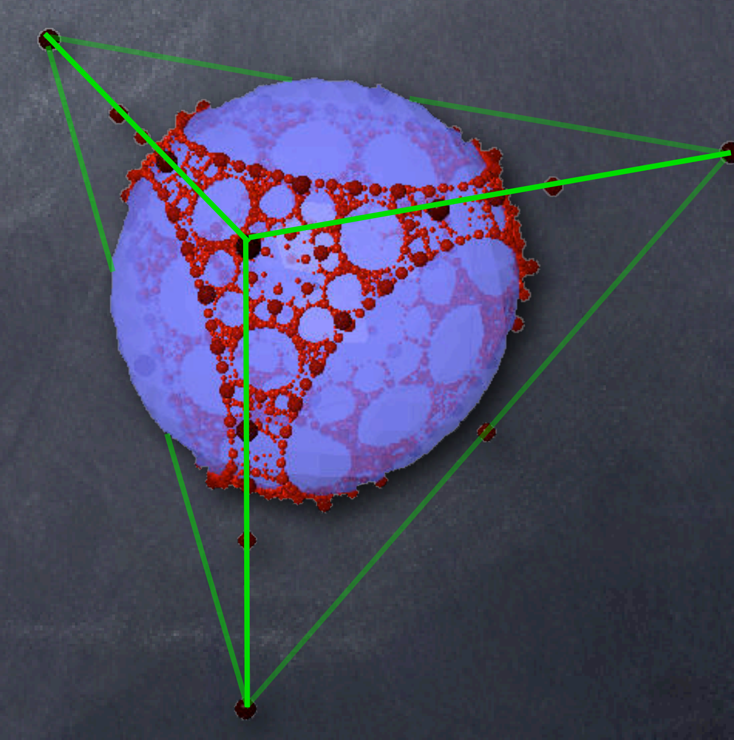

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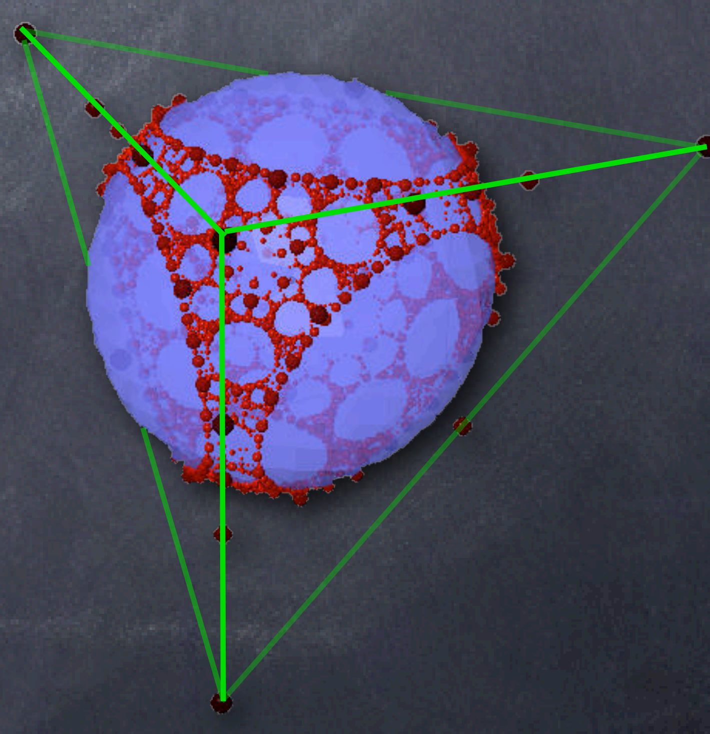
Corollary (Hohlweg-Labbé-R. 2011 ?)
If $(\widehat{\rho_n})_{n\in\mathbb{N}}$ converges to a limit ℓ, then

$$\ell \in \widehat{Q} \cap \mathrm{conv}(\Delta).$$



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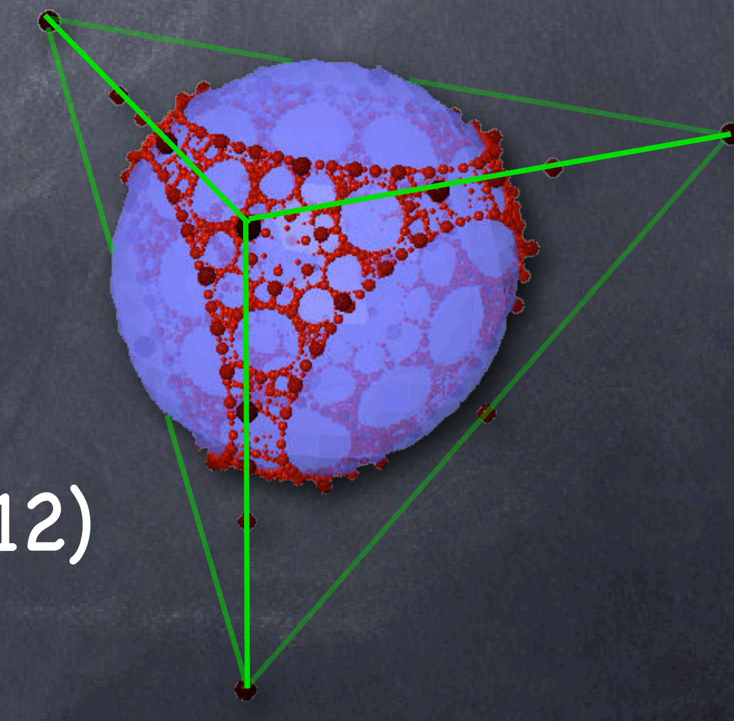
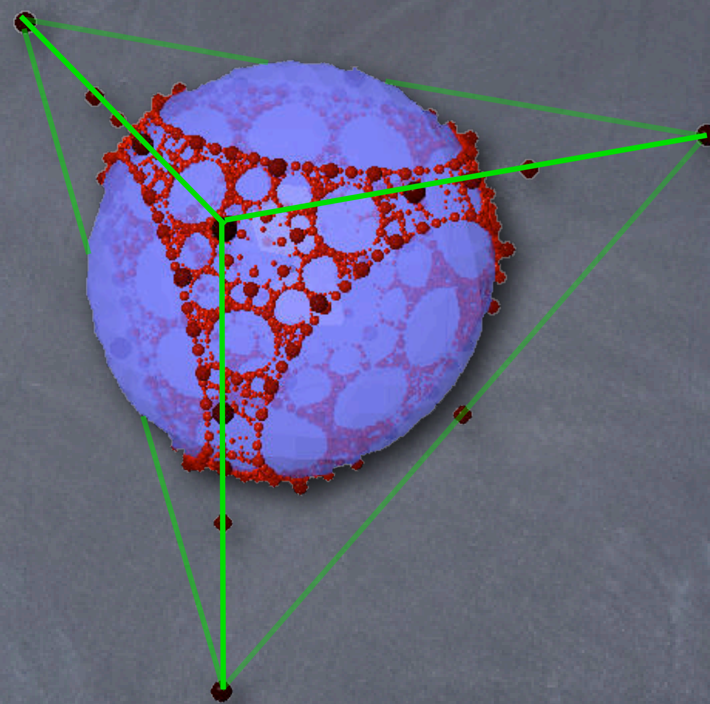


Remark: directions of roots converging in Q:
(i) Root systems of Lie algebras (Kac 1990).
(ii) Imaginary cone for Coxeter groups (Dyer, 2012)

Limits of normalized roots



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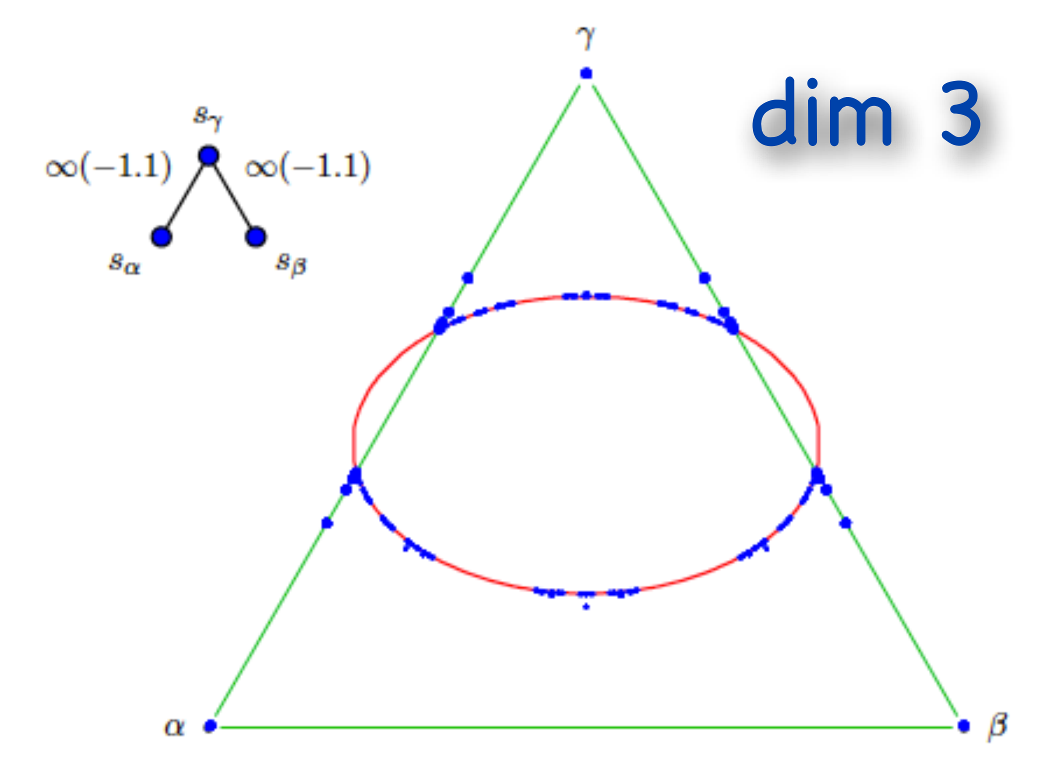
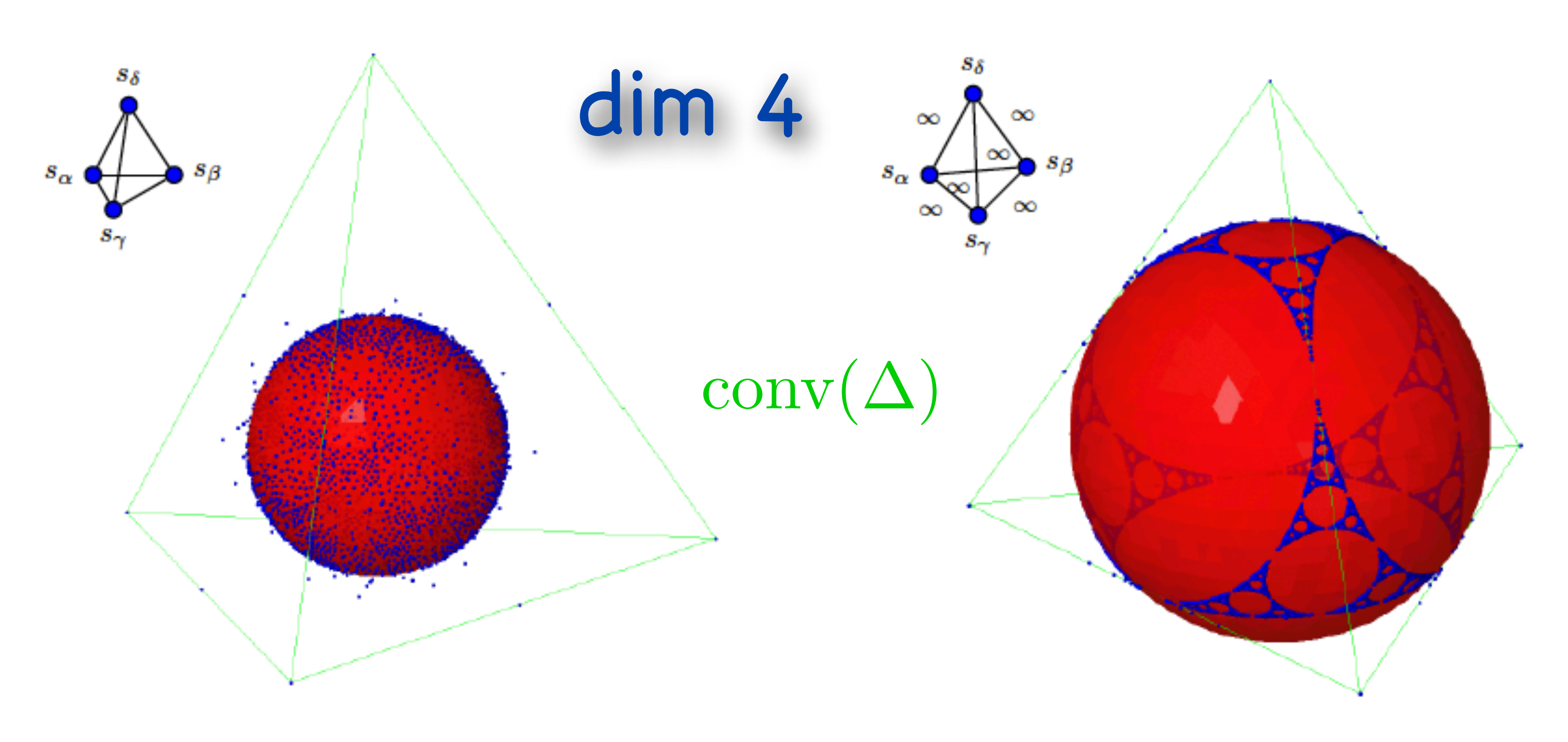
Problem: understand the set of accumulation points

('Limit roots') $E(\Phi) := \mathrm{Acc}(\widehat{\Phi})$ $\quad (\subseteq Q \cap \mathrm{conv}(\Delta))$

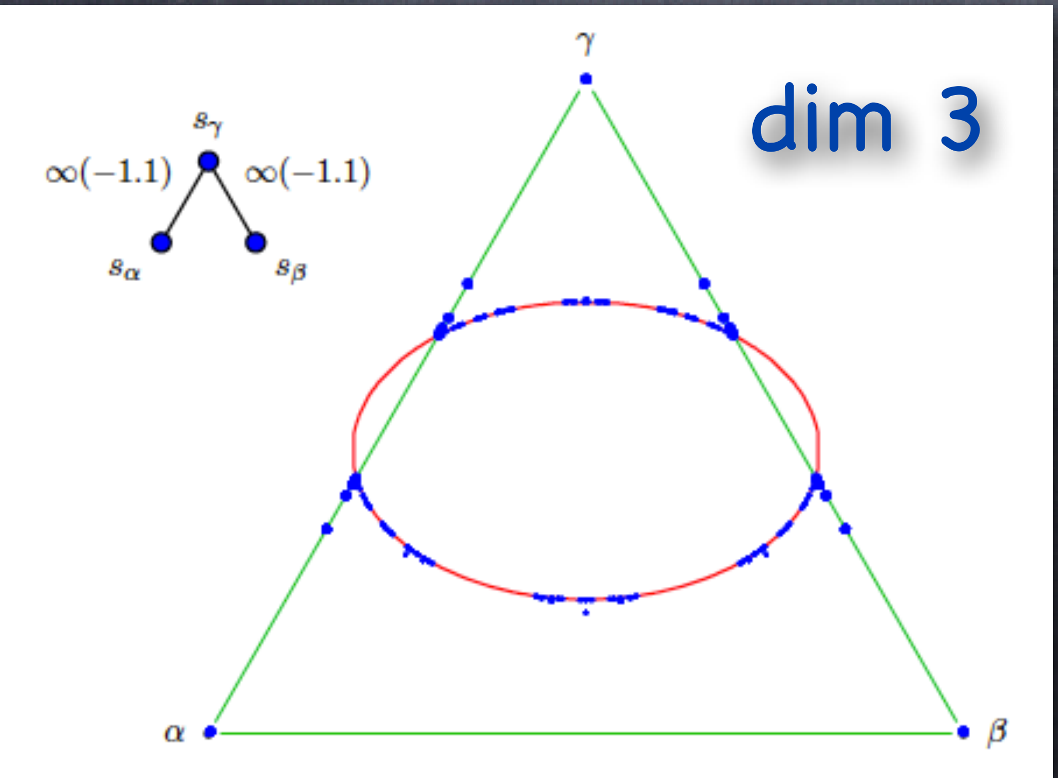


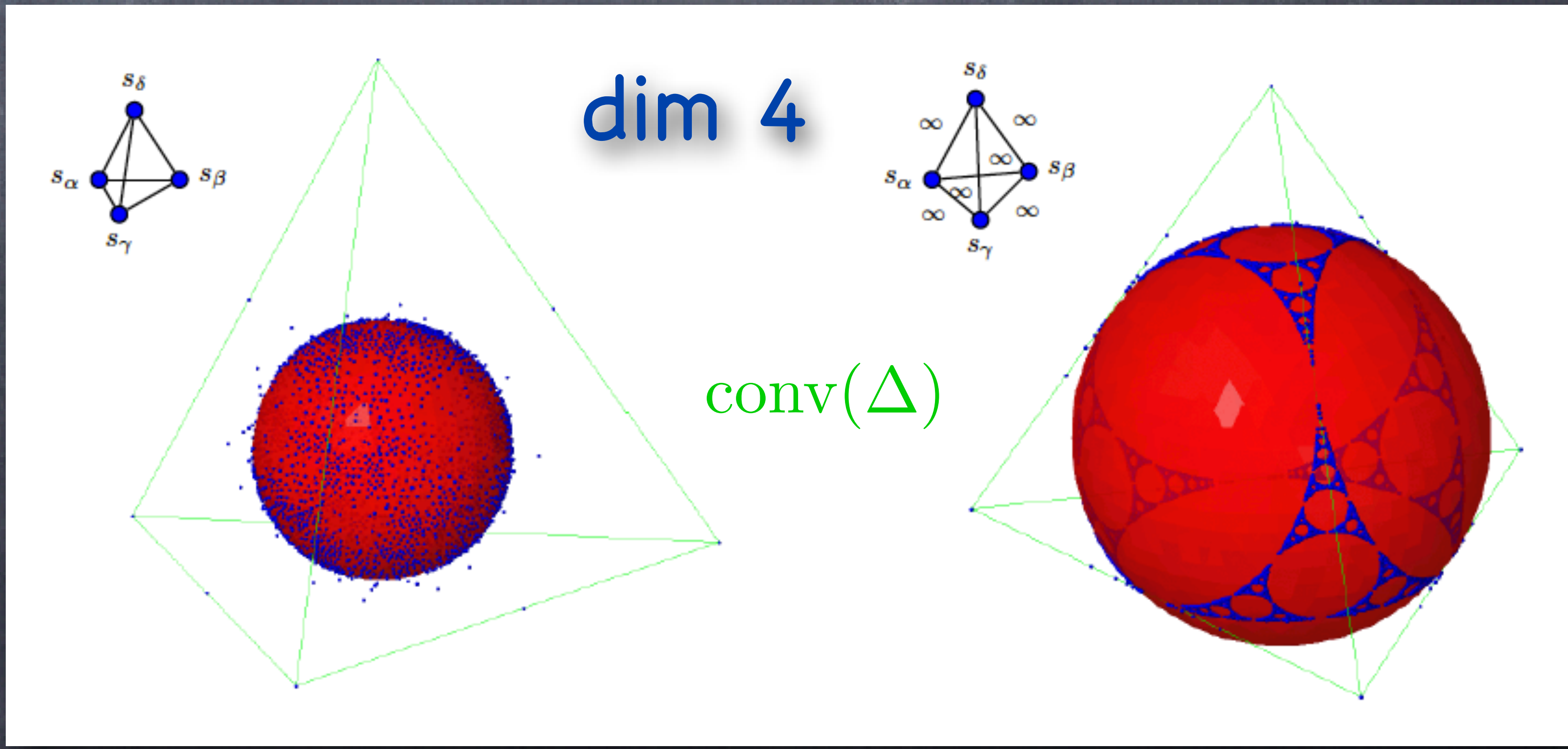


The set of limit roots $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

Some natural questions:

- A `fractal phenomenon'?
- Restriction to parabolic subgroups?
- How W acts on $E(\Phi)$?
- Link with Apollonian gasket (Kleinian groups) and sphere packings?





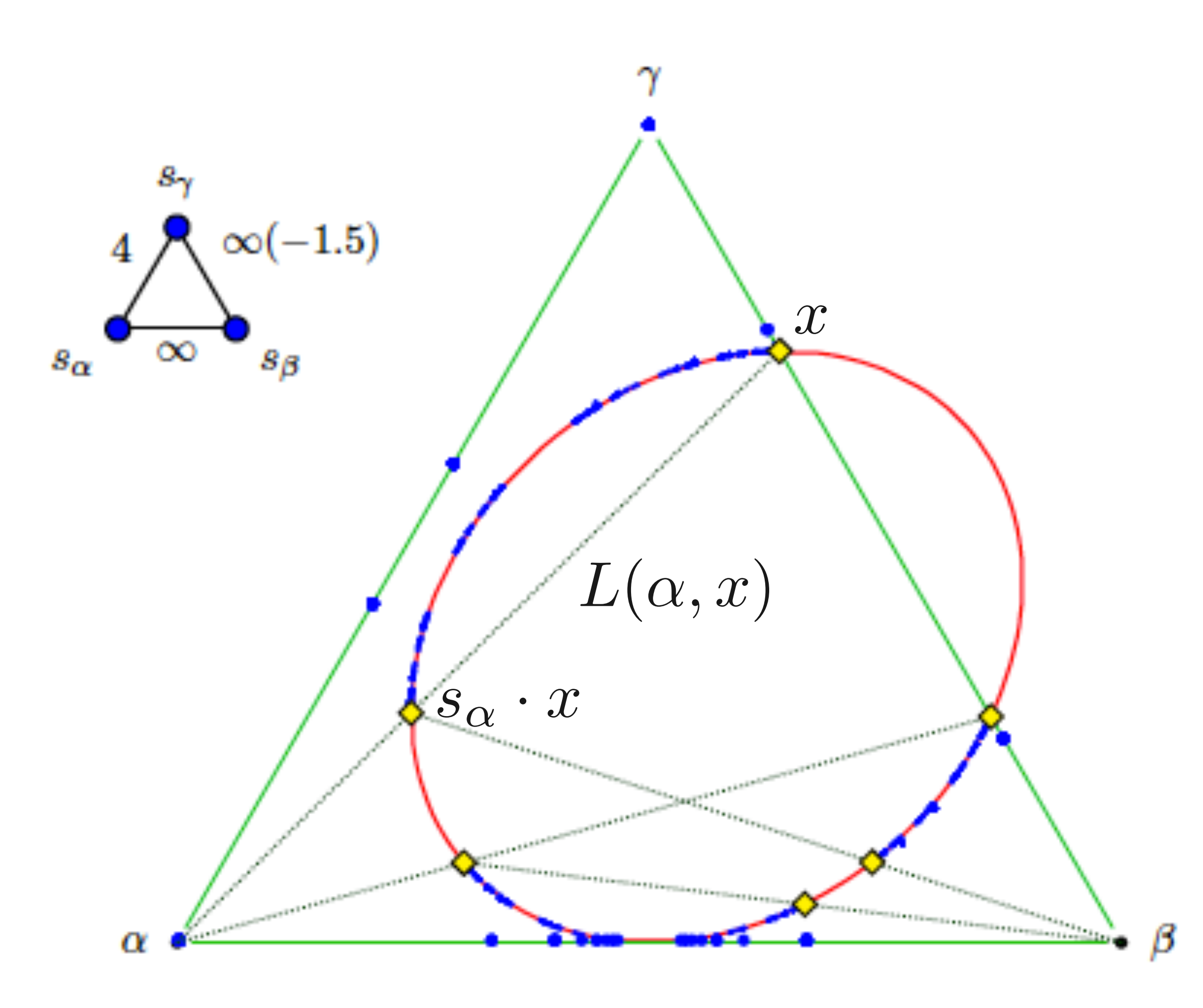
A geometric action on $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

Remark: V_1 is not stable under W.

- New action: $w \cdot v = \widehat{w(v)}$ on the set

$$D := \bigcap_{w \in W} w(V_0^+) \cap V_1 \text{ where } V_0^+ := \{v \in V \mid \sum_{\alpha \in \Delta} v_\alpha > 0\}$$

Proposition (Hohlweg-Labbé-R.)
(a) $E(\Phi) \subseteq D$ is stable under W ;
(b) $\widehat{Q} \cap L(\alpha, x) = \{x, s_\alpha \cdot x\}$.



Remarks:
(i) Action not faithful in general ($E(\Phi)$ finite in affine cases). Faithful if irreducible not affine of rk ≥ 3 ?
(ii) $D = \mathrm{conv} E(\Phi)$?

A geometric action on $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

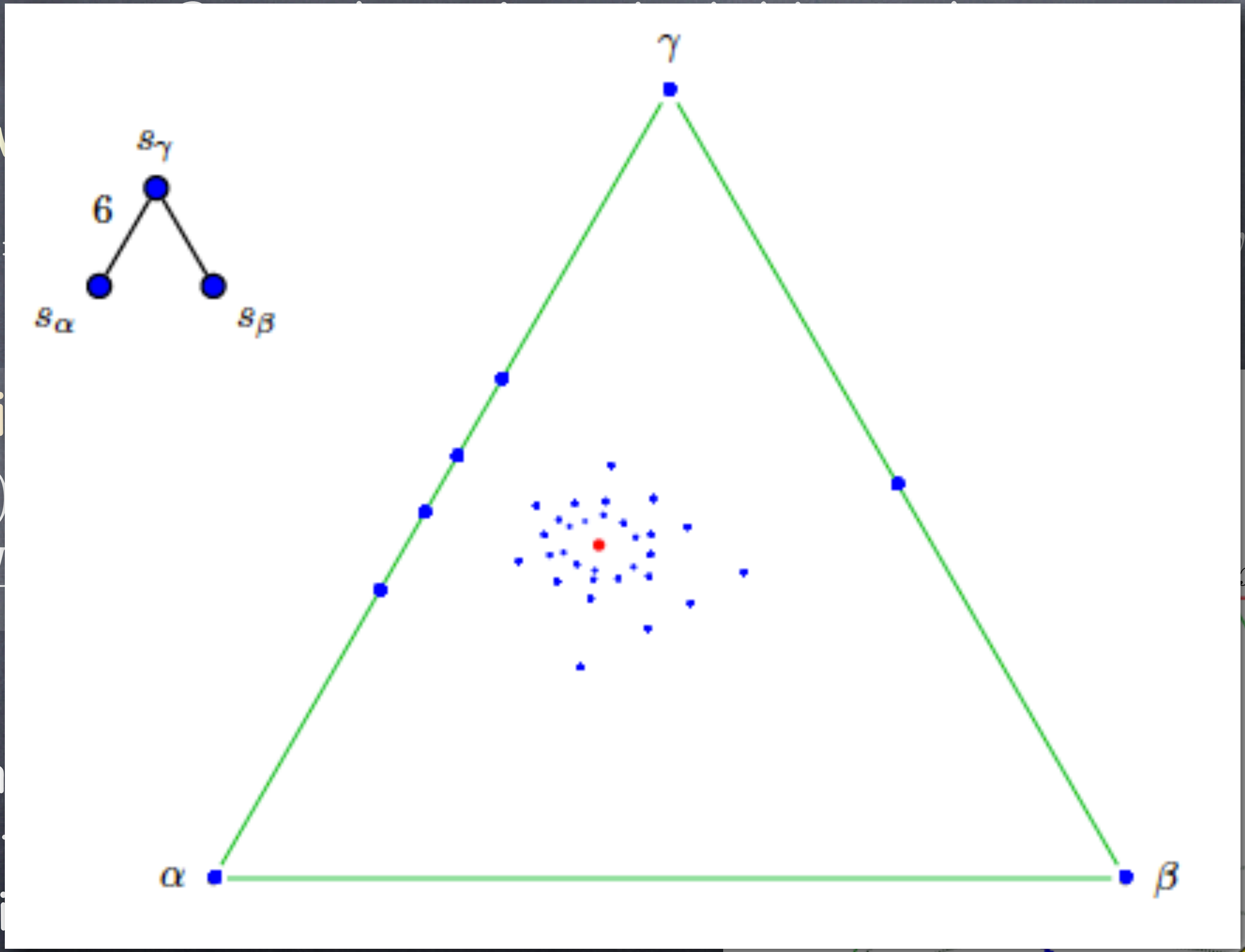
New

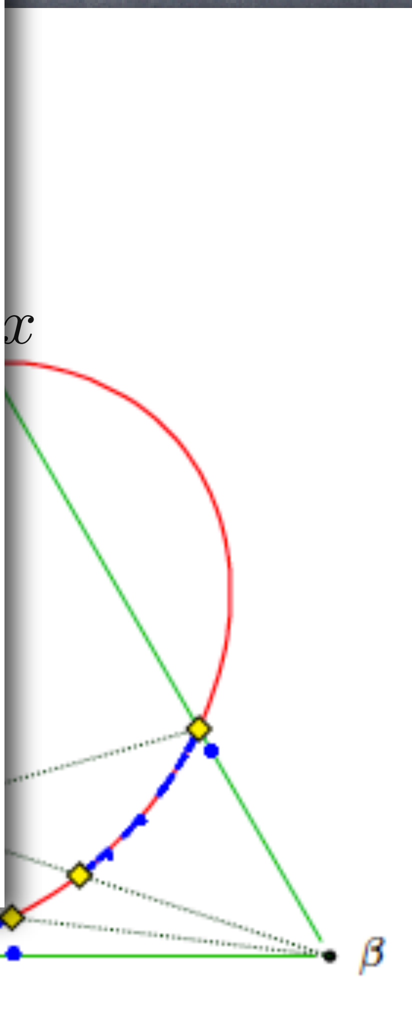
$D :$ ... $_\alpha > 0\}$

Propositi

(a) $E(\Phi)$

(b) $\widehat{Q} \cap I$





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$(E(\Phi)$ fini

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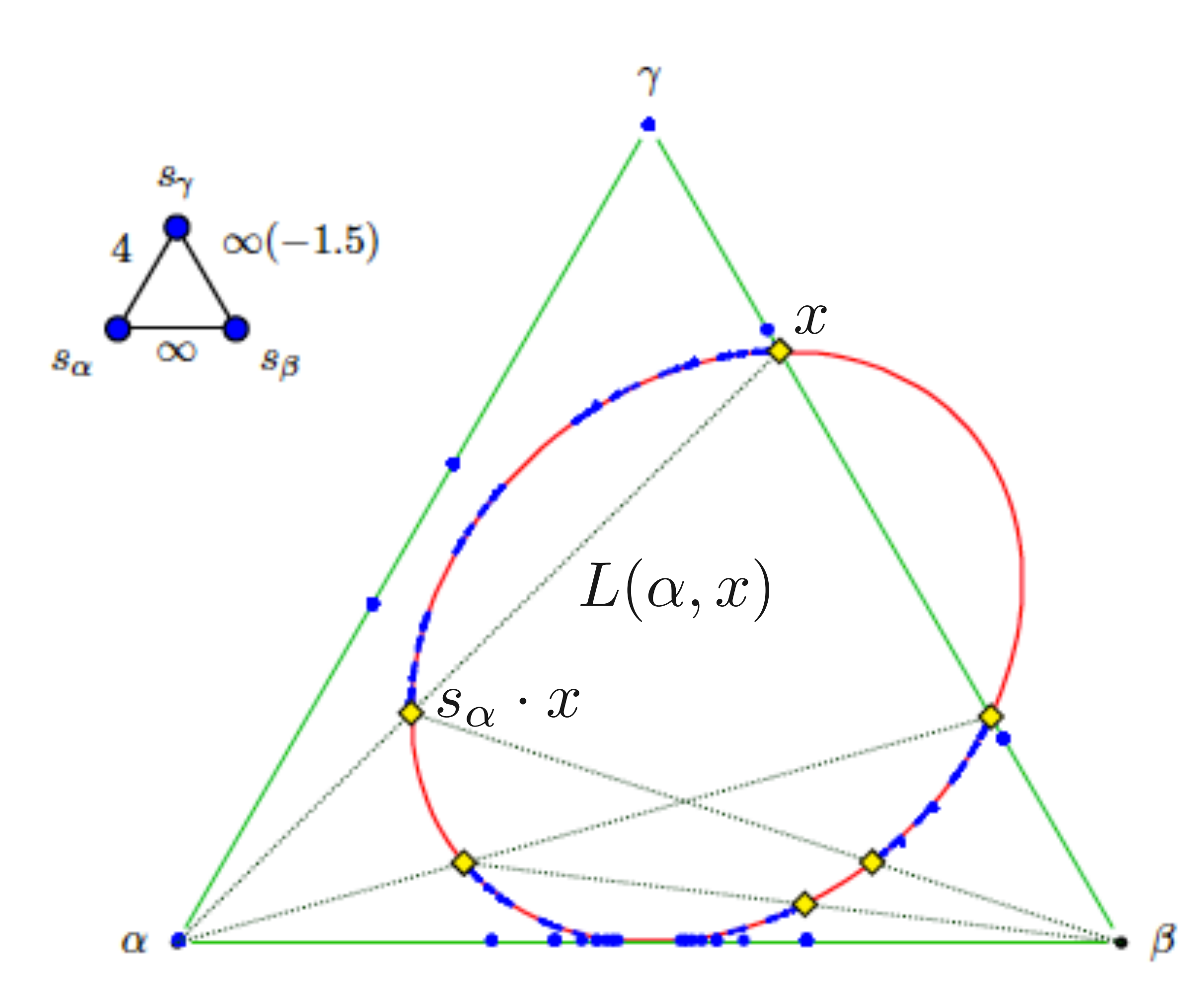

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Remarkable dense subsets of $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

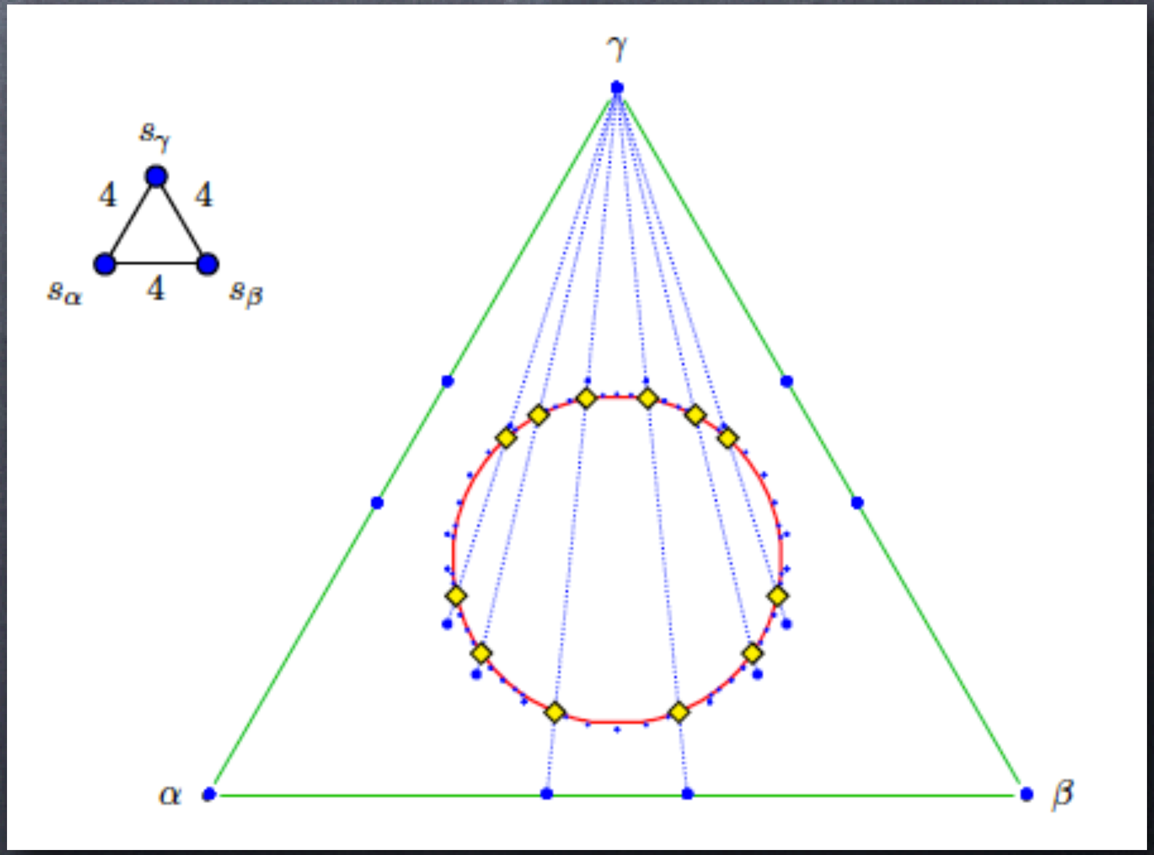
Dihedral reflection subgroup: $W' = \langle s_\rho, s_\gamma \rangle$, $\rho, \gamma \in \Phi^+$

Associated root system: $\Phi' = W'(\{\rho, \gamma\})$

Observation: $E(\Phi') = \widehat{Q} \cap L(\widehat{\rho}, \widehat{\gamma})$

Limits of normalized roots of dihedral reflection subgps:

$$E_2 := \bigcup_{\rho_1, \rho_2 \in \Phi^+} L(\widehat{\rho_1}, \widehat{\rho_2}) \cap \widehat{Q}$$



Theorem 2 (Hohlweg-Labbé-R. 2011)
The set E_2 is dense in $E(\Phi)$.

Remarkable dense subsets of $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

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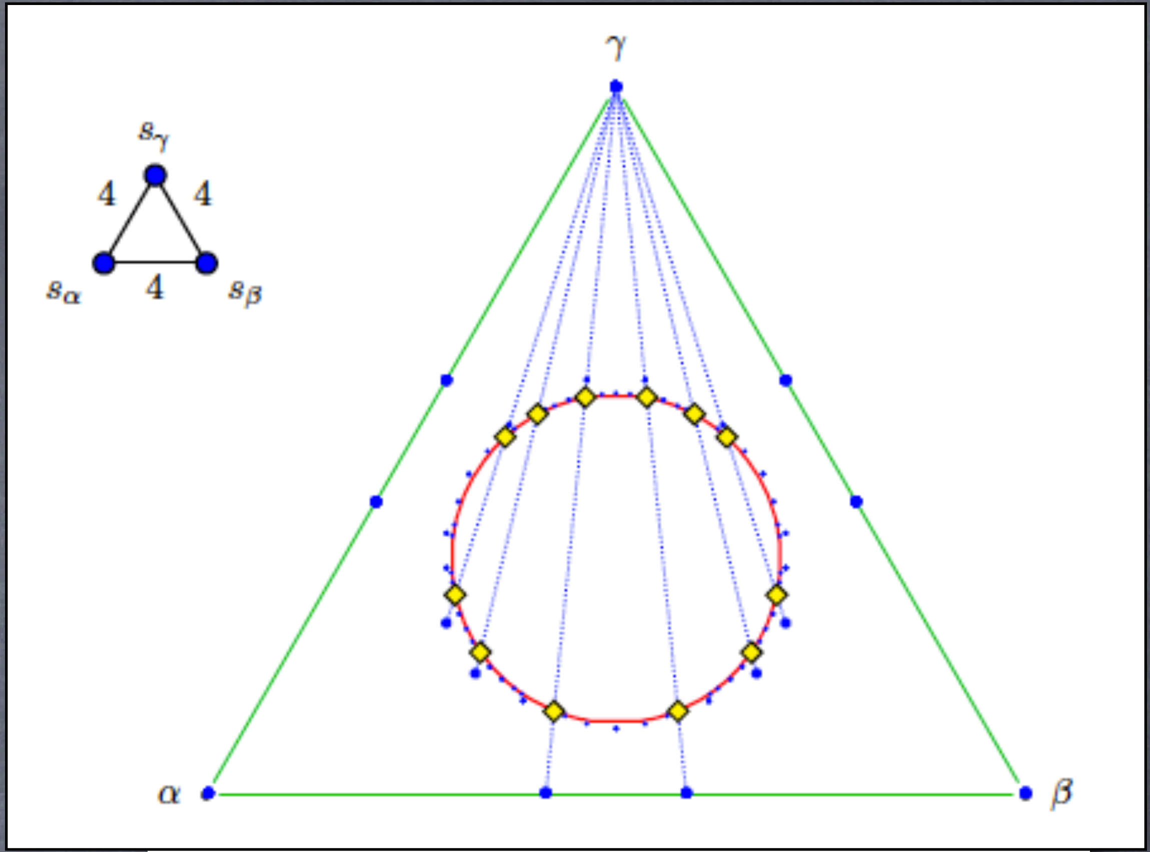
The set E_2 is dense in $E(\Phi)$.

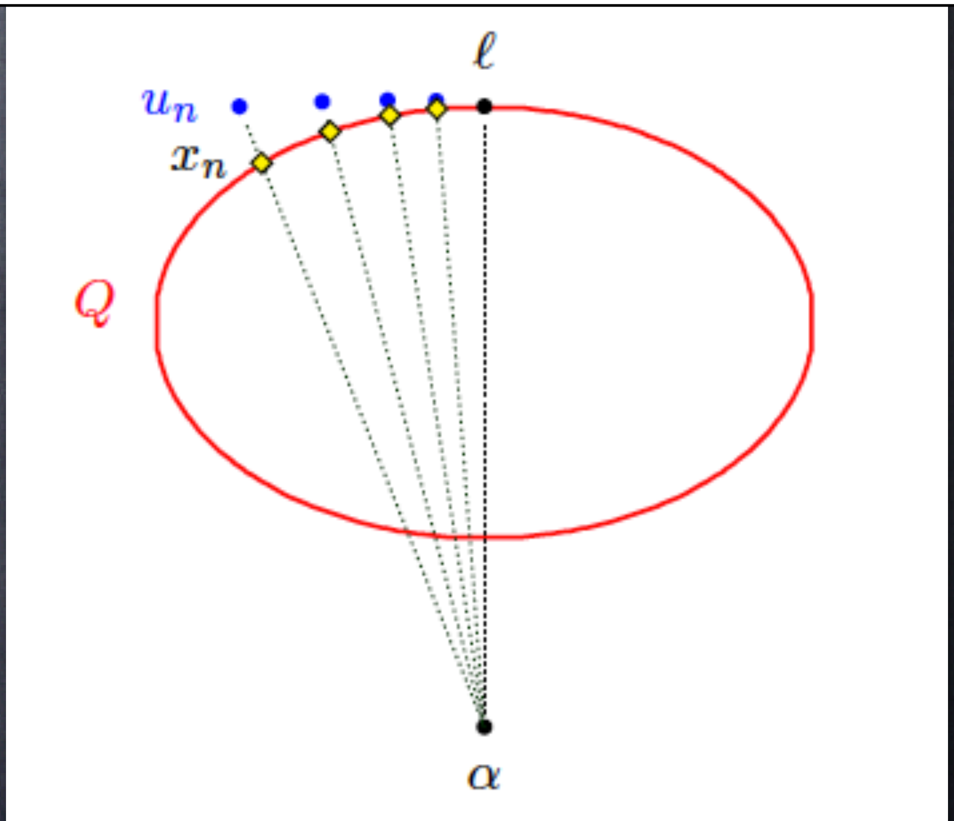
Proof (sketch):

- $E_2 = W \cdot E_2^\circ$ where

$$E_2^\circ := \bigcup_{\substack{\alpha\in\Delta \\ \rho\in\Phi^+}} L(\alpha,\widehat{\rho}) \cap \widehat{Q}$$



Proposition (Hohlweg-Labbé-R.)
The set E_2° is dense in $E(\Phi)$.



- Two cases: $l \notin V^\perp$ or $l \in V^\perp$ (which is dealt with by Perron-Frobenius)

A finite subset 'generating'

$$E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$$

Small root: root obtained from Δ along a path of finite dihedral reflection subgroups.

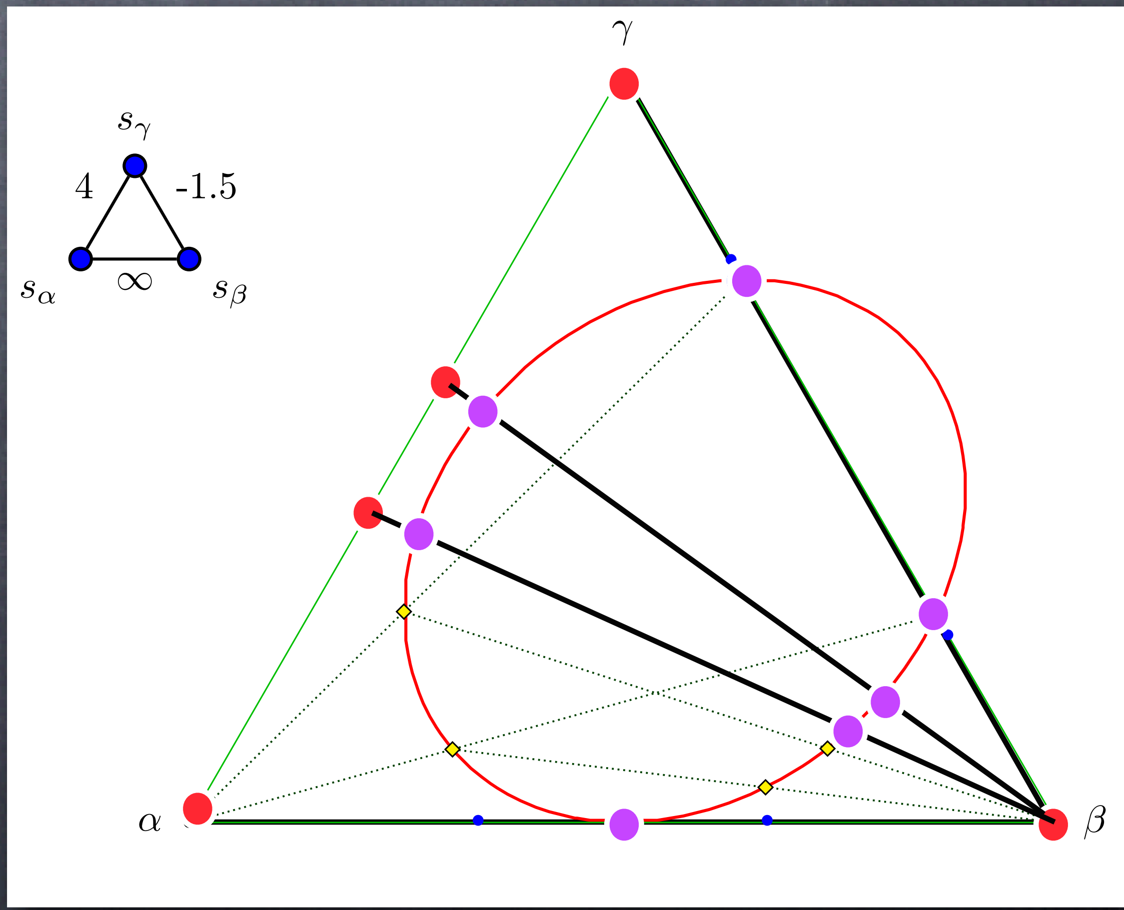
Theorem (Brink-Howlett, 1993)
The set Σ of small roots is finite.

Crucial tool for building a finite state automaton for Coxeter groups

Consider the finite subset:

$$E_f(\Phi) := \bigcup_{\gamma,\rho\in\Sigma} \widehat{Q} \cap L(\widehat{\gamma}, \widehat{\rho})$$

Theorem 3 (Dyer-Hohlweg-R. 2011)
The set $W \cdot E_f(\Phi)$ is dense in $E(\Phi)$.



Limit roots and parabolic subgroups

Consider $\Delta_I \subseteq \Delta$ and $V_I := \mathrm{span}(\Delta_I)$.

- Standard parabolic subgroup: $W_I := \langle s_\alpha \,|\, \alpha \in \Delta_I \rangle$;
- Associated root system: $\Phi_I := W_I(\Delta_I)$.

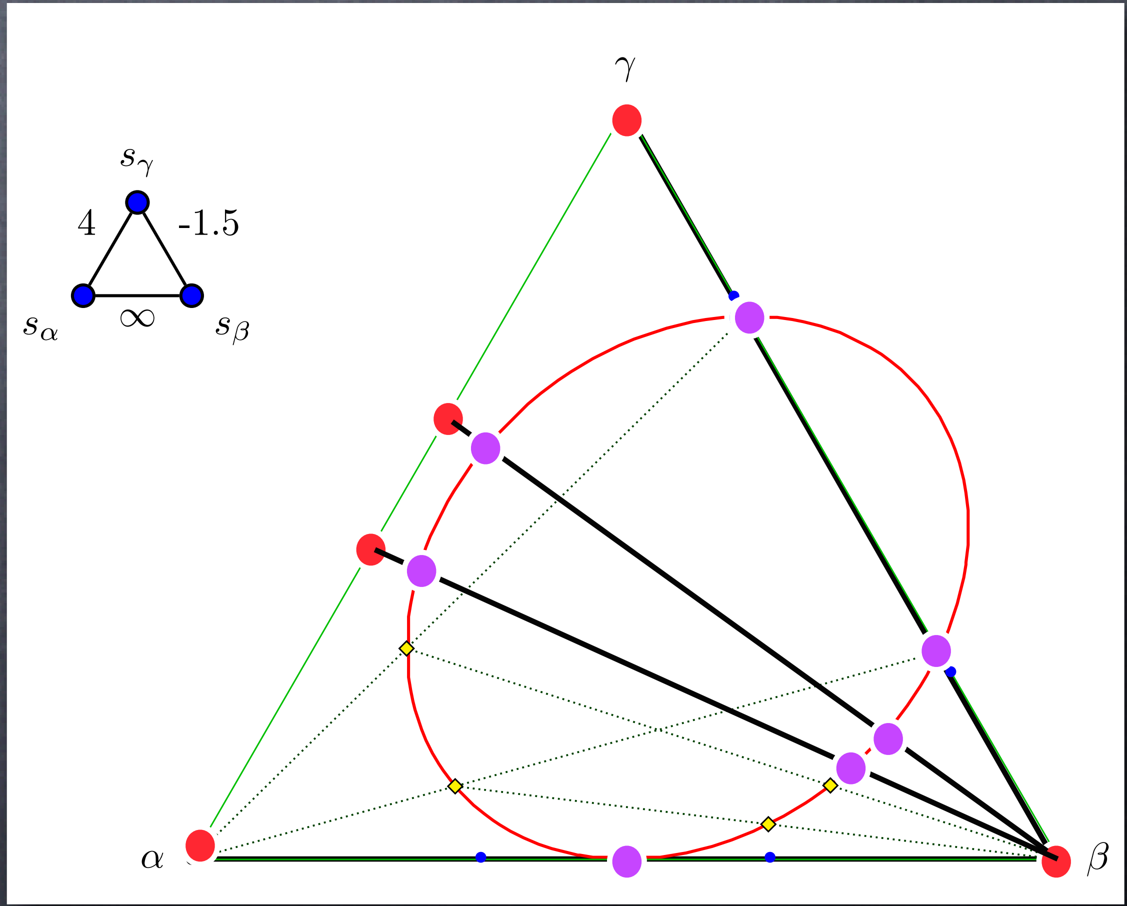
Remark: $E(\Phi_I) \neq E(\Phi) \cap V_I$ in general (e.g. rank 5).

Theorem 4 (Dyer-Hohlweg-R. 2011)

For $\Delta_I \subseteq \Delta$, we have

$$W_I \cdot E_f(\Phi_I) = (W \cdot E_f(\Phi)) \cap V_I$$

Remark: Same properties with a smaller set related to the fundamental coverings of the dominance order.

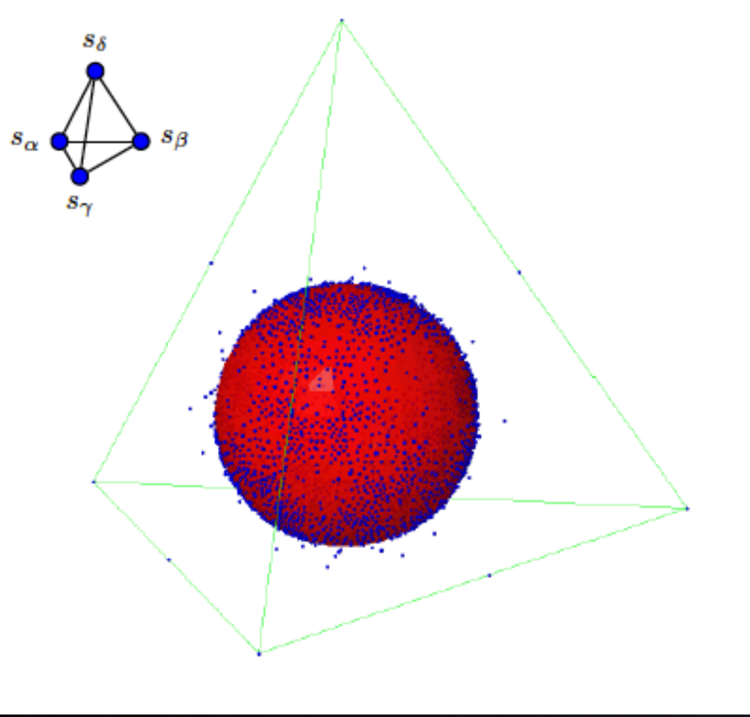


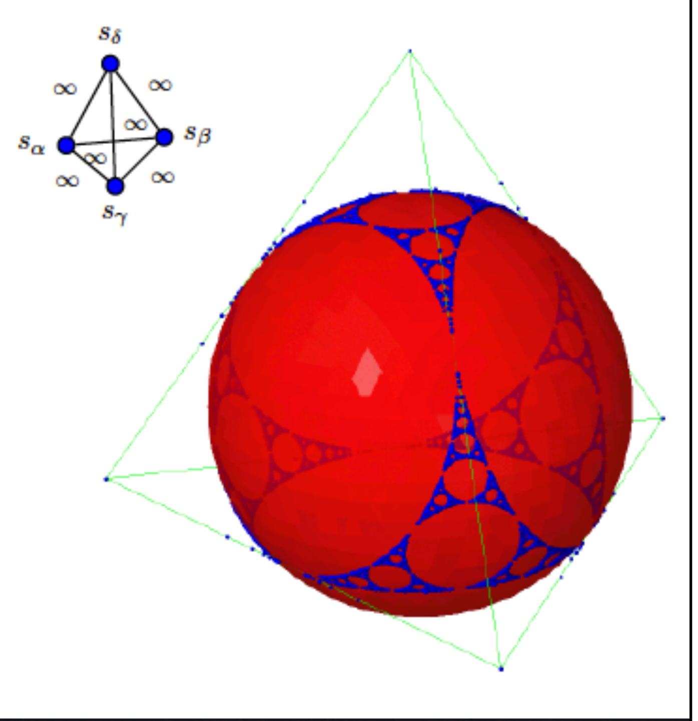
A fractal phenomenon?

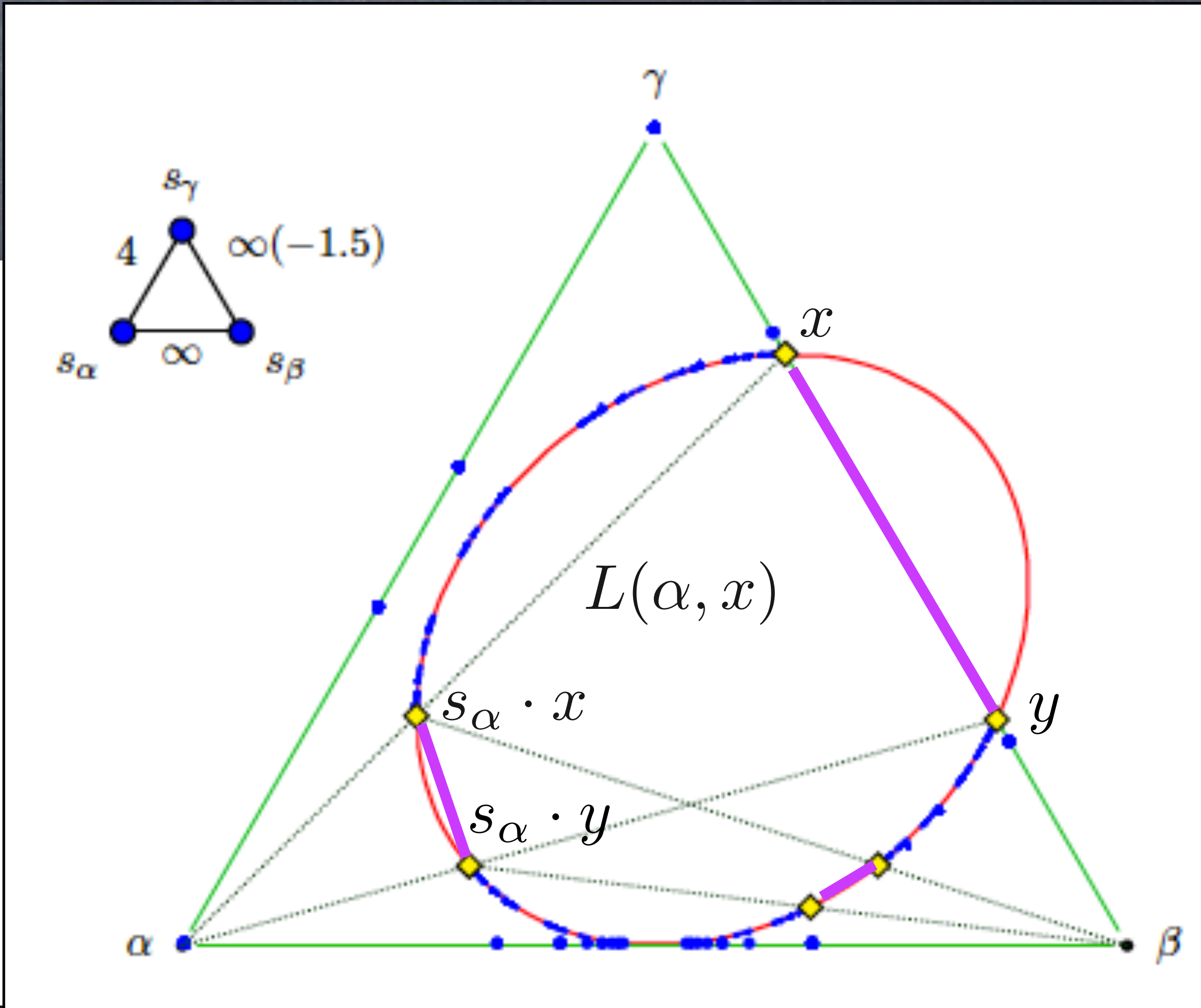
(conjectures/questions, work in progress with Ch. Hohlweg)

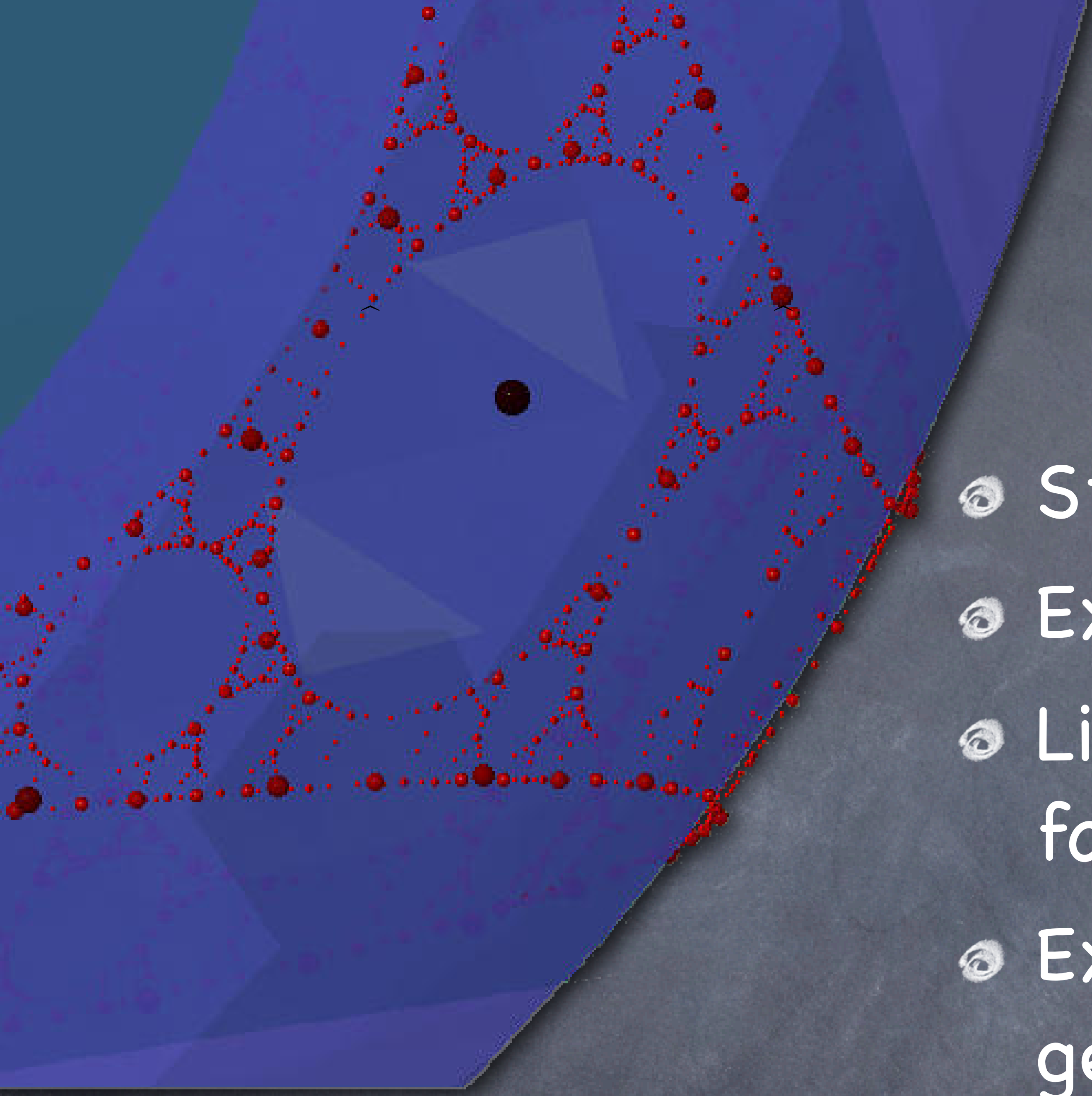
- If $\widehat{Q} \subseteq \mathrm{conv}(\Delta)$, then $E(\Phi) = \widehat{Q}$?
- In general : $E(\Phi) = \widehat{Q} \setminus$ all the images by the action of W of the parts of $\widehat{Q}$ outside the simplex, i.e.:

$$E(\Phi) = \widehat{Q} \cap \bigcap_{w \in W} w \cdot \mathrm{conv}(\Delta) \quad ?$$

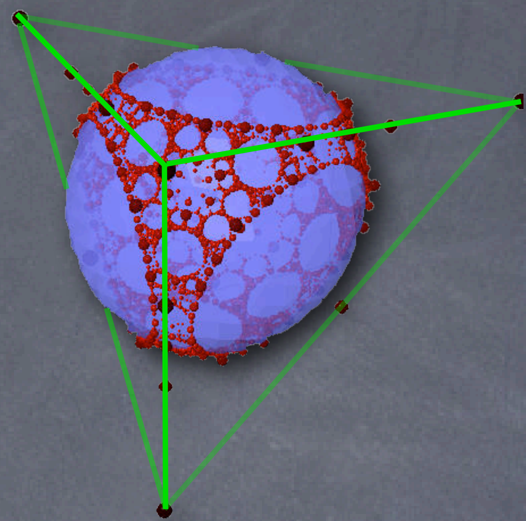




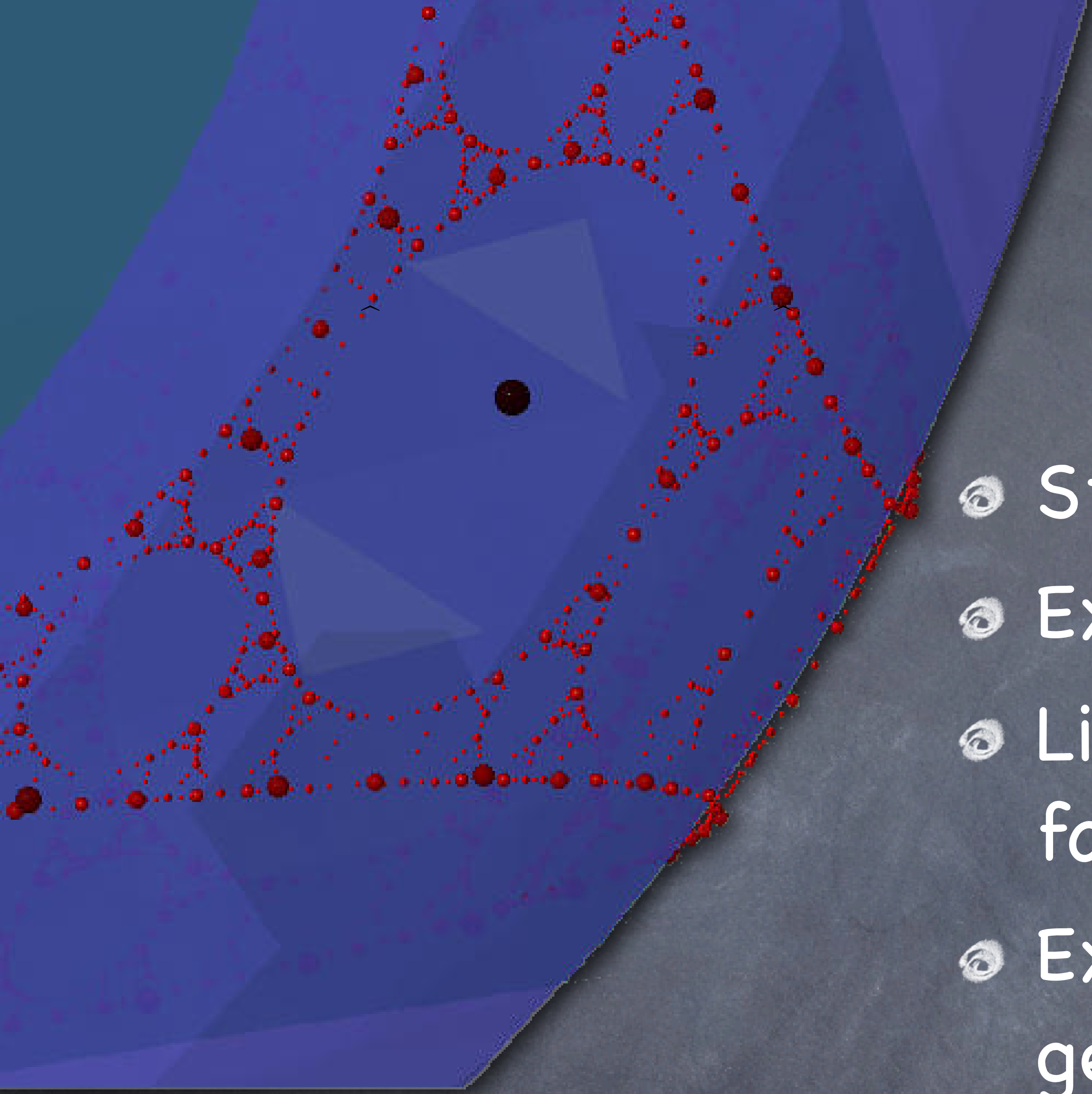




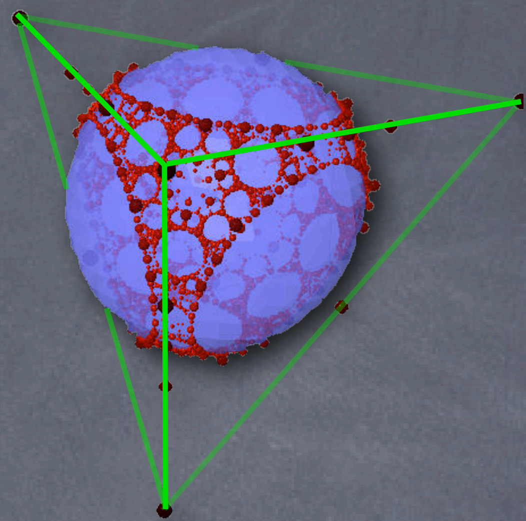
Further works



- Study the action of W on $E(\Phi)$.
- Explain the fractal phenomenon.
- Link with Dyer's imaginary cone for Coxeter groups.
- Extend the results to more general root systems.
- Applications to the study of `biclosed' and `biconvex' sets of roots?
- Links with Apollonian gaskets, Kleinian groups, sphere packings?
- ...



Further works



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Merci ! / Thank you!