Asymptotical behaviour of roots in infinite Coxeter groups

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Joint work with ***Christophe Hohlweg*** *(UQÀM) and* ***Jean-Philippe Labbé*** *(FU Berlin)*

What is a root system? (in this talk)

- V: a real vector space, of finite dimension n
- B: a symmetric bilinear form on V

Construction of a root system in (V, B):

1. Start with a simple system Δ:
   - Δ is a basis for V;
   - $\forall \alpha \in \Delta, B(\alpha, \alpha) = 1$;
   - $\forall \alpha \neq \beta \in \Delta$:
     - either $B(\alpha, \beta) = -\cos\left(\frac{\pi}{m}\right)$ for some $m \in \mathbb{Z}_{\geq 2}$,
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What is a root system?

2. For each $\alpha \in \Delta$, define the *B*-reflection s_α:

$$s_\alpha : \begin{array}{ccc} V & \to & V \\ v & \mapsto & v - 2B(\alpha, v)\,\alpha. \end{array}$$

Check: $s_\alpha(\alpha) = -\alpha$, and s_α fixes pointwise $\alpha^\perp$.

Notation: $S = \{s_\alpha,\ \alpha \in \Delta\}$.

3. Construct the *B*-reflection group $W := \langle S \rangle$.

4. Act by W on Δ to construct the root system

$$\Phi := W(\Delta)\ .$$

Note: if $\rho = w(\alpha)$ (with $\alpha \in \Delta$), $w s_\alpha w^{-1}$ is the *B*-reflection associated to the root ρ.

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Coxeter group and root system

Proposition (Krammer)

- *(W, S) is a Coxeter system.*
- *The order of $s_\alpha s_\beta$ is m if $B(\alpha, \beta) = -\cos(\pi/m)$, and ∞ if $B(\alpha, \beta) \leq -1$.*
- *Let $\Phi^+ := \Phi \cap \mathrm{cone}(\Delta)$. Then: $\Phi = \Phi^+ \sqcup (-\Phi^+)$.*

Note: Conversely, from any Coxeter system it is possible to construct a root system, using the classical geometric representation [Tits].

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Infinite root systems

Finite root systems are well studied :
Φ is finite $\Leftrightarrow$ W is finite ($\Leftrightarrow$ B is positive definite).

What happens when Φ is infinite?

Simplest example in rank 2:

Matrix of B in the basis (α, β): $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

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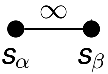

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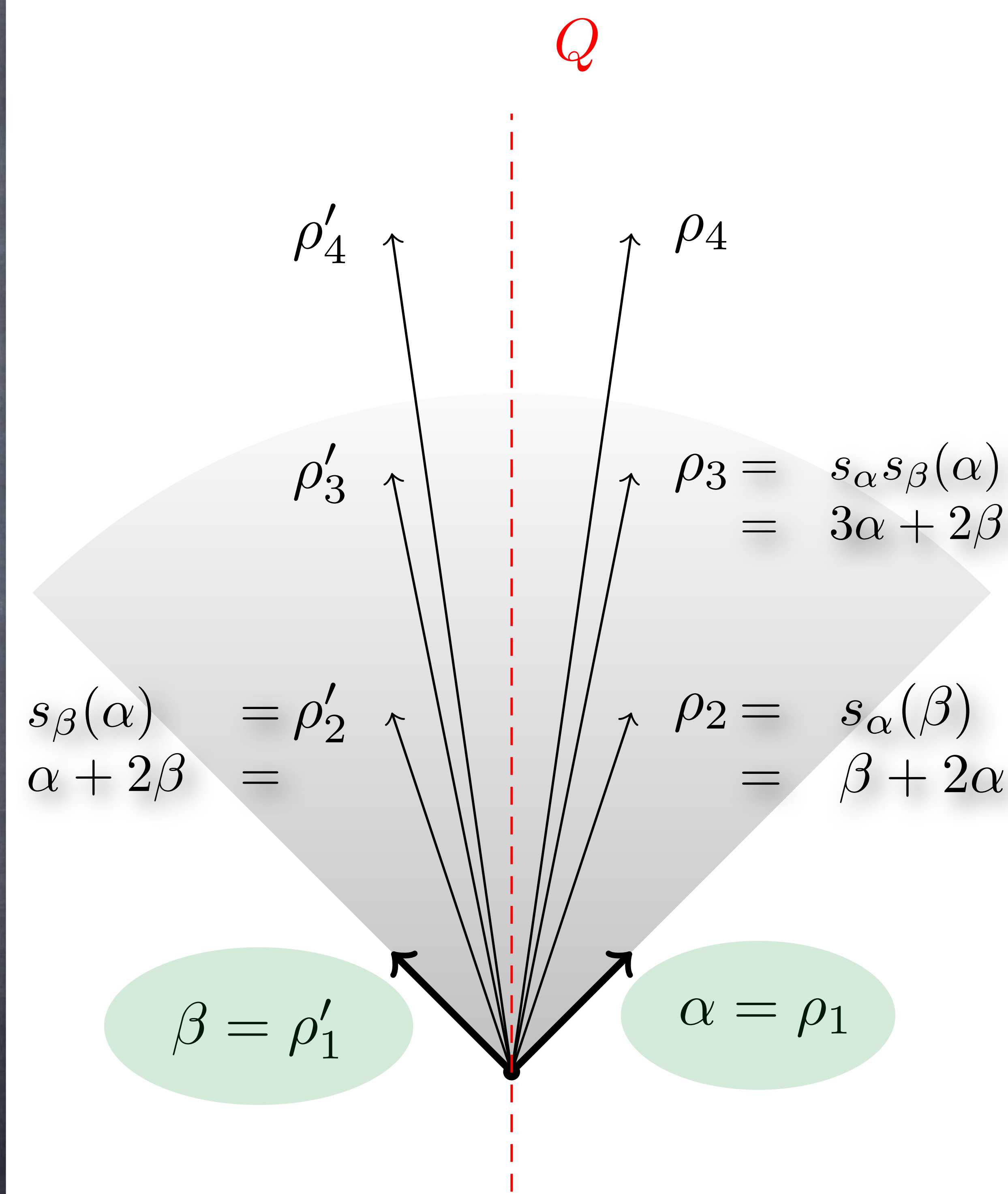
What is a root system ?

$\rho'_n = n\alpha + (n+1)\beta$

$\rho_n = (n+1)\alpha + n\beta$



(a) $B(\alpha, \beta) = -1$

$$s_\alpha(v) = v - 2B(v, \alpha)\alpha.$$

Observations

- The **norms** of the roots tend to ∞;
- The **directions** of the roots tend to the direction of the isotropic cone Q of B:

$$Q := \{v \in V,\ B(v,v) = 0\}.$$

(in the example the equation is $v_\alpha^2 + v_\beta^2 - 2v_\alpha v_\beta = 0$, and $Q = \text{span}(\alpha + \beta)$.)

What if $B(\alpha, \beta) < -1$?

- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write

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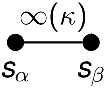
- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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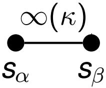
- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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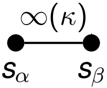
- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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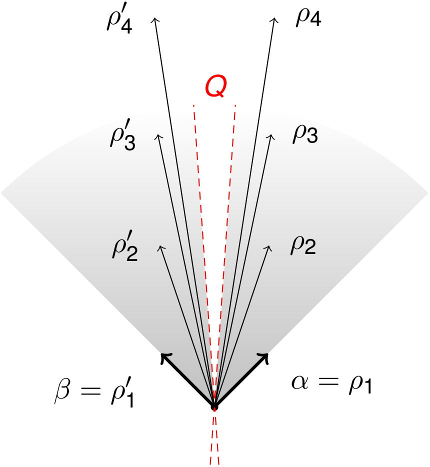
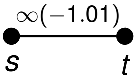
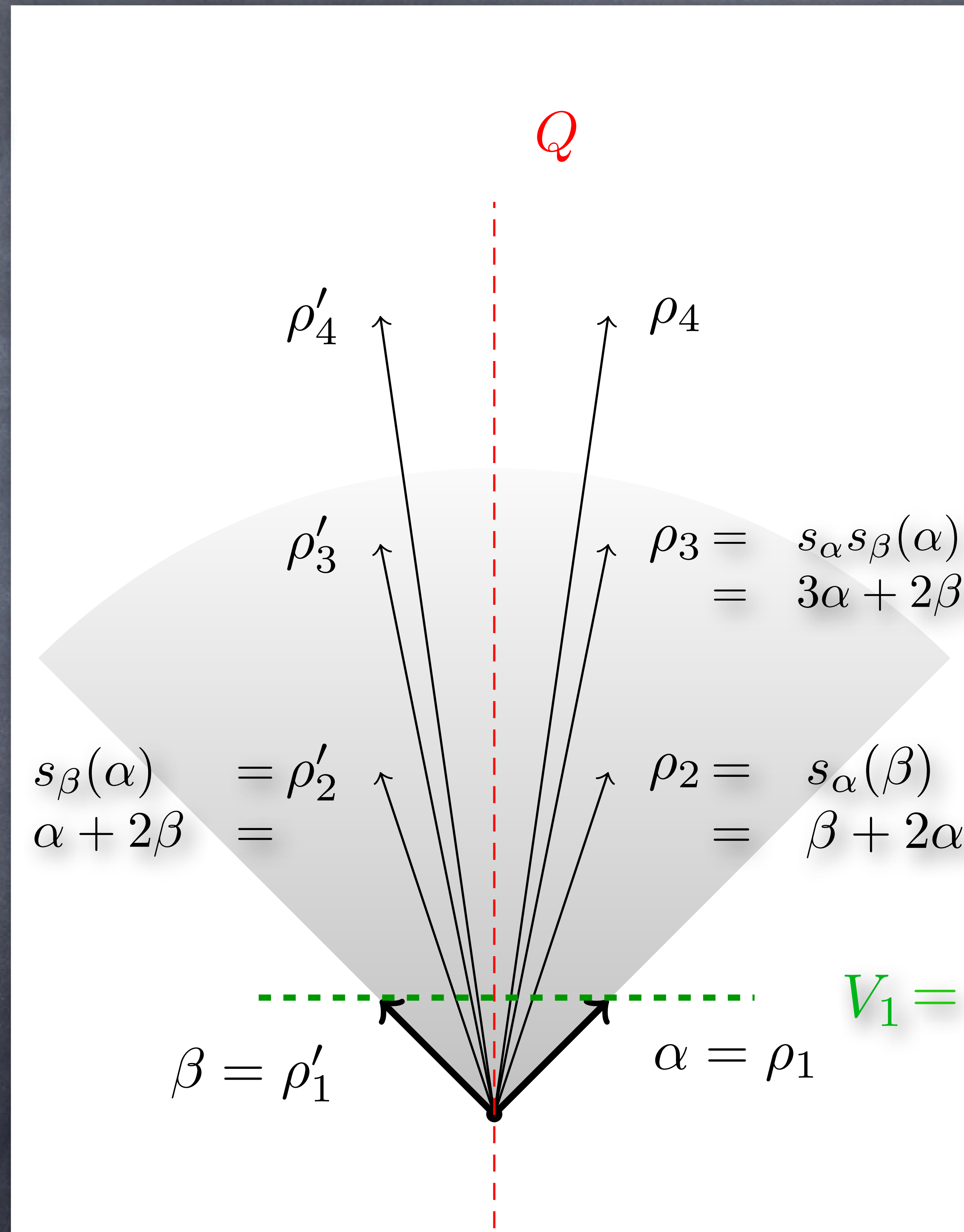




How to see examples of higher rank?

$\rho'_n = n\alpha + (n+1)\beta$

$\rho_n = (n+1)\alpha + n\beta$

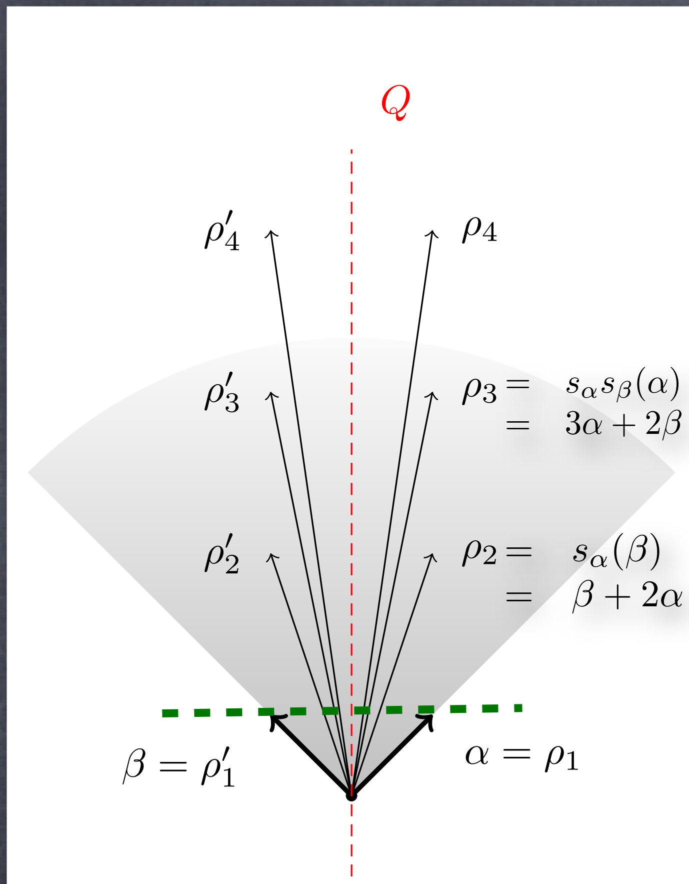


(a) $B(\alpha, \beta) = -1$

'Cut' the rays of Φ^+ by an affine hyperplane

$V_1 = \{v \in V \mid \sum_{\alpha \in \Delta} v_\alpha = 1\}$

How to see examples of higher rank?



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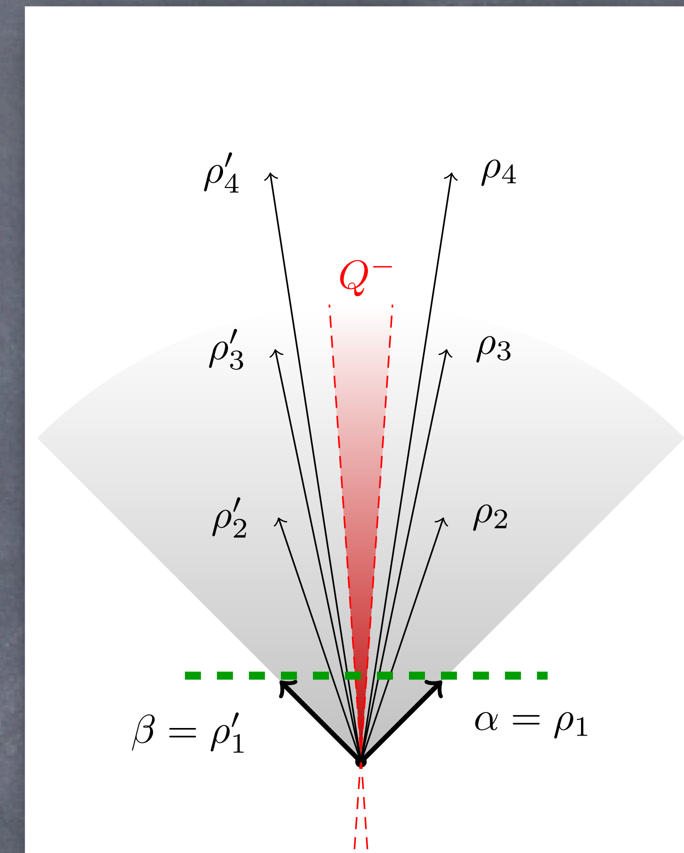
Affine hyperplane

$$V_1 = \{v \in V \mid \sum_{\alpha \in \Delta} v_\alpha = 1\}$$

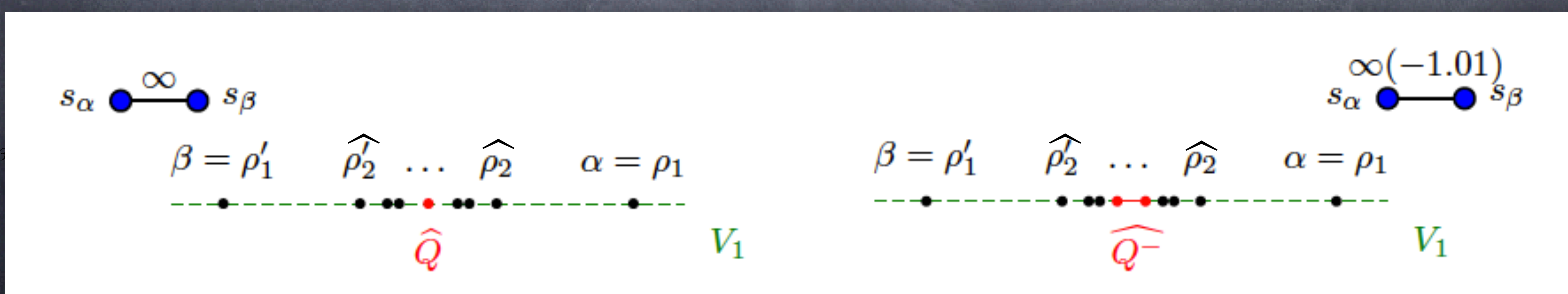
Normalized isotropic cone: $\widehat{Q} := Q \cap V_1$

Normalized roots

$$\widehat{\rho} := \rho / \sum_{\alpha \in \Delta} \rho_\alpha$$



(b) $B(\alpha, \beta) = -1.01 < -1$



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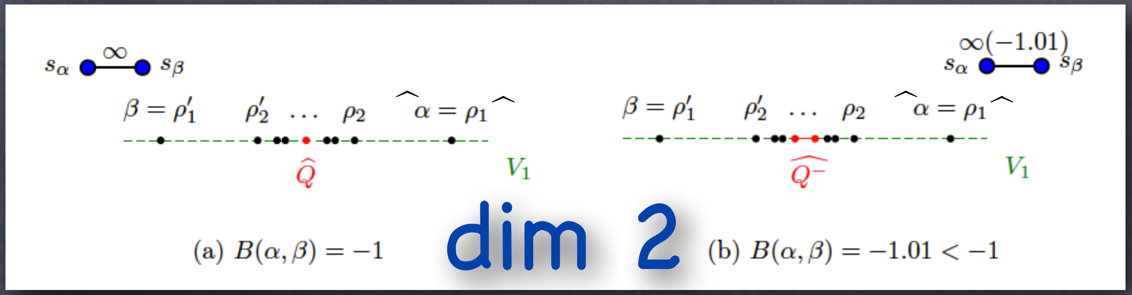
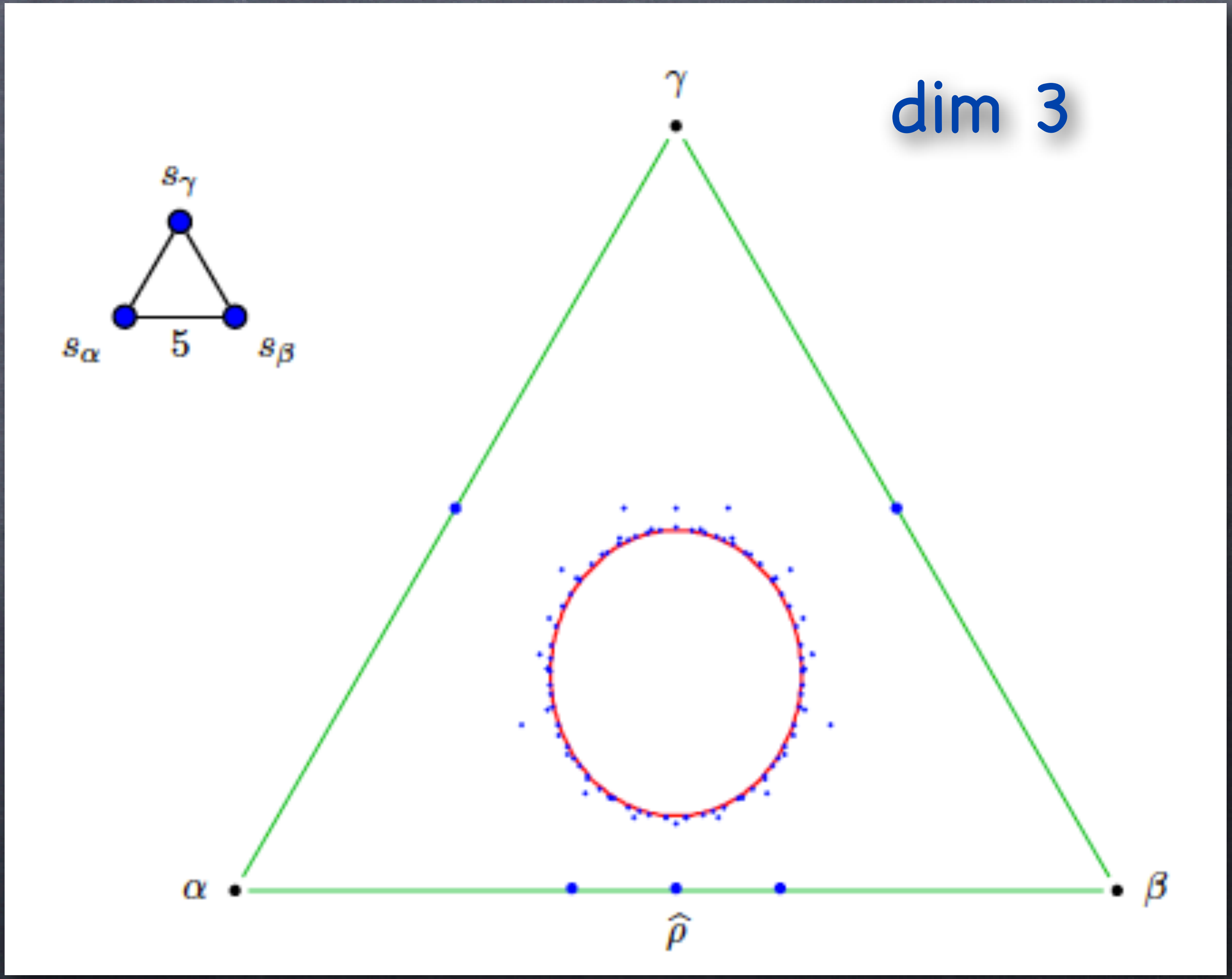
Other examples of infinite root systems in rank 3 and 4





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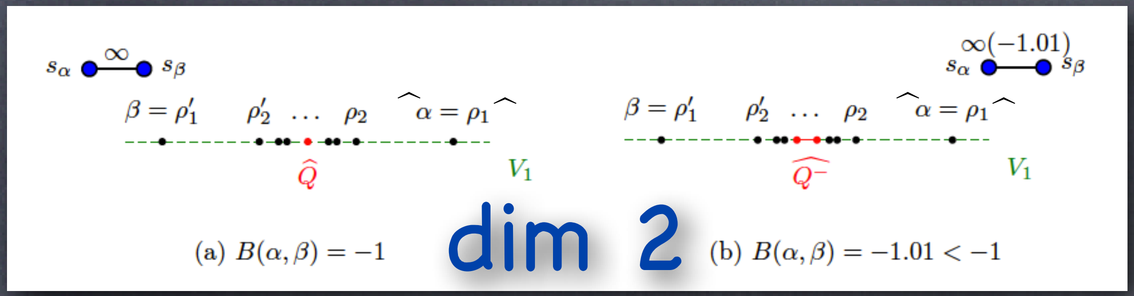

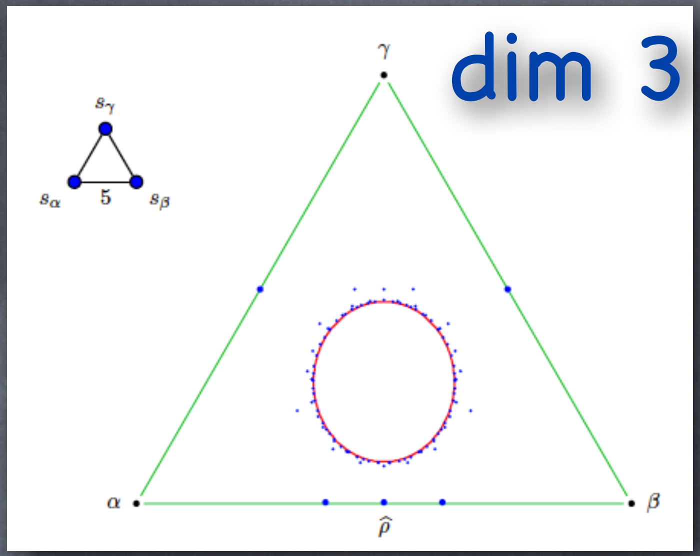

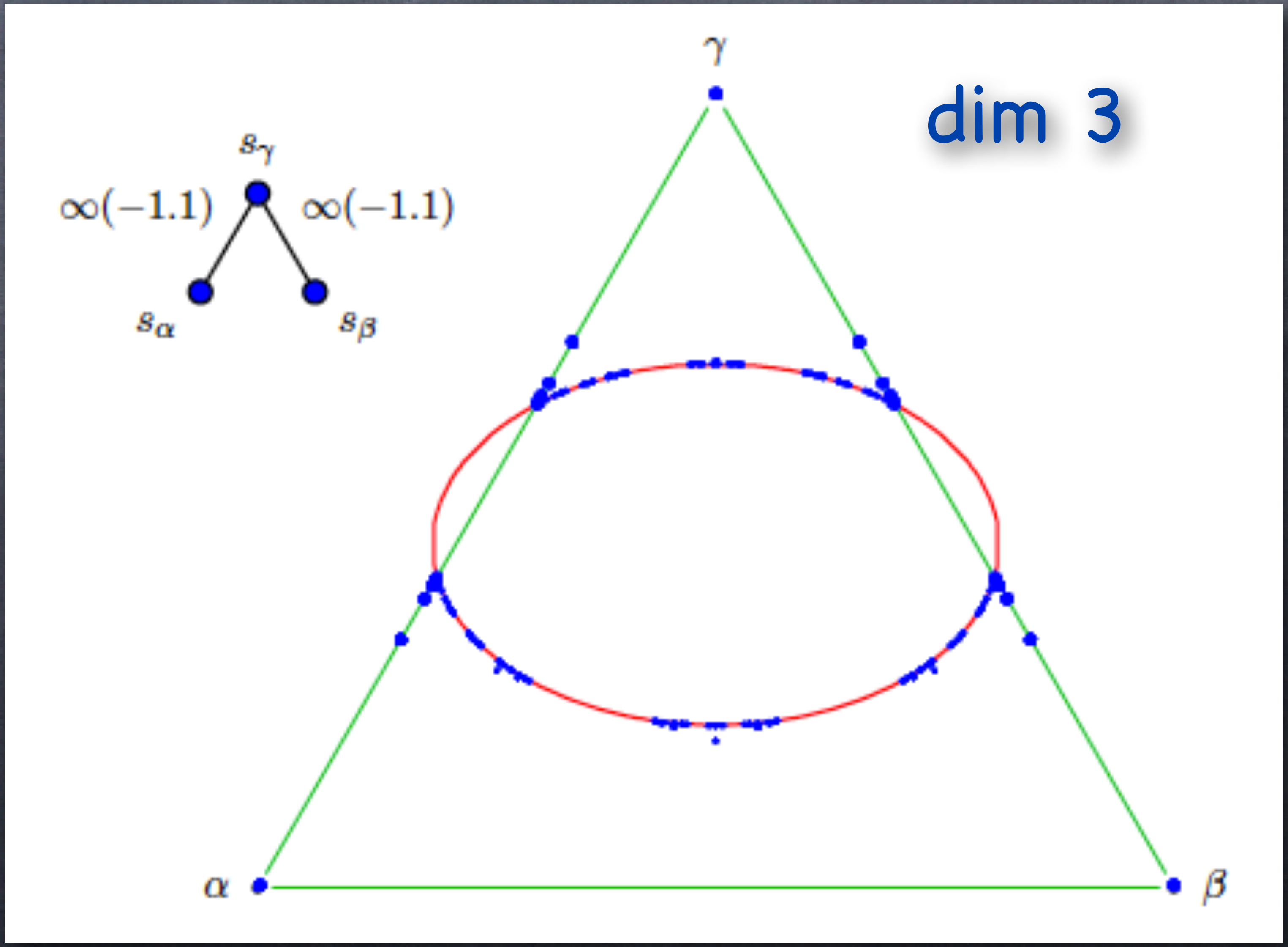

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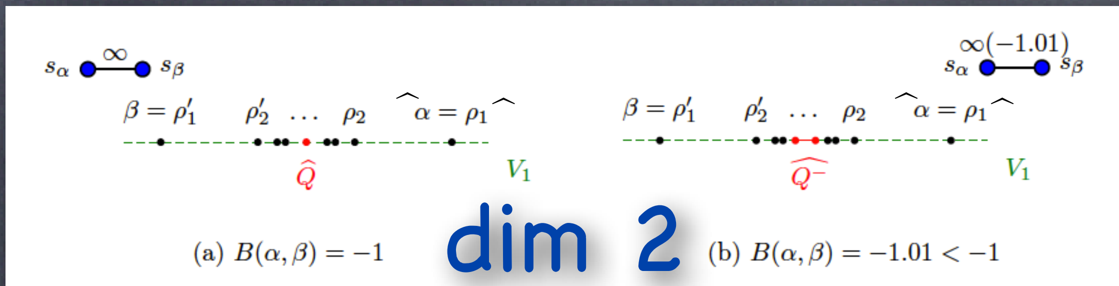

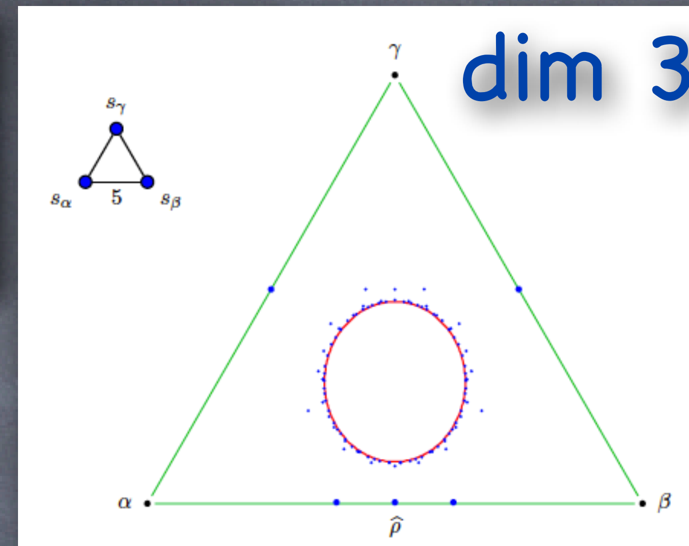

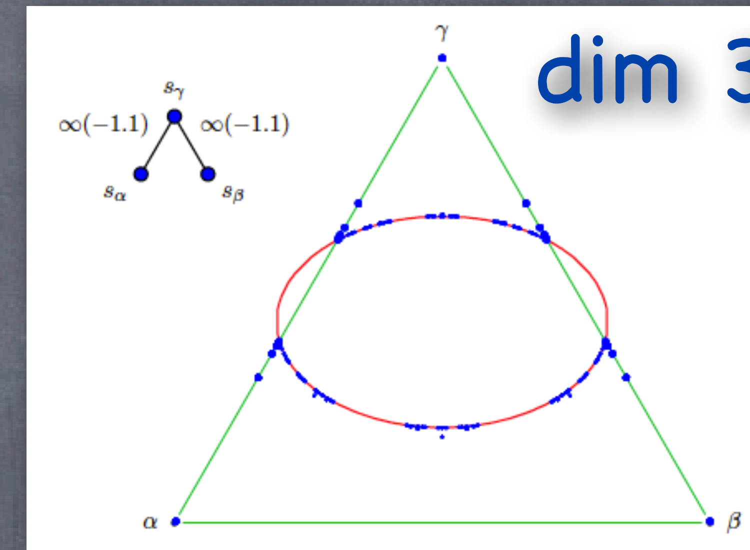

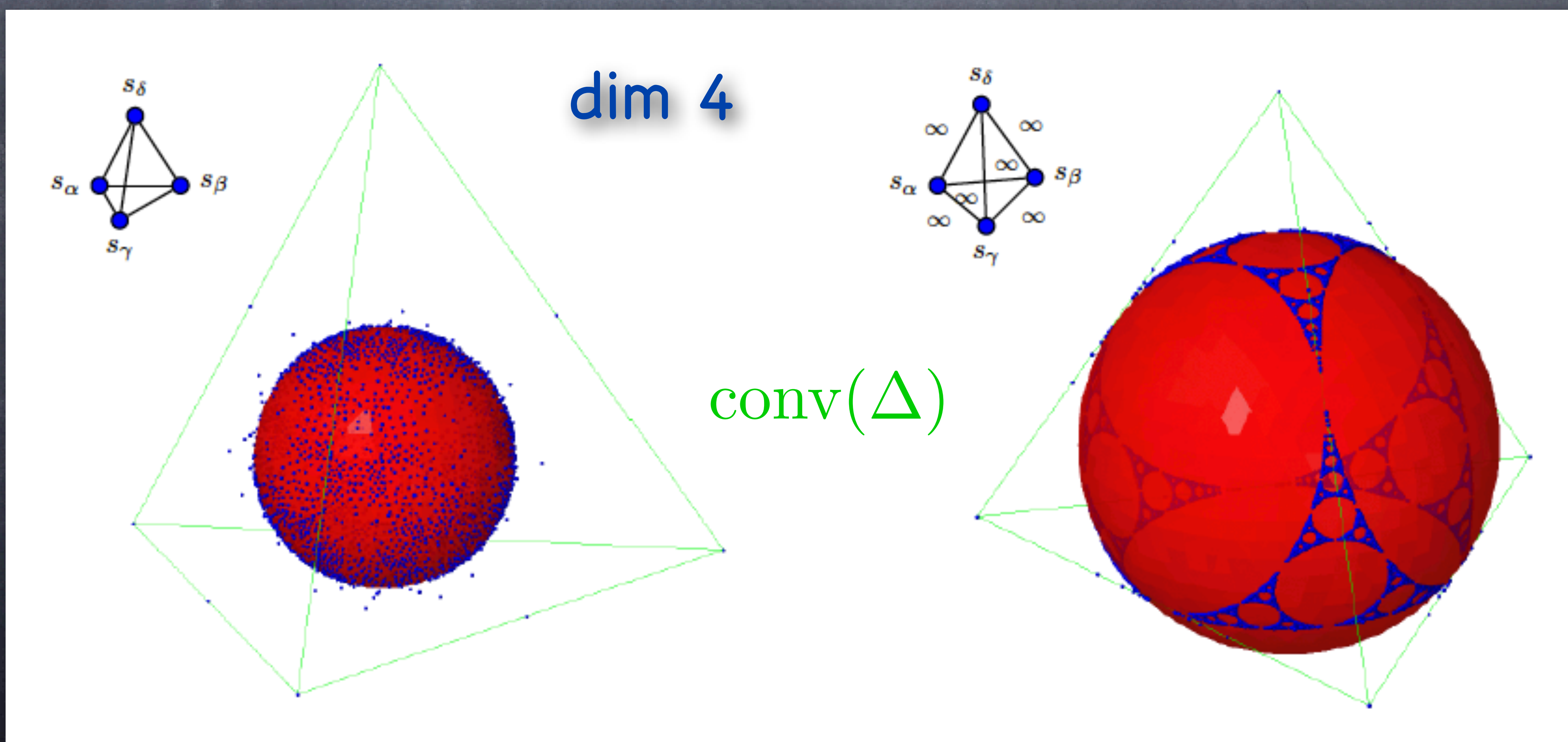

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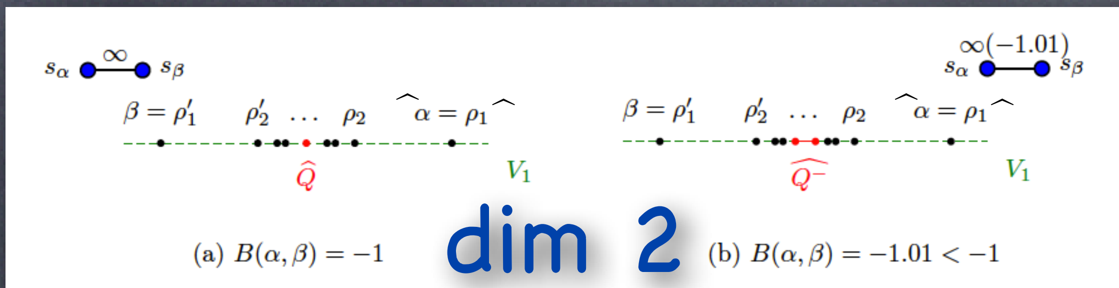

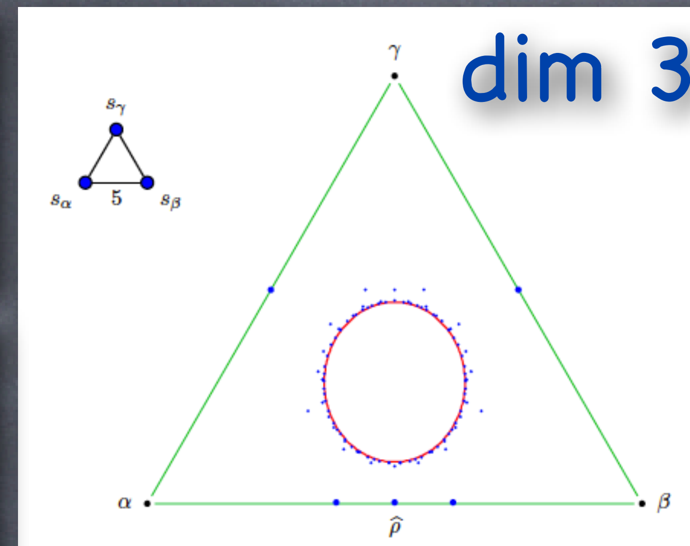

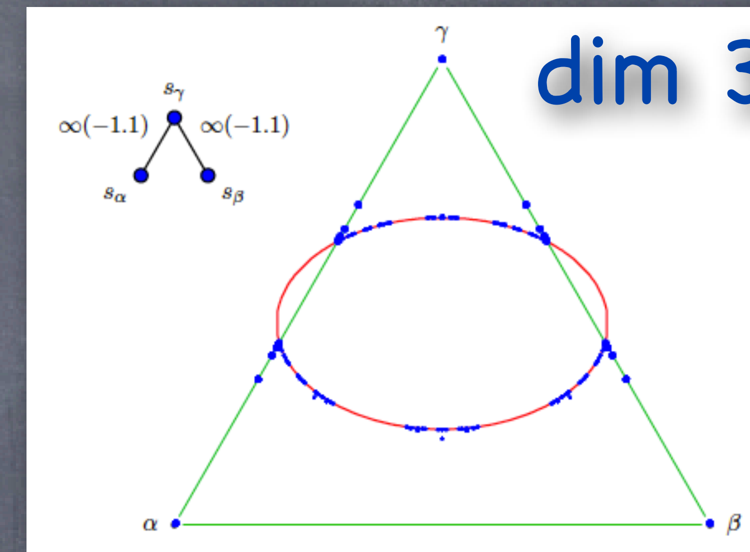

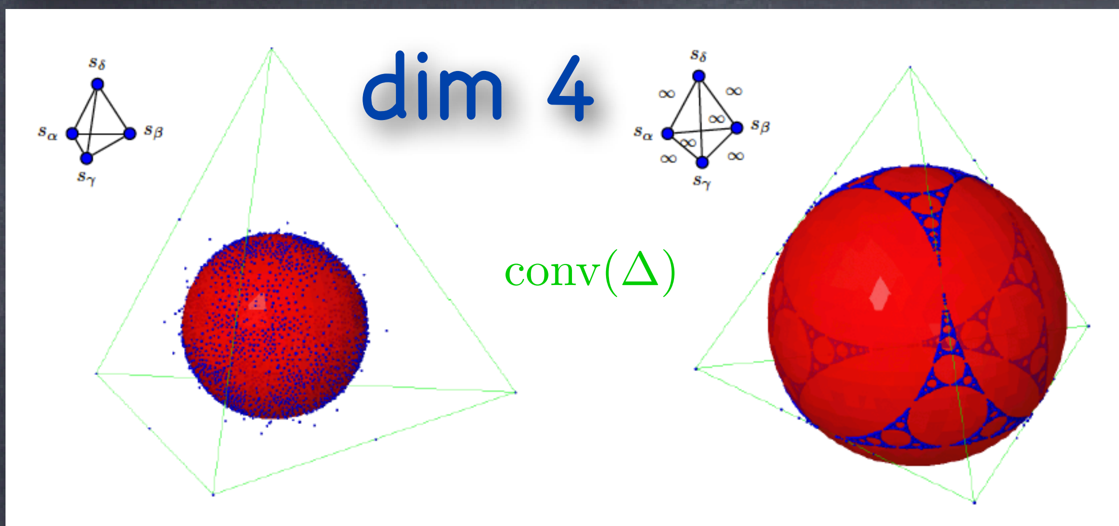

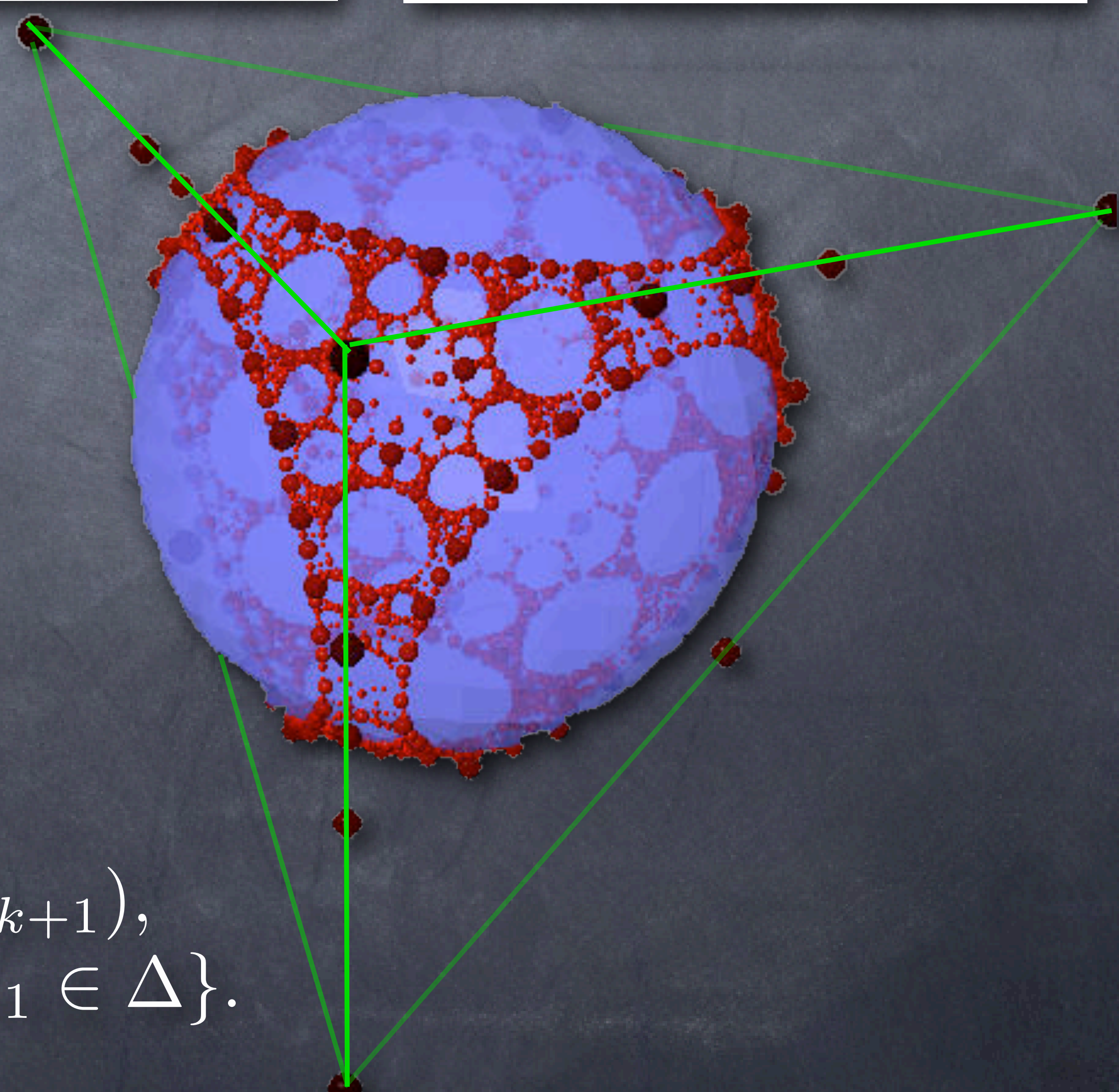

The displayed size of a normalized root (in red in this last picture) is decreasing as the depth of the root is increasing.

$$\mathrm{dp}(\rho) = 1 + \min\{k \mid \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}),\ \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}.$$

The "limit roots" lie in the isotropic cone Q

Theorem (Hohlweg-Labbé-R.)

Let Φ be a root system for an (infinite) Coxeter group, and $(\rho_n)_{n\in\mathbb{N}}$ an injective sequence in Φ. Then:

1. $||\rho_n||$ *tends to* ∞ *(for any norm on* V*);*
2. *if the sequence of normalized root* $\widehat{\rho_n}$ *has a limit* ℓ*, then*

$$\ell \in \widehat{Q} \cap \mathrm{conv}(\Delta).$$

Known in other contexts:

- Root systems of Lie algebras (Kac, 1990)
- Imaginary cone for Coxeter groups (Dyer, 2011)

$\rightsquigarrow$ **Problem:** understand the set of possible limits, i.e., the accumulation points of $\widehat{\Phi}$:

$$E(\Phi) := \mathrm{Acc}\left(\widehat{\Phi}\right) \qquad (\text{"limit roots"}).$$

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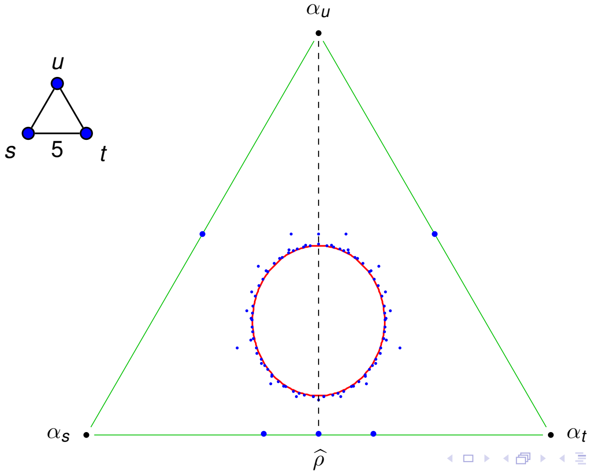
How to construct some particular limit roots

Take two roots ρ_1, ρ_2 in $\Phi \rightsquigarrow$ get a rank 2 reflection subgroup of W, and a root subsystem Φ'. Note:

- $\widehat{\Phi'} \subset L(\widehat{\rho_1}, \widehat{\rho_2})$;
- the isotropic cone for Φ' is $Q \cap \mathrm{span}(\rho_1, \rho_2)$;
- $\Rightarrow$ Limit roots for Φ': $E(\Phi') = Q \cap L(\widehat{\rho_1}, \widehat{\rho_2})$ (0,1 or 2 points).



The dihedral limit roots

Definition

We define the set $E_2(\Phi)$ of dihedral limit roots for the root system Φ as the subset of $E(\Phi)$ formed by the union of the $E(\Phi')$, for Φ' a root subsystem of rank 2 of Φ. Equivalently,

$$E_2(\Phi) := \bigcup_{\rho_1,\rho_2\in\Phi} L(\widehat{\rho_1},\widehat{\rho_2}) \cap Q.$$

Note: E_2 is countable.

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The set of dihedral limit roots E_2 is dense in E.

- E is closed, so $E = \overline{E_2}$;
- in general, E_2 is not equal to E. In fact sometimes $E = \widehat{Q}$!

The dihedral limit roots

Definition

We define the set $E_2(\Phi)$ of dihedral limit roots for the root system Φ as the subset of $E(\Phi)$ formed by the union of the $E(\Phi')$, for Φ' a root subsystem of rank 2 of Φ. Equivalently,

$$E_2(\Phi) := \bigcup_{\rho_1,\rho_2 \in \Phi} L(\widehat{\rho_1}, \widehat{\rho_2}) \cap Q.$$

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Other properties, further questions

- How does E behave in regard to restriction to parabolic subgroups ($E(\Phi_I) \neq E(\Phi) \cap V_I$ in general!)
- Natural action of W on E, easy to describe geometrically... Faithfulness?
- Explain the fractal, self-similar shapes of the pictures! We can use the action to interpret this, but we only have conjectures.
- Take $x \in E$. Is it true that $\overline{W \cdot x} = E$?
- Study conv(E), which equals the closure of Dyer's "imaginary cone".

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Thank you!

A fractal phenomenon?

(conjectures/questions, work in progress with Ch. Hohlweg)

- If $\widehat{Q} \subseteq \mathrm{conv}(\Delta)$, then $E(\Phi) = \widehat{Q}$?
- In general : $E(\Phi) = \widehat{Q} \setminus$ all the images by the action of W of the parts of $\widehat{Q}$ outside the simplex, i.e.:

$$E(\Phi) = \widehat{Q} \cap \bigcap_{w \in W} w \cdot \mathrm{conv}(\Delta) \quad ?$$