Coxeter elements in well-generated complex reflection groups

Vivien Ripoll

(Universität Wien)

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joint work with
Vic Reiner (Minneapolis) and
Christian Stump (Berlin)

Motivation

$\mathrm{NC}(n) := \{w \in \mathfrak{S}_n \mid \ell_T(w) + \ell_T(w^{-1}c) = \ell_T(c)\}$, where

- $T := \{$all transpositions of $\mathfrak{S}_n\}$, ℓ_T associated length function ("absolute length");
- c is a long cycle (n-cycle).

$\mathrm{NC}(n)$ is

- equipped with a natural partial order ("absolute order"), and is a lattice;
- isomorphic to the poset of NonCrossing partitions of an n-gon ("noncrossing partition lattice"), so it is counted by the Catalan number $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

Generalization to finite Coxeter groups (or reflection groups):

- replace $\mathfrak{S}_n$ with a Coxeter group W;
- replace T with $R := \{\text{all reflections of } W\}$, and ℓ_T with ℓ_R;
- replace c with a Coxeter element of W.

the W-noncrossing partition lattice

$$\mathrm{NC}(W, c) := \{w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c)\}$$

- also equipped with a "W-absolute order";
- counted by the W-Catalan number $\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{d_i+h}{d_i}$.

$\mathrm{Cat}(W)$ appears in other combinatorial objects attached to (W, c): cluster complexes, generalized associahedra, Cambrian fans and lattices, subword complexes...
$\rightsquigarrow$ "Coxeter-Catalan combinatorics".

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Outline

1. "Classical" definitions of a Coxeter elements
   - ... for a Coxeter system (W, S)
   - ... for a real reflection group
   - ... for a complex reflection group

2. Extended definitions
   - ... with alternative Coxeter structures
   - ... with reflection automorphisms
   - ... with other eigenvalues
   - Main result and consequences on Coxeter-Catalan combinatorics

3. Galois automorphisms
   - Field of definition of W and Galois automorphisms
   - Galois action on conjugacy classes of Coxeter elements

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Coxeter element of a Coxeter system

Definition

A Coxeter system (W, S) is a group W equipped with a generating set S of involutions, such that W has a presentation of the form:

$$W = \left\langle S \mid s^2 = 1 \ (\forall s \in S); \ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \right\rangle ,$$

with $m_{s,t} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t$.

Coxeter element, "Definition 0"

Write $S := \{s_1, \ldots, s_n\}$. A Coxeter element of (W, S) is a product of all the generators:

$$c = s_{\pi(1)} \cdots s_{\pi(n)} \quad \text{for } \pi \in \mathfrak{S}_n .$$

Fact: When W is **finite**, all Coxeter elements of (W, S) are conjugate. (ingredient: the Coxeter graph is a forest)

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Coxeter element of a real reflection group

- V real vector space of dimension n
- W finite subgroup of $\mathrm{GL}(V)$ generated by reflections

$\rightsquigarrow$ W admits a structure of **Coxeter system**:

- fix a chamber $\mathcal{C}$ of the hyperplane arrangement of W
- take $S := \{\text{reflections through the walls of } \mathcal{C}\}$

Definition ("Classical definition")

Let W be a finite real reflection group. A Coxeter element of W is a product (in any order) of all the reflections through the walls of a chamber of W.

Proposition

The set of Coxeter elements of W forms a conjugacy class.

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Complex reflection group

- V complex vector space of dimension n
- W finite subgroup of $\mathrm{GL}(V)$ generated by "reflections" ($r \in \mathrm{GL}(V)$ of finite order and fixing pointwise a hyperplane)

Finite *real* reflection groups can be seen as complex reflection groups.

But there are much more.
In general: no Coxeter structure, no privileged (natural, canonical) set of n generating reflections.
$\leadsto$ how to define a Coxeter element of W?

Digression: geometry of Coxeter elements in real groups

Assume W is real and irreducible.

Call $h :=$ Coxeter number $=$ the order of a Coxeter element.

Fact: $h = d_n$, the highest invariant degree of W.

$d_1 \leq \cdots \leq d_n$ degrees of homogeneous polynomials $f_1, \ldots, f_n$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.

Proposition (Coxeter)

If c is a Coxeter element, then there exists a plane $P \subseteq V$ stable by c and on which c acts as a rotation of angle $\frac{2\pi}{h}$.
In particular, c admits $e^{\frac{2i\pi}{h}}$ (and $e^{-\frac{2i\pi}{h}}$) as an eigenvalue.

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Geometry of Coxeter elements in real groups

Better: c is $e^{\frac{2i\pi}{h}}$-regular in the sense of Springer: it has a $e^{\frac{2i\pi}{h}}$-eigenvector $v \in V_{\mathbb{C}}$, which does not lie in the reflecting hyperplanes.

[Springer] : the set of ζ-regular elements (in a complex reflection group W) form a W-conjugacy class.

Proposition

$$c \text{ is a Coxeter element in } W$$
$$\Updownarrow$$
$$c \text{ admits } e^{\frac{2i\pi}{h}} \text{ as an eigenvalue}$$
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Coxeter element in a complex reflection group

Now W is a well-generated, irreducible complex reflection group: W can be generated by $n = \dim V$ reflections. Define the Coxeter number h of W as the highest invariant degree: $h := d_n$.

The set of elements of W having $e^{\frac{2i\pi}{h}}$ as eigenvalue

- is non-empty and forms a conjugacy class of W [Springer] ;
- = the set of elements having $e^{\frac{2i\pi}{h}}$ as eigenvalue.

Definition ("classical definition", Bessis '06)

Let W be a well-generated, irreducible complex reflection group. A Coxeter element of W is an element that admits $e^{\frac{2i\pi}{h}}$ as an eigenvalue.

Bessis' seminal work related to Coxeter-Catalan combinatorics and the dual braid monoid for complex groups uses this definition.

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Alternative Coxeter structures

In general a real reflection group does not have a unique Coxeter structure:

Example

Symmetry group of the regular hexagon $= I_2(6) \simeq A_1 \times A_2$

But unicity of the structure if "S **must consist of reflections**":

Rigidity Property (Observation/Folklore?)

Let W be a finite real reflection group, R the set of all reflections of W. Let $S, S' \subseteq R$ be such that (W, S) and (W, S') are both Coxeter systems.
Then (W, S) and (W, S') are isomorphic Coxeter systems.

proof not enlightening! (case-by-case check on the classification)

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New Coxeter elements

For a real reflection group W, one may be able to construct a set S of Coxeter generating reflections, which do not come from a chamber of the arrangement...

$\rightsquigarrow$ Isomorphic, but not conjugate structures!

Example of $I_2(5)$.

Definition

We call generalized Coxeter element of W a product (in any order) of the elements of some set S, where S is such that:

- S consists of reflections;
- (W, S) is a Coxeter system.

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Reflection automorphisms

(W, S) and (W, S') are isomorphic Coxeter systems $\implies$ there is an automorphism ψ of W mapping S to S'.

Fact: ψ is then a reflection automorphism of W, i.e., an automorphism of W stabilizing the set R of all reflections of W.

From the Rigidity Property we obtain:

Proposition

Let W be a finite real reflection group.

c is a generalized Coxeter element of W

$$\Updownarrow$$

$c = \psi(c_0)$ with ψ reflection automorphism and c_0 classical Coxeter element of W.

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Replace $e^{2i\pi/h}$ by another h-th root of unity

Definition ("Extended definition")

Let W be a well-generated, irreducible complex reflection group, and h its Coxeter number.

We call generalized Coxeter element an element of W that admits a primitive h-th root of unity as an eigenvalue.

Equivalently, c is a generalized Coxeter element if and only if $c = w^k$ where w is a *classical* Coxeter element and $k \wedge h = 1$.

Is this definition compatible with the extended definition for real groups ?

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Is this definition compatible with the extended definition for real groups ?

Four definitions

	Classical definition	Extended definition
W real	Product of reflections through the walls of a chamber	$\prod_{s\in S} s$, for some $S \subseteq R$, with (W, S) Coxeter
W complex	$e^{\frac{2i\pi}{h}}$ is eigenvalue	$e^{\frac{2ik\pi}{h}}$ is eigenvalue for some k, $k \wedge h = 1$

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Compatibility of the extended definitions

Theorem (Reiner-R.-Stump)

Let $c \in W$. The following are equivalent:

(i) *c has an eigenvalue of order h;*

(ii) *$c = \psi(w)$ where w is a classical Coxeter element and ψ is a reflection automorphism of W;*

(iii) *c is a Springer-regular element of order h.*

If W is real, this is also equivalent to:

(iv) *There exists $S \subseteq R$ such that (W, S) is a Coxeter system and c is the product (in any order) of elements of S.*

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Application to Coxeter-Catalan combinatorics

Corollary

W well-gen., irred. c.r.g., $R = \mathrm{Refs}(W)$. Any property

- *known for **classical** Coxeter elements, and*
- *"depending only on the combinatorics of the couple (W, R)",*

*extends to **generalized** Coxeter elements. This applies in particular to Coxeter-Catalan combinatorics, e.g.:*

- *the W-noncrossing partition lattices*

$$\mathrm{NC}(W, c) := \{w \in W \mid \ell_R(w) + \ell_R(w^{-1}c) = \ell_R(c)\}$$

  (for c a generalized Coxeter element) are isomorphic posets;
- *the number of reduced R-decompositions of a generalized Coxeter element into reflections is $\frac{n!h^n}{|W|}$;*
- *the Hurwitz action of the braid group B_n on reduced decompositions is transitive.*

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W well-gen., irred. c.r.g., $R = \mathrm{Refs}(W)$. Any property

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1. "Classical" definitions of a Coxeter elements
   - ... for a Coxeter system (W, S)
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   - ... with alternative Coxeter structures
   - ... with reflection automorphisms
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Definition

The field of definition K_W of W is

$$K_W := \langle \mathrm{tr}_V(w), w \in W \rangle .$$

Fact: the representation V of W can be realized over K_W, so K_W is the smallest field over which one can write all matrices of W.

Examples

- $K_W = \mathbb{Q}$ iff W crystallographic (Weyl group)
- $W = H_3$ or H_4 : $K_W = \mathbb{Q}(\sqrt{5})$
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Let $\Gamma := \mathrm{Gal}(K_W/\mathbb{Q})$. For $\gamma \in \Gamma$ an $w \in W$, define $\gamma(w)$ by acting on the coefficients of the matrix of w written in K_W.

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But: $\gamma(W)$ is the "same" reflection group as W in the classification, so they are conjugate: $\gamma(W) = aWa^{-1}$, for $a \in \mathrm{GL}(V)$.

$$W \xrightarrow{\gamma} \gamma(W) \xrightarrow{a^{-1}(-)a} a^{-1}\gamma(W)a = W$$

$\rightsquigarrow$ obtain a reflection automorphism ψ of W, associated to γ, defined modulo conjugation by an element of the normalizer $N_{\mathrm{GL}(V)}(W)$.

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The action of $\Gamma = \mathrm{Gal}(K_W/\mathbb{Q})$ *on* $\mathrm{Cox}(W)$ *is simply transitive :*

$$\forall C, C' \in \mathrm{Cox}(W), \exists! \gamma \in \Gamma, C' = \gamma \cdot C.$$

Consequence: $|\mathrm{Cox}(W)| = [K_W : \mathbb{Q}]$.

The proof?

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where $m_1,\ldots,m_n$ are the exponents of W $(m_i = d_i - 1)$

2. Prove that $[K_W : \mathbb{Q}] = \frac{\varphi(h)}{\varphi_W(h)}$ (*) ... case-by-case.

(*) is equivalent to Malle's characterization of K_W for W well-generated:

Theorem (Malle)

Let $\zeta = e^{2i\pi/h}$ and G_W be the setwise stabilizer of $\{\zeta^{m_1},\ldots,\zeta^{m_n}\}$ in the Galois group $\operatorname{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

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