Factorizations of a Coxeter element and discriminant of a reflection group

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Combinatorics Seminar
University of Minnesota
April 22nd 2011

Introduction

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Invariant theory of W (geometry of the discriminant Δ_W)	$\leftrightarrow$	Combinatorics of the noncrossing partition lattice of W (factorizations of a Coxeter element)

Outline

1. Noncrossing partition lattice and factorizations
   - The noncrossing partition lattice of type W
   - Factorizations of a Coxeter element
2. Geometry of the discriminant
   - Strata in the discriminant hypersurface
   - Bifurcation locus of the discriminant

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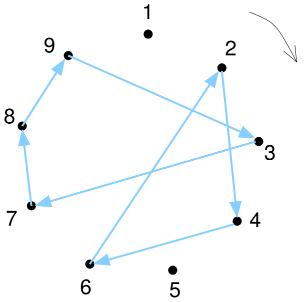

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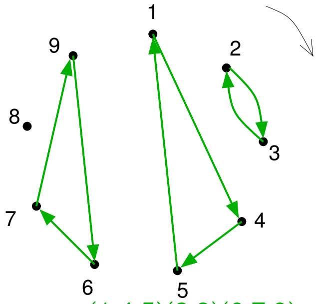

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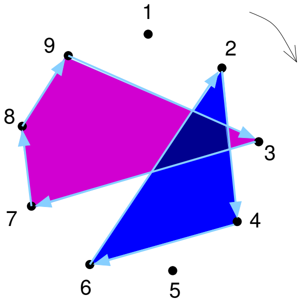

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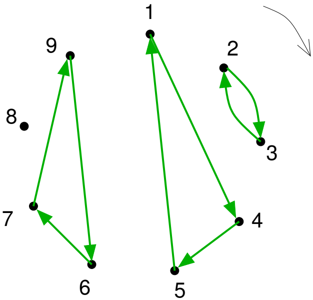

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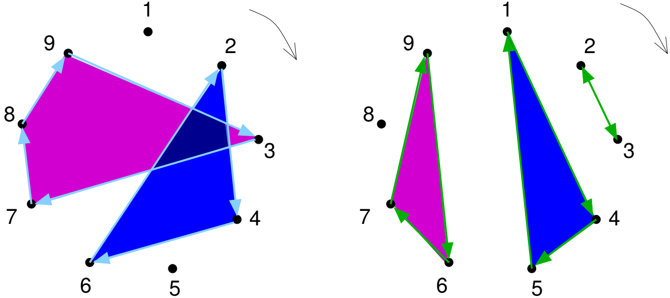

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Fuß-Catalan numbers

Kreweras's formula for multichains of noncrossing partitions

- $W := \mathfrak{S}_n$;
- c : an n-cycle.

The number of multichains $w_1 \preccurlyeq_R w_2 \preccurlyeq_R \dots \preccurlyeq_R w_p \preccurlyeq_R c$ in $\mathrm{NC}(W, c)$ is the **Fuß-Catalan number**

$$\mathrm{Cat}^{(p)}(n) = \prod_{i=2}^{n} \frac{i + pn}{i} = \frac{1}{pn+1}\binom{(p+1)n}{n}.$$

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Remark: $\mathrm{Cat}^{(1)}(W)$ (and $\mathrm{Cat}^{(p)}(W)$) appear in other contexts: Fomin-Zelevinsky cluster algebras, nonnesting partitions...

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- **Good news** : we can interpret some of them geometrically (and even refine them); in particular for $p = n$ or $n - 1$.

Submaximal factorizations of a Coxeter element

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Theorem (R.)

Let Λ be a conjugacy class of elements of length 2 of $\mathrm{NC}(W)$. Call submaximal factorizations of c of type Λ the block factorizations containing $n-2$ reflections and one element (of length 2) in the conjugacy class Λ. Then, their number is:

$$|\,\mathrm{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|} \deg D_{\Lambda}\ ,$$

where D_{Λ} is a homogeneous polynomial constructed from the geometry of the discriminant hypersurface of W.

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$\rightsquigarrow$ isomorphism:
$$\begin{array}{rcl} V/W & \xrightarrow{\sim} & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \dots, f_n(v)). \end{array}$$

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equation of $p(\bigcup_{H \in \mathcal{A}} H) = \mathcal{H}$, where $p : V \twoheadrightarrow V/W$.

Example $W = A_3$: discriminant ("swallowtail")

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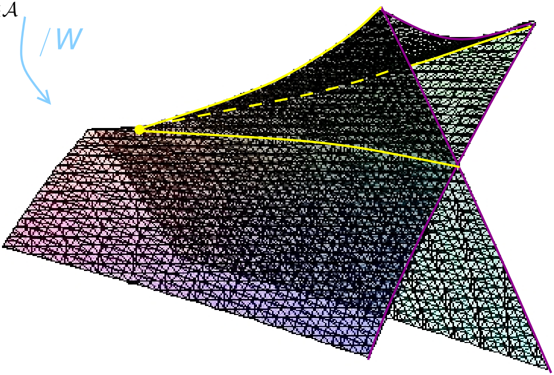

hypersurface $\mathcal{H}$ (discriminant) $\subseteq W \backslash V \simeq \mathbb{C}^3$

Intersection lattice and parabolic subgroups

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- Its Coxeter elements are called parabolic Coxeter elements.

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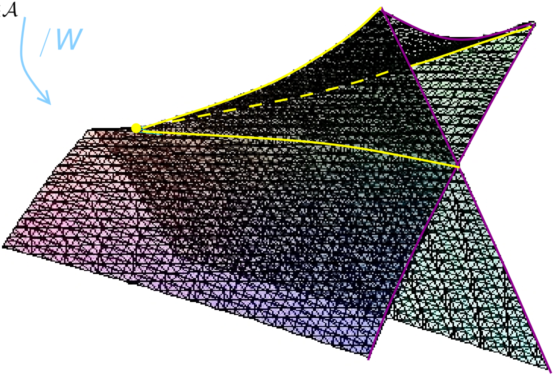
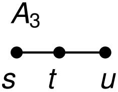
Example of $W = A_3$: stratification of the discriminant

$$\bigcup_{H \in \mathcal{A}} H \subseteq V$$





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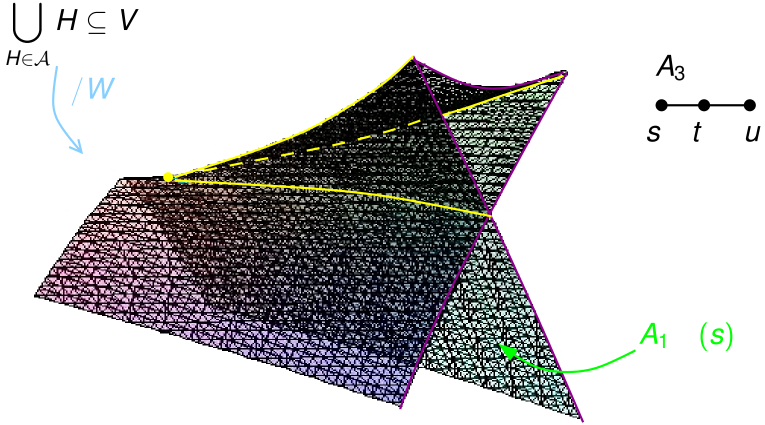

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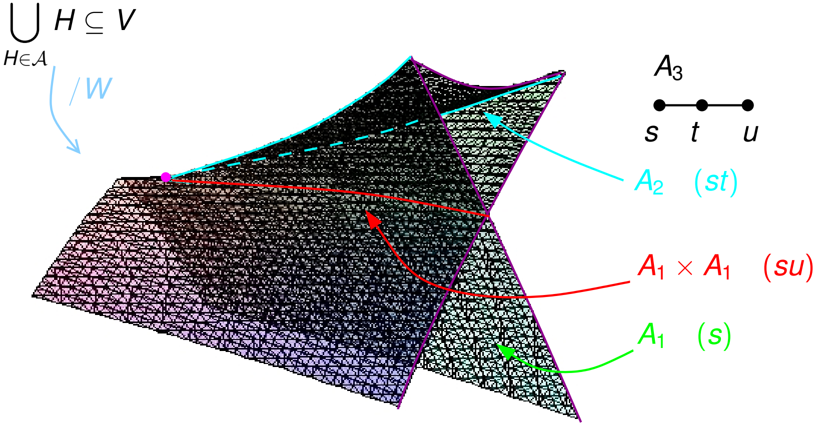

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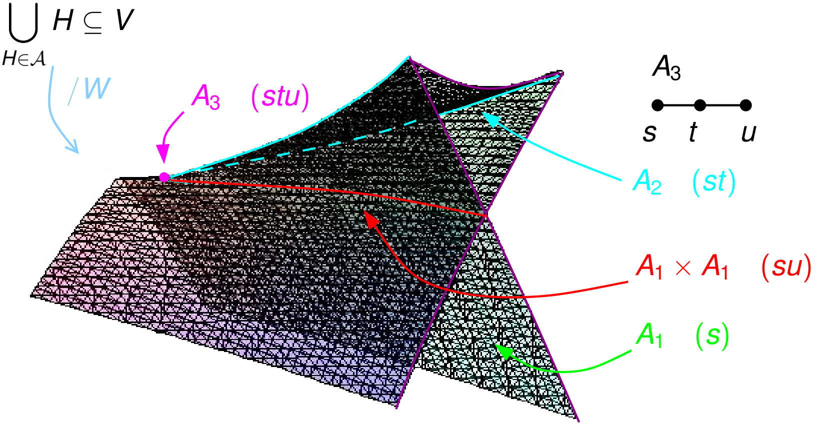

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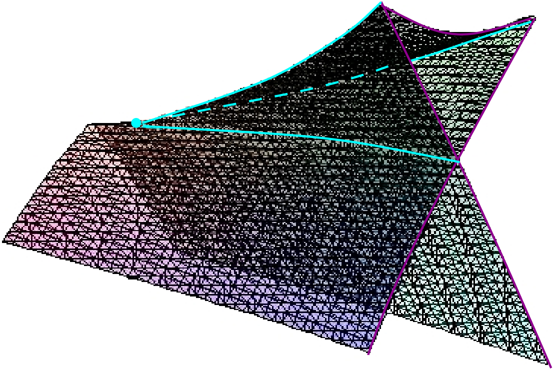
Definition

The bifurcation locus of Δ_W (w.r.t. f_n) is the hypersurface of $\mathbb{C}^{n-1}$:

$$\mathcal{K} := \{D_W = 0\}$$

Example of A_3: bifurcation locus $\mathcal{K}$



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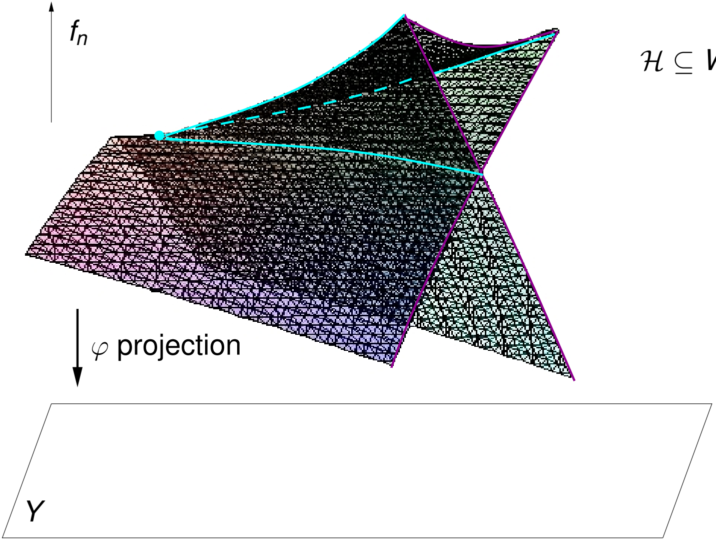

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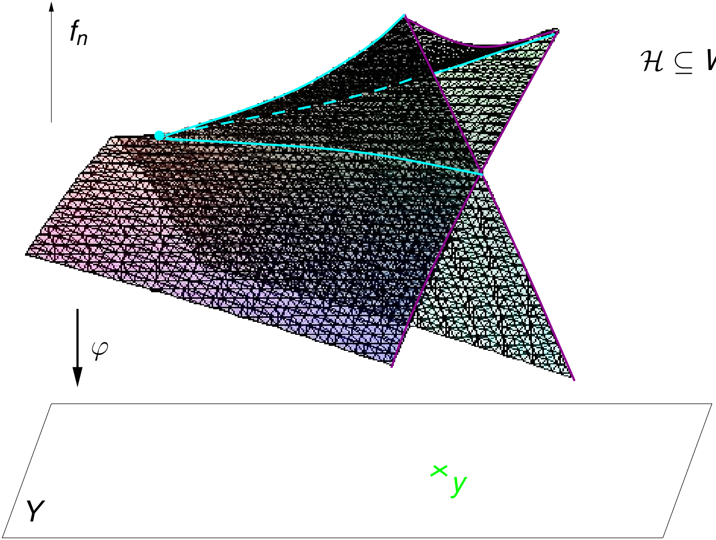

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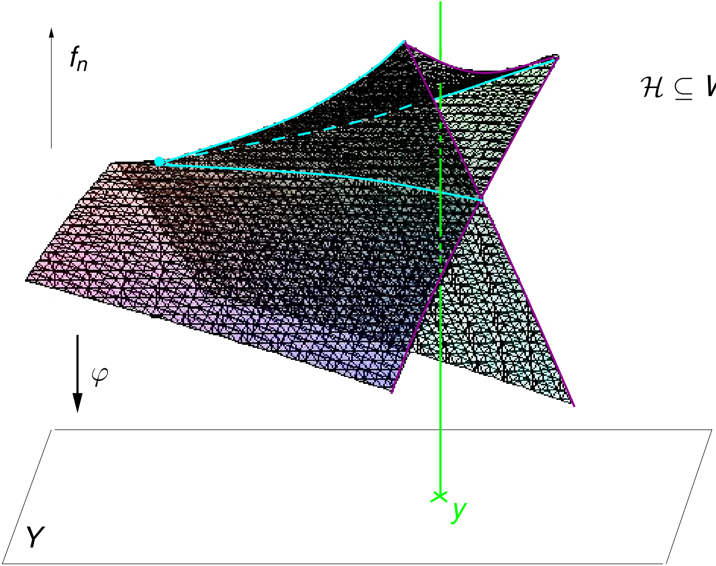

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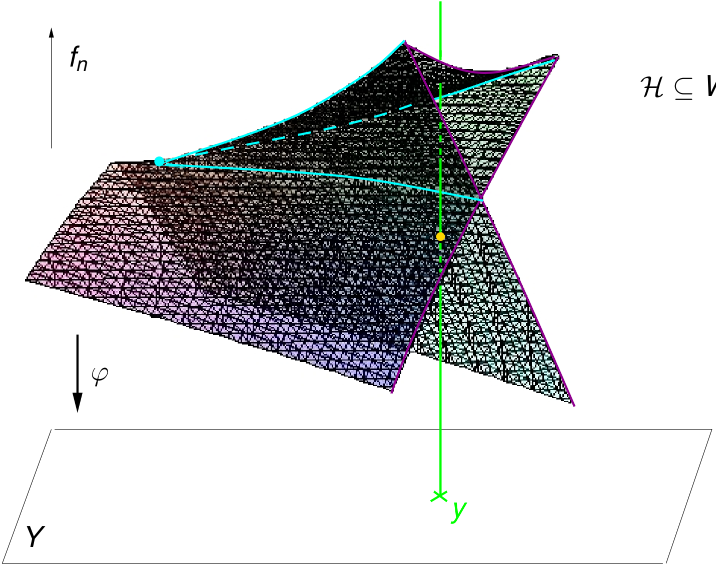

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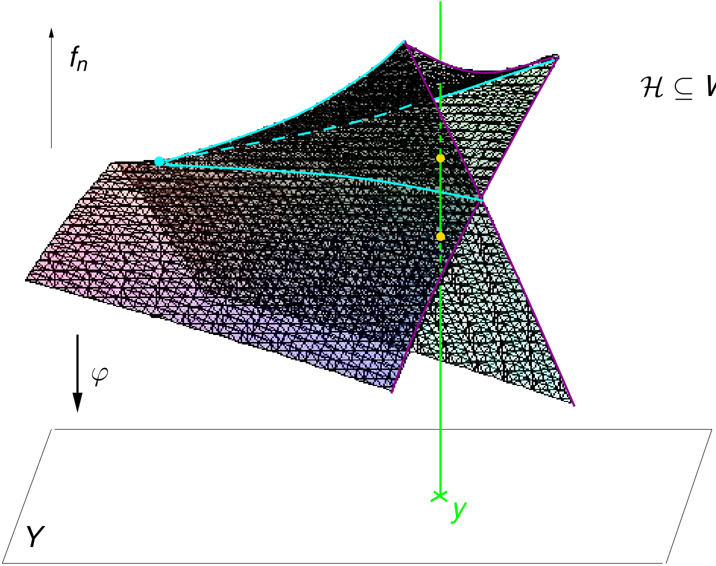

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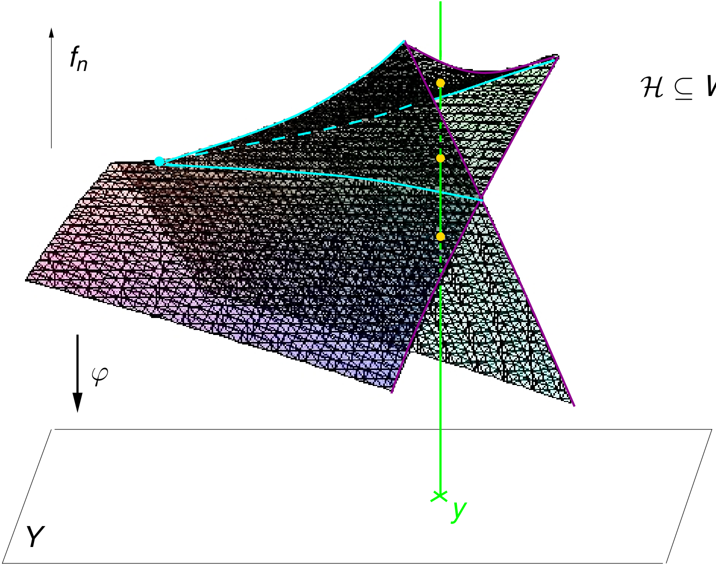

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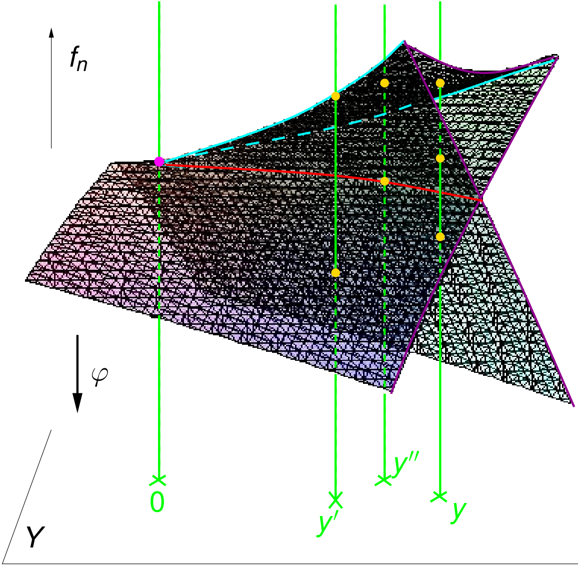

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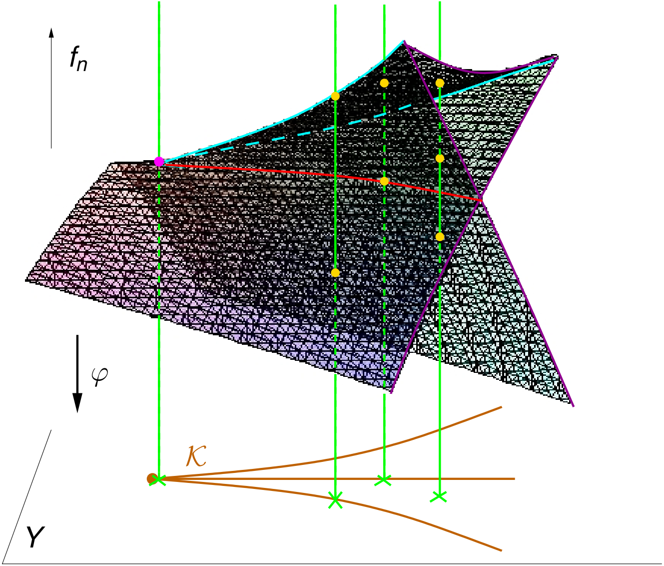

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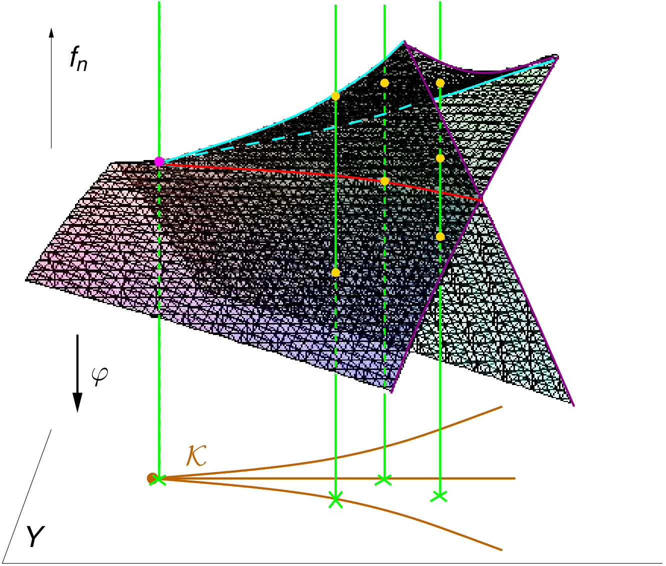

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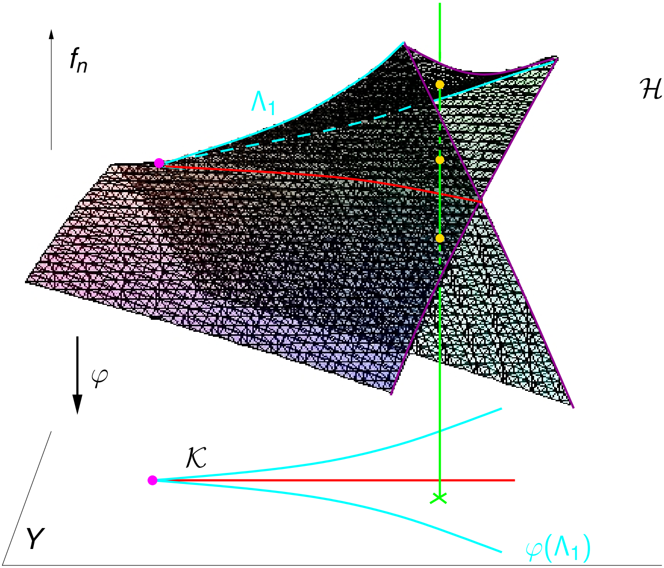

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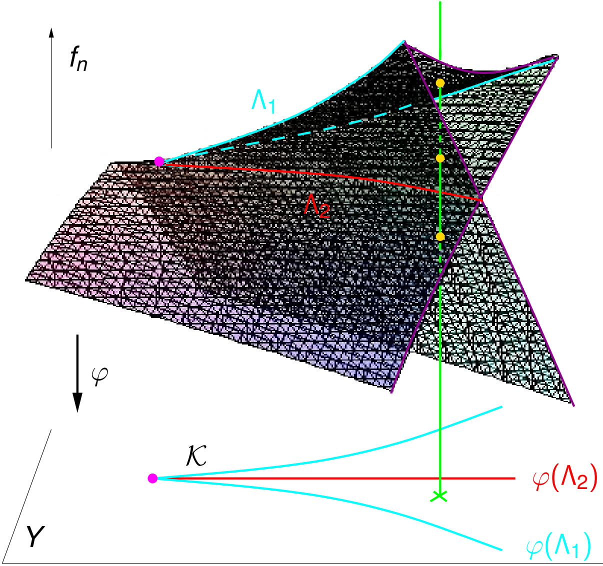

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Theorem (R.)

For $\Lambda \in \bar{\mathcal{L}}_2$, the number of submaximal factorizations of c of type Λ (i.e. , whose unique length 2 element lies in the conjugacy class Λ) is:

$$|\operatorname{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\; h^{n-1}}{|W|} \deg D_\Lambda \; .$$

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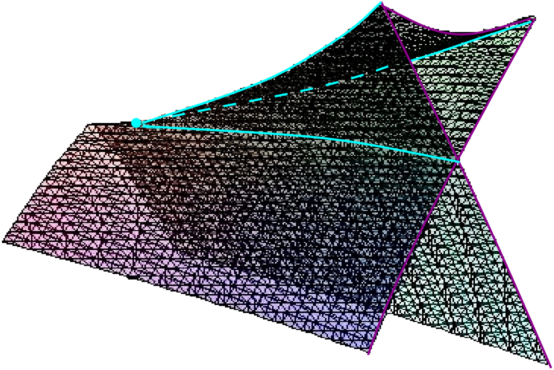
Corollary

The number of ***block factorisations of a Coxeter element*** *c* ***in*** *$n-1$* ***factors*** *is:*

$$|\operatorname{FACT}_{n-1}(c)| = \frac{(n-1)!\, h^{n-1}}{|W|}\left(\frac{(n-1)(n-2)}{2}h + \sum_{i=1}^{n-1} d_i\right) ,$$

where $d_1, \ldots, d_n = h$ are the invariant degrees of W.

The proof uses the Lyashko-Looijenga morphism and topological factorizations



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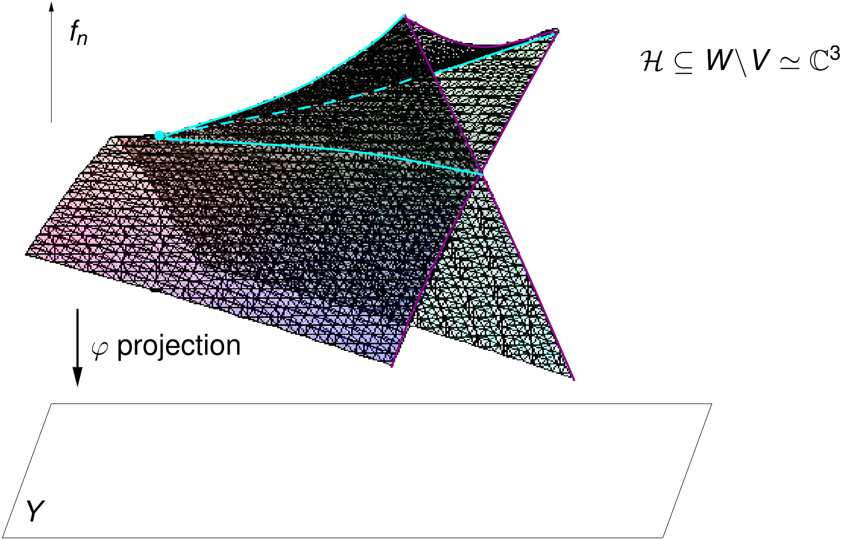

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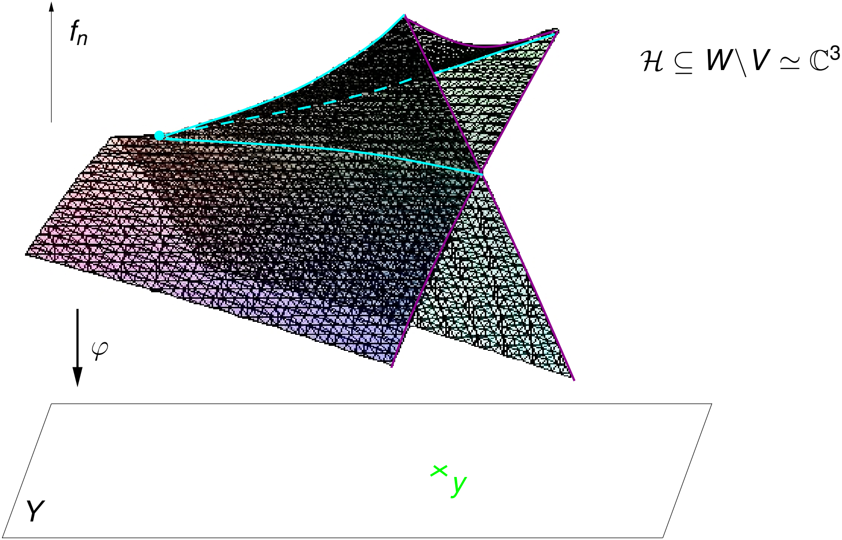

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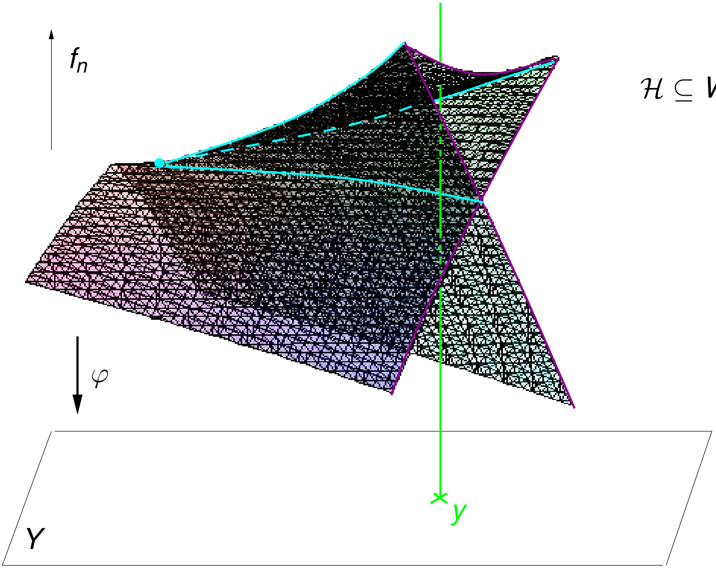

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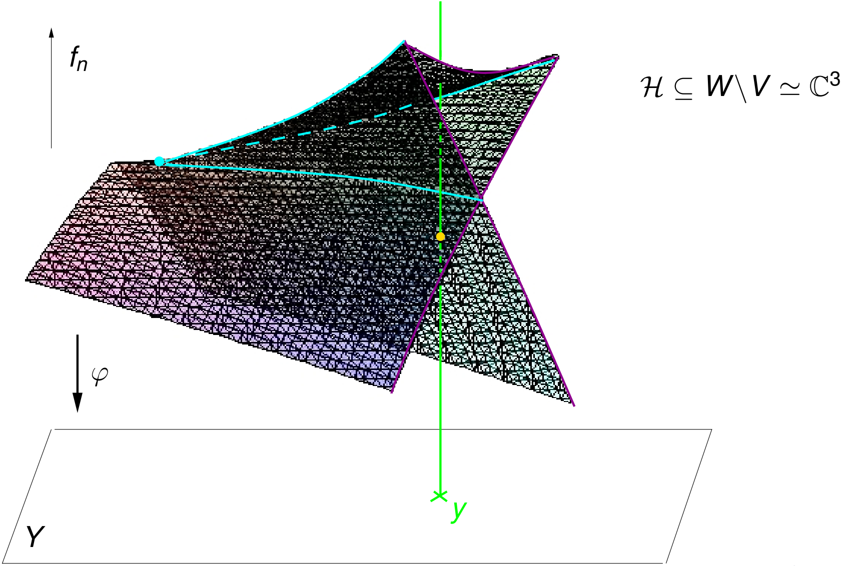

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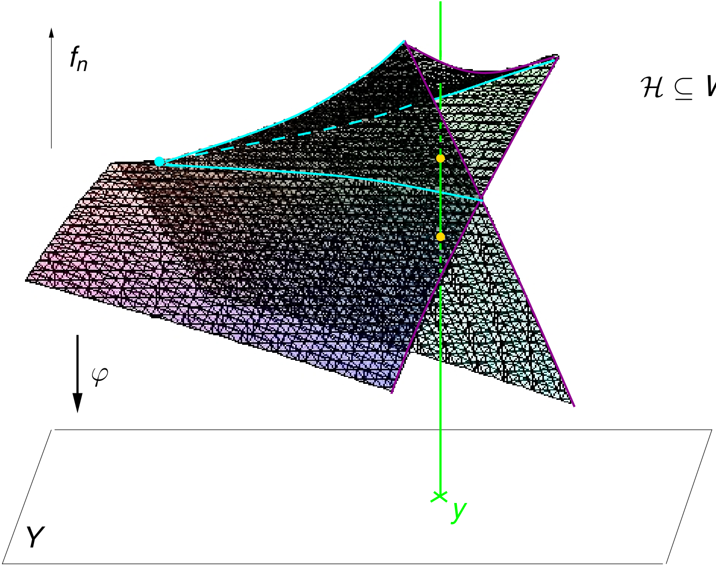

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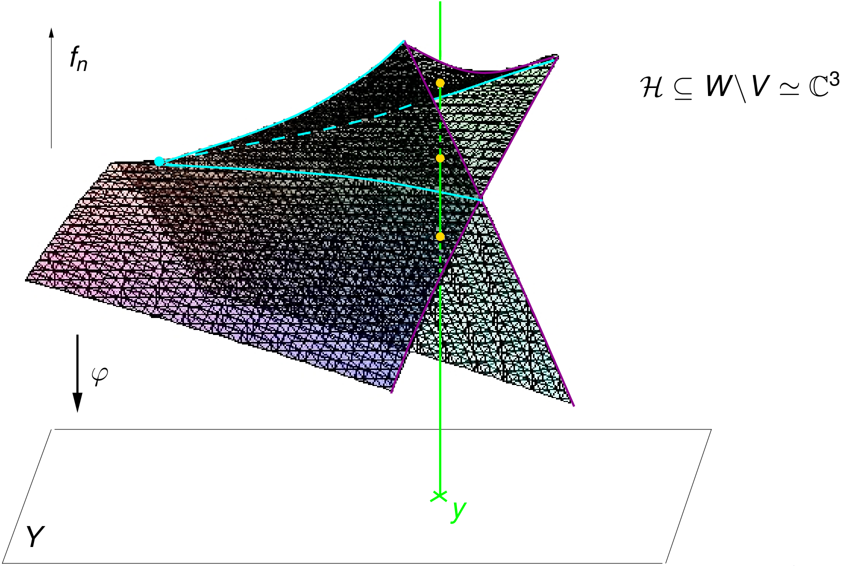

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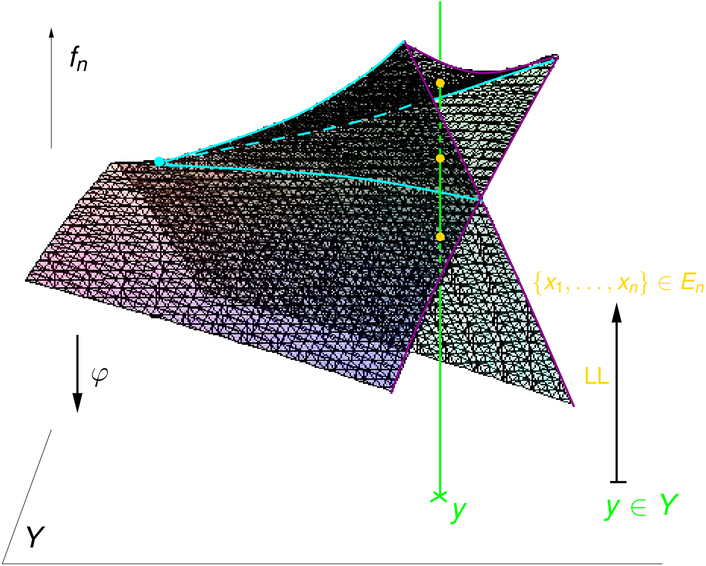

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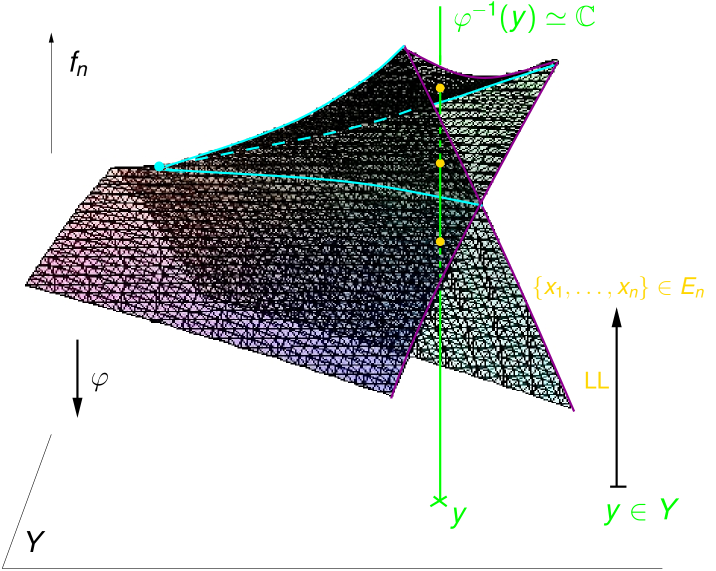

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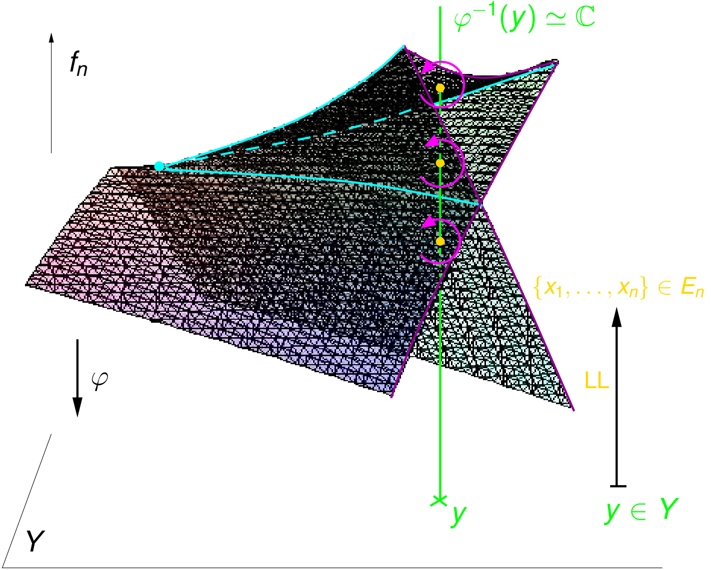

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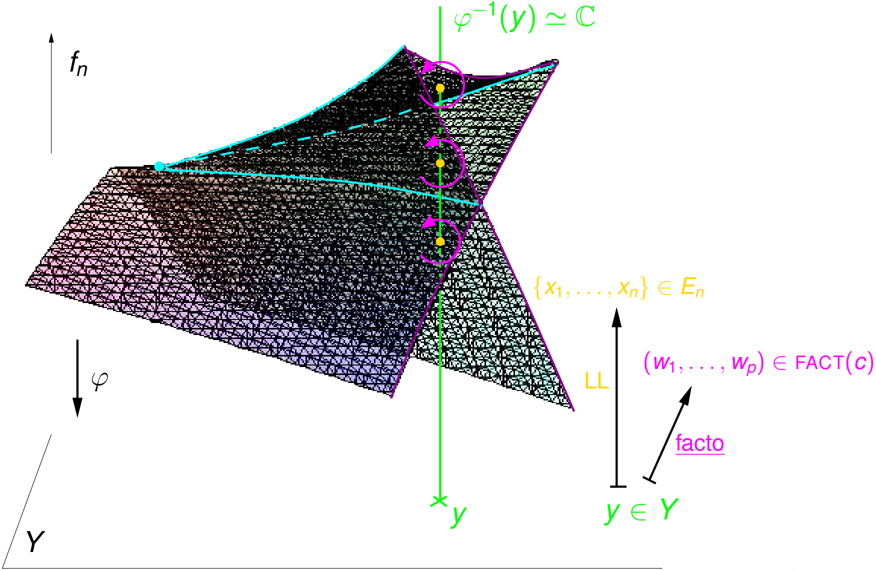

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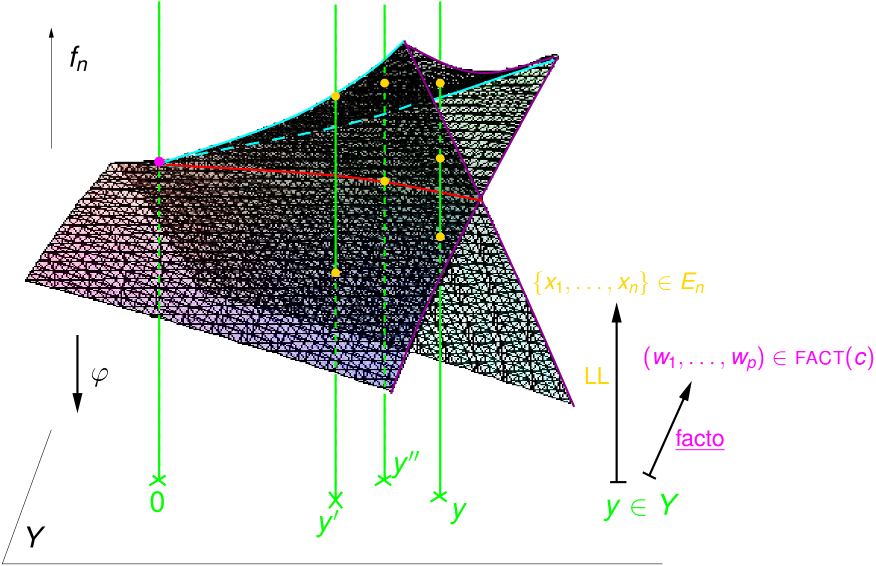

Fibers of LL and block factorizations of c Details

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Equivalently, the product map:

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is injective, and its image is the set of "compatible" pairs.

Fibers of LL and block factorizations of c ▸ Details

Let ω be a multiset in E_n.

"Compatibility" $\Rightarrow$ for all y in the fiber $\mathrm{LL}^{-1}(\omega)$, the distribution of lengths of factors of facto(y) is the same (composition of n).

Theorem (Bessis '07)

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$\leadsto$ a way to compute cardinalities of sets of factorizations using algebraic properties of LL.

Conclusion

- New manifestation of the deep connections between the geometry of W and the combinatorics of $\mathrm{NC}(W)$.
- Proof a bit more enlightening and satisfactory than the usual ones: we travelled from the numerology of $\textsc{fact}_n(c)$ to that of $\textsc{fact}_{n-1}(c)$, without adding any case-by-case analysis.
- We recover geometrically formulas for certain specific factorisations, known in the real case with combinatorial proofs [Krattenthaler].
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Outline

3. Appendix
   - Lyashlo-Looijenga morphism and topological factorizations
   - Jacobian
   - Comparison reflection groups / LL extensions

Lyashko-Looijenga morphism of type W

Definition

$$\begin{array}{rccl} \mathrm{LL}: & Y & \to & E_n := \{\text{multisets of } n \text{ points in } \mathbb{C}\} \\ & y & \mapsto & \{\text{roots, with multiplicities, of } \Delta_W(y, f_n) \text{ in } f_n\} \end{array}$$

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$$\Delta_W = f_n^n + a_2 f_n^{n-2} + a_3 f_n^{n-3} + \cdots + a_{n-1} f_n + a_n.$$

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$$\begin{array}{rccc} \mathrm{LL}: & \mathbb{C}^{n-1} & \to & \mathbb{C}^{n-1} \\ & (f_1, \ldots, f_{n-1}) & \mapsto & (a_2, \ldots, a_n) \end{array}$$

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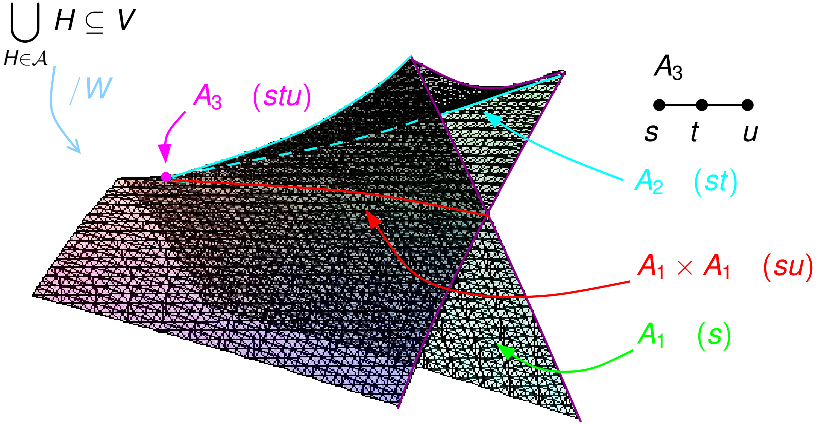
Example of $W = A_3$: stratification of the discriminant

End



$$\mathcal{H} = \{\Delta_W = 0\} \subseteq W \backslash V \simeq \mathbb{C}^3$$

An unramified covering

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Recall $D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$ (irreducible factors in $\mathbb{C}[f_1, \ldots, f_{n-1}]$).

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For any Λ in $\bar{\mathcal{L}}_2$,

- LL_Λ *is a finite morphism of degree* $\frac{(n-2)!\ h^{n-1}}{|W|} \deg D_\Lambda$*;*

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Set $J_{\mathrm{LL}} := \mathrm{Jac}((a_2, \ldots, a_n)/(f_1, \ldots, f_{n-1}))$. *Then:*

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So, $\sum \deg D_\Lambda = \deg D_{\mathrm{LL}} - \deg J_{\mathrm{LL}} = \ldots$

▸ Return

Reflection group *vs.* Lyashko-Looijenga extension

End

Reflection group W	Extension LL
$V \to V/W$ $\mathbb{C}[f_1,\dots,f_n] = \mathbb{C}[V]^W \subseteq \mathbb{C}[V]$ degree $\lvert W \rvert$	$Y \to \mathbb{C}^{n-1}$ $\mathbb{C}[a_2,\dots,a_n] \subseteq \mathbb{C}[f_1,\dots,f_{n-1}]$ degree $n!\ h^n / \lvert W \rvert$
$V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/W$ Generic fiber $\simeq W$	$Y - \mathcal{K} \twoheadrightarrow E_n^{\mathrm{reg}}$ $\simeq \mathrm{Red}_R(c)$
ramified on $\bigcup_{H \in \mathcal{A}} H \twoheadrightarrow \mathcal{H}$	$\mathcal{K} = \bigcup_{\Lambda \in \bar{\mathcal{L}}_2} \varphi(\Lambda) \twoheadrightarrow E_n - E_n^{\mathrm{reg}}$
$\Delta_W = \prod_{H \in \mathcal{A}} \alpha_H^{e_H}$	$D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$
$J_W = \prod \alpha_H^{e_H - 1}$ $e_H = \lvert W_H \rvert$	$J_{\mathrm{LL}} = \prod D_\Lambda^{r_\Lambda - 1}$ r_Λ = pseudo-order of elements of NCP_W of type Λ