Factorizations of a Coxeter element
in finite reflection groups

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Some integer sequences

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$(w_1, \ldots, w_p) \in (W - \{1\})^p$ is a block factorization of $c$ if

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What about $\text{FACT}_{n-1}(c)$?

Theorem (R.)

Let $\Lambda$ be a conjugacy class of elements of length 2 of $W$. Call submaximal factorizations of $c$ of type $\Lambda$ the block factorizations containing $n-2$ reflections and one element (of length 2) in the conjugacy class $\Lambda$. Then, their number is:

$$|\text{FACT}_\Lambda(n-1)(c)| = (n-1)! \frac{h^{n-1}}{|W| \deg D_{\Lambda}},$$

where $D_{\Lambda}$ is a homogeneous polynomial constructed from the geometry of the discriminant hypersurface of $W$. 

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The noncrossing partition lattice of type $W$

Absolute order $\preceq_R$ on $W$

$u \preceq_R v$ if and only if $\ell_R(u) + \ell_R(u^{-1}v) = \ell_R(v)$
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- the structure doesn’t depend on the choice of the Coxeter element (conjugacy) $\rightsquigarrow$ we write $\text{NC}(W)$.
- if $W = \mathfrak{S}_n$, $\text{NC}(W) \cong$ lattice of noncrossing partitions of an $n$-gon.
Proposition (Chapoton)

Suppose \( W \) irreducible of rank \( n \), and let \( c \) be a Coxeter element. The number of multichains \( w_1 \preceq_R w_2 \preceq_R \cdots \preceq_R w_p \preceq_R c \) is equal to the “Fuß-Catalan number of type \( W \)”

\[ \text{Cat}(p; W) = n \prod_{i=1}^{n} d_i + ph \]

where \( d_1, \ldots, d_n = h \) are the invariant degrees of \( W \).

Proof: [Athanasiadis, Reiner, Bessis...], case-by-case, using the classification of reflection groups.

\( \Rightarrow \) how to understand this formula uniformly?
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Interest of the theorem

- Formulas for block factorizations $\leftrightarrow$ formulas for multichains.

$|\text{FACT}\ n-1(c)| = (n-1)!h^n-1|W|\deg D\Lambda$ is (almost) case-free:

- Use of a bijection between some classes of factorizations and fibers of a ramified covering: the Lyashko-Looijenga covering LL.
- Computation of cardinalities of fibers via degrees of algebraic morphisms, restrictions of LL.

We can compute (in a case-free way) $\sum \Lambda \deg D\Lambda$.

$\Rightarrow$ We get an instance of Chapoton's formula, with a more enlightening proof:

**Corollary**

The number of block factorizations of a Coxeter element $c$ in $n-1$ factors is:

$|\text{FACT}\ n-1(c)| = (n-1)!h^n-1|W|\left( (n-1)(n-2)h + n-1\sum_{i=1}^{d_i} \right)$.
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Stratification of $V$ with the “flats” (intersection lattice):

$\mathcal{L} := \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\} \rightarrow \text{PSG}(W)$ (parabolic subgps of $W$)
Intersection lattice and parabolic subgroups

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\[ \mapsto \quad W_L \text{ (pointwise stabilizer of } L) \]

A parabolic subgroup is a reflection group [Steinberg].
Its Coxeter elements are called parabolic Coxeter elements.

$L_0 \in L \leftrightarrow W_0 \in \text{PSG}(W) \ni c_0 \text{ parabolic Coxeter element}$

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$L_0 \in \mathcal{L} \iff W_0 \in \text{PSG}(W) \ni c_0$ parabolic Coxeter element

$\text{codim}(L_0) = \text{rk}(W_0) = \ell_R(c_0)$
The quotient-space $V/W$

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There exist invariant polynomials $f_1, \ldots, f_n$, homogeneous and algebraically independent, s.t. $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$. 
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There exist invariant polynomials $f_1, \ldots, f_n$, homogeneous and algebraically independent, s.t. $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$. The degrees $d_1 \leq \cdots \leq d_n = h$ of $f_1, \ldots, f_n$ (called invariant degrees) do not depend on the choices of the fundamental invariants.
The quotient-space $V/W$

$W$ acts on the polynomial algebra $\mathbb{C}[V]$.

**Chevalley-Shephard-Todd’s theorem**

There exist invariant polynomials $f_1, \ldots, f_n$, homogeneous and algebraically independent, s.t. $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$.

The degrees $d_1 \leq \cdots \leq d_n = h$ of $f_1, \ldots, f_n$ (called invariant degrees) do not depend on the choices of the fundamental invariants.

$\sim$ isomorphism: $V/W \xrightarrow{\sim} \mathbb{C}^n$

$\tilde{v} \mapsto (f_1(v), \ldots, f_n(v))$. 

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Example $W = A_3$: discriminant ("swallowtail")

$$\bigcup_{H \in A} H \subseteq V$$
Example $W = A_3$: discriminant ("swallowtail")

$$\bigcup_{H \in \mathcal{A}} H \subseteq V / W$$
Example $\mathcal{W} = A_3$: discriminant ("swallowtail")

$$\bigcup_{H \in A} H \subseteq V$$

Hypersurface $\mathcal{H}$ (discriminant) $\subseteq V/\mathcal{W} \simeq \mathbb{C}^3$
Stratification of the discriminant hypersurface

\[ \bigcup_{H \in \mathcal{A}} H \subseteq V \]

\[ H \subseteq V/\mathcal{W} \]

\[ H = \{ \Delta W = 0 \} \subseteq \mathcal{W} \setminus V \simeq \mathbb{C}^3 \]
Stratification of the discriminant hypersurface

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\bigcup_{H \in A} H \subseteq V
\]

\[H = \{ \Delta_W = 0 \} \subseteq W \setminus V \cong \mathbb{C}^3\]
Stratification of the discriminant hypersurface

\[ \bigcup_{H \subseteq V} H \subseteq W \]

\[ H \in A \]

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Stratification of the discriminant hypersurface

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Theorem (Orlik-Solomon, Bessis)

*If* $W$ *is a real (or complex well-generated) reflection group, then the discriminant* $\Delta_W$ *is monic of degree* $n$ *in the variable* $f_n$.
Theorem (Orlik-Solomon, Bessis)

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Definition

The bifurcation locus of $\Delta_W$ (w.r.t. $f_n$) is the hypersurface of $\mathbb{C}^{n-1}$:

$$\mathcal{K} := \{D_W = 0\}$$
Bifurcation locus $\mathcal{K}$

$$\mathcal{H} \subseteq W \setminus V \cong \mathbb{C}^3$$
Bifurcation locus $\mathcal{K}$

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Factorizations of a Coxeter element
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$Y$

$f_n$

$\varphi$

$\times y$
Bifurcation locus \( \mathcal{K} \)

\[
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\]
Bifurcation locus $\mathcal{K}$
Bifurcation locus $\mathcal{K}$
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Factorizations of a Coxeter element
Bifurcation locus $\mathcal{K}$

$\mathcal{H} \subseteq \mathcal{W} \setminus \mathcal{V} \simeq \mathbb{C}^3$

$K \subseteq W \setminus V \simeq \mathbb{C}^3$
Submaximal factorizations of type $\Lambda$

$$\bar{\mathcal{L}}_2 := \{\text{strata of } \bar{\mathcal{L}} \text{ of codimension 2}\}$$

$$\leftrightarrow \{\text{conjugacy classes of elements of } \text{NC}(W)\}$$
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**Proposition**

The $\varphi(\Lambda)$, for $\Lambda \in \bar{L}_2$, are the *irreducible components* of $K$. 
Submaximal factorizations of type $\Lambda$

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**Proposition**

The $\varphi(\Lambda)$, for $\Lambda \in \bar{L}_2$, are the irreducible components of $\mathcal{K}$.

$\implies$ we can write $D_W = \prod_{\Lambda \in \bar{L}_2} D_{r^\Lambda}$, where $r^\Lambda \geq 1$ and the $D_{r^\Lambda}$ are polynomials in $f_1, \ldots, f_{n-1}$.
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Proposition

The $\varphi(\Lambda)$, for $\Lambda \in \bar{\mathcal{L}}_2$, are the irreducible components of $\mathcal{K}$.

$\leadsto$ we can write $D_W = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_{r_{\Lambda}}^\Lambda$, where $r_{\Lambda} \geq 1$ and the $D_{r_{\Lambda}}$ are polynomials in $f_1, \ldots, f_{n-1}$.

Theorem

For $\Lambda \in \bar{\mathcal{L}}_2$, the number of submaximal factorizations of $c$ of type $\Lambda$ (i.e., whose unique length 2 element lies in the conjugacy class $\Lambda$) is:

$$|\text{FACT}_{n-1}^\Lambda(c)| = \frac{(n - 1)! \ h^{n-1}}{|W|} \deg D_{r_{\Lambda}}.$$

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Submaximal factorizations of type $\Lambda$

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**Proposition**

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LL morphism and topological factorisations

\[ \varphi^{-1}(y) \simeq \mathbb{C} \]

\[ \{x_1, \ldots, x_n\} \in E_n \]

\[ (w_1, \ldots, w_p) \in \text{FACT}(c) \]

\[ f_n \]

\[ \varphi \]

\[ Y \]

\[ y \in Y \]

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Conclusion

- new manifestation of the mysterious connections between the geometry of $W$ and the combinatorics of NC($W$).
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proof more satisfactory than the case-by-case one.
Conclusion

- new manifestation of the mysterious connections between the geometry of $W$ and the combinatorics of $NC(W)$.
- proof more satisfactory than the case-by-case one.
- we recover geometrically formulas for certain specific factorisations, known in the real case with combinatorial proofs [Krattenthaler].

Todo: study further the geometrical setting (Lyashko-Looijenga morphism and its ramification) to obtain a global understanding of Chapoton's formula.

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Takk! - Merci! - Thank you!