Factorizations of a Coxeter element in finite reflection groups

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$\mathrm{FACT}_p(c) := \{\text{block factorizations of } c \text{ in } p \text{ factors}\}$.

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What about $\text{FACT}_{n-1}(c)$?

Theorem (R.)

Let Λ be a conjugacy class of elements of length 2 of W. Call submaximal factorizations of c of type Λ the block factorizations containing $n-2$ reflections and one element (of length 2) in the conjugacy class Λ. Then, their number is:

$$|\text{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|} \deg D_\Lambda\ ,$$

where D_Λ is a homogeneous polynomial constructed from the geometry of the discriminant hypersurface of W.

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- if $W = \mathfrak{S}_n$, $\mathrm{NC}(W) \simeq$ lattice of noncrossing partitions of an n-gon.

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Proposition (Chapoton)

Suppose W irreducible of rank n, and let c be a Coxeter element. The number of multichains $w_1 \preccurlyeq_R w_2 \preccurlyeq_R \dots \preccurlyeq_R w_p \preccurlyeq_R c$ is equal to the "Fuß-Catalan number of type W"

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$\rightsquigarrow$ how to understand this formula uniformly ?

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$\rightsquigarrow$ we get an instance of Chapoton's formula, with a more enlightening proof:

Corollary

The number of ***block factorisations of a Coxeter element*** *c* ***in*** *$n-1$* ***factors*** *is:*

$$|\,\mathrm{FACT}_{n-1}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|}\left(\frac{(n-1)(n-2)}{2}h + \sum_{i=1}^{n-1} d_i\right).$$

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$\rightsquigarrow$ isomorphism: $\begin{array}{ccc} V/W & \xrightarrow{\sim} & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \ldots, f_n(v)). \end{array}$

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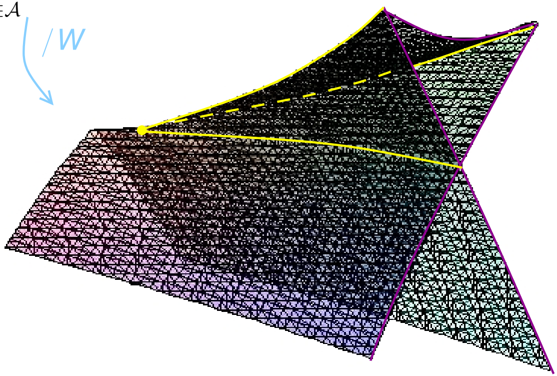

hypersurface $\mathcal{H}$ (discriminant) $\subseteq V/W \simeq \mathbb{C}^3$

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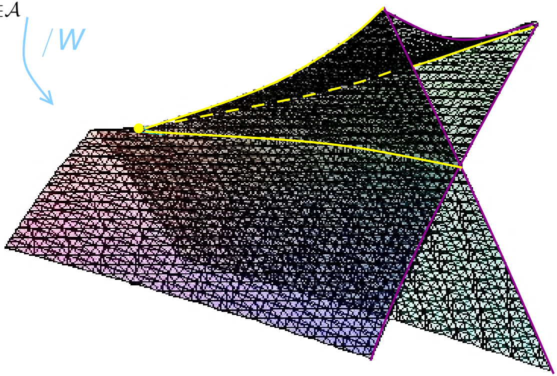
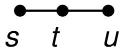


A_3



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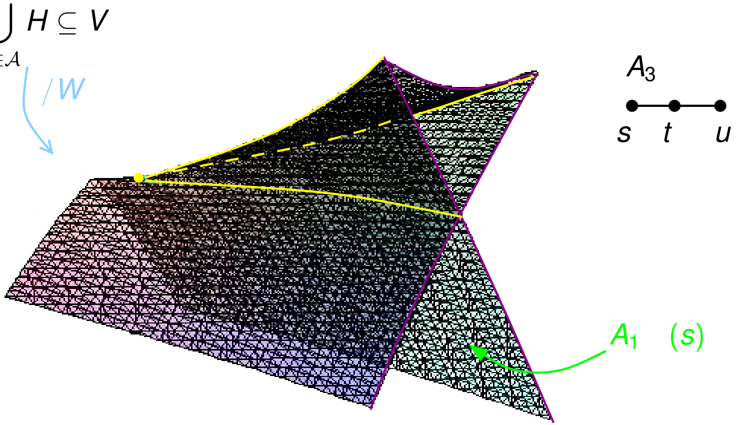

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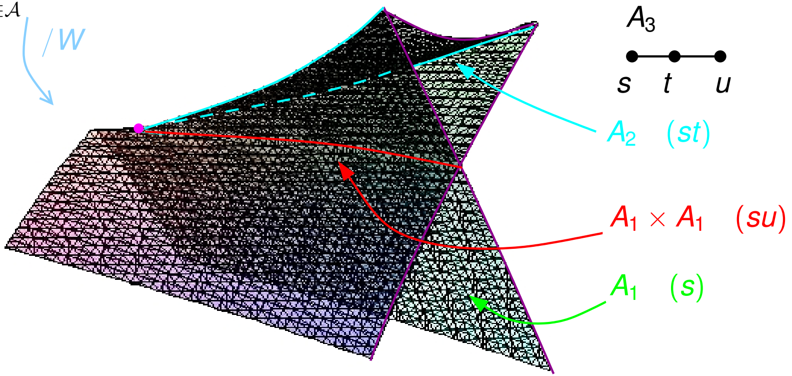

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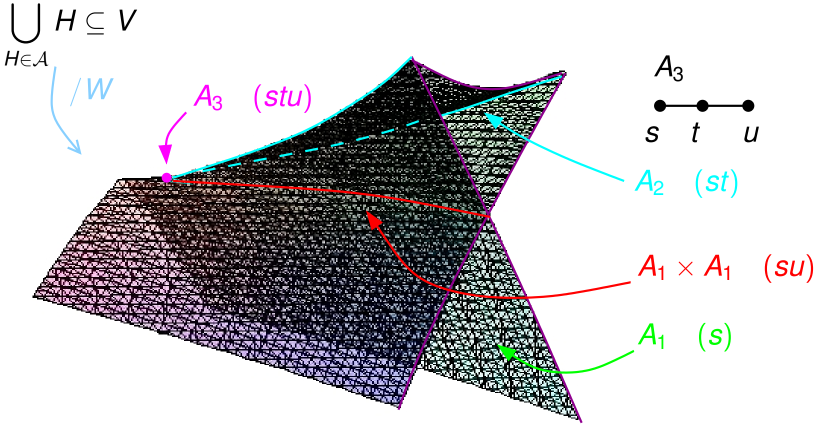
Stratification of the discriminant hypersurface



$$\mathcal{H} = \{\Delta_W = 0\} \subseteq W \backslash V \simeq \mathbb{C}^3$$

Bifurcation locus

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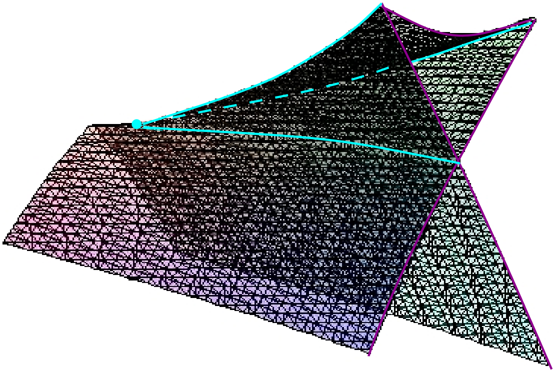
Definition

The bifurcation locus of Δ_W (w.r.t. f_n) is the hypersurface of $\mathbb{C}^{n-1}$:

$$\mathcal{K} := \{D_W = 0\}$$

Bifurcation locus $\mathcal{K}$



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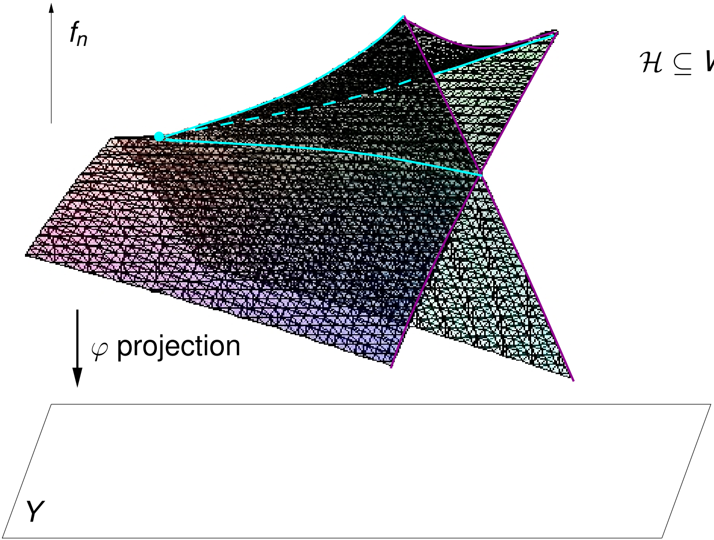

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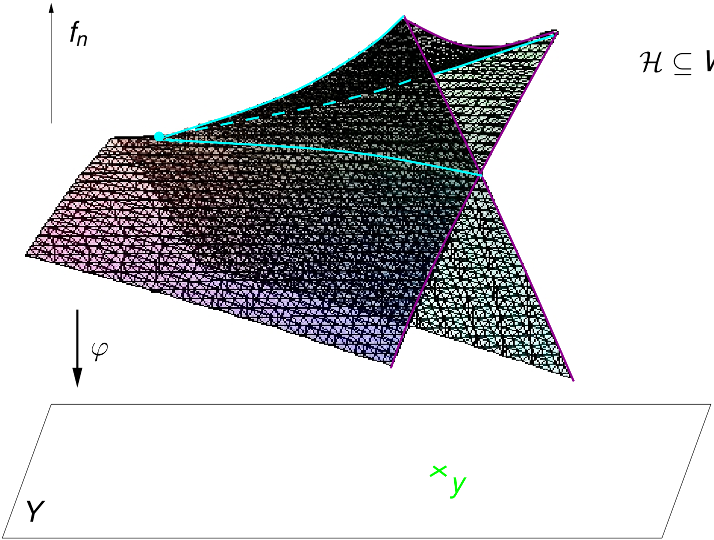

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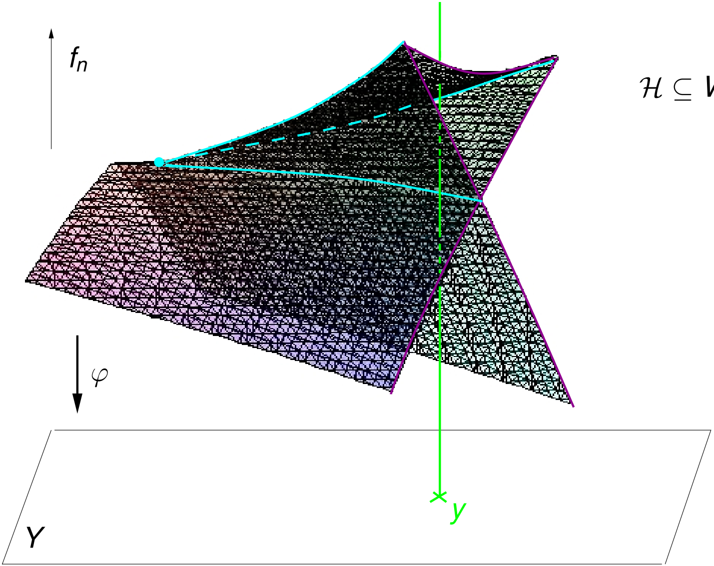

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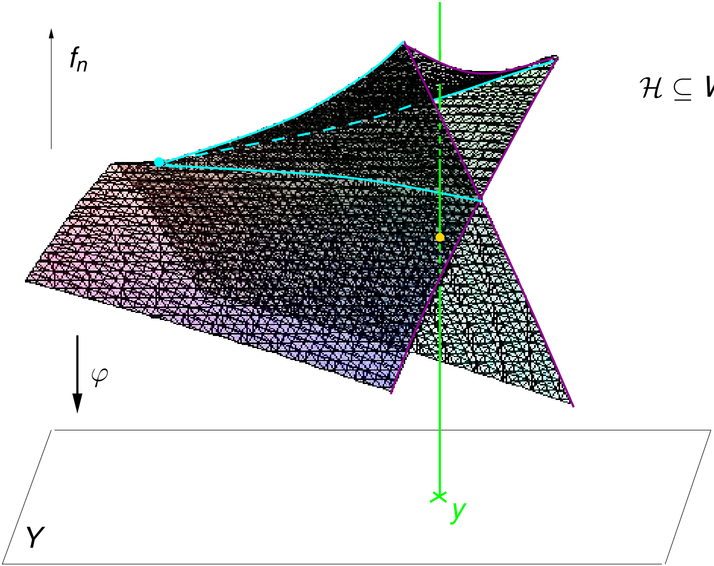

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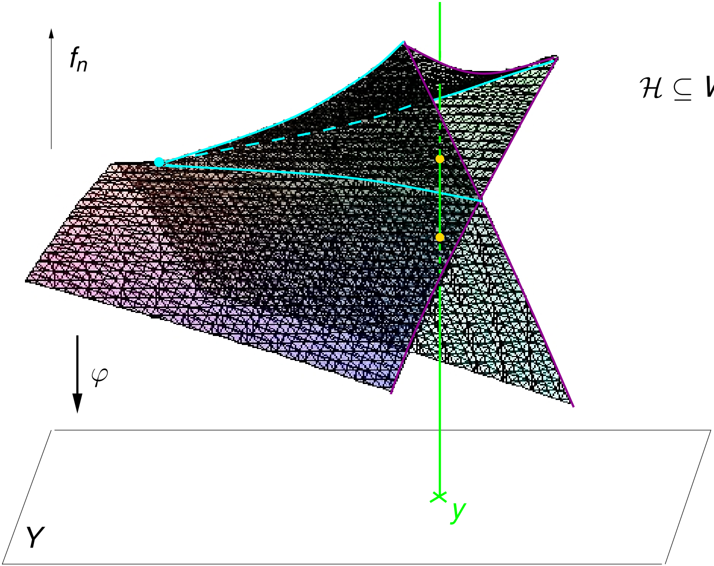

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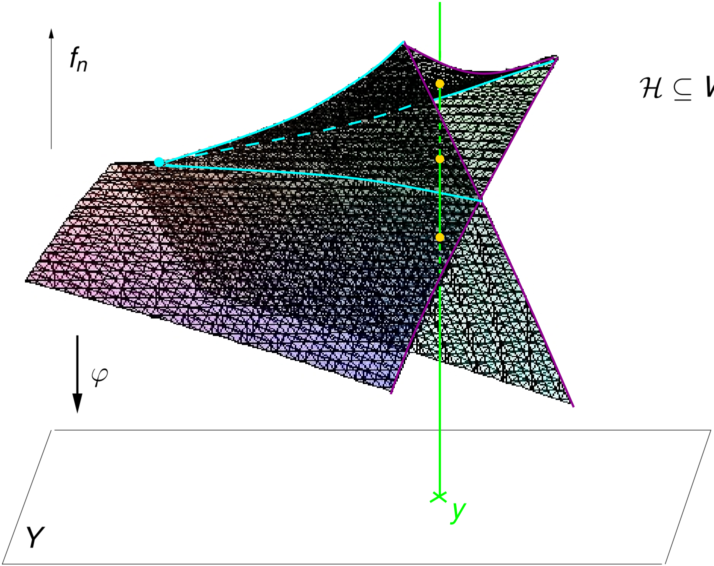

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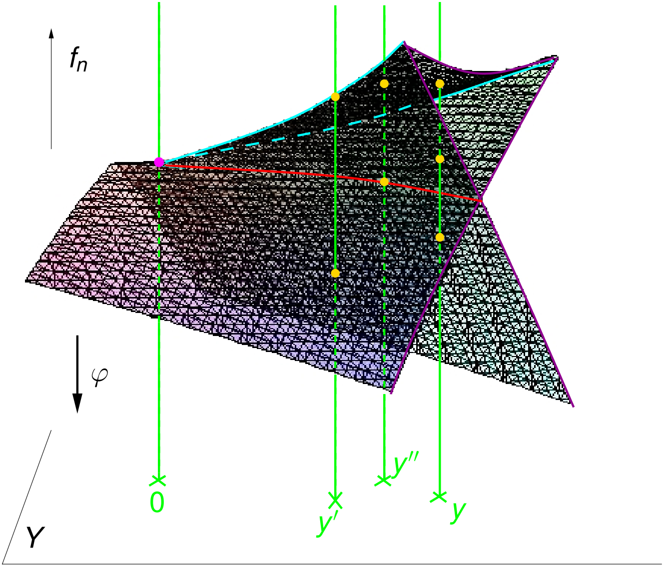

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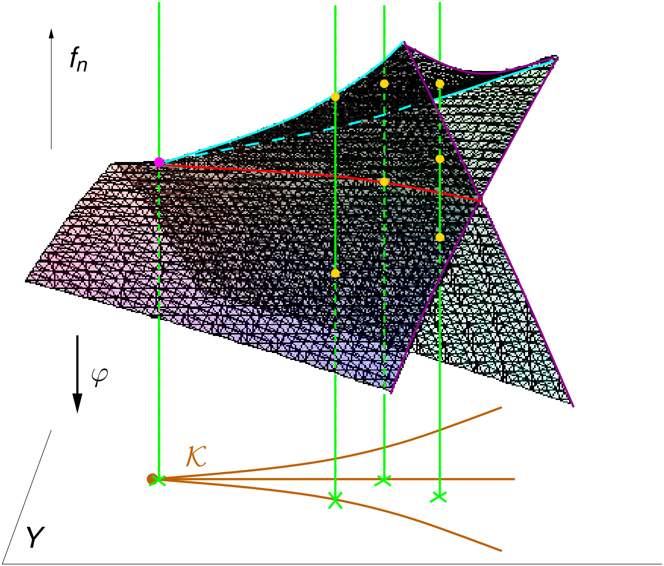

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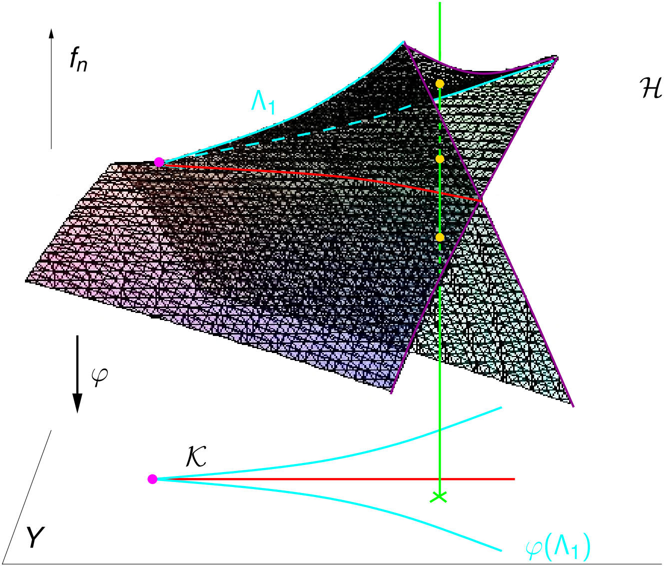

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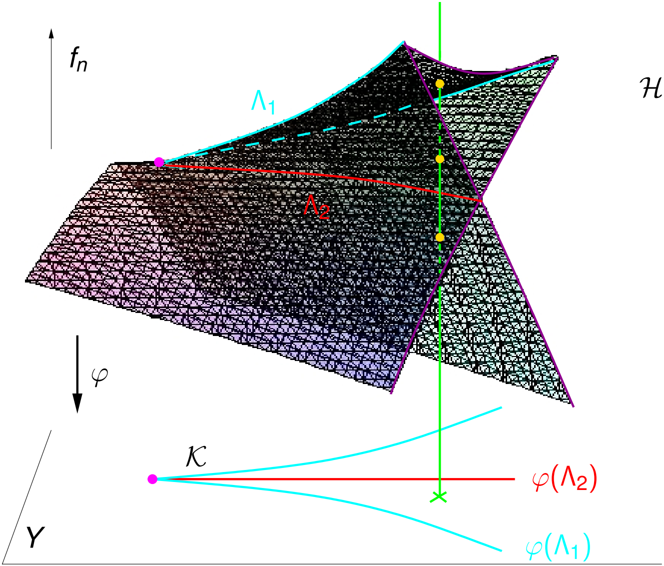

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Submaximal factorizations of type Λ

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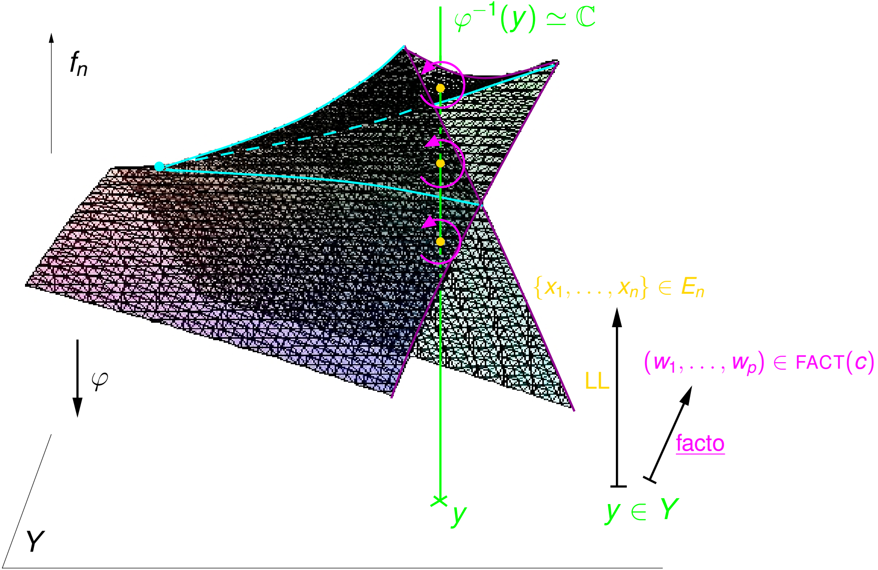
LL morphism and topological factorisations



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Takk! - Merci! - Thank you!

Reference: *Lyashko-Looijenga morphisms and submaximal factorisations of a Coxeter element*, arXiv:1012.3825.