On non-conjugate Coxeter elements in well-generated reflection groups

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Abstract. Given an irreducible well-generated complex reflection group W with Coxeter number h, we call a Coxeter element any regular element (in the sense of Springer) of order h in W; this is a slight extension of the most common notion of Coxeter element. We show that the class of these Coxeter elements forms a single orbit in W under the action of reflection automorphisms. For Coxeter and Shephard groups, this implies that an element c is a Coxeter element if and only if there exists a simple system S of reflections such that c is the product of the generators in S. We moreover deduce multiple further implications of this property. In particular, we obtain that all noncrossing partition lattices of W associated to different Coxeter elements are isomorphic. We also prove that there is a simply transitive action of the Galois group of the field of definition of W on the set of conjugacy classes of Coxeter elements. Finally, we extend several of these properties to regular elements of arbitrary order.

Résumé. Étant donnés un groupe de réflexion complexe W, irréductible et bien engendré, et h son nombre de Coxeter, nous appelons élément de Coxeter un élément régulier (au sens de Springer) d'ordre h; ceci est une extension de la notion la plus habituelle d'élément de Coxeter. Nous montrons que l'ensemble de ces éléments de Coxeter forme une seule orbite sous l'action des automorphismes de réflexion de W. Pour les groupes de Coxeter et de Shephard, ceci implique qu'un élément c est un élément de Coxeter si et seulement s'il existe un système simple S de réflexions tel que c soit le produit des générateurs dans S. Nous déduisons de cette propriété plusieurs autres résultats. En particulier, nous obtenons que tous les treillis de partitions non-croisées de W, associés à différents éléments de Coxeter, sont isomorphes. Nous montrons également qu'il existe une action simplement transitive du groupe de Galois du corps de définition de W sur l'ensemble des classes de conjugaison d'éléments de Coxeter. Enfin, nous étendons plusieurs de ces propriétés au cas des éléments réguliers d'ordre quelconque.

Keywords: reflection groups, Coxeter groups, Coxeter elements, noncrossing partitions, Shephard groups

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1 Background on reflection groups

Let $V = \mathbb{C}^n$, and consider a finite subgroup W of $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$. One calls W a *complex reflection group* if it is generated by its subset R of *reflections*, that is, the elements $r \in W$ for which the fixed space $\ker(r - \mathbb{1}) \subseteq V$ is a hyperplane. Results of G. C. Shephard and J. A. Todd [ST54] and of C. Chevalley [Che55] distinguish complex reflection groups as those finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ for which the invariant subalgebra of the action on $\mathrm{Sym}(V^*) \cong \mathbb{C}[x_1, \dots, x_n]$ yields again a polynomial algebra, $\mathrm{Sym}(V^*)^W = \mathbb{C}[f_1, \dots, f_n]$. While the basic invariants $f_1, \dots, f_n$ are not unique, they can be chosen homogeneous, and then their degrees $d_1 \leq \cdots \leq d_n$ are uniquely determined and called the *degrees* of W. The group W is called *irreducible* if it does not preserve a proper subspace of V. An important subclass of irreducible complex reflection groups are those that are *well-generated*, that is, for which there exists a

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subset of n reflections that generate W. In particular, this subclass contains all (complexifications of) irreducible *real* reflection groups inside $\mathrm{GL}_n(\mathbb{R})$, which have a well-known structure of *finite Coxeter groups*. It contains as well the subclass known as *Shephard groups*.

We briefly introduce these two subclasses below. In Section 2 we then define and give examples for Coxeter elements, and we explain some motivation coming from Coxeter-Catalan combinatorics and the study of generalized noncrossing partition lattices. Section 3 contains our first main result (Theorem 3.1), giving several equivalent characterizations of Coxeter elements. Two of these characterizations relate Coxeter elements to the product of generators for some specific generating sets, and we explain these in more detail in Sections 4.1 (for Coxeter and Shephard groups) and 4.2 (for arbitrary well-generated groups). In Section 5 we state our second main result (Theorem 5.2), describing a simply transitive action of the Galois group of the field of definition of W on the set of conjugacy classes of Coxeter elements. Finally, Section 6 presents an extension of some of the properties to Springer's regular elements.

This article is an extended abstract of [RRS14]; we refer the reader to the long version for details and complete proofs.

For a real reflection group W, let $\mathfrak{C}$ be a chamber of the arrangement of reflecting hyperplanes of W in $\mathbb{R}^n$. To this chamber, one can associate a distinguished set $S \subseteq R$ of *Coxeter generators* for W obtained by taking those reflections defined by the boundary hyperplanes of $\mathfrak{C}$. The pair (W, S) is then a *Coxeter system*, i.e., W has a Coxeter presentation with S as generating set. Moreover, every finite Coxeter system can be obtained this way from a finite real reflection group. We refer to [Hum90] for details.

A slight generalization of the Coxeter presentation allows one to deal with Shephard groups, i.e., subgroups of $\mathrm{GL}(V)$ that are symmetry groups of regular complex polytopes. These groups have been introduced and classified by G. C. Shephard in [She52]; we also refer to [Cox91] for more detail. H. S. M. Coxeter showed in [Cox67] that every Shephard group has a generalized Coxeter structure, as given in the following definition.

Definition 1.1 *A generalized Coxeter system (W, S) is a group W together with a subset $S \subseteq W$, such that W has a presentation with S as generating set and relations*

$$\underbrace{sts\ldots}_{m_{s,t}\text{ factors}} = \underbrace{tst\ldots}_{m_{t,s}\text{ factors}} \quad \text{for } s,t \in S \text{ with } s \neq t, \quad \text{and } s^{p_s} = 1 \text{ for } s \in S,$$

for some integers $m_{s,t} = m_{t,s}, p_s \geq 2$, where moreover $p_s = p_t$ whenever $m_{s,t}$ is odd.

Note that in this definition, we do not allow the labels $m_{s,t}$ and p_s to be infinite, unlike in the standard notion of Coxeter systems where the case $m_{s,t} = \infty$ is usually possible. However, this poses no restriction in our setting since we will only consider the case where such a generalized Coxeter system gives rise to a finite group.

Similarly to the situation for Coxeter systems, one can construct a (generalized) Coxeter graph for a generalized Coxeter system (W, S). The vertices are given by the generators and are labelled by their order, i.e., the vertex s is labelled by p_s. Moreover, whenever $m_{s,t} \geq 3$, the two vertices s and t are joined by an edge labelled by $m_{s,t}$. This yields the definition of irreducible generalized Coxeter systems as those for which the associated Coxeter graph is connected. Work of H. S. M. Coxeter [Cox67] and of D. W. Koster [Kos75] provide the relations between the combinatorics of finite generalized Coxeter system and the geometry of real reflection groups and Shephard groups.

- Any Shephard group has a structure of generalized Coxeter system (this is analogous to the well-known property for real reflection groups). Moreover, as with the chamber geometry in the real case, there is a reasonably natural way to construct a set of generalized Coxeter generators consisting of reflections, using the geometry of the Shephard group [Cox67]. We call such a presentation constructed from the geometry a *standard generalized Coxeter presentation*.

- Given a generalized Coxeter system (W, S) and an $|S|$-dimensional complex vector space V, there exist a representation $\rho : W \to \mathrm{GL}(V)$ and an Hermitian form on V which is invariant under $\rho(W)$, such that for $s \in S$, the element $\rho(s)$ is a reflection of V of order p_s.

- If the generalized Coxeter system is irreducible and finite, there is such a representation ρ which is faithful and such that $\rho(W)$ is a Shephard group or a real reflection group.

The last point yields that finite, irreducible generalized Coxeter systems correspond to the class of complex reflection groups which is the union of (complexified) real irreducible reflection groups and Shephard groups. The latter are known to be all real reflection groups with unbranched Coxeter graph, together with the infinite family $G(r, 1, n)$ with $r \geq 3$, and 15 of the non-real irreducible, exceptional groups.

2 Coxeter elements and noncrossing partition lattices

Consider first an irreducible real reflection group W with a fixed chamber $\mathfrak{C}$ of its reflection arrangement. Denote by $S = \{s_1, \ldots, s_n\}$ the set of reflections associated to the chamber $\mathfrak{C}$, so that (W, S) is a Coxeter system. A *Coxeter element* in W is classically defined as the product of the reflections in S in any order, see [Cox51]. It thus depends on the choice of the chamber $\mathfrak{C}$ and on the order of the factors. It is well known that, however, the set of such Coxeter elements forms a single conjugacy class in W, and that the order of a Coxeter element is equal to the highest degree d_n (for basic results on Coxeter elements, we refer to [Hum90, Ch. 3.16–3.19] or [Kan01, Ch. 29]). Coxeter elements play an important role in the theory of (finite) Coxeter groups. In particular, they are crucial in Coxeter-Catalan combinatorics, namely in the context of noncrossing partitions, cluster complexes, generalized associahedra, Cambrian fans and lattices, and subword complexes. For details about these concepts, we refer to [Arm06, Rea07, PS15] and the references therein. Work of T. Brady and C. Watt [BW02] and of D. Bessis [Bes03] on the braid group of W show the importance of the *W-noncrossing partition lattice* $\mathrm{NC}(W, c)$ associated to a Coxeter element $c \in W$. This poset $\mathrm{NC}(W, c)$ is defined as the principal order ideal generated by any Coxeter element c, that is,

$$\mathrm{NC}(W, c) = [\mathbb{1}, c]_W = \{w \in W \mid \mathbb{1} \leq_R w \leq_R c\}. \tag{1}$$

Here, $\leq_R$ denotes the *absolute order* on W given by

$$x \leq_R y \iff \ell_R(x) + \ell_R(x^{-1}y) = \ell_R(y) \tag{2}$$

where $\ell_R(x)$ is the *absolute length* of x with respect to the set R of all reflections in W, and $\mathbb{1} \in W$ denotes the identity element.

Further work of D. Bessis [Bes15] shows how to generalize the noncrossing partition lattice to all irreducible well-generated groups. The definitions of absolute length and absolute order in (2) still make sense in the complex case, but one needs a replacement for the notion of Coxeter element. This is provided by the following notion of regularity from T. A. Springer [Spr74]. An element $w \in W$ is called *regular* if w has an eigenvector v in the complement of the reflecting hyperplanes, so that W acts freely on the orbit of v. Say that w is ζ-regular if $w(v) = \zeta v$ in this situation. The multiplicative order d of w within W is then the same as that of ζ within $\mathbb{C}^\times$, and one calls d a *regular number* for W. A simple characterization of regular numbers was obtained by G. I. Lehrer and T. A. Springer [LS99], and later proven uniformly by G. I. Lehrer and J. Michel [LM03]. This characterization implies that for irreducible well-generated groups, the *Coxeter number* $h = d_n$ is always a regular number, and that it is the highest regular number possible. It turns out that for a real reflection group, the class of usual Coxeter elements corresponds to the class of $e^{2i\pi/h}$-regular elements. D. Bessis thus replaced the Coxeter element c in a real reflection group with an $e^{2i\pi/h}$-regular element in an irreducible well-generated group, see [Bes15, Definition 7.1].

In the present paper, we consider the following more general definition.

Definition 2.1 *Let W be an irreducible well-generated complex reflection group W. A* Coxeter element *in W is a regular element in W of order $h = d_n$.*

As mentioned above, the usual definition is more restrictive than the one given here: a Coxeter element is classically taken to be regular for the specific eigenvalue $e^{2i\pi/h}$, and this notion is, for real reflection groups, equivalent to the definition using the product of the reflections through the walls of a chamber in the reflection arrangement. Both definitions (the classical one, and the extended one from Definition 2.1) have been used in the literature, but their subtle distinction has been sometimes source of confusion. For example, the statement of Theorem C in [Kan01, Ch. 32.2] is erroneous. Also, some results of [BR11]

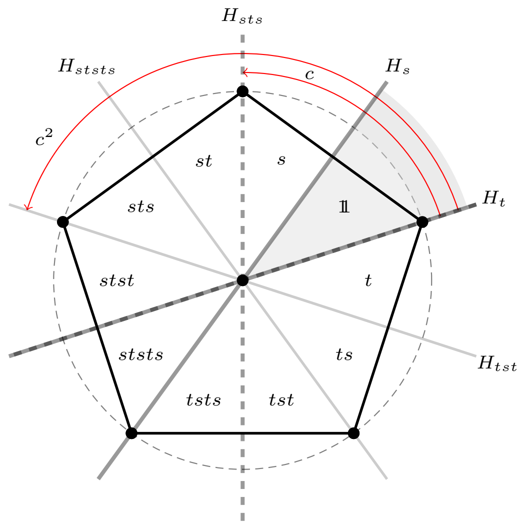

Fig. 1: The symmetry group of the regular pentagon.

used the extended definition, although they relied on results from [Bes15] which used only the restrictive definition. One of the purposes of this work is to clarify these confusions. We will next see that the more general definition provides new insights even for real reflection groups.

Example 2.2 *(cf. [Mil14, Example 2.7]) An example (actually the smallest for which our generalization is nontrivial) to bear in mind is the real reflection group W of type $H_2 = I_2(5)$ shown in Figure 1. It is the dihedral group of order 10 consisting of all symmetries of a regular pentagon. The degrees of W are 2 and 5, and the Coxeter number is thus $h = 5$. Set s and t to be the reflections through the boundary hyperplanes of a given chamber in the reflection arrangement, and let $c = st$ be a corresponding Coxeter element. It is given by a rotation of angle $2\pi/5$. Its eigenvalues are ζ and $\zeta^4 = \zeta^{-1}$, where $\zeta := e^{\frac{2i\pi}{5}}$. Observe that c^2 is again regular of order h. It is a rotation of angle $4\pi/5$ and has eigenvalues ζ^2 and $\zeta^3 = \zeta^{-2}$. In particular, c and c^2 are not conjugate in W. In the classical notion of Coxeter elements, c and c^4 are Coxeter elements (both can be written as products of reflections through the walls of a chamber of the reflection arrangement), while c^2 and c^3 are not. Observe though that $c^2 = sts \cdot t$, that the reflections sts and t also generate W, and that $(W, \{sts, t\})$ is again a Coxeter system, isomorphic to the Coxeter system $(W, \{s, t\})$. So c^2 and c^3 are still products of the generators of a Coxeter system, but this Coxeter system $(W, \{sts, t\})$ does not come from the walls of a chamber, as in the usual Coxeter generators of a reflection group.*

For an irreducible well-generated group W and a Coxeter element c in W, we define the *poset of W-noncrossing partitions* $NC(W, c)$ as in (1). Since conjugation by elements of W preserves the set of reflections R, it also preserves the absolute length ℓ_R and respects the absolute order $\leq_R$. Hence, whenever two Coxeter elements c and c' are W-conjugate, the posets $\mathrm{NC}(W, c)$ and $\mathrm{NC}(W, c')$ are isomorphic. T. A. Springer showed in [Spr74, Theorem 4.2] that, for a fixed ζ in $\mathbb{C}$, the ζ-regular elements in W form a single W-conjugacy class. It is known to experts that the poset structure on $\mathrm{NC}(W, c)$ does **not** depend on the choice of the Coxeter element c, even in the more general notion of Coxeter elements in which there is not necessarily a single conjugacy class. But it seems to have only been mentioned so far in [CS14], where the poset isomorphisms were checked explicitly using a computer. As a result of the considerations in this paper, we obtain a conceptual reason for the existence of a poset isomorphism between $\mathrm{NC}(W, c)$ and $\mathrm{NC}(W, c')$ for two regular elements of order h that are **not** W-conjugate, see Corollary 3.3 below.

3 Characterizations of Coxeter elements

The first main result of this paper, Theorem 3.1 below, provides several alternative characterizations of Coxeter elements. We show that the set of Coxeter elements forms a single orbit in W for the action of *reflection automorphisms*, that is, those automorphisms of W that preserve the set R of reflections. This allows one to characterize Coxeter elements using only $e^{2i\pi/h}$-regular elements and reflection automorphisms as described in characterization (iv) below. It also implies that Coxeter elements are exactly the products of generators of W for some well-behaved generating subsets of reflections; see characterization (v), for which we need to explain beforehand the terminology. We call a *regular generating set* of W such a well-behaved generating set: it is a minimal generating set of reflections having some additional properties as will be made precise in Section 4.2. We will see in Proposition 4.4 that any irreducible well-generated group admits a regular generating set. Note that most explicit presentations for W using diagrams "à la Coxeter" feature a regular generating set. We say that two generating sets for W are *isomorphic* if there exists a bijection between them which extends to an automorphism of W.

Moreover we obtain the purely combinatorial characterization (vi) for Coxeter elements in finite groups that admit a *generalized Coxeter system*. As we have explained in Section 1, these are exactly the real reflection groups and the Shephard groups, and for a real reflection group, a generalized Coxeter system is a Coxeter system in the usual sense.

Theorem 3.1 *Let W be an irreducible well-generated complex reflection group with Coxeter number h, and let $c \in W$. The following statements are equivalent.*

(i) *c is a Coxeter element (i.e., c is regular of order h);*

(ii) *$c = w^p$ for an $e^{2i\pi/h}$-regular element w and an integer p coprime to h;*

(iii) *c has an eigenvalue of order h;*

(iv) *$c = \psi(w)$ for an $e^{2i\pi/h}$-regular element w and a reflection automorphism ψ.*

Fix a regular generating set S_0 for W. Then the following statement is as well equivalent.

(v) *There exists a generating set S contained in R and isomorphic to S_0, such that c is the product (in some order) of the elements in S.*

If W admits a generalized Coxeter system, the following statement is as well equivalent.

(vi) *There exists a subset S of the set R of reflections in W such that (W, S) is a generalized Coxeter system and c is the product (in some order) of the elements in S.*

The proof of Theorem 3.1 is derived as follows. The assertions (i)⇔(ii) and (i)⇒(iii) are direct consequences of the definition of Springer's regularity. The implication (iii)⇒(i) comes from a counting argument using Pianzola-Weiss' formula, and can be found in [Kan01, Theorem 32-2C] for the real case and in [CS14, Proposition 2.1] for the general case. In our work, we prove the three remaining characterizations (iv), (v) and (vi). Characterization (iv) is a consequence of Proposition 3.2 below, and is the first main result of this work. Characterization (vi) for real reflection groups and Shephard groups will be treated in Section 4.1. It relies on the study of classical Coxeter elements in these groups, on characterization (iv) and on a rigidity property of the generalized Coxeter presentations of these groups (see Proposition 4.1). Characterization (v) will also follow in Section 4.2 from characterization (iv) and from the definition of regular generating sets.

The first consequence of Theorem 3.1 is that characterization (iv) allows one to transfer properties that are known for "classical" Coxeter elements to the more general notion of Coxeter elements, see e.g. Corollary 3.3 below. Moreover it shows that the two natural generalizations of the usual definition of Coxeter elements (one in the real setting, the other for the complex setting) coincide in the real case. Property (vi) can indeed be taken as a natural generalized definition of Coxeter elements in finite real reflection groups. We will see that this generalization brings new elements whenever the Coxeter group is non-crystallographic (see Remark 5.4). Note that a slightly more restrictive version of this definition already appears in D. Bessis' work on the dual braid monoid of Coxeter groups, see [Bes03, Definition 1.3.2]. This generalized definition for Coxeter elements was also used for general finite or infinite Coxeter groups in [BDSW14]. Table 1 records the "classical" and the more general definition of Coxeter elements for an irreducible well-generated group, either real or complex.

	"classical" definition	general definition
W real	product of the reflections through the walls of some chamber	product of elements of S for some Coxeter system (W, S) with $S \subseteq R$
W complex	$e^{\frac{2i\pi}{h}}$-regular	regular of order h

Tab. 1: Different notions of Coxeter elements in a reflection group W

The statement below is a reformulation of characterization (iv) in Theorem 3.1. We denote by $\mathrm{Aut}_R(W)$ the group of reflection automorphisms of W.

Proposition 3.2 *The action of* $\mathrm{Aut}_R(W)$ *on* W *preserves the set of Coxeter elements, and is transitive on it.*

The proof of this property involves the study of Galois automorphisms of W, and makes use of a result from I. Marin and J. Michel (Proposition 5.1); see [RRS14, §2] for details.

Proposition 3.2 implies that, for any two Coxeter elements c and c', there is a reflection automorphism ψ mapping c to c'. Since ψ sends reflections to reflections, the absolute length ℓ_R and the absolute order $\leq_R$ are respected by ψ. Hence, ψ restricts to a poset isomorphism from the interval $[\mathbb{1}, c]_W$ to the interval $[\mathbb{1}, c']_W$ in the absolute order, and we obtain the following corollary.

Corollary 3.3 *Let* W *be an irreducible well-generated complex reflection group, and let* c *and* c' *be two Coxeter elements. Then the two posets* $\mathrm{NC}(W, c)$ *and* $\mathrm{NC}(W, c')$ *are isomorphic.*

This corollary implies that all properties of the poset of noncrossing partitions that have only been proven for the classical definition of Coxeter elements also hold for the more general definition. In particular, for a Coxeter element $c \in W$, the poset $\mathrm{NC}(W, c)$ is an EL-shellable, self-dual lattice whose elements are counted by the *W-Catalan number* $\prod_{i=1}^{n} \frac{d_i+h}{d_i}$. We refer to [Arm06, BR11, KM13, Müh15] for these and several other properties of the noncrossing partitions that were so far only proven for the restricted definition of Coxeter elements.

More generally, Proposition 3.2 implies that all the properties of Coxeter elements relative only to the combinatorics of the group W equipped with its set of generators R, do not depend on the choice of a Coxeter element. For example, the transitivity of the Hurwitz action of the n-strands braid group B_n on the reduced R-decompositions of $e^{2i\pi/h}$-regular elements (see [Bes15, Definition 6.19, Proposition 7.6]) implies the same property for all regular elements of order h. This was proven for Coxeter elements of infinite Coxeter groups as well in [BDSW14, Theorem 1.3].

Remark 3.4 *Theorem 3.1(vi) provides a purely combinatorial description of Coxeter elements in reflection groups admitting generalized Coxeter systems. Nevertheless, this property is rather difficult to check for a given element. We do not know of any better purely combinatorial description, nor do we know of any combinatorial description for Coxeter elements in arbitrary well-generated groups. Coxeter elements have a number of simple properties: they have absolute length* n *(the rank of the group), order* h*, and the Hurwitz action is transitive on the set of reduced* R*-decompositions of a Coxeter element. However, these properties are not sufficient to characterize Coxeter elements. For example, there exist elements in the group of type* D_4 *that are of absolute length* 4 *and for which the Hurwitz action is transitive on reduced* R*-decompositions, but which do not have a primitive* 6*-th root of unity as an eigenvalue*[(i)]*. Similarly, there exist elements in type* B_6 *that are of absolute length* 6 *and order* 12*, but which also do not have a primitive* 6*-th root of unity as an eigenvalue.*

(i) We thank Patrick Wegener for pointing out the existence of such elements.

4 Coxeter elements and generating sets

4.1 Characterization of Coxeter elements in real groups and Shephard groups

Let W be an irreducible real reflection group, and $S = \{s_1, \ldots, s_n\}$ be a set of fundamental reflections determined by the choice of a chamber of the reflection arrangement of W. We obtain a Coxeter system (W, S), and it is well known that the product, *in any order*, of the reflections in S, is a regular element for the eigenvalue $e^{2i\pi/h}$. The proof goes by studying the action of such an element on a specific plane called the Coxeter plane, see e.g. [Hum90, Ch. 3.17].

For a Shephard group W, we explained in Section 1 that there is an analog of the chamber geometry, that allows one to construct a set S of standard generating reflections (giving rise to a "standard generalized Coxeter presentation"). It can be checked that in this case as well, the product, in any order, of the reflections in S is $e^{2i\pi/h}$-regular.

We can now explain how to prove characterization (vi) of Coxeter elements in Theorem 3.1, i.e., that c is a regular element of order h on a Coxeter or Shephard group W if and only if c can be written as $s_1 \ldots s_n$ where $S = \{s_1, \ldots, s_n\}$ consists of reflections and (W, S) is a generalized Coxeter system. The "only if" part follows directly from the discussion above and from Proposition 3.2 (transitivity of reflection automorphisms on Coxeter elements), since a reflection automorphism transports any (generalized) Coxeter system consisting of reflections to another one. The "if" part follows from Proposition 4.1 below that exhibits a rigidity property of generalized Coxeter presentations, allowing one to construct a reflection automorphism between any two standard generating sets.

Proposition 4.1 *Let W be a complex reflection group, and let R be the set of all its reflections. Assume S, S' are two subsets of R such that (W, S) and (W, S') are generalized Coxeter systems. Then (W, S) and (W, S') are isomorphic generalized Coxeter systems.*

In particular, rewritten in the context of abstract Coxeter systems, Proposition 4.1 implies the following. Let (W, S) be a *finite* Coxeter system, and let T denote the conjugacy closure of S in W. Then any Coxeter system (W, S') for W, with $S' \subseteq T$, is isomorphic to (W, S). Note that however, S' is not necessarily W-conjugate to S, as seen in the example of the dihedral group $I_2(5)$ (Example 2.2).

Remark 4.2 *It is well known that Proposition 4.1 does not hold if one does not assume $S \subseteq R$. Some classical counterexamples arise from the existence of a group isomorphism between $I_2(2m)$ and $A_1 \times I_2(m)$ (for any $m \geq 3$). This property was moreover shown not to hold in general, even with $S \subseteq R$, for irreducible* infinite *Coxeter groups, see [Müh00].*

Proposition 4.1, particularly for classical Coxeter systems, is known to experts, but we have not been able to find a proof in the literature (for example, the property is stated in [Bes03, §1.1] without proof). The only proof we can give is case-by-case (reducing the problem to irreducible groups, then checking that there is no abstract group isomorphism between two different Coxeter systems; see [RRS14, §4.3] for details).

4.2 Regular generating sets for well-generated reflection groups

In order to obtain an analogous characterization of Coxeter elements for arbitrary well-generated groups (not only Coxeter and Shephard groups), we need to introduce some well-behaved generating sets of these groups. Any irreducible well-generated group of rank n can be minimally generated by n reflections. Much work has been devoted to finding well-behaved presentations by generators and relations such that the generating set consists of n reflections. For the general case of an arbitrary irreducible well-generated group, there is no canonical presentation analogous to the Coxeter presentation discussed above. However, a uniform approach has been given by D. Bessis [Bes15], using the geometry of the braid group of W and a construction known as the *dual braid monoid*. From Bessis' presentations, one can obtain a minimal generating set for W, consisting of n reflections $r_1, \ldots, r_n$ such that the product $r_1 \cdots r_n$ is $e^{2i\pi/h}$-regular (see [RRS14, §4.2] for details).

The presentations of W obtained in earlier works can be obtained from Bessis' presentations by removing redundancies. In particular, the standard presentations of Coxeter groups and of Shephard groups can be obtained this way. In the general case, one can also recover the presentations described by Coxeter in [Cox67], and the presentations given in [BMR98] (see also [BM04] and [MM10a, §6]). Such well-behaved presentations are gathered in the table [Mic14], and are implemented explicitly in the package CHEVIE [GHL+96] of GAP.

It turns out that some of these presentations feature a generating set which has a stronger property defined below.

Definition 4.3 *Let W be an irreducible well-generated group of rank n, together with a set S of n generating reflections. We call S a* regular generating set *for W if:*

(1) any reflection in W is conjugate to a power of a reflection in S;

(2) the product, in any *order, of all the elements in S is a Coxeter element, i.e., a regular element of order h.*

This definition might appear to be somewhat artificial; it is introduced because it contains everything needed to state and prove characterization (v) of Theorem 3.1 (see [RRS14, §4.2] for details). The following proposition ensures that such regular presentations indeed exist for any well-generated group.

Proposition 4.4 *Any irreducible well-generated group W admits a presentation whose set of generators is a regular generating set. More precisely, the following presentations satisfy Properties (1) and (2):*

(i) for W real: the standard Coxeter presentation (arising from the chamber geometry);

(ii) for W Shephard group: the standard "generalized Coxeter presentation" for W;

(iii) for any W: the explicit presentations in [Mic14] implemented in CHEVIE.

Remark 4.5 *Given our construction of a generating set from Bessis' presentations, Proposition 4.4 implies that for an $e^{2i\pi/h}$-regular element c, there exists a reduced decomposition $(r_1,\ldots,r_n)$ of c such that the product, in* any *order, of $r_1,\ldots,r_n$ is a Coxeter element (i.e., regular of order h). This property does not depend on the choice of c as an $e^{2i\pi/h}$-regular element, or even as a Coxeter element (this follows easily from Proposition 3.2). Note however that it* does *depend in general on the chosen reduced decomposition. There may exist reduced decompositions $(r'_1,\ldots,r'_n)$ of c such that for some $\sigma\in S_n$, the product $r'_{\sigma(1)}\cdots r'_{\sigma(n)}$ is not a Coxeter element. For example, in type D_4, consider the reduced decomposition $c = s\cdot t\cdot uvu\cdot u$ (where s,t,u,v are the standard Coxeter generators, u being the central one in the diagram). Then the product $s\cdot uvu\cdot t\cdot u$ has order $4\neq 6=h$ and is thus not a Coxeter element.*

Remark 4.6 *Alternative presentations, other than the ones in [Mic14], have been given for the five exceptional non-Shephard well-generated groups; see in particular [MM10a, §6], where the new generating sets are obtained from an initial one by applying Hurwitz action. It is natural to ask whether these alternative presentations are also regular, i.e., whether the product of the generators, in any order, is a regular element of order h. It turns out that for G_{24}, G_{27} and G_{29}, all the alternative presentations are also regular, whereas for G_{33} and G_{34}, among the five presentations P_1 through P_6 described by G. Malle and J. Michel in [MM10a, §6.4], only the initial presentation P_1 is regular.*

5 Conjugacy classes of Coxeter elements

Our second main theorem concerns the connection between Coxeter elements and the field of definition of W. Recall that the ***field of definition*** K_W is the subfield of $\mathbb{C}$ generated by the traces of the elements in $W\subseteq \mathrm{GL}(V)$. It is known, see e.g. [Ben93, Proposition 7.1.1], that the representation V of W can be realized over K_W. We thus assume from now on that $W\subseteq \mathrm{GL}_n(K_W)$. We also denote by $\Gamma_W := \mathrm{Gal}(K_W/\mathbb{Q})$ the Galois group of the field extension K_W over $\mathbb{Q}$. We next describe an action of Γ_W on the conjugacy classes of Coxeter elements, and show that this action is simply transitive, see Theorem 5.2.

The group Γ_W naturally acts on $\mathrm{GL}_n(K_W)$ by Galois conjugation of the matrix entries. For $\gamma\in\Gamma_W$, we denote by $\bar\gamma$ the associated automorphism of $\mathrm{GL}_n(K_W)$. However, this action does not necessarily preserve W, so we cannot always associate an automorphism of W to γ. For example, if W is the dihedral group of type $H_2 = I_2(5)$, then $K_W=\mathbb{Q}(\sqrt{5})$ and the Galois group Γ_W has order 2. But one can check that the only involutive automorphisms of $I_2(5)$ are inner automorphisms, i.e., are given by conjugation by an element of W.

Nevertheless, $\bar{\gamma}(W)$ is obviously again a complex reflection group, and it turns out that it is always conjugate to W in $\mathrm{GL}_n(\mathbb{C})$ (cf. [LT09, Theorem 8.32]). Let $g \in \mathrm{GL}_n(\mathbb{C})$ be such that $\bar{\gamma}(W) = gWg^{-1}$. We then obtain from γ an automorphism of W given by $w \mapsto g^{-1}\bar{\gamma}(w)g$. An automorphism obtained this way is called *Galois automorphism* of W attached to γ. These automorphisms have been studied in detail by I. Marin and J. Michel in [MM10b], and we will rely on one of their results, see Proposition 5.1 below.

We first record the following straightforward properties.

(i) The character of a Galois automorphism attached to γ (seen as a representation of W) is given by $w \mapsto \gamma(\mathrm{tr}_V(w))$.

(ii) There are several choices for such an automorphism but one can pass from one to another via conjugation by some element in the normalizer $N_W = N_{\mathrm{GL}_n(\mathbb{C})}(W)$.

(iii) Any Galois automorphism of W is a reflection automorphism.

(iv) Let ψ be a reflection automorphism of W. Then ψ is a Galois automorphism attached to γ if and only if the character of ψ (seen as a representation of W) is the image by γ of the character V, i.e.,

$$\forall w \in W, \mathrm{tr}_V(\psi(w)) = \gamma(\mathrm{tr}_V(w)).$$

We do not necessarily have a natural action of Γ_W on W, but using (ii), we get an action of Γ_W on the set of N_W-conjugacy classes of elements in W. Note that Springer's theory (see [Spr74, Theorem 4.2.(iv)]) implies that two regular elements in W having the same eigenvalues are W-conjugate. Therefore, N_W-conjugacy classes of regular elements (and in particular of Coxeter elements) are the same as W-conjugacy classes. Denote by $\mathcal{C}(W)$ the set of conjugacy classes of Coxeter elements. By Proposition 3.2, reflection automorphisms stabilize the set of Coxeter elements, so the action of Γ_W stabilizes $\mathcal{C}(W)$. Theorem 5.2 below states that the action of Γ_W on $\mathcal{C}(W)$ is simply transitive. The transitivity follows from the following result from [MM10b].

Proposition 5.1 (cf. [MM10b, Theorem 1.2]) *Any reflection automorphism of an irreducible complex reflection group W is a Galois automorphism (attached to some $\gamma \in \Gamma_W$).*

Combined with Proposition 3.2, this implies that the action of Γ_W is transitive on $\mathcal{C}(W)$. The proof of the *simple* transitivity exhibits several other properties related to the field of definition, that we collect in Theorem 5.2. We use the following notation:

- $m_1 = d_1 - 1, \ldots, m_n = d_n - 1$ are the *exponents* of W,
- $\varphi(j)$ (for $j \in \mathbb{N}$) is the number of integers in $\{1, \ldots, j\}$ that are coprime to j, and
- $\varphi_W(j)$ is the number of integers coprime to j among the set of exponents $\{m_1, \ldots, m_n\}$.

For an irreducible, well-generated group W with Coxeter number h, we also denote by G_W the setwise stabilizer of $\{\zeta^{m_1}, \ldots, \zeta^{m_n}\}$ in the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, where ζ is a primitive h-th root of unity.

Theorem 5.2 *Let W be an irreducible well-generated complex reflection group, K_W be its field of definition, and $\mathcal{C}(W)$ be the set of conjugacy classes of Coxeter elements of W. The following properties hold:*

(i) the transitive action of $\Gamma_W = \mathrm{Gal}(K_W/\mathbb{Q})$ on $\mathcal{C}(W)$ is free;

(ii) $[K_W : \mathbb{Q}] = |\mathcal{C}(W)| = \varphi(h)/\varphi_W(h)$;

(iii) K_W is equal to the fixed field $\mathbb{Q}(\zeta)^{G_W}$;

(iv) K_W is generated by the coefficients of the characteristic polynomial of any Coxeter element of W.

Remark 5.3 *The characterization of K_W in (iii) (or in (iv), which is easily seen to be equivalent) has already been obtained by G. Malle in [Mal99, Theorem 7.1], his proof using a case-by-case check via the classification. We found it independently by other means but we also need a case-by-case analysis.*

The equality $|\mathcal{C}(W)| = \varphi(h)/\varphi_W(h)$ is a direct consequence of Springer's theory (describing the eigenvalues of a regular elements in terms of the exponents) and is included in the theorem for the sake of clarity. The proof of Theorem 5.2 goes as follows. We first prove that the four properties (i)–(iv) are equivalent; this is a case-free proof, except for the use of Proposition 3.2. The theorem is then derived by checking via the classification that the equality $[K_W : \mathbb{Q}] = \varphi(h)/\varphi_W(h)$ is satisfied for any irreducible well-generated group.

Remark 5.4 *Recall that a complex reflection group is a finite Weyl group if and only if its field of definition is $\mathbb{Q}$. Theorem 5.2(i) thus implies that all regular elements of order h are $e^{2i\pi/h}$-regular if and only if W is a Weyl group. We also recover the well-known fact that for Weyl groups, the equality $\varphi_W(h) = \varphi(h)$ holds, see [Hum90, Proposition 3.20].*

Remark 5.5 *The intriguing relation in Theorem 5.2(ii) between the field of definition and the residues of the exponents modulo h yields the question of what happens for badly-generated groups. There are 8 of them in the exceptional types. For all but G_{15}, the highest invariant degree d_n is still a regular number, so we could define a Coxeter element as a regular element of order d_n as for well-generated groups. For badly-generated groups in the infinite series, d_n is regular only for the groups of type $G(2d, 2, 2)$. In these cases, the equality $|\mathcal{C}(W)| = \varphi(d_n)/\varphi_W(d_n)$ still holds for the same reasons as for well-generated groups. Moreover, we have*

(i) for $G(2d, 2, 2)$, G_7, G_{11}, G_{12}, G_{19}, G_{22} and G_{31}, one still has $[K_W : \mathbb{Q}] = \varphi(d_n)/\varphi_W(d_n)$, whereas

(ii) for G_{13} (and also for G_{15}), one has $[K_W : \mathbb{Q}] = 2\varphi(d_n)/\varphi_W(d_n)$.

We do not know of an explanation for these observations. This actually implies that for all the groups listed in (i), Theorem 5.2 still holds.

6 Regular elements and reflection automorphisms

Some of our results can be extended to regular elements of arbitrary order. In particular, statement (i) below is a generalized version of Proposition 3.2 (the notations φ and φ_W are defined before Theorem 5.2).

Theorem 6.1 *Let W be an irreducible complex reflection group, K_W be its field of definition, and d be a regular number for W. Denote by $\mathcal{C}_d(W)$ the set of conjugacy classes of regular elements of order d. Then*

(i) the action of reflection automorphisms on W preserves the set of regular elements of order d, and is transitive on it;

(ii) the natural action of $\Gamma_W = \mathrm{Gal}(K_W/\mathbb{Q})$ on $\mathcal{C}_d(W)$ is transitive;

(iii) the cardinality of $\mathcal{C}_d(W)$ is $\varphi(d)/\varphi_W(d)$;

(iv) the integer $\varphi(d)/\varphi_W(d)$ divides $[K_W : \mathbb{Q}]$.

Unlike for Coxeter elements, in the general case the action in statement (ii) may be not free. In that case, $\varphi(d)/\varphi_W(d)$ is a proper divisor of $[K_W : \mathbb{Q}]$.

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