Limit points of root systems of infinite Coxeter groups

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From joint works with

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- **Jean-Philippe Labbé** (FU Berlin)

Outline

1. Root systems and “limit roots” of a Coxeter group W
2. Normalized roots, limit roots and isotropic cone
3. Action of W on the limit roots and topological properties

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A definition of root system

- V: a real vector space, of finite dimension n
- B: a symmetric bilinear form on V

Construction of a root system in (V, B):

1. Start with a simple system Δ:
   - Δ is a basis for V;
   - $\forall \alpha \in \Delta, B(\alpha, \alpha) = 1$;
   - $\forall \alpha \neq \beta \in \Delta$:
     - either $B(\alpha, \beta) = -\cos\left(\frac{\pi}{m}\right)$ for some $m \in \mathbb{Z}_{\geq 2}$,
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2. For each $\alpha \in \Delta$, define the *B*-reflection s_α:

$$\begin{array}{rccc} s_\alpha : & V & \to & V \\ & v & \mapsto & v - 2B(\alpha, v)\,\alpha. \end{array}$$

Check: $s_\alpha(\alpha) = -\alpha$, and s_α fixes pointwise $\alpha^\perp$.

Notation: $S = \{s_\alpha,\ \alpha \in \Delta\}$.

3. Construct the *B*-reflection group $W := \langle S \rangle$.

4. Act by W on Δ to construct the root system

$$\Phi := W(\Delta)\ .$$

Note: if $\rho = w(\alpha)$ (with $\alpha \in \Delta$), $w s_\alpha w^{-1}$ is the *B*-reflection associated to the root ρ.

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Coxeter group and root system

Proposition (Krammer)

- *(W, S) is a Coxeter system, with Coxeter presentation:*

$$W = \left\langle S \;\middle|\; s^2 = 1 \ (\forall s \in S);\ (st)^{m_{s,t}} = 1 \ (\forall s \neq t \in S) \right\rangle,$$

where $m_{s_\alpha, s_\beta} = \begin{cases} m & \text{if } B(\alpha, \beta) = -\cos(\pi/m), \\ \infty & \text{if } B(\alpha, \beta) \leq -1. \end{cases}$

- *Let* $\Phi^+ := \Phi \cap \mathrm{cone}(\Delta)$. *Then:* $\Phi = \Phi^+ \sqcup (-\Phi^+)$.

Note: Conversely, from any Coxeter system it is possible to construct a root system, using the classical geometric representation [Tits].

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Infinite root systems

For finite root systems:
Φ is finite $\Leftrightarrow$ W is finite ($\Leftrightarrow$ B is positive definite).

What does an infinite root system look like?

Simplest example, in rank 2:

Matrix of B in the basis (α, β): $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

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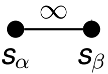

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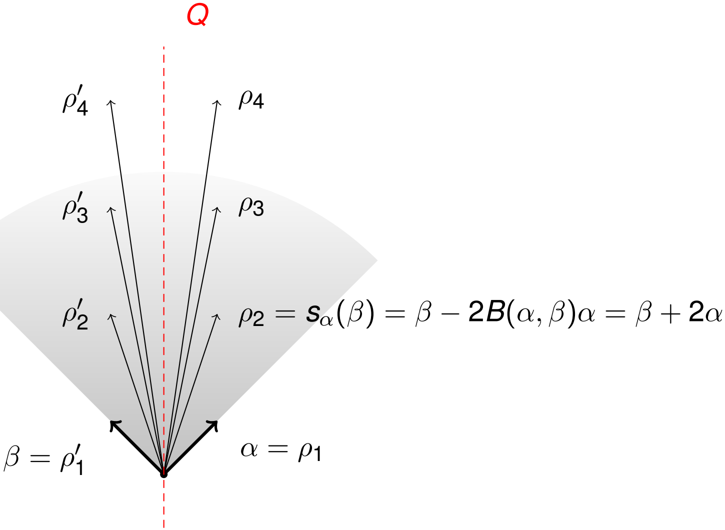
Infinite dihedral group, case $B(\alpha, \beta) = -1$



$\rho'_n = n\alpha + (n+1)\beta \quad ; \quad \rho_n = (n+1)\alpha + n\beta$

Observations

- The **norms** of the roots tend to ∞;
- The **directions** of the roots tend to the direction of the isotropic cone Q of B:

$$Q := \{v \in V,\ B(v,v) = 0\}.$$

(in the example the equation is $v_\alpha^2 + v_\beta^2 - 2v_\alpha v_\beta = 0$, and $Q = \mathrm{span}(\alpha + \beta)$.)

What if $B(\alpha, \beta) < -1$?

- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write

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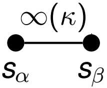
- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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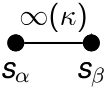
- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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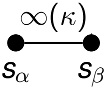
- Matrix of B: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write 

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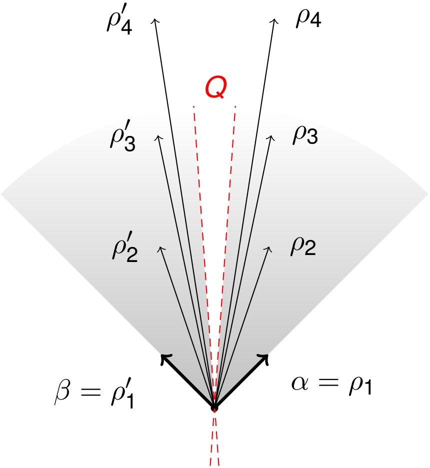

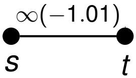

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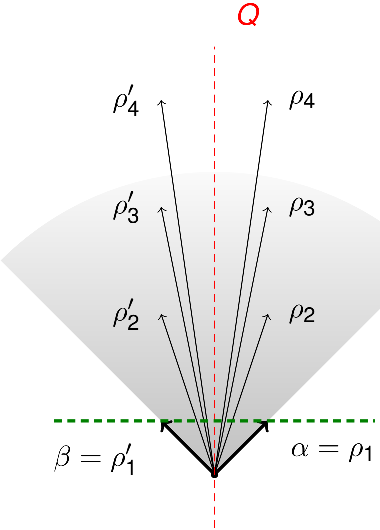
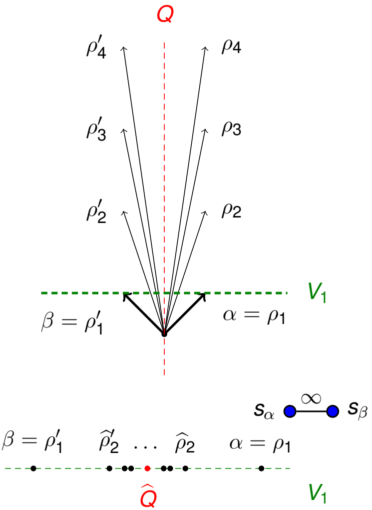
Let's see examples of higher rank

We cut the directions of the roots with an affine hyperplane.

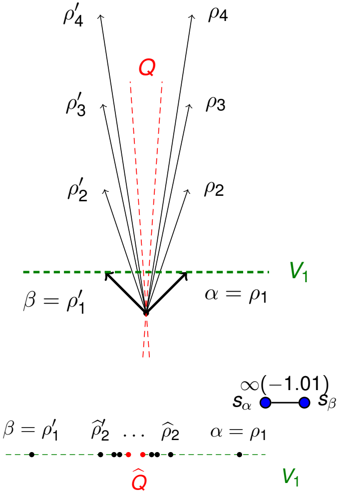


$$V_1 = \{v \in V \mid \sum_{\alpha \in \Delta} v_\alpha = 1\}$$

“Normalization” of roots

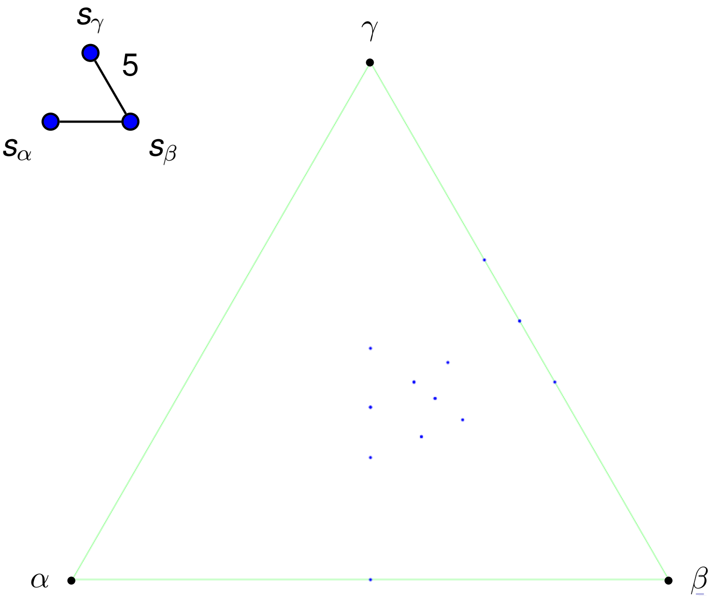


case $B(\alpha, \beta) = -1$

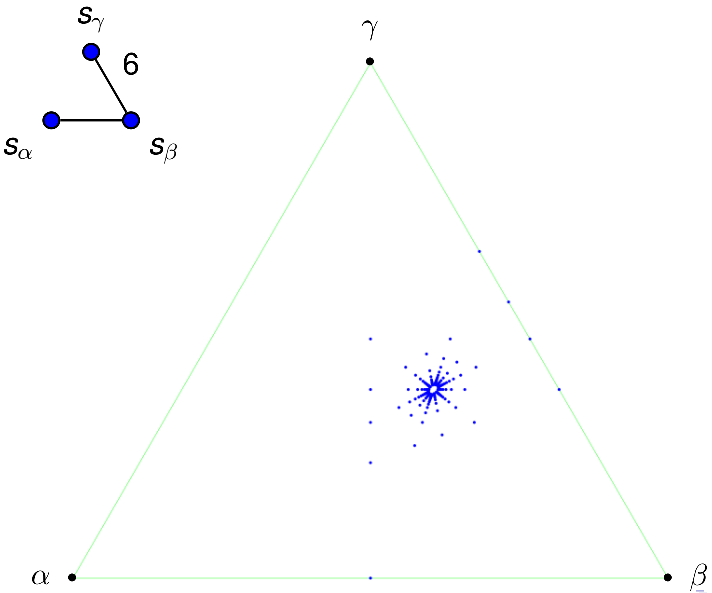


case $B(\alpha, \beta) = -1.01 < -1$

Examples in rank 3: finite group, sgn $B = (3, 0)$. (H_3)



Examples in rank 3: affine group, sgn $B = (2,0)$ $(\widetilde{G_2})$



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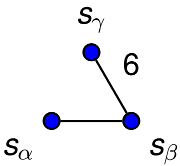

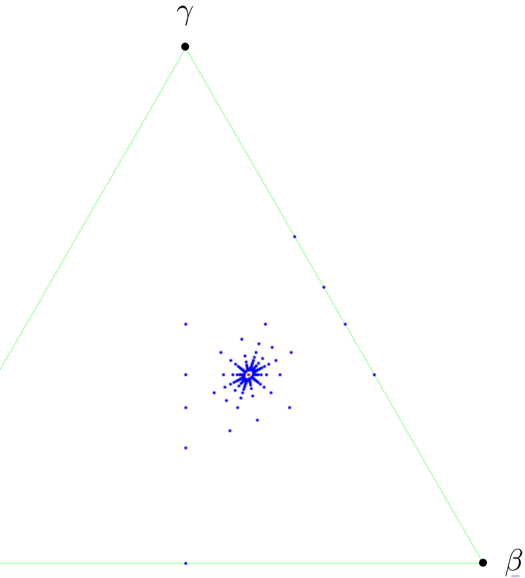

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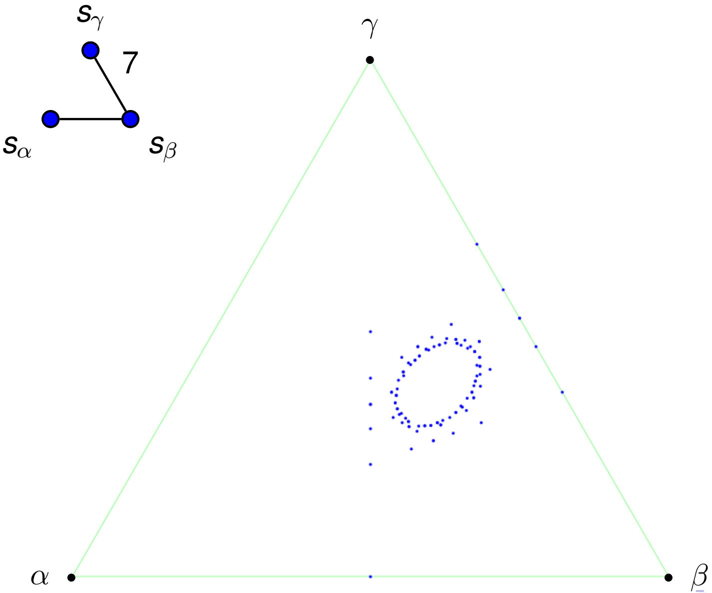

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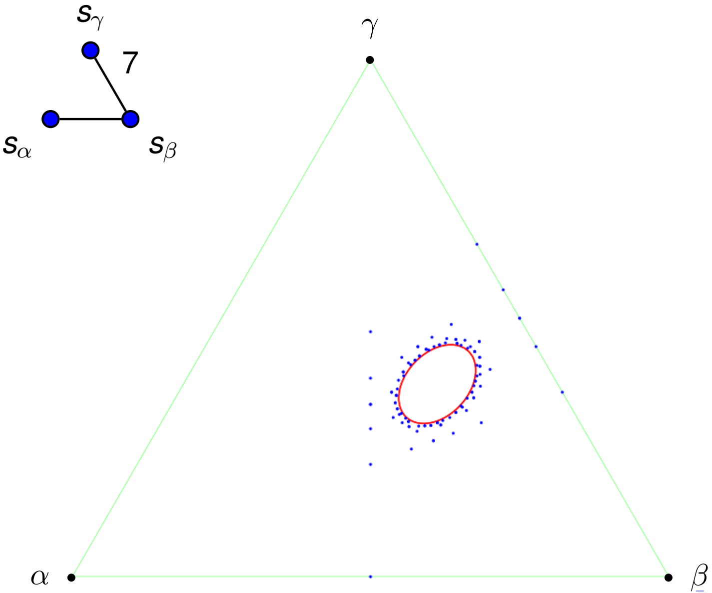

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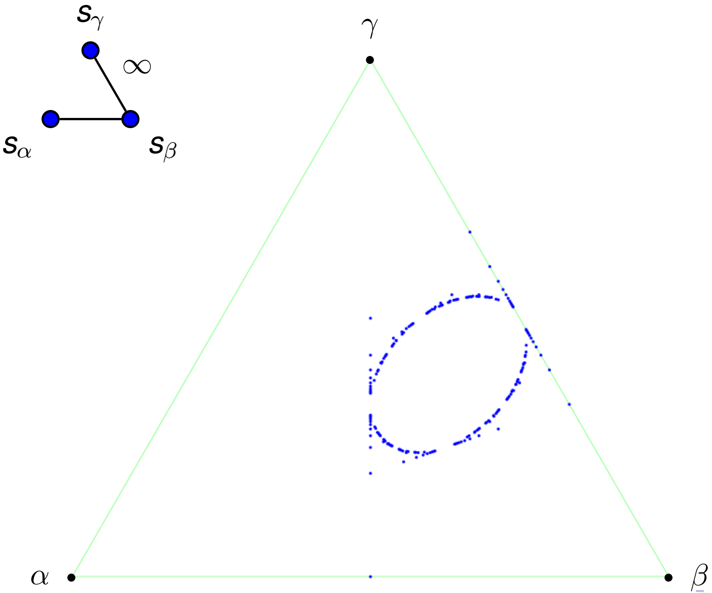

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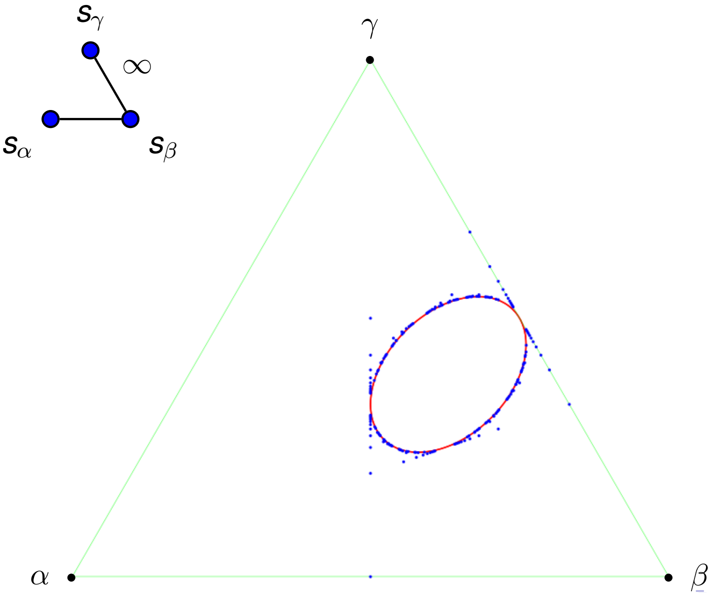

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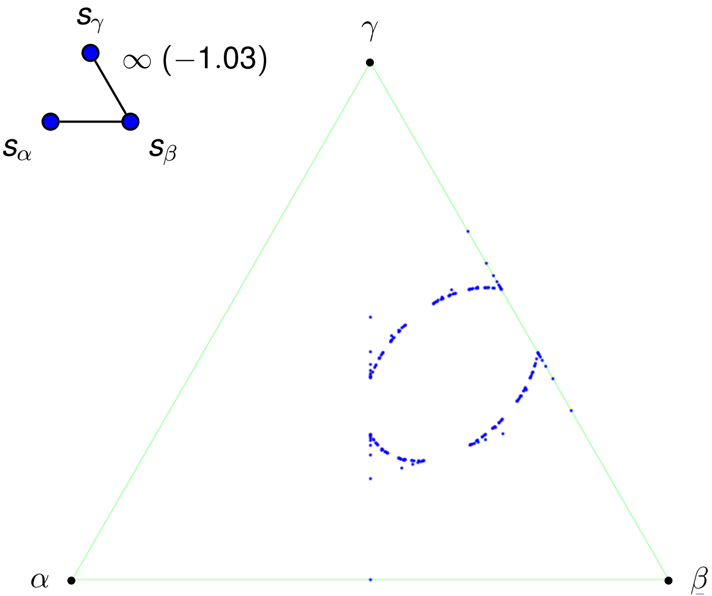

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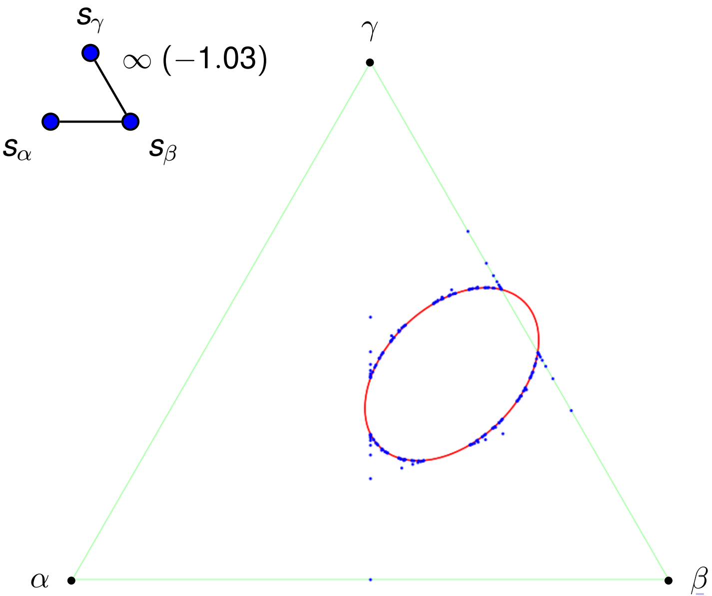

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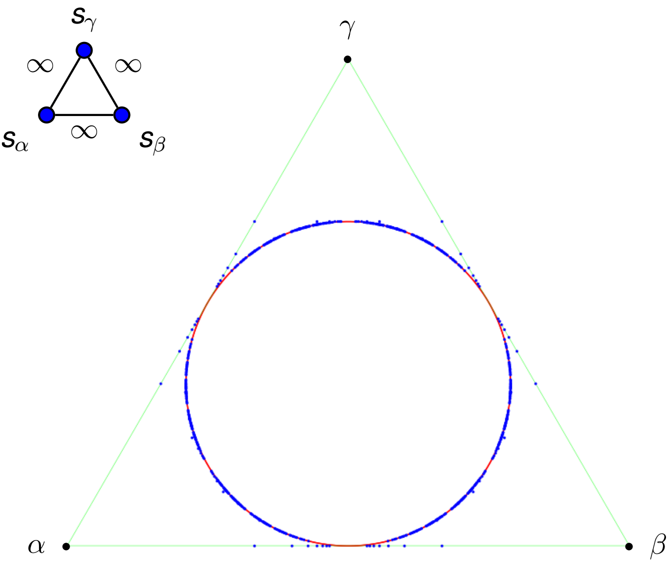

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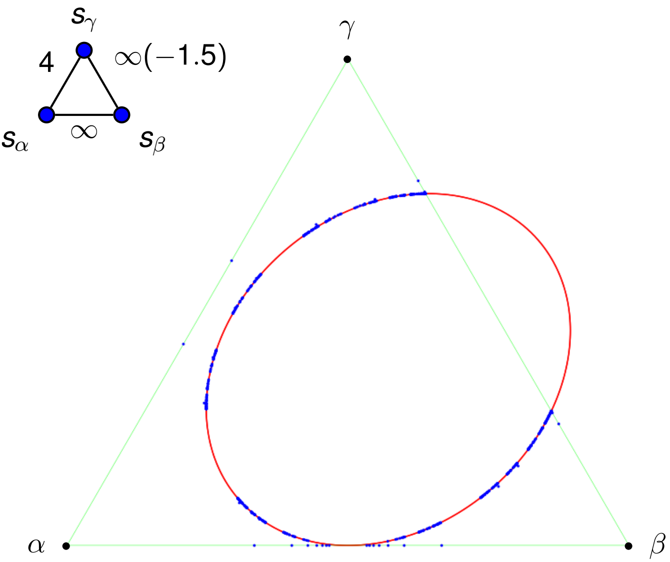

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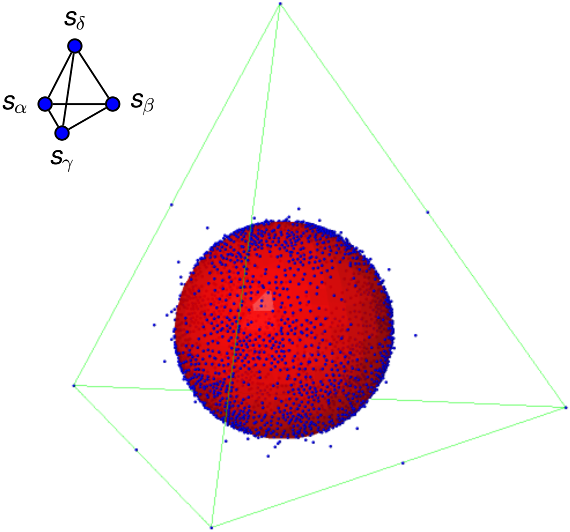

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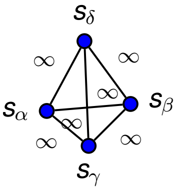

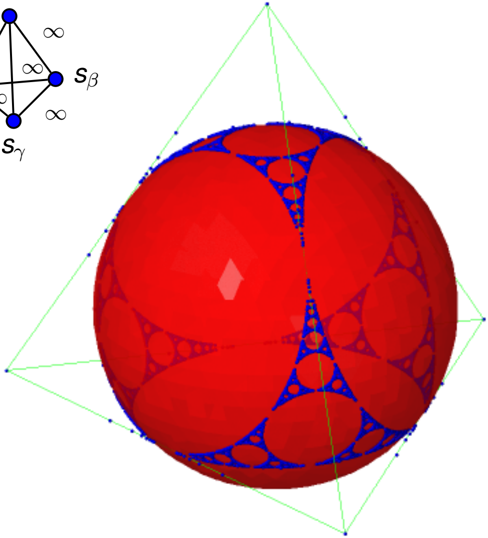

The limit roots lie in the isotropic cone Q

Theorem (Hohlweg-Labbé-R. '11)

Let Φ be a root system for an (infinite) Coxeter group, and $(\rho_n)_{n\in\mathbb{N}}$ an injective sequence in Φ. Then:

- $||\rho_n||$ *tends to* ∞ *(for any norm on* V*);*
- *if the sequence of normalized root* $\widehat{\rho_n}$ *has a limit* ℓ*, then*

$$\ell \in \widehat{Q} \cap \mathrm{conv}(\Delta).$$

Property proved independently in other contexts:

- [Kac 90] for Weyl groups of Kac-Moody algebras,
- generalized by [Dyer 2012] (work on the imaginary cone of a Coxeter group).

$\rightsquigarrow$ **Problem:** understand the set of possible limits, i.e., the accumulation points of $\widehat{\Phi}$:

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A natural group action of W on E

Geometric action of W on a part of V_1: $w \cdot v := \widehat{w(v)}$.

Defined on $D = V_1 \cap \bigcap_{w \in W} w(V \setminus V_0)$, where $V_0 = \overrightarrow{V_1}$.

Proposition

- $E(\Phi) \subseteq D$ *and* $E(\Phi)$ *is stable under the action of* W.
- *For* $\alpha \in \Phi$ *and* $x \in E$, $\widehat{Q} \cap L(\widehat{\alpha}, x) = \{x, s_\alpha \cdot x\}$.

A natural group action of W on E

Geometric action of W on a part of V_1: $w \cdot v := \widehat{w(v)}$.

Defined on $D = V_1 \cap \bigcap_{w \in W} w(V \setminus V_0)$, where $V_0 = \overrightarrow{V_1}$.

Proposition

- *$E(\Phi) \subseteq D$ and $E(\Phi)$ is stable under the action of W.*
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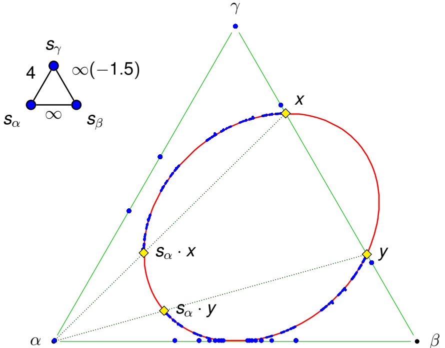

Faithfulness of the action

If W affine, then $E =$ singleton $\rightsquigarrow$ non faithful action.

Theorem (Dyer-Hohlweg-R. '12)

If W is infinite, non-affine and irreducible, then the action of W on E is faithful.

- we prove that E is not contained in a finite union of affine subspaces of V_1.
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If W is irreducible infinite, then for all $x \in E$, the orbit of x under the action of W is dense in E:

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The proof uses:

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Start with the intersections of $\widehat{Q}$ with the faces of conv(Δ), and act by W...

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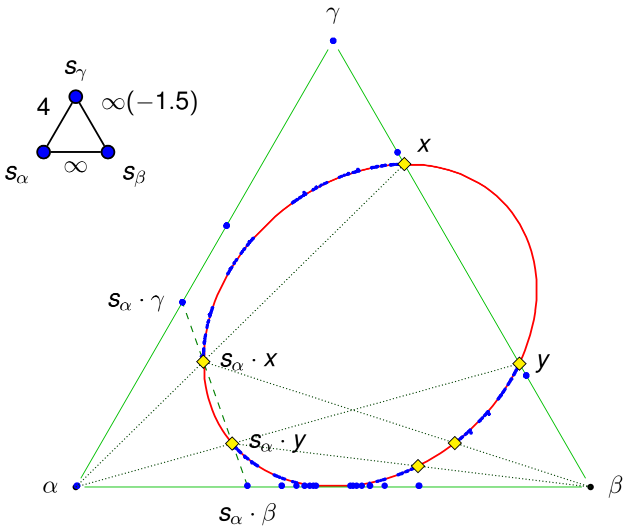

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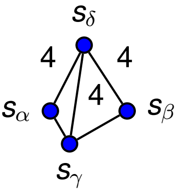
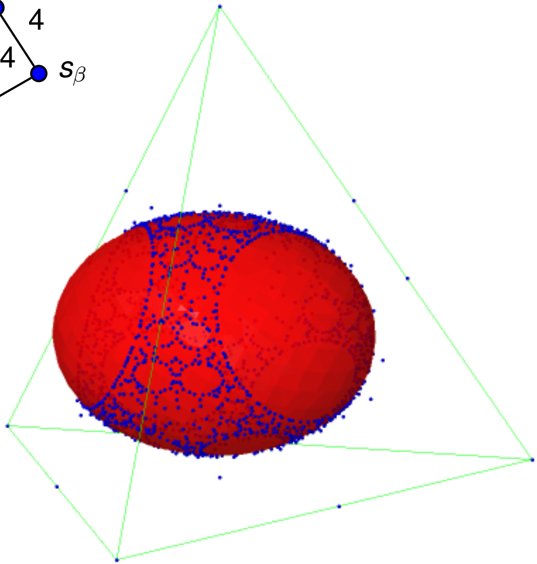


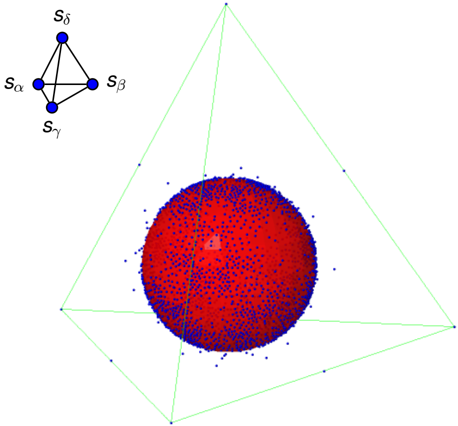


How to describe E directly?

Special case:

Theorem

Suppose W irreducible, infinite non affine. If $\widehat{Q} \subseteq \operatorname{conv}(\Delta)$, then $\operatorname{sgn} B = (n-1, 1)$ and $E(\Phi) = \widehat{Q}$.

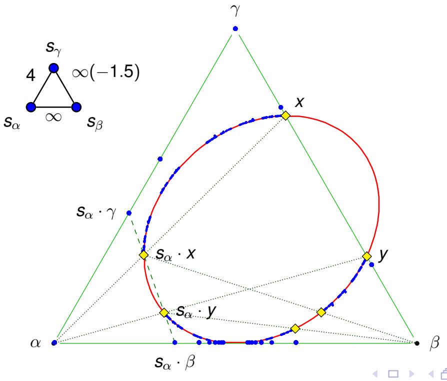


How to describe E directly? (general case)

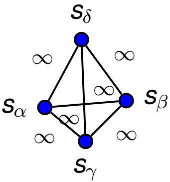
Conjecture

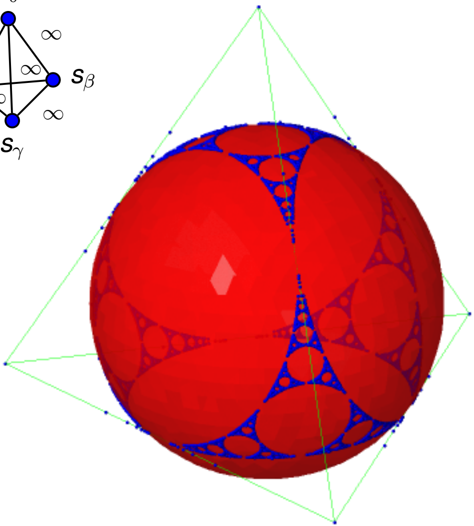
If W is irreducible, $E(\Phi)$ is equal to $\widehat{Q}$ minus all the images by W of the parts of $\widehat{Q}$ which are outside $\mathrm{conv}(\Delta)$*, i.e. :*

$$E(\Phi) = \widehat{Q} \cap \bigcap_{w \in W} w \cdot \mathrm{conv}(\Delta).$$



General case: $\widehat{Q}$ cut the faces





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$$\text{Conjecture} \Leftrightarrow E = \mathrm{conv}(E) \cap \widehat{Q}.$$

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Definition (Kac, Hée, Dyer...)

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