Valette I

On a-T-menability and its permanence prop

1. Basic definitions

Let G be a loc cpt σ-cpt group

$\pi \in \mathrm{Rep}\, G$ unitary rep on a Hilbert space $\mathcal{H}_\pi$

<u>Def</u> 1) π is <u>C_0</u> if $\forall \xi, \eta \in \mathcal{H}_\pi$

$$\lim_{g \to \infty} \langle \pi(g)\xi \mid \eta \rangle = 0$$

2) π <u>almost has invariant vectors</u>

(notation: $\pi \succ 1_G$) if $\exists (\xi_n)_{n \geq 1}$

s.t $\|\xi_n\| = 1$ $\forall n$ and

$\left(g \longmapsto \langle \pi(g)\xi_n \mid \xi_n \rangle\right)_{n \geq 1}$ converges

to 1 unif. on compact subsets of G.

<u>Examples</u> 1) Denote by λ_G the left-reg rep on

$L^2(G)$. λ_G is always C_0

$\xi, \eta \in C_c(G)$ (compact support), then

$g \longmapsto \langle \lambda_G(g)\xi \mid \eta \rangle$ has compact support.

Using density of $C_c(G)$ in $L^2(G)$,

get that λ_G is C_0.

2) $\lambda_G \succ 1_G$ iff G is amenable

(Reiter, $\sim$ 1960)

Note: if π is C_0 and $\pi \succ 1_G$, $(\xi_n)_{n\geq 1}$ as in def, then

$$\lim_{g\to\infty} \lim_{n\to\infty} \langle \pi(g)\xi_n | \xi_n \rangle = 1$$

$$\lim_{n\to\infty} \lim_{g\to\infty} \langle \pi(g)\xi_n | \xi_n \rangle = 0$$

<u>Def</u> G is <u>a-(T)-menable</u> (or <u>has the Haagerup property</u>) if G admits a rep π which is C_0 and $\pi \succ 1_G$.

<u>Thm</u> G is a-(T)-menable $\Longleftrightarrow$ G admits an affine isometric action α on a Hilbert space which is metrically proper (i.e. $\lim\limits_{g\to\infty} \|\alpha(g)v\| = +\infty \quad \forall v \in \mathcal{H}$)

[without proof]

<u>Examples</u> 1) From def: amenable groups

2) Bieberbach's thm: G acts isometrically properly on $\mathbb{E}^n$ iff (up to a compact kernel) G is a closed subgroup of $O(n) \ltimes \mathbb{R}^n$

<u>Def</u> Let X be a closed subset of G.
The pair (G, X) has <u>relative property (T)</u> if the following equivalent conditions are satisfied:

i) $\forall \pi \in \mathrm{Rep}\, G$, if $\pi \succ 1_G$, with a sequence $(\xi_n)_{n\geq 1}$ of almost invariant vectors, the sequence
$$g \longmapsto \langle \pi(g)\xi_n \mid \xi_n \rangle$$
converges to 1 uniformly on X.

ii) $\forall \alpha$ affine isometric action on a Hilbert space, $\alpha(X)$ has bounded orbits (i.e. $\forall v \in \mathcal{H}$, $\{\alpha(g)v : g \in X\}$ is bounded).

<u>Examples</u> (Kazhdan '67, Margulis '73)
$$(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2), \quad (SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$$
have relative property (T).

<u>Clear</u> If G is a-(T)-menable and (G, X) has relative (T), then X is compact.

<u>Lemma</u> (Akemann - Walter '81, Peterson 2009)

Let G be a countable discrete group

TFAE: i) G is not a-(T)-menable

ii) $\forall (\varphi_n)_{n\geq 1}$ sequence of positive-definite functions on G, converging pointwise to 1, then $\exists$ subsequence $(n_k)_{k\geq 1}$ and an infinite subset $X \subset G$ st. $\varphi_{n_k}|_X \xrightarrow[k\to\infty]{} 1$ uniformly.

2. What is a-(T)-menability good for?

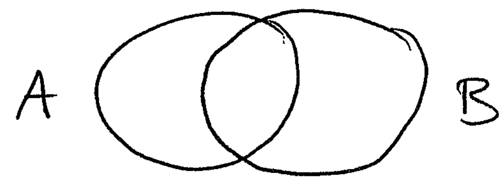
<u>Thm</u> (Higson - Kasparov '97) If G is a-(T)-menable, then the strong ~~free~~ form of the Baum-Connes conjecture holds for G (i.e. conjecture with coefficients, computing K-theory of reduced crossed products $A \rtimes_r G$).

<u>Thm</u> (Haagerup) (G discrete)

Haagerup property for G is equivalent to some approx property of the von Neumann algebra LG.

3. How to construct examples

Notation : for $A, B \subset X$: A <u>cuts</u> B (notation : $A \vdash B$) if $A \cap B \neq \emptyset \neq A^c \cap B$



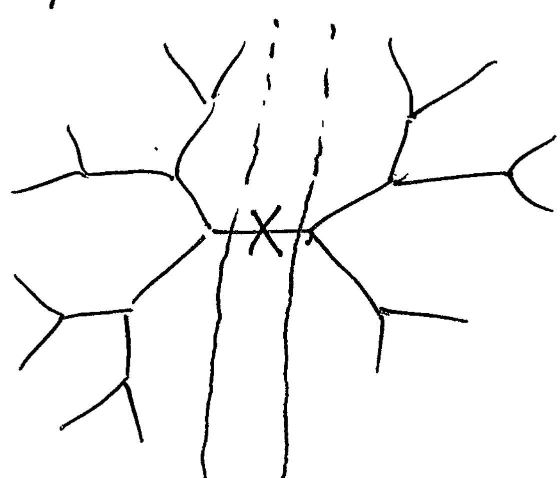
<u>Def</u> Let X be a set. A <u>measured walls structure</u> is a Radon measure μ on the locally compact space $2^X \setminus \{\emptyset, X\}$ s.t. $\forall x, y \in X$, $x \neq y$,

$$d_\mu(x,y) := \mu\{A \subset X : A \vdash \{x,y\}\} < +\infty$$

<u>Remark</u> d_μ is a pseudo-metric

$$d_\mu(x,y) \leq d_\mu(x,z) + d_\mu(z,y)$$

<u>Example 1</u> Let X be the vertex set of a tree. Define an half-tree as the vertex set of a connected component of tree $\setminus$ edge



For $A \subset 2^X \setminus \{\emptyset, X\}$, define

$\mu(A) = \#\{\text{half-trees contained in } A\}$

In other words $\mu = \frac{1}{2} \sum_{H \text{ Half-trees}} \delta_H$

$d_\mu(x,y) = d(x,y)$ (combinatorial distance on the tree)

<u>Example 2</u> Let X be the set of vertices of a cubical CAT(0) complex. By Sageev, hyperplanes disconnect the complex into two connected components $\rightsquigarrow$ half spaces

Measure μ on $2^X \setminus \{\emptyset, X\}$

$$\mu = \frac{1}{2} \sum_{H \text{ half-space}} \delta_H$$

$d_\mu(x,y) = d_1(x,y)$ = combinatorial distance from x to y in the 1-skeleton of the complex.

<u>Example 3</u> X = real hyperbolic space $\mathbb{H}^n$

The group $G = \mathrm{Isom}(X) (= O(n,1))$ acts transitively on the set of open half-spaces with an invariant measure m.

The set of open half-spaces separating x from y is relatively compact in the space of all open half-spaces, hence it has finite measure for m. For A a Borel subset of $2^X \setminus \{\emptyset, X\}$, set

$$\mu(A) = \frac{1}{2}\, m\,\{\text{half-spaces contained in } A\}$$

$$d_\mu(x,y) = \lambda\, d(x,y) \quad (\text{hyperbolic dist})$$

(Crofton formula)

Def Let (X,μ) be a measured walls structure. A group G acts on (X,μ) if $G \curvearrowright X$ and μ is G-invariant. The G-action is proper if, for some $x_0 \in X$, $\lim_{g\to\infty} d_\mu(gx_0, x_0) = +\infty$ (orbits go to infinity in the wall metric).

<u>Remarks</u> 1) If (X,μ), (X,ν) are measured walls structures, then $(X,\mu+\nu)$ is still a measured walls structure and

$$d_{\mu+\nu} = d_\mu + d_\nu$$

2) If (Y,ν) is a measured walls structure, $f : X \longrightarrow Y$ a map, then $(X, f^*\nu)$ is a measured walls structure, using

$f^{-1} : 2^Y \longrightarrow 2^X$, then pushing-forward ν.

<u>Thm</u> (Chatterji-Drutu-Haglund 2008
previous result for discrete groups
Robertson-Steger, 1998
Cherix-Martin-Valette, 2003)

G is loc cpt σ-cpt. TFAE:

(i) G has the Haagerup property

(ii) G acts properly on some measured walls structure

(iii) There exists a measured walls structure (G,μ) which is G-inv and proper

Proof of easy parts

(iii) $\Rightarrow$ (ii) trivial

(ii) $\Rightarrow$ (iii) Use orbit map to pull back measured walls structure on G

(ii) $\Rightarrow$ (i) G acts properly on (X, μ)

For $x \in X$, set χ_x = char function of $\{A \in 2^X : x \in A\}$, set $c(x,y) = \chi_x - \chi_y$

μ is a measure on $2^X \setminus \{\emptyset, X\}$

$c(x,y) \in L^2(2^X \setminus \{\emptyset, X\}, \mu)$

$\|c(x,y)\|^2 = d_\mu(x,y)$

$c(x,y) + c(y,z) = c(x,z)$

If π denotes the natural ~~map~~ rep of G on $L^2(\mu)$,

$$\pi(g)\, c(x,y) = c(gx, gy)$$

For a base point $x_0 \in X$, set

$$b(g) \overset{\text{def}}{=} c(gx_0, x_0)$$

$b \in Z^1(G, \pi)$ $\quad b(gh) = \pi(g) b(h) + b(g)$

$$\alpha(g)\, v \underset{\text{def}}{=} \pi(g) v + b(g) \qquad (v \in L^2(\mu))$$

is an affine isom action of G on $\mathcal{H}$

$$\alpha(gh) = \alpha(g)\,\alpha(h)$$

$$\|\alpha(g)(0)\|^2 = \|b(g)\|^2 = \|c(gx_0, x_0)\|^2 = d_\mu(gx_0, x_0) \xrightarrow[g\to\infty]{} \infty$$

thus α is metrically proper

thus G is a-(T)-menable. $\square$

Examples of a-(T)-menable groups

From example 1: every group ~~action~~ acting properly on a tree is a-(T)-menable (e.g. free groups, $SL_2(\mathbb{Z})$, $SL_2(\mathbb{Q}_p)$)

From example 2: every group acting properly on a CAT(0) cube complex is a-(T)-menable (e.g. Thompson's group F (Farley))

From example 3: every closed subgroup of $O(n,1)$ is a-(T)-menable (e.g. π_1(hyperbolic manifold)).

4. Permanence properties

Class of a-(T)-menable groups is stable under:

- closed subgroups
- direct products
- free products (or amalgamated products over finite groups)
- If H is closed in G, H is a-(T)-menable, and if G/H is G-amenable in the sense of Eymard, then G is a-(T)-menable.

⚠ a-(T)-menability is <u>not</u> preserved by semi-direct products.

E.g. $SL_2(\mathbb{Z})$, $\mathbb{Z}^2$ are a-(T)-menable but $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative (T) so $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is not a-(T)-menable.

<u>Open question</u> If $H \curvearrowright N$ by autom, under which conditions is ~~$H \ltimes N$~~ $H \ltimes N$ a-(T)-menable?

5 Permutational wreath products

Let H, G be non-trivial groups, Λ a homogeneous space of G

Def The <u>permutational wreath product</u>

$$H \wr_\Lambda G \text{ is } \left(\bigoplus_\Lambda H\right) \rtimes G$$

Assume that Λ is a quotient group of G

Thm Consider the two conditions

(i) H, G and Λ are a-(T)-menable

(ii) $H \wr_\Lambda G$ is a-(T)-menable

Then (i) $\Rightarrow$ (ii) (Cornulier-Stalder-Valette, 2008)

(ii) $\Rightarrow$ (i) (Chifan - Ioana, 2009)

Pf (ii) $\Rightarrow$ (i) Assume that $H \wr_\Lambda G$ is Haagerup.

Then so are H and G (which are subgroups of $H \wr_\Lambda G$).

Fix $h \in H \setminus \{1\}$, set $A = \bigoplus_\Lambda \langle h \rangle$

(abelian G-invariant subgroup)

$A \rtimes G \subset H \wr_\Lambda G$, so $A \rtimes G$ is Haagerup.

Now, Λ acts on A "quasi-freely".

If $a \in A \setminus \{1\}$, $\mathrm{Stab}_\Lambda(a)$ is finite

<u>Prop</u> (Chifan-Ioana) Let $G \twoheadrightarrow \Lambda$

$\Lambda \curvearrowright A$ (abelian) by automorphisms, assume $\exists a \in A \setminus \{1\}$ st $\mathrm{Stab}_\Lambda(a)$ is finite. Then, if $A \rtimes G$ is a-(T)-menable, then so is Λ. [Denote $\alpha : \Lambda \to \mathrm{Aut}(A)$
$p : G \twoheadrightarrow \Lambda$]

<u>Sketch of proof</u> By contrapositive, assume Λ is not Haagerup. Let π be a unitary rep of $A \rtimes G$, $\pi > 1_{A \rtimes G}$. We must show that π is <u>not</u> C_0.

Use spectral thm:

$$\pi|_A = \int_{\hat{A}}^{\oplus} \chi \, dE(\chi)$$

If $\xi \in \mathcal{H}_\pi$, $\|\xi\| = 1$,

$$\langle \pi(a)\xi, \xi \rangle = \int_{\hat{A}} \chi(a) \, d\mu_\xi(\chi) \quad (a \in A)$$

where μ_ξ is a proba measure on $\hat{A}$

- $\underbrace{\|\mu_\xi - \mu_\eta\|}_{\text{total var distance}} \leq 2 \|\xi - \eta\|$

- $\mu_{\pi(g)\xi} = g_* \mu_\xi \quad (g \in G)$
- $\| \pi((\alpha \circ p)(g) a) \xi - \xi \|^2 \leq$

$$\leq \|\pi(a)\xi - \xi\|^2 + 2\| g_* \mu_\xi - \mu_\xi \| \quad (g \in G,\ a \in A)$$

Let $(\xi_n)_{n \geq 1}$ be a sequence of almost-invar vectors in $\mathcal{H}_\pi$, $\|\xi_n\| = 1$. Then

$$\| g_* \mu_{\xi_n} - \mu_{\xi_n} \| \longrightarrow 0 \quad (g \in G).$$

Let X be ~~the~~ some infinite subset of Λ (to be specified), $a \in A \setminus \{1\}$ with $\mathrm{Stab}_\Lambda(a)$ finite. For $\gamma \in X$,

$$\| \pi(\alpha(\gamma) a) \xi_n - \xi_n \|^2 \leq \|\pi(a) \xi_n - \xi_n\|^2 + 2 \sup_{\lambda \in X} \| \lambda_* \mu_{\xi_n} - \mu_{\xi_n} \|$$

Assume that, for some subsequence $(n_k)_{k \geq 1}$, we have $\sup_{\lambda \in X} \| \lambda_* \mu_{\xi_{n_k}} - \mu_{\xi_{n_k}} \| \xrightarrow[k \to \infty]{} 0$

Then $\sup_{\gamma \in X} \| \pi(\underbrace{\alpha(\gamma) a}) \xi_{n_k} - \xi_{n_k} \| = 0$

runs into an infinite subset of X.

Conclusion: π is <u>not</u> C_0 (because $\langle \pi(\alpha(\gamma)a)\xi_{n_k} | \xi_{n_k} \rangle \longrightarrow 1$ unif on some infinite subset of A).

To find X and (n_k), want to apply Peterson's lemma. Need some rep's of Λ. Set $\mu_n = \mu_{\xi_n}$. Assume there is a proba measure ν_n on $\hat{A}$ Λ-quasi-invar, s.t

$$\|\mu_n - \nu_n\| \leq \frac{1}{2^{n-1}}$$

$$\|\gamma_* \nu_n - \nu_n\| \xrightarrow[n \to \infty]{} 0 \quad \forall \gamma \in \Lambda$$

On $L^2(\hat{A}, \nu_n)$, get a unitary rep σ_n on Λ by $(\sigma_n(\lambda) f)(\chi) =$

$$= \frac{d\lambda_*^{-1}\mu_n}{d\mu_n}(\chi)\, f(\lambda^{-1}\chi) \qquad (\chi \in \hat{A})$$

$$\eta_n \equiv 1 \in L^2(\hat{A}, \nu_n)$$

Computation: $\langle \sigma_n(\lambda)\eta_n | \eta_n \rangle \longrightarrow 1 \quad (\lambda \in \Lambda)$

Then Peterson's lemma applies to provide X and (u_k).

Finally, let us write ν_n:

$$\Lambda \setminus \{1\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots \}$$

$$\nu_n = \left(1 - \frac{1}{2^n}\right)\mu_n + \frac{1}{2^n}\sum_{k=1}^{\infty} \frac{1}{2^k}\left((\lambda_k)_* \mu_n\right)$$

$\square$