Assaf Naor · L^1 embeddings of the Heisenberg group and fast estimation of graph isoperimetry
joint work with Cheeger, Kleiner

Notation (X, d_X), (Y, d_Y) metric spaces.
$C_Y(X)$ the infimum over $D>0$ such that $\exists \lambda > 0$ (scaling factor)
and $f: X \to Y$ s.t. $\lambda d_X(x,y) \leq d_Y(f(x), f(y)) \leq D \lambda d_X(x,y)$
$\forall x, y \in X$

For $Y = L_p \,(= L_p(0,1))$, $C_Y(X) = C_p(X)$
$C_1(X)$ = the L^1 distorsion of X.
$C_2(X)$ = the euclidian distorsion of X.

Consider the discrete Heisenberg group $H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{Z} \right\}$
d_W = the word metric associated to a fixed symmetric set of generators.

Th (CKN). $C_1 \left(\left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}, a, b, c \in \{1..n\}, d_W \right\} \right) \geq (\log n)^c$

where $c > 0$ is a universal constant.

$f: H \to L_1$ $\quad \omega_f(t)$ the largest function such that
$$\|f(x) - f(y)\| \geq \omega_f(d_W(x,y))$$
Th If $f: H \to L_1$ is lipschitz in the d_W metric
then $\omega_f(t_n) \lesssim \dfrac{t_n}{(\log t_n)^c}$, c universal constant.
for some $t_n \to \infty$

Why 1) L_1 is a "test case"
2) Application to combinatorial optimization.

The sparsest cut problem $\quad C, D : \{1 \ldots n\} \times \{1 \ldots n\} \to [0, \infty[$
two symmetric functions. let $\emptyset \subsetneq S \subsetneq \{1 \ldots n\}$

Define
$$\Phi(S) = \frac{\sum_{i,j=1}^{n} C(i,j) \, |1_S(i) - 1_S(j)|}{\sum_{i,j=1}^{n} D(i,j) \, |1_S(i) - 1_S(j)|}$$

$\cancel{\Phi^*(S) = \min}$

$$\Phi^*(C,D) = \min_{\emptyset \neq S \neq \{1\dots n\}} \Phi(S).$$

<u>Different formulation</u> $G = (\{1 \dots n\}, E)$ a graph.

$\forall e \in E$, $\exists\, C(e) > 1$, $\forall i,j$, $\exists\, D(i,j) \geq 0$.

$$\frac{\sum_{ij \in E} C(i,j)\, |1_S(i) - 1_S(j)|}{\sum_{i,j=1}^{n} D(i,j)\, |1_S(i) - 1_S(j)|}$$

The case of uniform demand: $C(e) = 1$, $D(i,j) = 1$.

Then $\Phi(S) = \dfrac{\#\text{ edges joining } S \text{ and } S^c}{|S| \cdot |S^c|}$

Computing Φ^* is solving the isoperimetric problem.

<u>The sparsest cut problem</u> Given C, D, compute (estimate) $\Phi^*(C,D)$ in polynomial time.

<u>Old</u> Computing $\Phi^*(C,D)$ exactly is NP-hard.

<u>2001</u> Chuzhoy-Khana: $\exists\, \varepsilon_0 > 0$ such that if you could compute $\Phi^*(C,D)$ up to a factor $\leq 1 + \varepsilon_0$ then $P = NP$.

<u>Khot-Vishnoi</u> Gave evidence that you cannot compute $\Phi^*(C,D)$ up to any constant.

<u>Linial-London-Rabinovic (95)</u>

<u>lemma</u> $\forall C, D$, $\Phi^*(C,D) = \min_{f_1 \dots f_n \in L_1} \dfrac{\sum_{i,j} C_{i,j} \|f_i - f_j\|_1}{\sum_{i,j} D(i,j) \|f_i - f_j\|_1}$

Naor 2 | Pf Cut cone representation: A metric space (X,d) is isometric to a subset of L_1 iff $\forall S \subset X$, $\exists \lambda_S \geq 0$ s.t. $\forall x,y \in X$, $d(x,y) = \sum_{S \subseteq X} \lambda_S |1_S(x) - 1_S(y)|$

Linial - London - Rabinovich, Aumann - Rabani

Define $M^*(C,D) = \min \left(\sum_{i,j} c(i,j)\, d_{ij} \right)$

given $\sum_{i,j} D(i,j)\, d_{ij} = 1$

d_{ij} is a semi-metric : $d_{ii} = 0$, $d_{ij} = d_{ji}$, $d_{ij} \leq d_{ik} + d_{kj}$

$M^*(C,D)$ can be computed in polynomial time (linear program).

$$? \leq M^*(C,D) \leq \phi^*(C,D)$$

Bourgain embedding theorem If (X,d) is an n-point metric space then $C_1(X) < 100 \log n$.

If d^*_{ij} is the metric for which M^* is attained, we get $M^*(C,D) \geq \dfrac{\phi^*(C,D)}{\log n}$

Bourgain's thm is tight.

Goemans-Linial : lemma The metric space $(L_1, \sqrt{\|x-y\|})$ is isometric to a subset of L_2.

Define $M^{**}(C,D)$ to be the minimum of

$\sum_{i,j} C(i,j)\, d_{ij}$ given $\sum D(i,j)\, d_{ij} = 1$

(d_{ij}) is a semi-metric such that $\sqrt{d_{ij}}$ is a isometric to a subset of L_2.

i.e. $d_{ij} = \|x_i - x_j\|_2^2$, $x_i \in L_2$

Minimize a quadratic function under quadratic constraints

$\Rightarrow M^{**}(C,D)$ can be computed in polynomial time.

$M^*(C,D) \leq M^{**}(C,D) \leq \Phi^*(C,D)$.

Thm (Arora, Lee, N.) If (X,d) is an n-point metric space such that $(X,\sqrt{d})$ is isometric to a subset of L_2, then $C_2(X) \leq (\log n)^{\frac{1}{2}+o(1)}$ + sharp up to $o(1)$.

Ex: Hilbert space is isometric to a subset of L_1

Cor $$M^{**}(C,D) \geq \frac{\Phi^*(C,D)}{(\log n)^{\frac{1}{2}+o(1)}}$$

Best known polynomial ~~algom~~ algorithm today.

Goemans-Linial conjecture: If $(X,\sqrt{d})$ is isometric to a subset of L_2 then $C_1(X) = O(1)$.

Khot-Vishnoi: No. $C_1(X) \gtrsim \log\log n$ for some X

Th (CKN) $\exists\ C, D : \{1 \ldots n\} \to [0,\infty[$ s.t.

$$M^{**}(C,D) \leq \frac{\Phi^*(C,D)}{(\log n)^c}$$

Pf uses the following fact:

Lee, N.: $\exists$ a metric d on H which is 100-equivalent to d_W and $\sqrt{d}$ is isometric to a subset of L_2

Hint for the proof of non-amenability of the Heisenberg group.

Pansu proved that H does not admit a bi-Lipschitz embedding into L_2. But there is no differentiability theory of L_1-valued functions

Ex $t \mapsto 1_{[0,t]} \in L_1$ is an isomorphic embedding but is not differentiable.

Naor 3) Cheeger and Kleiner developped such a theory.

if $f : H \to L_1$, $\exists$ measure such that

$$\|f(x) - f(y)\|_1 = \int_{\substack{S \text{ measurable} \\ \text{subset of } H}} |\mathbb{1}_S(x) - \mathbb{1}_S(y)| \, d\underbrace{\Sigma_f}_{\text{cut measure of } f}(S)$$

differentiation : "locally Σ_f must be supported on half spaces".