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G countable group X infinite countable set

<u>Def</u> $G \curvearrowright X$ is amenable if $\exists$ a G-invariant mean on X

i.e. $\mu : \mathcal{P}(X) \to [0,1]$ s.t. $\mu(X) = 1$

$\mu(A \cup B) = \mu(A) + \mu(B) \quad \forall A, B \subset X$ s.t. $A \cap B = \emptyset$

$\mu(gA) = \mu(A) \quad \forall g \in G, \ \forall A \subset X$

A group G is amenable if $G \curvearrowright G$ by multiplication is amenable

<u>Rem</u> Different definition from yesterday (Zimmer, Yu, ...)

Equivalent definition (Fölner, 1955).

$G \curvearrowright X$ is amenable $\Leftrightarrow$ $\exists$ a Folner sequence for $G \curvearrowright X$

i.e. a sequence $\{A_n\}_{n \geq 1}$ of finite non-empty subset of X

s.t. $\dfrac{|A_n \triangle g A_n|}{|A_n|} \xrightarrow[n \to \infty]{} 0$

<u>Rem</u> G amenable $\Leftrightarrow$ $\forall G \curvearrowright X$ amenable

$? \Leftarrow \exists$ ————

It is true a priori only for free actions.

<u>Question</u> (Greenleaf, 1969). If there exists a G-inv. mean on X, where G acts on X "<u>reasonably</u>", is G amenable.

* faithful : otherwise it is enough to study the quotient group
* transitive : otherwise $G \curvearrowright X$ amenable $\Rightarrow$ $G \curvearrowright G \sqcup X$ amenable faithful.

$\mathcal{A} = \{ G$ countable, G admits an amenable, faithful, transitive action $\}$

<u>Ex</u> Amenable groups $\subset \mathcal{A}$

<u>Non-Ex</u> 1) If G has property (T) of Kazhdan, $G \notin \mathcal{A}$ (e.g. $SL_3(\mathbb{Z})$)

2) $H \triangleleft G$, H not of finite exponent then if (G, H) has relative property (T), then $G \notin \mathcal{A}$ (e.g. $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative (T) $\Rightarrow SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \notin \mathcal{A}$

Th $\mathbb{F}_n \in \mathcal{A}$, $\forall n$. (van Douwen 90, Glasner–Monod 07, Grigorchuk–Nekrashevych 07)

Th (Glasner–Monod 07): $G * H \in \mathcal{A}$ unless G has the fixed point property (i.e. ~~any~~ every amenable action has a fixed point) and H has virtually fixed point property

Properties of $\mathcal{A}$

① $G, H \in \mathcal{A} \Leftrightarrow G \times H \in \mathcal{A}$

② $G, H \in \mathcal{A} \Rightarrow G * H \in \mathcal{A}$ ($\Leftarrow$ is false)

③ H is coamenable in G ($G \curvearrowright G/H$ is amenable)

$H \in \mathcal{A} \Rightarrow G \in \mathcal{A}$ ($\Leftarrow$ is false)

open if $H < G$ is of finite index, $G \in \mathcal{A} \overset{?}{\Rightarrow} H \in \mathcal{A}$

④ $G, H \in \mathcal{A}$, $G \ltimes H \notin \mathcal{A}$

(e.g. $G = SL_2(\mathbb{Z})$, $H = \mathbb{Z}^2$, $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \notin \mathcal{A}$)

⑤ $G, H, A \in \mathcal{A} \not\Rightarrow G \underset{A}{*} H \in \mathcal{A}$ ex $SL_2(\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/4\mathbb{Z}$

$G = \mathbb{Z}/6\mathbb{Z} \ltimes \mathbb{Z}^2$, $H = \mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}^2$ $A = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}^2$

$G \underset{A}{*} H = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \notin \mathcal{A}$.

Def G is cycl pinched 1-relator group if

$G = \langle a_1 \ldots a_n\ b_1 \ldots b_m \mid c = d \rangle$

where c is a cycl. reduced non primitive word on $\langle a_1 \ldots a_n \rangle = \mathbb{F}_n$

d ——— $\langle b_1 \ldots b_m \rangle = \mathbb{F}_m$

Therefore $G = \mathbb{F}_n \underset{\mathbb{Z}}{*} \mathbb{F}_m$ where $\mathbb{Z} \simeq \langle c \rangle < \mathbb{F}_n$
$= \langle d \rangle < \mathbb{F}_m$

Th (M, 09) • $\mathbb{F}_n \underset{\mathbb{Z}}{*} \mathbb{F}_m \in \mathcal{A}$, $\forall n, m$

• $\forall H < \mathbb{F}_n \underset{\mathbb{Z}}{*} \mathbb{F}_m$ finite index, $H \in \mathcal{A}$.

Cor Surface groups are in $\mathcal{A}$.

Room 2 Indeed $\Gamma_g = \pi_1(\Sigma_g) = \langle a_1 b_1 \dots a_g b_g \mid \underbrace{[a_1,b_1]\dots[a_{g-1},b_{g-1}]}_{c} = \underbrace{[a_g,b_g]}_{d}$

Σ_g closed orientable of genus g

$$\Gamma_g = \mathbb{F}_{2g-2} \underset{\mathbb{Z}}{*} \mathbb{F}_2 .$$

Cor M 3-manifold fibers over S^1.

$$0 \to \Gamma_g \to \pi_1(M) \to \mathbb{Z} \to 0$$

Γ_g coamenable in $\pi_1(M)$ and $\mathcal{A}$ is stable under taking extensions of coamenable subgroups $\Rightarrow \pi_1(M) \in \mathcal{A}$.

Recall G, H, A amenable groups $\not\Rightarrow G \underset{A}{*} H \in \mathcal{A}$.

Question ~~For~~ Under which conditions is it true?

Th (M10) If $G \overset{\pi}{\twoheadrightarrow} H$ amenable, $A < G$ s.t. $\pi|_A$ is injective and $[H:A] \geq 2$ Then $G \underset{A}{*} H \in \mathcal{A}$

In particular $G \underset{A}{*} G \in \mathcal{A}$ is G is amenable, $A \subsetneq G$.

Th (M10) If $A < G, H$ A finite, G infinite amenable. $H \curvearrowright K$ amenable, $A \curvearrowright Y$ is free $\Rightarrow G \underset{A}{*} H \in \mathcal{A}$.

Def G, H countable gp, A finite common subgroup of G, H

$(G, H, A) \in \mathcal{A}'$ if $\exists\, G \curvearrowright X$ and $H \curvearrowright Y$ s.t.

① $G \curvearrowright X$ is transitive ② $\forall g \in G \setminus A$, $\forall h \in H \setminus A$, the sets

$\{x \in X,\ Ax \cap gAx = \emptyset\}$
$\{x \in Y,\ Ax \cap hAx = \emptyset\}$ are infinite

③ $\exists$ Følner sequence $\{C_n\}_n$ for $G \curvearrowright X$
$\{D_n\}_n$ for $H \curvearrowright Y$

s.t. . $|C_n| = |D_n|$ $\forall n \geq 1$

. the sets $\{AC_n\}_{n\geq 1}$, $\{AD_n\}_{n\geq 1}$ are pairwise disjoint.

④ $A \curvearrowright X$ and $A \curvearrowright Y$ are free

Prop If $(G,H,A) \in \mathcal{A}'$, then $G \underset{A}{*} H \in \mathcal{A}$.

Idea of the pf Thm of Baire.

X infinite countable $\quad Sym(X)$ gr of permutations of X

Topology of pointwise convergence on $Sym(X)$

i.e. $\alpha_n \to \alpha$ if $\forall F$ finite $\subset X$, $\exists n_0$, $\alpha_n|_F = \alpha_{n_0}|_F \ \forall n \geq n_0$.

Fact $Sym(X)$ is a Baire space.

To find a permutation $\alpha \in Sym(X)$ satisfying the properties $\{P_i\}_{i \geq 1}$ it is enough to show that the sets $U_i = \{\alpha \in Sym(X) \mid \alpha$ satisfies $P_i\}$ is generic $\forall i$

Then take $\alpha \in \bigcap_i U_i$

~~Pf of proposition~~ let $Z = \{\sigma \in Sym(X),\ a\sigma = \sigma a \ \forall a \in A\}$

Baire space.

For $\sigma \in Z$, denote $H^\sigma = \sigma^{-1} H \sigma$.

We consider $G \underset{A}{*} H^\sigma \curvearrowright X$ by $g \cdot x = gx \ \forall g \in G$

$h \cdot x = \sigma^{-1} h \sigma, \ \forall h \in H.$

Lemma 1 : The set $O_1 = \{\sigma \in Z,\ G \underset{A}{*} H^\sigma \curvearrowright X$ faithfully$\}$ is generic in Z.

Lemma 2 : The set $O_2 = \{\sigma \in Z,\ \exists \{n_k\}$ s.t. $\sigma(C_{n_k}) = D_{n_k}\}$ is generic in Z.

$G \underset{A}{*} H^\sigma \curvearrowright X$ is transitive, faithful (lemma 1)

amenable with $\{C_{n_k}\}$ Følner sequence of $G \underset{A}{*} H^\sigma \curvearrowright X$

$$\frac{|C_{n_k} \Delta g C_{n_k}|}{|C_{n_k}|} \xrightarrow[k \to \infty]{} 0 \ \forall g \in G$$

Proof 3 $\dfrac{|C_{nk} \Delta hC_{nk}|}{|C_{nk}|} = \dfrac{|C_{nk} \Delta \sigma^{-1}h\sigma C_{nk}|}{|C_{nk}|}$

$$= \frac{|\sigma C_{nk} \Delta h\sigma C_{nk}|}{|C_{nk}|} \underset{③ \text{ in def}}{\overset{\text{lemma 2}}{=}} \frac{|D_{nk} \Delta h D_{nk}|}{|D_{nk}|} \xrightarrow{n\to\infty} 0$$

<u>pf of the thm</u>: $A \underset{\text{finite}}{<} G, H$, G infinite amenable,

$H \curvearrowright Y$ amenable, $A \curvearrowright Y$ free

$\{C_n\}_{n\geq 1}$ Følner for $G \curvearrowright G$

$\{D_n\}_{n\geq 1}$ —— $H \curvearrowright Y$

We can suppose that the sets $\{AC_n\}_{n\geq 1}$, $\{A.D_n\}_{n\geq 1}$ are pairwise disjoint.

<u>lemma</u>: If there exists an amenable G-action and amenable H-action, then $\exists$ G-action and H-action with Følner sequences $\{C_n\}_{n\geq 1}$ and $\{D_n\}_{n\geq 1}$ s.t.

$|C_n| = |D_n|$ $\forall n\geq 1$

$\left.\begin{array}{l} H \text{ action on } Y = H \cup Y \\ G \text{ action on } X = G \end{array}\right)$ satisfies the conditions in the definition.

<u>Cor</u> G inf amenable, $\exists N < H$ of finite index s.t.

$N \cap A = \{1\} \Rightarrow G \underset{A}{*} H \in \mathcal{A}$

<u>ex</u> If H is residually finite eg $SL_3(\mathbb{Z}) \underset{A}{*} G \in \mathcal{A}$.

<u>Cor</u> G, H amenable groups, A finite $\Rightarrow G \underset{A}{*} H \in \mathcal{A}$