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G countable group X infinite countable set

Def $G \curvearrowright X$ is amenable if \exists a G -invariant mean on X

i.e. $\mu: \mathcal{P}(X) \rightarrow [0, 1]$ s.t. $\mu(X) = 1$

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \forall A, B \subset X \text{ s.t. } A \cap B = \emptyset$$

$$\mu(gA) = \mu(A) \quad \forall g \in G, \forall A \subset X$$

A group G is amenable if $G \curvearrowright G$ by multiplication is amenable

Rem Different definition from yesterday (Zimmer, Yu, ...)

Equivalent definition (Følner, 1955).

$G \curvearrowright X$ is amenable $\iff \exists$ a Følner sequence for $G \curvearrowright X$

i.e. a sequence $\{A_n\}_{n \geq 1}$ of finite non-empty subset of X

$$\text{s.t. } \frac{|A_n \triangle gA_n|}{|A_n|} \xrightarrow{n \rightarrow \infty} 0$$

Rem G amenable $\iff \forall G \curvearrowright X$ amenable

? $\iff \exists$ _____

It is true a priori only for free actions.

Question (Greenleaf, 1969). If there exists a G -inv. mean on X , where G acts on X "reasonably", is G amenable.

* faithful: otherwise it is enough to study the quotient group

* transitive: otherwise $G \curvearrowright X$ amenable $\Rightarrow G \curvearrowright G \sqcup X$ amenable faithful.

$\mathcal{A} = \{ G \text{ countable, } G \text{ admits an amenable, faithful, transitive action} \}$

Ex Amenable groups $\subset \mathcal{A}$

Non-Ex 1) If G has property (T) of Kazhdan, $G \notin \mathcal{A}$

(e.g. $SL_3(\mathbb{Z})$)

2) $H \triangleleft G$, H not of finite exponent then if (G, H) has relative property (T), then $G \notin \mathcal{A}$ (e.g. $(SL_2(\mathbb{Z}) \times \mathbb{Z}^2, \mathbb{Z}^2)$ has relative (T) $\Rightarrow SL_2(\mathbb{Z}) \times \mathbb{Z}^2 \notin \mathcal{A}$)

Th $F_n \in \mathcal{A}$, $\forall n$. (van Poumen 90, Glasner-Ponod 07, Grigorchuk-Nekrashevych 07)

Th (Glasner-Ponod 07): $G \times H \in \mathcal{A}$ unless G has the fixed point property (i.e. ~~any~~ every amenable action has a fixed point) and H has virtually fixed point property

Properties of \mathcal{A} (1) $G, H \in \mathcal{A} \Leftrightarrow G \times H \in \mathcal{A}$

(2) $G, H \in \mathcal{A} \Rightarrow G \times H \in \mathcal{A}$ (\Leftarrow is false)

(3) H is coamenable in G ($G \curvearrowright G/H$ is amenable)

$H \in \mathcal{A} \Rightarrow G \in \mathcal{A}$ (\Leftarrow is false)

open if $H < G$ is of finite index, $G \in \mathcal{A} \stackrel{?}{\Rightarrow} H \in \mathcal{A}$

* (4) $G, H \in \mathcal{A}$, $G \times H \in \mathcal{A}$

(e.g. $G = SL_2(\mathbb{Z})$, $H = \mathbb{Z}^2$, $SL_2(\mathbb{Z}) \times \mathbb{Z}^2 \notin \mathcal{A}$)

* (5) $G, H, A \in \mathcal{A} \not\Rightarrow G \times_A H \in \mathcal{A}$ ex $SL_2(\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$

$G = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}^2$, $H = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^2$, $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2$

$G \times_A H = SL_2(\mathbb{Z}) \times \mathbb{Z}^2 \notin \mathcal{A}$.

Def G is cyl pinched 1-relator group if

$G = \langle a_1, \dots, a_m, b_1, \dots, b_m \mid c = d \rangle$

where c is a cyl. reduced non primitive word on $\langle a_1, \dots, a_m \rangle$, F_n

$\langle b_1, \dots, b_m \rangle = F_m$

Therefore $G = F_m \times_{\mathbb{Z}} F_m$ where $\mathbb{Z} = \langle c \rangle c F_m = \langle d \rangle c F_m$

Th $(\pi, \text{og}) \times F_m \times_{\mathbb{Z}} F_m \in \mathcal{A}$, $\forall m$

* $\forall H < F_m \times_{\mathbb{Z}} F_m$ finite index, $H \in \mathcal{A}$.

Cor Surface groups are in \mathcal{A} .

Prop 2] Indeed $\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid \underbrace{[a_1, b_1] \dots [a_{g-1}, b_{g-1}]}_c = \underbrace{[a_g, b_g]}_d \rangle$
 Σ_g closed orientable of genus g

Cor M 3-manifold fibers over S^1 .

$$0 \rightarrow \pi_1(F) \rightarrow \pi_1(M) \rightarrow \mathbb{Z} \rightarrow 0$$

$\pi_1(F)$ amenable in $\pi_1(M)$ and it is stable under taking extensions of amenable subgroups $\Rightarrow \pi_1(M) \in \mathcal{A}$.

Recall G, H, A amenable groups $\Rightarrow G \times_A H \in \mathcal{A}$.

Question ~~For~~ Under which conditions is it true?

Th (M10) If $G \twoheadrightarrow H$ amenable, $A < G$ s.t. $\pi|_A$ is injective and $[H:A] \geq 2$ then $G \times_A H \in \mathcal{A}$

In particular $G \times_A G \in \mathcal{A}$ if G is amenable, $A \neq G$.

Th (M10) If $A < G, H$ finite, G infinite amenable, $H \triangleleft K$ amenable, $A \triangleleft Y$ is free $\Rightarrow G \times_A H \in \mathcal{A}$.

Def G, H countable gp, A finite common subgroup of G, H
 $(G, H, A) \in \mathcal{A}'$ if $\exists G \triangleleft X$ and $H \triangleleft Y$ s.t.

① $G \triangleleft X$ is transitive ② $\forall g \in G \setminus A, \forall h \in H \setminus A$, the sets

$\{x \in X, A x \cap g A x = \emptyset\}$
 $\{y \in Y, A y \cap h A y = \emptyset\}$ are infinite

③ \exists Følner sequence $\{C_n\}_n$ for $G \triangleleft X$
 $\{D_n\}_n$ for $H \triangleleft Y$
 s.t. $|C_n| = |D_n| \forall n \geq 1$

the sets $\{A C_n\}_{n \geq 1}, \{A D_n\}_{n \geq 1}$ are pairwise disjoint.

④ $A \triangleleft X$ and $A \triangleleft Y$ are free

Prop If $(G, H, A) \in \mathcal{A}'$, then $G \rtimes H \in \mathcal{A}$.

Idea of the pf Thm of Baire.

X infinite countable $\text{Sym}(X)$ gr of permutations of X

Topology of pointwise convergence on $\text{Sym}(X)$

i.e. $\alpha_n \rightarrow \alpha$ if $\forall F$ finite $\subset X$, $\exists n_0$, $\alpha_n|_F = \alpha|_F \forall n \geq n_0$.

Fact $\text{Sym}(X)$ is a Baire space.

To find a permutation $\alpha \in \text{Sym}(X)$ satisfying the properties $\{P_i\}_{i \geq 1}$ it is enough to show that the sets

$U_i = \{ \alpha \in \text{Sym}(X) \mid \alpha \text{ satisfies } P_i \}$ is generic $\forall i$

Then take $\alpha \in \bigcap U_i$

Pf of proposition let $Z = \{ \sigma \in \text{Sym}(X), \alpha \sigma = \sigma \alpha \forall \alpha \in A \}$ Baire space.

For $\sigma \in Z$, denote $H^\sigma = \sigma^{-1} H \sigma$.

We consider $G \rtimes_A H^\sigma \curvearrowright X$ by $g \cdot x = gx \forall g \in G$
 $h \cdot x = \sigma^{-1} h \sigma, \forall h \in H$.

Lemma 1 The set $O_1 = \{ \sigma \in Z, G \rtimes_A H^\sigma \curvearrowright X \text{ faithfully} \}$ is generic in Z .

Lemma 2 The set $O_2 = \{ \sigma \in Z, \exists \{m_k\} \text{ s.t.}$

$\sigma(C_{m_k}) = D_{m_k} \}$ is generic in Z .

$G \rtimes_A H^\sigma \curvearrowright X$ is transitive, faithful (lemma 1)

amenable with $\{m_k\}$ Følner sequence of $G \rtimes_A H^\sigma \curvearrowright X$

$$\frac{|C_{m_k} \Delta g C_{m_k}|}{|C_{m_k}|} \xrightarrow{k \rightarrow \infty} 0 \quad \forall g \in G$$

Prop 3

$$\frac{|C_{n_k} \Delta h C_{n_k}|}{|C_{n_k}|} = \frac{|C_{n_k} \Delta \sigma^{-1} h \sigma C_{n_k}|}{|C_{n_k}|}$$

$$= \frac{|\sigma C_{n_k} \Delta h \sigma C_{n_k}|}{|C_{n_k}|} \stackrel{\text{lemma 2}}{=} \frac{|D_{n_k} \Delta h D_{n_k}|}{|D_{n_k}|}$$

$\downarrow n \rightarrow \infty$
0

pf of the thm: $A \subset G, H$, G infinite amenable, $H \cap Y$ amenable, $A \cap Y$ free

$\{C_n\}_{n \geq 1}$ Følner for $G \cap G$

$\{D_n\}_{n \geq 1}$ $H \cap Y$

We can suppose that the sets $\{A C_n\}_{n \geq 1}$, $\{A \cdot D_n\}_{n \geq 1}$ are pairwise disjoint.

lemma: If there exists an amenable G -action and amenable H -action, then $\exists G$ -action and H -action with Følner sequences $\{C_n\}_{n \geq 1}$ and $\{D_n\}_{n \geq 1}$ s.t.

$$|C_n| = |D_n| \quad \forall n \geq 1$$

H action on $Y = H \cup Y$ } satisfies the conditions in the definition.
 G action on $X = G$

Cor G inf amenable, $\exists N < H$ of finite index s.t.

$$N \cap A = \{1\} \Rightarrow G \times_A H \in \mathcal{A}$$

ex If H is residually finite eg $SL_3(\mathbb{Z}) \times_A G \in \mathcal{A}$.

Cor G, H amenable groups, A finite $\Rightarrow G \times_A H \in \mathcal{A}$

