Gilbert Levitt : Mc Cool Groups

Joint w. Guirardel

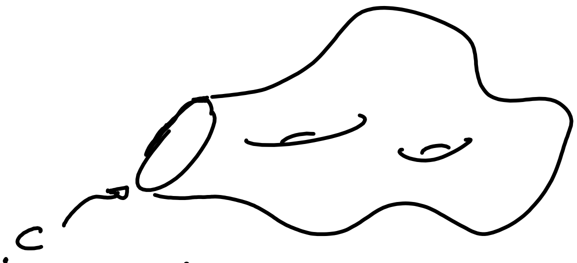
Take F_n a free group, $Out(F_n) = Aut(F_n)/_{In(F_n)}$

$\mathcal{C}$ a finite set of conjugacy classes in F_n

$Out_{\mathcal{C}}(F_n)$ pointwise stabiliser of $\mathcal{C}$

$\mathcal{C} = \{ \underbrace{[x_1, x_2][x_3, x_4]}_{c} \}$ in F_4

$Out_{\mathcal{C}}(F_n)$ = Mapping class group of



<u>THEOREM</u> (Mc Cool 1975) $Out_{\mathcal{C}}(F_n)$ is finitely presented.

<u>THEOREM 1</u> (Guirardel-Levitt) $Out_{\mathcal{C}}(F_n)$ is VFL (NOTE: should be VLF "virtually libre finie") : $\exists$ finite index subgroup with finite $K(\pi,1)$.

Theorem1 is valid if $\mathcal{C}$ infinite or replace F_n by torsion-free hyperbolic G $Out_{\mathcal{C}}(G)$

<u>Motivation</u> : Thm 2 The stabilizer of a point in the boundary of outer space is build out of Mc Cool groups. So it's VFL.

Two steps in understanding $Out_{\mathcal{C}}(F_n)$ and $Out_{\mathcal{C}}(G)$

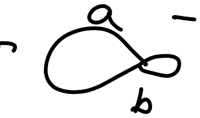
(1) Use outer space (Culler - Vogtmann 1986) to reduce to one ended case

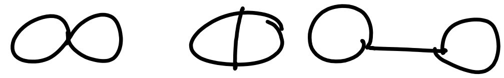
(2) Use JSJ (Rips - Sela) to do one ended case.

$\boxed{\text{PROOF}}$ that $Out(F_n)$ is VFL (Culler - Vogtmann)

$Out(F_n)$ acts on an outer space CV_n. A point in outer space is a free isometric minimal action of F_n on a normalized metric tree.

e.g. F_2 acts on a Cayley tree.

Other viewpoint : A point in CV_n is a metric graph T 
with $\pi_1 T \cong F_n$

Example $n=2$ possible T's are 

CV_n is a complex : get a "simplex" by :
- varying lengths
- normalizing so that the total length of T is 1

Out (F_n) acts on CV_n by precomposition and change of markings

Properties : (1) Action is "cocompact" finitely many orbits of simplices
- in case where $n=2$, have 3 orbits.

(2) Stabilisers are finite.

(3) CV_n is contractible (that's the hard part)

(1), (2) (3) $\Longrightarrow$ VFL

Note : $\dim CV_n = 3n-3-1$

spine of CV_n has smaller dimension and is a deformation retract.

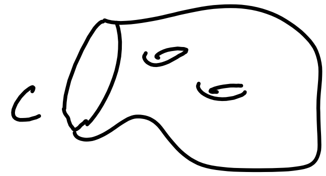
RELATIVE OUTER SPACE

Definition: An action of G on a tree T is relative to $\mathcal{C}$ ($\mathcal{C}$ a set of conjugacy classes) if elements of $\mathcal{C}$ are elliptic in T (they fix a point).

$CV_G(\mathcal{C})$: set of "free action relative to $\mathcal{C}$"

Precisely, the action is relative to $\mathcal{C}$
- edge stabilisers are trivial ($\Longrightarrow$ vertex stabilizers are free factors)
- vertex stabilisers G_v are freely indecomposable relative to $\mathcal{C}$: G_v does not act on a tree with trivial edge stabiliser and elements of $\mathcal{C}$ contained in G_v are being elliptic.

Example: $\mathcal{C} = \{ [c] \}$ If T is a tree in $CV_c(\mathcal{C})$ only are non-trivial vertex stabilizers (up to conjugacy): smallest free factor containing c

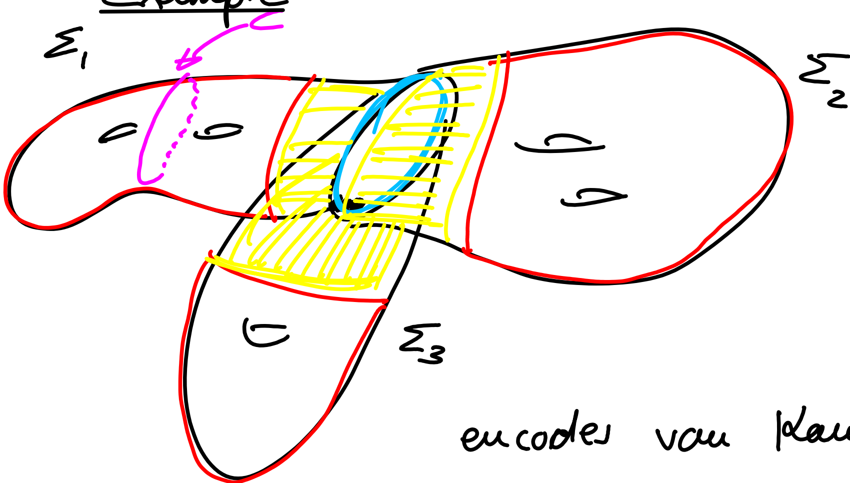


$CV_c(\mathcal{C})$ trivial

Trying to do ⊛ with varying the group & the space, if we know that the stabilisers are VFL, then we know that the whole group is VFL as well, but otherwise we don't know much.

From now on suppose that G is freely indecomposable relative to $\mathcal{C}$.

Example



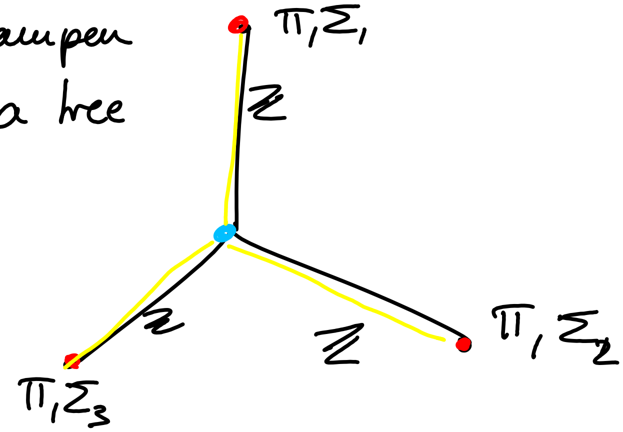
$G = \pi_1$

Understand Out(G)

G is a torsion-free one-ended hyperbolic group

encodes van Kampen

also action on a tree



Understand Out(G)

(1) Every splitting of G over $\mathbb{Z}$ may be read from T: come from edges of T as simple closed curves in Σ_i.

(2) T is Out(G) invariant: every automorphism is visible.

Structure of Out(G)

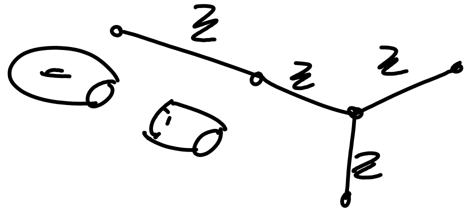
$$1 \longrightarrow \mathcal{C} \longrightarrow \mathrm{Out}^0(G) \longrightarrow \prod MCG(\Sigma_i) \longrightarrow 1$$

finite index subgroup: disregard permutations, reflexions

$\mathcal{C}$: Dehn twists in yellow annuli $\mathbb{Z}^2$

Example: $\mathcal{C} = \{$purple curve$\} = c$

$T_{\mathcal{C}}$



Is $\mathrm{Out}_{\mathcal{C}}(G)$ invariant

- every splitting over $\mathbb{Z}$ rel $\mathcal{C}$ is visible in $T_{\mathcal{C}}$.
- $T_{\mathcal{C}}$ is $\mathrm{Out}_{\mathcal{C}}(G)$-invariant

get similar exact sequence for $\mathrm{Out}_{\mathcal{C}}(G)$.

In GENERAL:

Have to use relative JSJ relative to $\mathcal{C}$.