Lecureux 1: Boundary amenability of groups acting on buildings

Def (Γ_n) Γ has property (A) if $\exists H$ Hilbert, $\rho_{1,2} \uparrow \infty$
$f : \Gamma \to H$
$\rho(d(x,y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x,y))$

Motivation: If Γ has property A then it satisfies the Novikov conjecture.

Thm (Higson Roe) Γ has property A if $\exists K$ compact space $\Gamma \curvearrowright K$ which is topologically amenable i.e. $\exists \mu_n : K \to \mathrm{Prob}(\Gamma)$ s.t. $\|\mu_n(\gamma x) - \gamma \mu_n(x)\| \xrightarrow[n\to\infty]{} 0$ uniformly ~~in x~~

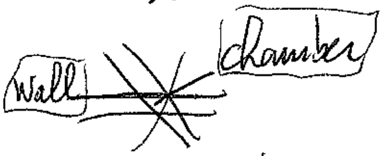
Groups which have property A.

1) Amenable groups.
2) Hyperbolic groups (Adams)
3) linear groups (Guetner, Higson, Weinberger)
4) groups acting properly on f.dim. CAT(0) cube complexes (Brodzki, Campbell, Guetner, Niblo, Wright)

Th G acting properly on a locally finite building then it has property A. In fact its action on nice boundary is amenable.

Buildings

Coxeter groups $W = \langle s \in S, (st)^{m_{st}} = 1 \rangle$
$m_{st} \in \mathbb{N}^* \cup \{\infty\}$, $m_{ss} = 1$.

Examples 

Affine Coxeter groups.


Hyperbolic

Fact (Davis-Moussong)
Every Coxeter group acts properly discontinuously on some CAT(0) space Σ.

Reflection conjugate of ~~s~~ $s \in S$

Wall : $\{$fixed points of a reflection$\}$

Chamber connected composant of $\Sigma \setminus \{\text{walls}\}$

Panel : Intersection of two adjacent chambers

Def A building is a gluing of chambers along their panels, covered by apartments, such that

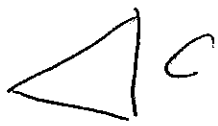
1) every apartment is $\simeq \Sigma$
2) any two points are in some apartment
3) For any two apartements A, A', $\exists$ isom $A \simeq A'$ fixing $A \cap A'$.

Example A tree without end points, $W = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

- Buildings associated to semi-simple groups over local fields (e.g. $SL_2(\mathbb{Q}_p)$). class of Euclidian buildings.
- Kac-Moody groups.

II Combinatorial boundary (joint with Caprace)

Fact For every chamber C, panel σ, there is a unique chamber adjacent to σ, and at a minimal distance from C, denoted $\mathrm{proj}_\sigma(C)$.

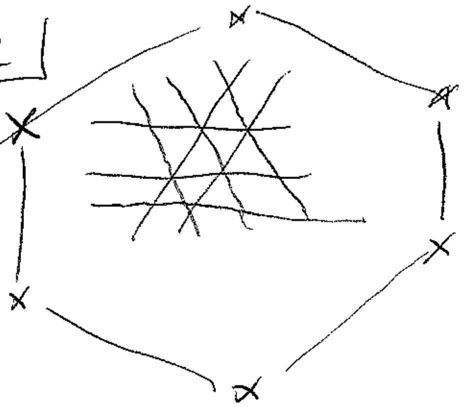


$$i_{ch} \; ch(X) \longrightarrow \prod_{\sigma \text{ panel}} \mathrm{link}(\sigma) \quad \text{is injective.}$$

$$C \mapsto (\sigma \mapsto \mathrm{proj}_\sigma(C))$$

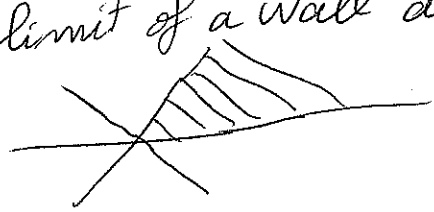
Def $\mathcal{C}_{ch}(X) = \overline{i_{ch}(ch(X))}$ is the combinatorial compactification.

Lecureux 2

Ex  compactification of the appartment.

Hyperbolic case 

$\mathcal{C}h(X) \to \partial_\infty \mathbb{H}^2$
is surjective with cardinality 1 or 2
(2 for a limit of a wall at infinity)

Tool Sectors : 

$\xi \in \mathcal{C}h(X)$
$x \in ch(X)$ $Q(x, \xi)$ is a "convex hull of x and ξ."
More precisely, if $C_n \to \xi$, $\mathrm{Conv}(x, C_n)$ converges pointwise
and $Q(x,\xi) = \lim \mathrm{conv}(x, C_n)$.

Th If $\xi \in \mathcal{E}_{ch}(X)$, $x, y \in ch(X)$
then $Q(x,\xi) \cap Q(y,\xi) \supset Q(z,\xi)$ for some z.

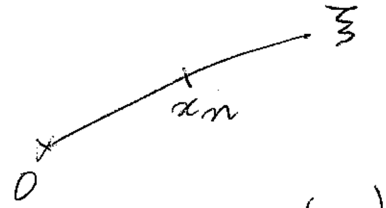
Cor Every point in $\mathcal{C}_{ch}(X)$ is in the boundary of some apartment.
(because every sector is contained in an apartment).

Th (Caprace–L.) G amenable group acting properly on X, then G is amenable $\Leftrightarrow$ G fixes a point (up to finite index).

III Boundary ~~and~~ amenability

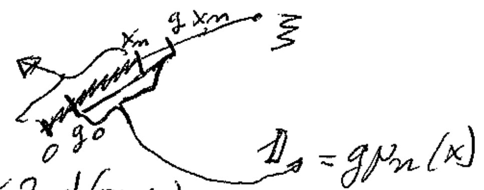
Th $G \circlearrowright X$ properly. Then $\exists\, \mu_n : \mathcal{C}_{ch}(X) \to \mathrm{Prob}(G)$
such that $\| \mu_n(gx) - g\mu_n(x) \| \longrightarrow 0$.

Basic construction $\quad G=\mathbb{F}_2 \curvearrowright T$ Cayley graph.



$$\mu_n(\xi) = \frac{\mathbb{1}_{[0,x_n]}}{n}$$

$\mu_n(gx) = \mathbb{1}$



$$\|\mu_n(g\xi) - g\mu_n(\xi)\| \le \frac{2\,d(o,go)}{n} \longrightarrow 0$$

(1) Choose a base point $o \in ch(X)$

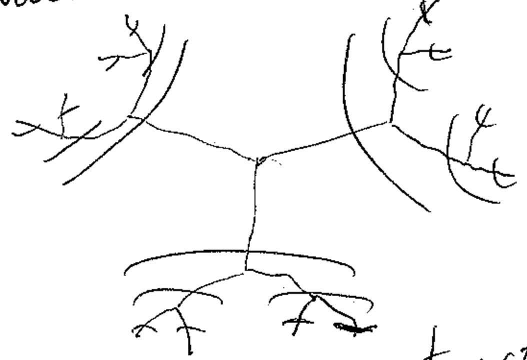
We construct $\mu_n(\xi)$ with support in $Q(o,\xi)$

$\rightsquigarrow$ reduce the problem to the construction of μ_n in one apartment.

(2) (Januszkiewicz) W embeds equivariantly in a finite product of trees

(in fact $\mathcal{C}_{ch}(\Sigma) \hookrightarrow \mathcal{C}_{ch}(T_1 \times \dots \times T_\ell) = \mathcal{C}_{ch}(T_1) \times \dots \times \mathcal{C}_{ch}(T_\ell)$.

Choose $W_0 \triangleleft W$ torsion free, finite index



(3) Use the basic construction on each T_i $\;i \in \{1, \dots, \ell\}$ and take the product. Beware that $\mathrm{Supp}(\mu_n(\xi))$ should be included in $Q(o,\xi)$.