Property (T) for groups graded by root systems.

Based on

1. M. Ershov, A. Jaikin-Zapirain, Property (T) for noncommutative universal lattices, Inventiones Mathematicae 179 (2010), 303–347.

2. M. Ershov, A. Jaikin-Zapirain, M. Kassabov, Property (T) for groups graded by root systems, in preparation.

I. Property (T): Kazhdan's result.

Definition. *For a topological group G, by $\mathfrak{Rep}(G)$ we will denote the class of continuous unitary representations of G in Hilbert spaces, and by $\mathfrak{Rep}_0(G)$ the class of continuous unitary representations of G without nonzero invariant vectors. Let Q be a subset of G.*

(a) Let $V \in \mathfrak{Rep}(G)$. A nonzero vector $v \in V$ will be called (Q, ε)-invariant if

$$\|qv - v\| \leq \varepsilon \|v\| \text{ for any } q \in Q.$$

(b) Let $V \in \mathfrak{Rep}_0(G)$. The Kazhdan constant $\kappa(G, Q, V)$ is the infimum of the set

$$\{\varepsilon > 0 : V \text{ contains an } (Q, \varepsilon)\text{-invariant vector.}\}$$

(c) The Kazhdan constant $\kappa(G, Q)$ of G with respect to Q is the infimum of the set $\{\kappa(G, Q, V)\}$ where V runs over $\mathfrak{Rep}_0(G)$.

Definition. *A topological group* G *is said to have property* (T) *if* $\kappa(G, Q) > 0$ *for some compact subset* Q *of* G.

The property (T) has the following two consequences for a locally compact group G:

(a) G is compactly generated (if G is discrete G is finitely generated);

(b) $G/\overline{[G,G]}$ is compact (if G is discrete $G/[G,G]$ is finite).

Theorem 1. *(D. Kazhdan 67, ...) The following holds*

1. *if G is locally compact group and Γ is a lattice in G, then G has property (T) if and only if Γ has property (T);*

2. *if G is a simple Lie group of rank ≥ 2 over a local field, then G has property (T).*

Corollary 2. *A lattice in a simple Lie group of rank ≥ 2 over a local field is finitely generated and FAb.*

How to prove that a given group has (T)?

Definition. *Let G be a discrete group and B a subset of G. The pair (G, B) has relative property (T) if for any $\varepsilon > 0$ there are a finite subset S of G and $\mu > 0$ such that if V is any unitary representation of G and $v \in V$ is (S, μ)-invariant, then v is (B, ε)-invariant.*

G has property (T) $\iff$ (G, G) has relative property (T)

if $\kappa(G, B) > 0$ and (G, B) has relative property (T), then G has property (T)

Our strategy to prove the property (T) for a group G:

Present G as $G = \langle H_1, \ldots, H_k \rangle$ and prove that $\kappa(G, \cup H_i) > 0$ and $(G, \cup H_i)$ has relative property (T).

Codistance

V^K is the subspace of K-invariant vectors

Let $G = \langle H_1, \ldots, H_k \rangle$.

$X = \cup H_i$ is a Kazhdan subset of G if and only if for every $V \in \mathfrak{Rep}_0(G)$, any $0 \neq v \in V$ cannot be arbitrarily close to each of the subspaces V^{H_i}

If $k = 2$ it is equivalent to asserting that the angle between the subspaces V^{H_1} and V^{H_2} must be bounded away from 0.

Definition. *Let V be a Hilbert space, and let $\{U_i\}_{i=1}^n$ be subspaces of V. The quantity*

$$\rho(\{U_i\}) = \sup \left\{ \frac{\|u_1 + \cdots + u_n\|^2}{n(\|u_1\|^2 + \cdots + \|u_n\|^2)} : u_i \in U_i \right\}$$

will be called the codistance between the subspaces $\{U_i\}_{i=1}^n$.

Consider the Hilbert space V^n. Then $\rho(\{U_i\})$ is the cosine of the angle between the subspaces $U_1 \times U_2 \times \ldots \times U_n$ and $diag(V) = \{(v, v, \ldots, v) : v \in V\}$.

$1/n \leq \rho(U_1, \ldots, U_n) \leq 1$

Definition. *Let $\{H_i\}_{i=1}^n$ be subgroups of G that generate G. The codistance between $\{H_i\}$ in G, denoted $\rho(\{H_i\})$, is defined to be the supremum of the set*

$$\{\rho(V^{H_1}, \ldots, V^{H_n}) : V \in \mathfrak{Rep}_0(G)\}.$$

$\kappa(G, \cup H_i) > 0$ if and only if $\rho(H_1, \ldots, H_n) < 1$

Let $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$ be a finite conected graph without loops.

if $e = (x, y) \in \mathcal{E}(\Gamma)$, then also $\bar{e} = (y, x) \in \mathcal{E}(\Gamma)$.

If $e = (x, y)$, we let $e^- = x$ be the initial vertex of e and by $e^+ = y$ the terminal vertex of e.

Let V be a Hilbert space and $\Omega^0(\Gamma, V)$ be the Hilbert space of functions $f : \mathcal{V}(\Gamma) \to V$ with the scalar product

$$\langle f, g \rangle = \sum_{y \in \mathcal{V}(\Gamma)} \langle f(y), g(y) \rangle \tag{1}$$

and let $\Omega^1(\Gamma, V)$ be the Hilbert space of functions $f : \mathcal{E}(\Gamma) \to V$ with the scalar product

$$\langle f, g \rangle = \frac{1}{2} \sum_{e \in \mathcal{E}(\Gamma)} \langle f(e), g(e) \rangle. \tag{2}$$

Define the linear operator

$d : \Omega^0(\Gamma, V) \to \Omega^1(\Gamma, V)$ by $(df)(e) = f(e^+) - f(e^-)$.

Then the adjoint operator $d^* : \Omega^1(\Gamma, V) \to \Omega^0(\Gamma, V)$ is given by formula

$$(d^* f)(y) = \sum_{y=e^+} \frac{1}{2} (f(e) - f(\bar{e}))\,.$$

The symmetric operator $\Delta = d^* d : \Omega^0(\Gamma, V) \to \Omega^0(\Gamma, V)$ is called the *Laplacian of* Γ and is given by the formula

$$(\Delta f)(y) = \sum_{y=e^+} (f(y) - f(e^-)) = \sum_{y=e^+} df(e).$$

The smallest positive eigenvalue of Δ is commonly denoted by $\lambda_1(\Delta)$ and called the *spectral gap* of the graph Γ (clealry, it is independent of the choice of V).

Definition. Let G be a group and Γ a finite graph without loops. A *decomposition of* G *over* Γ is a choice of a vertex subgroup $G_\nu \subseteq G$ for every $\nu \in \mathcal{V}(\Gamma)$ and an edge subgroup $G_e \subseteq G$ for every $e \in \mathcal{E}(\Gamma)$ such that

(a) The vertex subgroups $\{G_\nu : \nu \in \mathcal{V}(\Gamma)\}$ generate G;

(b) $G_e = G_{\bar{e}}$ and $G_e \subseteq G_{e^+} \cap G_{e^-}$ for any $e \in \mathcal{E}(\Gamma)$.

We will say that the decomposition of G over Γ is *regular* if for each $\nu \in \mathcal{V}(\Gamma)$ the vertex group G_ν is generated by edge subgroups $\{G_e : e^+ = \nu\}$

Theorem 3. *Let* Γ *be a connected* k*-regular graph and let* G *be a group with a given regular decomposition over* Γ*. Let*

$$p = \max_{\nu \in \mathcal{V}(\Gamma)} \rho(G_e : \ e^+ = \nu).$$

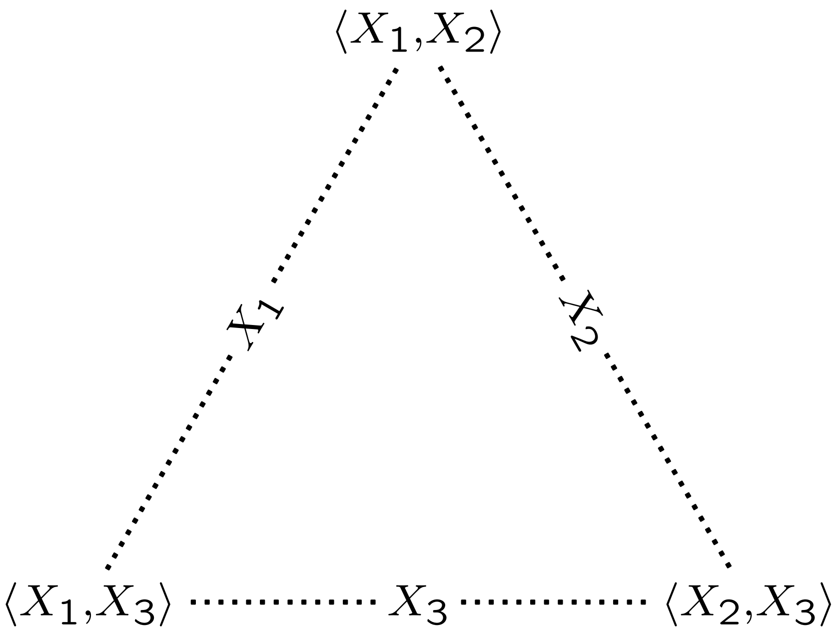
Let Δ *be the Laplacian of* Γ*, and assume that*

$$p < \frac{\lambda_1(\Delta)}{2k}.$$

Then $\cup_{\nu \in \mathcal{V}(\Gamma)} G_\nu$ *is a Kazhdan subset of* G*, and moreover*

$$\kappa(G, \cup G_\nu) \geq \sqrt{\frac{2(\lambda_1(\Delta) - 2pk)}{\lambda_1(\Delta)(1-p)}}.$$

Groups over triangles (J. Dymara and T. Januszkiewicz)



Corollary 4. *Let G be a group generated by X_1, X_2, X_3. Assume that $\rho(X_i, X_j) < \frac{3}{4}$ for all pairs i, j. Then $\kappa(G, X_1 \cup X_2 \cup X_3) > 0$. In particular, if X_1, X_2, X_3 are finite, G has property (T).*

Proof. In this situation $k = 2$ and $\lambda_1(\Delta) = 3$. Hence $\frac{\lambda_1(\Delta)}{2k} = \frac{3}{4}$. □

Let $R = \langle T \rangle$ ($T = \{t_0 = 1, t_1, \cdots, t_d\}$)

$EL_n(R) = \langle \{Id + rE_{ij} : r \in R, 1 \leq i \neq j \leq n\} \rangle$

if $n \geq 3$, $EL_n(R) = \langle \{Id \pm tE_{ij} : t \in T, 1 \leq i \neq j \leq n\} \rangle$.

If $R = \mathbb{Z}[x_1, \ldots, x_d]$, then $SL_n(R) = EL_n(R)$.

Corollary 5. *Let F be a finite field of order greater than 4 and $n \geq 3$. Let R be free associative algebra on $t_1, \ldots, t_k$. Then $G = EL_n(R)$ has property (T).*

Proof. For simplicity let us assume that $n = 3$. We put $A = \{\alpha_0 + \alpha_1 t_1 + \cdots + \alpha_k t_k : \ \alpha_s \in F\}$ and let

$$X_1 = \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} : \ a \in A\}, X_2 = \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix} : \ a \in A\}, \ X_3 = \{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \ a \in A\}.$$

Then X_1, X_2, X_3 generate G. Moreover $\rho(X_i, X_j) = \frac{\sqrt{|F|}+1}{2\sqrt{|F|}}$. (This is because $< X_i, X_j >$ is a finite group isomorphic to

$$\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : \ a, c \in A, b \in A^2\}$$

and we can use the basic representation theory of finite groups in order to calculate $\rho(X_i, X_j)$.) $\square$

The critical case $p = \frac{\lambda_1(\Delta)}{2k}$.

For each $\nu \in \mathcal{V}(\Gamma)$ we fix a normal subgroup CG_ν of G_ν (called the *core subgroup* of G_ν).

Let $V \in \mathfrak{Rep}(G)$. Then we can write

$\Omega^1(\Gamma, V) = W_1 \oplus W_2 \oplus W_3$, where

$W_1 = \{f \in \Omega^1(\Gamma, V) : \; f(e) \in V^{G_{e^+}}\}$.

$W_2 = \{f \in \Omega^1(\Gamma, V) : \; f(e) \in (V^{G_{e^+}})^{\perp} \cap V^{CG_{e^+}}\}$.

$W_3 = \{f \in \Omega^1(\Gamma, V) : \; f(e) \in (V^{CG_{e^+}})^{\perp}\}$.

Let $\rho_i : \Omega^1(\Gamma, V) \to W_i$ be the projections.

Theorem 6. *Let G be a group with a chosen regular decomposition over a connected k-regular graph Γ, and choose a normal subgroup CG_ν of G_ν for each $\nu \in \mathcal{V}(\Gamma)$. Suppose that*

(i) For each vertex ν of Γ,

$$\rho(G_e : e^+ = \nu) \leq \frac{\lambda_1(\Delta)}{2k}.$$

(ii) For each $\nu \in \mathcal{V}(\Gamma)$ and any representation V of the vertex group G_ν without CG_ν-invariant vectors,

$$\rho(V^{G_e} : \ e^+ = \nu) < \frac{\lambda_1(\Delta)}{2k}.$$

(iii) There are constants A, B such that for any representation V of G and for any function $g \in \Omega^0(\Gamma, V)$ one has

$$\|\rho_2(dg)\|^2 \leq A\|\rho_1(dg)\|^2 + B\|\rho_3(dg)\|^2.$$

Then $\cup G_\nu$ is a Kazhdan subset of G.

Definition. Let E be real vector space. A finite non-empty subset Φ of E is called a *root system in* E if

(a) Φ spans E;

(b) Φ does not contain 0;

(c) Φ is closed under inversion, that is, if $\alpha \in \Phi$ then $-\alpha \in \Phi$.

The dimension of E is called the *rank of* Φ.

Φ is called *irreducible* if it cannot be represented as a disjoint union of two non-empty subsets, whose $\mathbb{R}$-spans have trivial intersection.

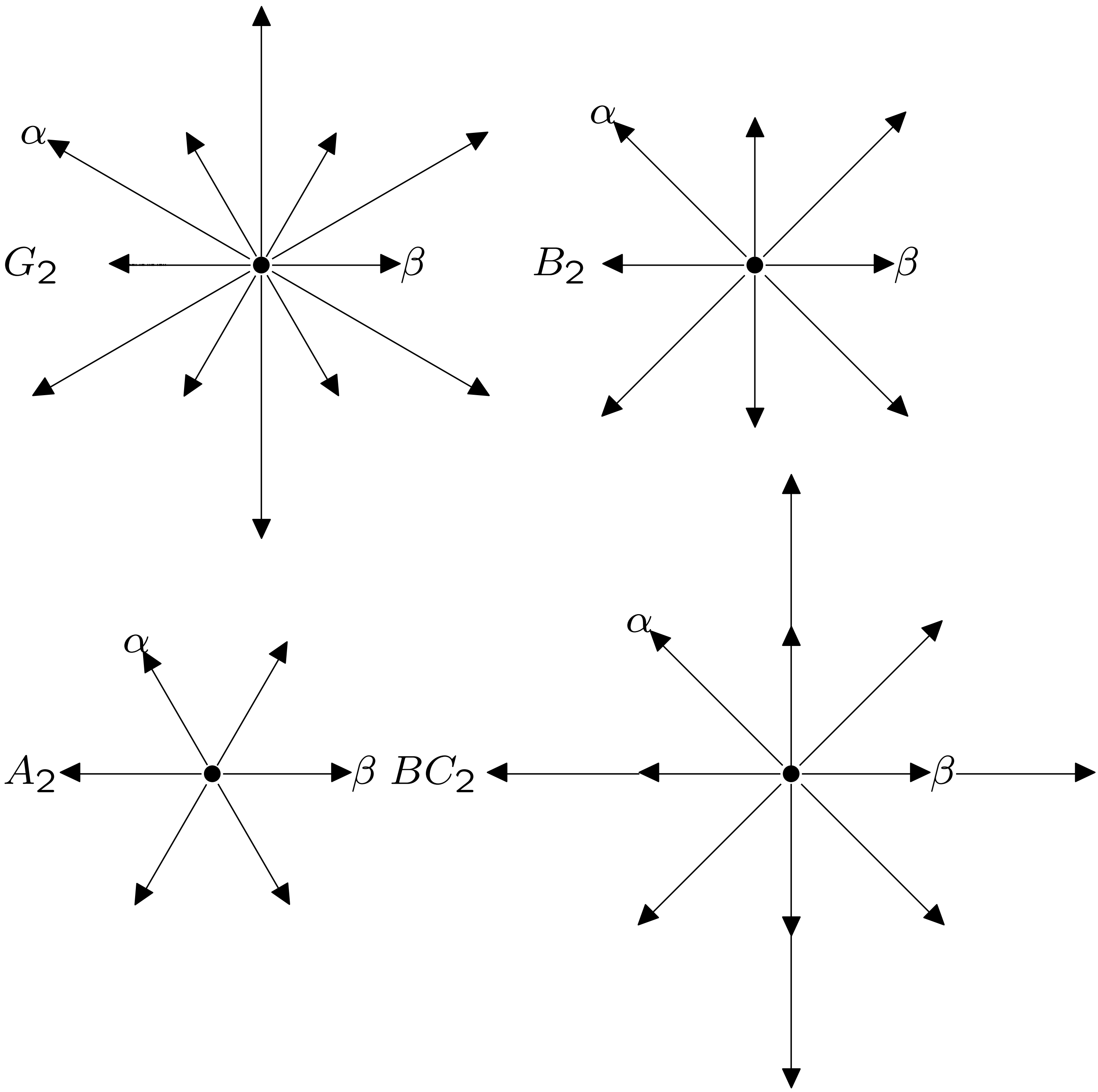
Definition. A root system Φ in a space E will be called *classical* if E can be given the structure of a Euclidean space such that

(a) For any $\alpha, \beta \in \Phi$ we have $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$;

(b) If $\alpha, \beta \in \Phi$, then $\alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta \in \Phi$

Remark: Every irreducible classical root system is isomorphic to one of the following:

$A_n, B_n (n \geq 2), C_n (n \geq 3), BC_n (n \geq 1), D_n (n \geq 4), E_6, E_7, E_8, F_4, G_2$. The only non-reduced systems on this list are the ones of type BC_n.



Classical irreducible root systems of rank 2.

Definition. Let Φ be a root system and G a group. A Φ-*grading of* G is a collection of subgroups $\{X_\alpha\}_{\alpha\in\Phi}$ of G, called *root subgroups* such that

(i) $\{X_\alpha\}_{\alpha\in\Phi}$ generate G.

(ii) For any $\alpha, \beta \in \Phi$, with $\alpha \notin \mathbb{R}_{<0}\beta$, we have

$$[X_\alpha, X_\beta] \subseteq \langle X_\gamma \mid \gamma = a\alpha + b\beta \in \Phi,\ a, b \geq 1\rangle$$

If $\{X_\alpha\}_{\alpha\in\Phi}$ is a collection of subgroups satisfying (ii) but not necessarily (i), we will simply say that $\{X_\alpha\}_{\alpha\in\Phi}$ is a Φ-*grading* (without specifying the group).

We want to show that some group G having Φ-grading $\{X_\alpha\}_{\alpha\in\Phi}$ has property (T). Our strategy is the following

I) Prove that $\kappa(G, \cup_{\alpha\in\Phi}X_\alpha) > 0$.

II) Prove the relative property (T) for the pair $(G, \cup_{\alpha\in\Phi}X_\alpha)$.

Graph of groups corresponding to Φ-gradings

Definition. Let Φ be a root system in a space E. Let $\mathfrak{F} = \mathfrak{F}(\Phi)$ denote the set of all linear functionals $f : E \to \mathbb{R}$ such that $f(\alpha) \neq 0$ for all $\alpha \in \Phi$.

For $f \in \mathfrak{F}$, the set $\Phi_f = \{\alpha \in \Phi \,|\, f(\alpha) > 0\}$ is called the *Borel set of* f. The sets of this form will be called *Borel subsets of* Φ. We will say that two elements $f, f' \in \mathfrak{F}$ are equivalent and write $f \sim f'$ if $\Phi_f = \Phi_{f'}$.

Let G be a group and $\{X_\alpha\}_{\alpha \in \Phi}$ a grading of G. For any subset A of Φ denote by X_A the group generated by $\{X_a : \ a \in A\}$.

Let Γ be the following graph:

$$\mathcal{V}(\Gamma) = \mathfrak{F}/\sim,\ \mathcal{E}(\Gamma) = \{(f,g):\ \Phi_f \neq -\Phi_g\}.$$

We obtain a decomposition of G over Γ if we put

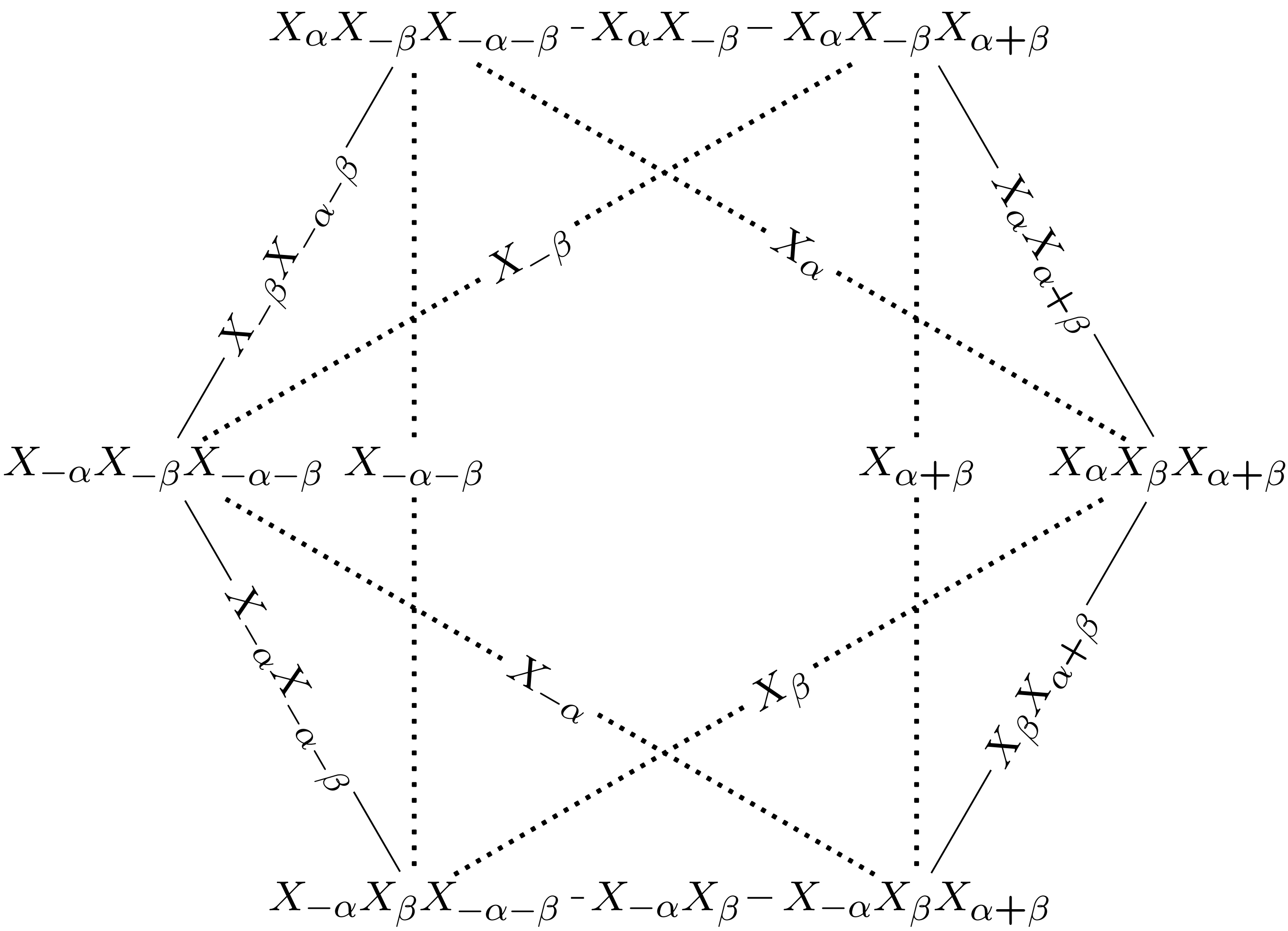
$$G_f = X_{\Phi_f},\ G_e = X_{\Phi_f \cap \Phi_g}\ (e = (f,g)).$$

Proposition 7. *Let $N = |V(\Gamma)|$. Then*

a) Γ is $(N-2)$-regular graph and $\lambda_1(\Delta) = N-2$.

b) For every $f \in \mathcal{V}(\Gamma)$ $\rho(G_e:\ e^+ = f) \leq 1/2$.

Thus, the condition (i) of Theorem 6 holds.



Weyl graph of groups for a groups graded by a root system of type A_2.

Construction of the core subgroups

Definition. Let Φ be a root system. Two Borel sets Φ_f and Φ_g will be called

- *co-maximal* if an inclusion $\Phi_h \supset \Phi_f \cap \Phi_g$ implies that $\Phi_h = \Phi_f$ or $\Phi_h = \Phi_g$;

- *The boundary of a Borel set* Φ_f is the set

$$B_f = \bigcup_g (\Phi_f \setminus \Phi_g),$$

  where Φ_g and Φ_f are co-maximal.

- *The core of a Borel set* Φ_f is the set

$$C_f = \Phi_f \setminus B_f$$

Put $CG_f = X_{C_f} = \langle X_g : \ g \in C_f \rangle$. (The group CG_f is normal in G_f.)

Definition. Let Φ be a root system in E. Φ is called *regular* if for each root $\alpha \in \Phi$ there are $\beta, \gamma \in \Phi \setminus \mathbb{R}\alpha$, such that

$$\mathbb{R}\alpha + \mathbb{R}\beta = \mathbb{R}\beta + \mathbb{R}\gamma$$

Definition. A grading $\{X_\alpha\}_{\alpha \in \Phi}$ will be called *strong* if for any functional $f \in \mathfrak{F}(\Phi)$ and any root $\gamma \in C_f$ we have

$$X_\gamma \subseteq \langle X_\beta \mid \beta \in \Phi_f \text{ and } \beta \notin \mathbb{R}\gamma \rangle.$$

Theorem 8. *Assume that Φ is regular and the grading $\{X_\alpha\}_{\alpha \in \Phi}$ of G is strong. Then the conditions (ii) and (iii) of Theorem 6 hold and so $\kappa(G, \cup_\alpha X_\alpha) > 0$.*

Corollary 9. *Let $n \geq 2$ and $A_n = \{e_i - e_j : 1 \leq i \neq j \leq n+1\} \subset \mathbb{R}^{n+1}$. Let R be a finitely generated ring with 1 and $G = EL_{n+1}(R)$ the elementary linear group. Put*

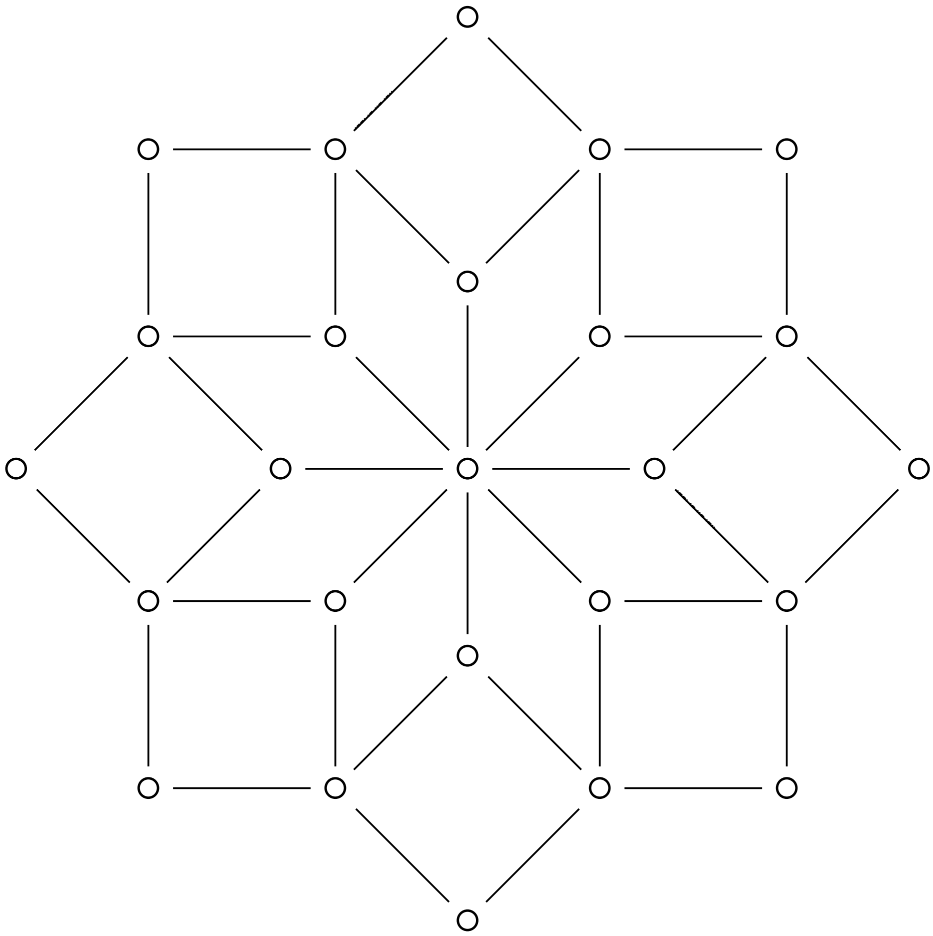
$$X_{e_i - e_j} = \{Id + rE_{ij} : r \in R\}.$$

Then $\kappa(G; \cup X_{e_i-e_j}) > 0$.

Corollary 10. *The group $G = EL_{n+1}(R)$ has property (T) for $n \geq 2$.*

Proof. The relative property (T) for the pair $(G; \cup X_{e_i-e_j})$ was proved by M. Kassabov (2007). □

Theorem 11. *Let R be a finitely generated commutaive ring with 1 and Φ a classical root system of rank ≥ 2. Then $St_\Phi(R)$ has property (T).*



Also we have proved analogous results for twisted groups. Here, for example the root system which appears in the case of $St_{{}^2F_4}(R)$.

How to give a good estimation for the Kazhdan constant?

For example, $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$, whence $SL_n(\mathbb{Z})$ has property (T) if $n \geq 3$.

What is $\kappa(SL_n(\mathbb{Z}), S)$ for a natural generating set S (for example when $S = \{Id \pm E_{ij} : 1 \leq i \neq j \leq n\}$)?

Kazhdan's proof is not effective

Burger, Shalom, Kassabov

$$\kappa(SL_n(\mathbb{Z}), S) = O\left(\frac{1}{\sqrt{n}}\right)$$

Definition. Let Φ be a root system in a space $V = \mathbb{R}\Phi$. A *reduction of* Φ is a surjective linear map $\eta : V \to V'$ where V' is another non-trivial real vector space. The set $\Phi' = \eta(\Phi) \setminus \{0\}$ is called the *reduced root system*. We will also say that η is a *reduction of* Φ *to* Φ' and symbolically write $\eta : \Phi \to \Phi'$.

Lemma 12. *Let Φ be a root system, η a reduction of Φ, and Φ' the reduced root system. Let $\{X_\alpha\}_{\alpha\in\Phi}$ be a Φ-grading. For any $\alpha' \in \Phi'$ put*

$$Y_{\alpha'} = \langle X_\alpha \mid \eta(\alpha) = \alpha' \rangle.$$

Then $\{Y_{\alpha'}\}_{\alpha'\in\Phi'}$ is a Φ'-grading, which will be called the induced grading.

A reduction $\eta : \Phi \to \Phi'$ enables us to replace a grading of a given group G by the "large" root system Φ by the induced grading by the "small" root system Φ' which may be easier to analyze.

Proposition 13. *Let Φ be a root system, η be a 2-good reduction of Φ, and $\Phi' = \eta(\Phi) \setminus \{0\}$ the reduced root system. Let $\{X_\alpha\}_{\alpha \in \Phi}$ be a 2-strong grading of a group G. Then the induced grading $\{Y_{\alpha'}\}_{\alpha' \in \Phi'}$ is a strong grading of G.*

Proposition 14. *Every irreducible classical root system of rank > 2 admits a 2-good reduction to an irreducible classical root system of rank 2.*

Theorem 15. *Let Φ be a reduced irreducible classical root system of rank at least 2, R a finitely generated commutative ring with 1 generated by $T = \{1 = t_0, t_1, \ldots, t_d\}$ and Σ the standard set of generators of $St_\Phi(R)$. Then* $\kappa(St_\Phi(R), \Sigma) \geq$

$$\begin{cases} O\left(\frac{1}{\sqrt{n+d}}\right) & \Phi = A_n, B_n (n \geq 3), D_n \\ O\left(\frac{1}{\sqrt{d}}\right) & \Phi = E_6, E_7, E_8, F_4 \\ O\left(\frac{1}{2^{d/2}}\right) & \Phi = B_2, G_2 \\ O\left(\frac{1}{\sqrt{n+2^d}}\right) & \Phi = C_n \end{cases}$$