# Luc Guyot: Limits of metabelian groups

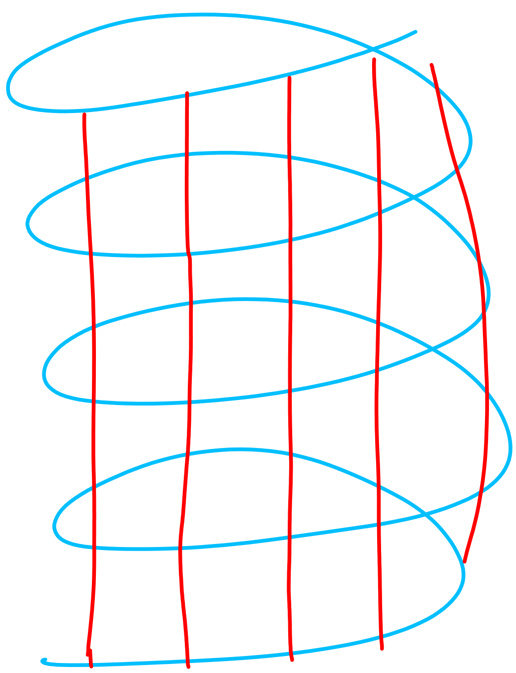
<u>Definition:</u> $\lambda \in \mathbb{C} \setminus \{0\}$ $G(\lambda) = \mathbb{Z}[\lambda^{\pm 1}] \rtimes_{\lambda} \mathbb{Z}$ whenever $\lambda$ is transcendental $G(\lambda) \cong \mathbb{Z} \wr \mathbb{Z}$

<u>Problem</u> $X \subseteq \mathbb{C} \setminus \{0\}$, describe
$\{G(\lambda) \ \lambda \in X\} \subseteq \mathcal{M}_2$ the space of marked groups on 2 generators.

## ① The space of marked groups

$n \geq 1$ $G = \langle S \rangle$ $S \in G^n$ $(G, S) =$ Cayley graph of $G$ w.r.t $S$ colored and oriented and pointed

$\mathcal{M}_n = \{(G,S), \{ {S \in G^n \atop G = \langle S \rangle} \} / \sim_m$



$T_1 = \mathbb{Z}, (,(1, 3))$

$B_{T_1}(G) \simeq B_{(\mathbb{Z}^2, (e_1, e_2))}(G)$

$d(T_1, T_2) = e^{6}$

$\mathcal{M}_m$ is compact

## 2. Convergence and logic

$[x,y] = xyx^{-1}y^{-1}$ $\quad \forall x\, \forall y\, \forall z,\, \forall t \quad [[x,y],[z,t]]=1$

**Definition**: $Th_\forall(G) = \{$ universal sentence true in $G$ $\}$

**Theorem F** (Sela, Champetier-Guirardel)
let $G$ be a finitely generated group $Th_\forall(G) = Th_\forall(\overset{M_n}{F_n})$ $(n \geq 2)$ iff $G$ is a limit of groups isomorphic to $F_2$ $\mathbb{Z} \wr \mathbb{Z}$

**Definition**: $M_m = F_m / F_m''$

**Thm G''** $(Th_\forall(G) = Th_\forall(F_n)$ $n \geq 2$ & $rk\, G \leq 3)$

$$\Updownarrow$$

$G$ belongs to the list $F_2, F_3$ $\quad F_2 * \langle z \rangle = \langle a, b, c, \mid c v c^{-1} = v \rangle$
$v \in F(a,b) \setminus \{1\}$ and is not a proper power
$G$ belongs to $\{M_2, \mathbb{Z} \wr \mathbb{Z}\}$ (Chapuis 97)

What about are relator groups?

### RESULTS

(n=2) $1 \to M' \to \overset{M}{\overset{\shortparallel}{M_2}} \longrightarrow \mathbb{Z}^2 \longrightarrow 1$

$R = \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$

$\quad \mathfrak{q} \underset{\text{closed}}{\subset} R \qquad \mathfrak{q} \lhd M$

**Definition**: $M(\mathfrak{q}) = M / \mathfrak{q}$

$\lambda \in \mathbb{C}$ $\quad P_\lambda(x) = \mathbb{Z}[x]$ the minimal polynomial of $\lambda$ over $\mathbb{Z}$
$[G] = \{(H,S) \in M_2 \mid H$ is abstractly isomorphic to $G\}$

**Theorem 1** $\overline{[G(\lambda)]} = [G(\lambda)] \sqcup \bigsqcup_{k>0} M(P_{\lambda^k}) \sqcup \{M\}$

Corollary 2 Let $G$ be a 2 generator group

$$Th_{\forall}(G) = Th_{\forall}(G(\lambda)) \iff G \in \{G(\lambda), M(P_\lambda)\}$$

Definition

$X \subseteq \mathbb{C} \setminus \{0\}$ is tame if either $X$ is finite or

$$[G(\lambda) \mid \lambda \in X] = \bigcup_{\lambda \in X} \overline{[G(\lambda)]} \cup [\mathbb{Z} \wr \mathbb{Z}]$$

Thm 3

Let $X \subset \mathbb{C} \setminus \{0\}$. Assume that there exists $\varepsilon > 0$ s.t. $|\overline{\lambda}| > 1 + \varepsilon$ for every algebraic unit in $X$. Then $X$ is tame.

$\lambda$ is an algebraic unit if $P_\lambda(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1 x \mp 1$

$|\overline{\lambda}| = \max_{1 \le i \le d} |\lambda_i|$ $\lambda_i$ conjugate to $\lambda$.

Corollary 4

(a) $X = \{\text{Pisot numbers}\} \cup \{\text{totally real algebraic numbers}\}$ is tame

(b) Assume $[\mathbb{Q}(\lambda) : \mathbb{Q}] < d$ for every algebraic unit $\lambda \in X$, then $X$ is tame

PROOF (a) By thm Salem, Pisot numbers closed in $\mathbb{R}$

$|\overline{\lambda}|$ real part of $x^2 - x - 1 > 1{,}3$

(b) Thm Schinzel - Zassenhaus ($\lambda$ not a rt of unity)

$$|\overline{\lambda}| > 1 + \frac{1}{e(d)}$$

## (3) Ideas for the proofs

(1) Consider converging sequence $(G(\lambda_n), S_n)$

(2) Reduce the data

Proposition 5

Proposition 5

Let $S_1, S$ be generating pairs of $G(\lambda) = \mathbb{Z}[\lambda^{\pm 1}] \rtimes \mathbb{Z} \xrightarrow{\sigma} \mathbb{Z}$

If $\sigma(S) = S(S')$ then $(G(\lambda), S) = (G(\lambda), S)$ as marked groups

(3) $(\lambda_n) \in (\mathbb{I} \setminus \{0\})^{\mathbb{N}}$ $(k_n, l_n)$ coprime integers

(4) Translation in ring theory

Ⓐ Given $(\lambda_n)$ $(k_n, l_n)$ identifying the ring $R \subset \mathbb{C}^{\mathbb{N}}$ generated by $\underline{\lambda} = (\lambda_n^{k_n})$ and $\underline{\mu} = (\lambda_n^{l_n})$

Ⓑ Given $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}] \subset \mathbb{C}$ find sequence $(\lambda_n) \in (\mathbb{C} \setminus \{0\})^{\mathbb{N}}$ and $(k_n, l_n)$ such $\mathbb{Z}[\underline{\lambda}^{\pm 1}, \underline{\mu}^{\pm 1}] \cong R$

$$\underline{\lambda} \longmapsto \lambda$$
$$\underline{\mu} \longmapsto \mu$$

Hints for Ⓐ

$\lambda_n$ is not a root of unity, there is an absolute value $|\cdot|_n$ on $\mathbb{Q}(\lambda_n)$ such that $|\lambda_n| \neq 1$

* Compare the growth of $|\lambda_n|_n^{k_n}$ and $|\lambda_n|_n^{l_n}$ to show that $\mathbb{Z}[\underline{\lambda}^{\pm 1}, \underline{\mu}^{\pm 1}] \simeq \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.