

Luc Guyot: Limits of metabelian groups

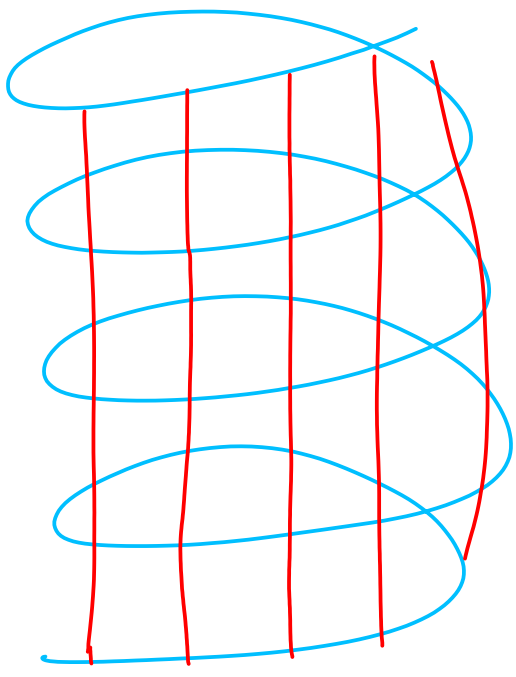
Definition: $\lambda \in \mathbb{C} \setminus \{0\}$ $G(\lambda) = \mathbb{Z}[\lambda^{\pm 1}] \rtimes_{\lambda} \mathbb{Z}$ whenever λ is transcendental $G(\lambda) \cong \mathbb{Z} \wr \mathbb{Z}$

Problem $X \subseteq \mathbb{C} \setminus \{0\}$, describe $\{G(\lambda) \mid \lambda \in X\} \subseteq \mathcal{M}_2$ the space of n -marked groups on 2 generators.

① The space of marked groups

$n \geq 1$ $G = \langle S \rangle$ $S \in G^n$ $(G, S) =$ Cayley graph of G w.r.t S colored and oriented and pointed

$$\mathcal{M}_n = \{ (G, S) \mid \begin{matrix} S \in G^n \\ G = \langle S \rangle \end{matrix} \} / \sim_m$$



$T_1 = \mathbb{Z}, (1, 1, 1)$
 $B_{T_1}(G) \cong B_{(\mathbb{Z}^2, (e_1, e_2))}(G)$
 $d(T_1, T_2) = \bar{e}^6$
 \mathcal{M}_n is compact

2. Convergence and logic

$[x, y] = xyx^{-1}y^{-1} \quad \forall x \forall y \forall z, \forall t \quad [[x, y], [z, t]] = 1$

Definition : $Th_{\forall}(G) = \{ \text{universal sentence true in } G \}$

Theorem F (Sela, Champetier-Guivardel)

let G be a finitely generated group $Th_{\forall}(G) = Th_{\forall}(F_n)$ ($n \geq 2$) iff G is a limit of groups isomorphic to F_2 $\cong_{wr} \mathbb{Z}$

Definition : $M_n = F_n / F_n''$

Thm G' ($Th_{\forall}(G) = Th_{\forall}(F_n)$ $n \geq 2$ & $rk G \leq 3$)



G belongs to the list F_2, F_3 $F_2 \neq \langle 2 \rangle = \langle a, b, c, [c, c^{-1}] \rangle$
 $v \in F(a, b) \setminus \{1\}$ and is not a proper power

G belongs to $\{M_2, \cong_{wr} \mathbb{Z}\}$ (chapuis 97)

What about are retractor groups ?

RESULTS

$(n=2) \quad 1 \rightarrow M' \rightarrow \overset{M}{\mathbb{H}_2} \rightarrow \mathbb{Z}^2 \rightarrow 1$

$R = \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[z^{\pm 1}, y^{\pm 1}]$

$\square \subset R \quad \square \triangleleft M$
closed

Definition : $M(\square) = M / \square$

$\lambda \in \mathbb{C} \quad P_{\lambda}(z) = \mathbb{Z}[z]$ the minimal polynomial of λ over \mathbb{Z}

$[G] = \{ (H, S) \in \mathcal{M}_2 \mid H \text{ is abstractly isomorphic to } G \}$

Theorem 1 $\overline{[G(\lambda)]} = [G(\lambda)] \coprod \coprod_{k>0} M(P_k) \coprod \{M\}$

Corollary 2 Let G be a 2 generator group

$$\text{Thy}(G) = \text{Thy}(G(A)) \iff G \in \{G(A), M(P_A)\}$$

Definition

$X \subseteq \mathbb{C} \setminus \{0\}$ is tame if either X is finite or

$$[G(A) | A \in X] = \bigcup_{A \in X} \overline{[G(A)]} \cup [\mathbb{Z}w\mathbb{Z}]$$

Thm 3

Let $X \subset \mathbb{C} \setminus \{0\}$. Assume that there exists $\epsilon > 0$ s.t. $|A| > 1 + \epsilon$ for every algebraic unit in X . Then X is tame.

A is an algebraic unit if $P_A(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1x + 1$

$$|A| = \max_{1 \leq i \leq d} |a_i|, A, \text{ conjugate to } A.$$

Corollary 4

- (a) $X = \{\text{Pisot numbers}\} \cup \{\text{totally real algebraic numbers}\}$ is tame
- (b) Assume $[\mathbb{Q}(A) : \mathbb{Q}] < d$ for every algebraic unit $A \in X$, then X is tame

PROOF

(a) By the Salem, Pisot numbers closed in \mathbb{R}
 $|A|$ real part of $x^2 - 2x - 1 > 1, 3$

(b) Thm Schmitzel - Zassenhaus (A not a set of unity)
 $|A| > 1 + \frac{1}{\epsilon(d)}$

③ Ideas for the proofs

- (1) Consider converging sequences $(G(A_n), S_n)$
- (2) Reduce the data

Proposition 5

Proposition 5

Let S', S be generating pairs of $G(\mathbb{Z}) = \mathbb{Z}[\mathbb{Z}^{\pm 1}] \rtimes \mathbb{Z} \xrightarrow{\sigma} \mathbb{Z}$

If $\sigma(S) = S(S')$ then $(G(\mathbb{Z}), S) = (G(\mathbb{Z}), S')$ as marked groups

(3) $(\lambda_n) \in (\mathbb{Z} \setminus \{0\})^{\mathbb{N}}$ (k_n, l_n) coprime integers

(4) Translation in ring theory

(A) Given (λ_n) (k_n, l_n) identifying the ring $R \subset \mathbb{C}^{\mathbb{N}}$ generated by $\lambda = (\lambda_n^{k_n})$ and $\mu = (\lambda_n^{l_n})$

(B) Given $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}] \subset \mathbb{C}^{\mathbb{N}}$ find sequences $(\lambda_n) \in (\mathbb{Z} \setminus \{0\})^{\mathbb{N}}$ and (k_n, l_n) such $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}] \cong R$

$$\begin{array}{ccc} \lambda & \longmapsto & \lambda \\ \mu & \longmapsto & \mu \end{array}$$

Hints for (A)

λ_n is not a root of unity, there is an absolute value $|\cdot|_n$ on $\mathbb{Q}(\lambda_n)$ such that $|\lambda_n|_n \neq 1$

* Compare the growth of $|\lambda_n|_n^{k_n}$ and $|\lambda_n|_n^{l_n}$ to show that $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}] \subset \mathbb{Z}[\alpha^{\pm 1}, \gamma^{\pm 1}]$.