

Breuilard III/1

Th (Hrushovski 09) A is a K -approximate group.

Then $\exists B_n \subset B_{n-1} \subset \dots \subset B_1 \subset A$,

• $|B_i| \geq c_i(K) |A|$

• $B_i^2 \subset B_{i-1}$

• $[B_i, B_j] \subset B_{i+j-1}$

Rem $\Rightarrow A$ has a large subset B_1 s.t. $[B_1, B_1] \subset B_1$

Thm (Helfgott 05) If $A \subset SL_2(\mathbb{Z}/p\mathbb{Z})$ is a generating subset, then $|AAA| \geq \min(|SL_2(\mathbb{Z}/p\mathbb{Z})|, |A|^{1+\epsilon})$ where $\epsilon > 0$ is an absolute constant.

Thm (Hrushovski 09) G is a simple alg group over \mathbb{F} an algebraically closed field.

$\forall K \geq 1, \exists c(K) > 0$ s.t. $A \subset G(\mathbb{F})$ a finite set

• either $|A| \leq c(K)$

• or $A \subset H(\mathbb{F}), H \subset G$ a proper alg subgroup, of complexity $\chi(H) \leq c(\dim G)$

• or $|AAA| \geq \min\left(\frac{|A|^K}{c(K)}, K|A|\right)$

\Leftarrow there are no non trivial "sufficiently Zariski dense" approximate subgroups of simple algebraic groups \Rightarrow

Th (Green-Tao-B, Pyber-Szabo)

One can take $c(K) = K^c$ where $c = c(\dim G)$

in Hrushovski's thm.

« Every sufficiently Zariski-dense K -approximate subgroup of $G(K)$ is $K^{c(d)}$ -controlled by the finite group it generates.

Cor $\exists \epsilon = \epsilon(d) > 0$ such that if G is a finite group of Lie type and rank at most d , then for every generating set A we have $|AAA| \geq \min\{|G|, |A|^{1+\epsilon}\}$.

Open question: Classify approximate subgroups of A_n .

Pyber's example a generating approximate subgroup of A_n which is "non-trivial" (large and not almost all of A_n)

let $H \subset A_n$ given by $H = \langle \tau_{12}, \tau_{34}, \dots, \tau_{2m-1, 2m} \rangle$

$$H \cong \mathbb{F}_2^m \quad n = 2m+1$$

let $a = \sigma^2$ where $a = (1, 2, \dots, n)$ the long cycle

then $\langle a, H \rangle = A$ (because n is odd $\Rightarrow \langle a \rangle = \langle a \rangle$)

Take $A = a^{\pm 1} \cup H \cup \{1\}$. It is a C -approximate group (because H is almost normalized by a):

$$aHa^{-1} = \langle \tau_{34}, \dots, \tau_{2m-1, 2m}, \tau_{2m+1, 1} \rangle$$

$$\text{and so } |aHa^{-1} \cap H| = \frac{|H|}{2}$$

In fact A is controlled by H .

Breuil-Laud III/2

In Harshenko's proof, a key point is an estimate of Larsen-Pink (95).

G a simple algebraic group, $k = \text{alg closed}$,

$T \subset G(k)$ a finite subgroup

then either T is not sufficiently Zariski-dense

($T \subset H(k)$, $H \neq G$, $\chi(H) \leq c(G)$)

or $\forall V$ ~~variety~~ proper subvariety of G ,

$$|T \cap V| \ll_{\chi(V)} |T| \frac{\dim V}{\dim G}$$

Approximate Larsen-Pink inequality

G, k as above. $A \subset G(k)$ a finite k -approximate subgroup

then either A is not sufficiently Zariski dense

($A \subset H(k)$, $H \neq G$, $\chi(H) \leq c(G)$)

or $\forall V$ proper variety of G ,

$$|A \cap V| \leq K^c |A| \frac{\dim V}{\dim G}$$

where $c = c(\dim(G))$

Idea for the proof of LP inequality: induction on $\dim V$.

~~induction in $\dim V$~~

$\dim V = 0$ or $\dim G \rightarrow$ clear

$\dim V^- \leq \dim V^+$ $A \cap V^-$ huge, $A \cap V^+$ huge.

idea use simplicity of G

$\hookrightarrow \exists a \in A$ s.t. $V^- \cdot a \cdot V^+ \cdot a^{-1}$ has $\dim > \dim V^+$

We would like to say $(A \cap V^-) \cdot a \cdot (A \cap V^+) \cdot a^{-1}$ has at least

$|A \cap V^-| |A \cap V^+|$ elements.

Sketch of pf of thm modulo LP

$A \in G(k)$, K -approx.

A is sufficiently Zariski-dense

Have to show: either $|A| \leq K^c$

Idea ~~Take A act on the variety G/T in K^c~~ or $|A| \geq \frac{|KA|}{K^c}$

Act on the variety of tori: G/T , T = maximal torus in $G(k)$

2 consequences of LP (1) if $a \in A$ is regular semi-simple

$$|A \cap Z(a)| \approx |A| \frac{\dim Z(a)}{\dim G}$$

$$Z(a) = \frac{K^c}{T} \text{ maximal torus.}$$

Use $v = T$ and $v = \mathcal{U}(a)$ (conjugacy class of A).

$$\rightarrow |A^2 \cap Z(a)| \cdot |a^A| \approx |A|.$$

by "gp action Lemma".

$$\text{Crucial Lemma: } T \cap A \neq \{1\} \Rightarrow a T a^{-1} \cap A \neq \{1\}$$