Breuillard III/1

Th (Hrushowski 09) A is a K-approximate group.
Then $\exists$ $B_n \subset B_{n-1} \subset \dots \subset B_1 \subset A$,

- $|B_i| \geq c_i(K)\,|A|$
- $B_i^2 \subset B_{i-1}$
- $[B_i, B_j] \subset B_{i+j-1}$

Rem $\Rightarrow$ A has a large subset B_1 s.t. $[B_1, B_1] \subset B_1$

Thm (Helfgott 05) If $A \subset SL_2(\mathbb{Z}/p\mathbb{Z})$ is a generating subset, then $|AAA| \geq \min(|SL_2(\mathbb{Z}/p\mathbb{Z})|, |A|^{1+\varepsilon})$
where $\varepsilon > 0$ is an absolute constant.

Thm (Hrushowski 09) $\mathbb{G}$ a simple alg group over k an algebraically closed field.
$\forall K \geq 1$, $\exists c(K) > 0$ s.t. $A \subset G(k)$ a finite set

- either $|A| \leq c(K)$
- or $A \subset \mathbb{H}(k)$, $\mathbb{H} \subset \mathbb{G}$ a proper alg subgroup, of complexity $\chi(\mathbb{H}) \leq c(\dim \mathbb{G})$
- or $|AAA| \geq \min\left(\frac{|\langle A \rangle|}{c(K)}, K|A|\right)$

«there are no non trivial "sufficiently Zariski dense" approximate subgroups of simple algebraic groups»

Th (Green-Tao-B, Pyber-Szabo)
One can take $c(K) = K^c$ where $c = c(\dim G)$ in Hrushowki's thm.

« Every sufficiently Zariski-dense K-approximate subgroup of $G(k)$ is $K^{c(d)}$-controlled by the finite group it generates.

Cor $\exists\ \varepsilon = \varepsilon(d) > 0$ such that if G is a finite group of Lie type and rank at most d, then for every generating set A we have $|AAA| \geqslant \min\{|G|, |A|^{1+\varepsilon}\}$.

Open question: Classify approximate subgroups of A_n.

Pyber's example a generating approximate subgroup of A_n which is "non-trivial" (large and not almost all of A_n)

let $H \subset A_n$ given by $H = \langle \tau_{12}, \tau_{34}, \dots, \tau_{2m-1,2m} \rangle$

$n = 2m+1$

$H \simeq \mathbb{F}_2^m$. let $a = \sigma^2$ where $a = (1, 2, \dots, n)$ the long cycle

then $\langle a, H \rangle = A$ (because n is odd $\Rightarrow \langle a \rangle = \langle \sigma \rangle$)

Take $A = a^{\pm 1} \cup H \cup \{1\}$. It is a C approximate group (because H is almost normalized by a):

$a H a^{-1} = \langle \tau_{34}, \dots, \tau_{2m-1,2m}, \tau_{2m+1,1} \rangle$

and so $|a H a^{-1} \cap H| = \frac{|H|}{2}$)

In fact A is controlled by H.

Breuillard III/2

In Hrushovski's proof, a key point is an estimate of Larsen-Pink (95).

G a simple algebraic group, k = alg closed,

$T \subset G(k)$ a finite subgroup

then • either T is not sufficiently Zariski-dense

$(T \subset \mathbb{H}(k), \mathbb{H} \subsetneq G, \chi(\mathbb{H}) \leq c(G))$

• or $\forall V$ ~~variety~~ proper subvariety of G,

$$|T \cap V| \ll_{\chi(V)} |T|^{\frac{\dim V}{\dim G}}$$

<u>Approximate Larsen-Pink inequality</u>

G, k as above. $A \subset G(k)$ a finite K-approximate subgroup

then • either A is not sufficiently Zariski dense

$(A \subset \mathbb{H}(k), \mathbb{H} \subsetneq G, \chi(\mathbb{H}) \leq c(G))$

• or $\forall V$ proper subvariety of G,

$$|A \cap V| \leq_{\chi(V)} K^{C} |A|^{\frac{\dim V}{\dim G}}$$

where $C = c(\dim(G))$

<u>Idea for the proof of LP inequality</u>: induction on $\dim V$.

~~induction in dim V~~

$\dim V = 0$ or $\dim G$ $\rightarrow$ clear

$\dim V^- \leq \dim V^+$ $\quad A \cap V^-$ huge, $A \cap V^+$ huge.

<u>idea</u> use simplicity of G

$\hookrightarrow \exists a \in A$ s.t. $V^- \cdot a V^+ a^{-1}$ has $\dim > \dim V^+$

We would like to say $(A \cap V^-) \cdot a (A \cap V^+) a^{-1}$ has at least $|A \cap V^-| \, |A \cap V^+|$ elements.

Sketch of pf of thm modulo LP

$A \subset G(k)$, K-approx.

A is sufficiently Zariski-dense

Have to show: . either $|A| \leq K^C$

or $|A| \geq \frac{|\langle A\rangle|}{K^C}$

Idea ~~Take~~ ~~Act on the variety of tori~~

Act on the variety of tori: G/T, T = maximal torus in $G(k)$

2 consequences of LP ① if $a \in A$ is regular semi-simple

$$|A \cap Z(a)| \underset{K^C}{\asymp} |A|^{\frac{\dim Z(a)}{\dim G}}$$

$Z(a) = T$ maximal torus.

Use $V = T$ and $V = Cl(a)$ (conjugacy class of A).

$$\rightarrow |A^2 \cap Z(a)| \cdot |a^A| \simeq |A|.$$

by "gp action lemma".

Crucial lemma: $T \cap A \neq \{1\} \Rightarrow aTa^{-1} \cap A \neq \{1\}$