Breuillard II/1

We will prove the theorem (that there are no non-trivial approximate fields).

<u>Lemma</u> If A is a k approximate field, $\forall d \geq 1$ then $|Alg_d(A)| \leq k^{C(d)} |A|$.

where $C(d)$ is a constant depending on d.

Lemma (Balog - Szemeredi - Gowers - Tao)

If $A \subset G =$ group, if $|AA| \leq K|A|$, then $\exists A' \subset A$, $|A'| \geq K^{-c}|A|$ s.t. $(A' \cup A'^{-1} \cup \{1\})^2$ is a cK^c approximate group.

(II) Rough classification

Assume $|AA| \leq K|A|$.

If $K < \frac{3}{2}$, A group (at the beginning of lecture 1).

" $\frac{3}{2} \leq K < 2 \xrightarrow{\text{(Sanders)}} A \subset$ at most $C(K)$ cosets of a finite subgroup of $\# \leq C(K)|A|$.

$K = 2 \to$ arithmetic progressions

$A = [-N, N]$, $|A+A| \leq 2|A|$.

Question what ~~we~~ can we say about A if $K \geq 2$?

$G = \mathbb{Z}$: Th (Freiman, 60's) If $A \subset \mathbb{Z}$ is such that $|A+A| \leq K|A|$ then $A \subset P =$ "generalized arithmetic progression" of rank $O(K^c)$ and size

$$|P| \leq e^{O(K^c)}|A|.$$

A generalized arithmetic progression:

$$P = \pi\left(\prod_{i=1}^{d} [0, L_i]\right) \quad (\text{box in } \mathbb{Z}^d)$$

where $\pi : \mathbb{Z}^d \to \mathbb{Z}$ is affine. Ex ~~xxxx xxxx xxxx xxxx~~

d is called the rank of the progression.

$$P+P = \pi\left(\prod_{i=0}^{d} [0, 2L_i]\right) \Rightarrow |P+P| \leq 2^d |P|.$$

Breuillard II/2 | **Thm** (Green - Ruszaj, 90's)

$A \subset G$ abelian group, $|A+A| \leq K|A|$

$\Rightarrow A \subset H + P$, H finite subgroup

P a generalized arithmetic progression.

Def (Control). A, B in G.

Say A is K-controlled by B if

- $|B| \leq K|A|$
- $A \subset BX \cap XB$ for some X, $|X| \leq K$.

Goal Classify approximate subgroups up to control.

In nilpotent groups, $\exists$ K-approximate subgroups

~~+~~ controlled by generalized arithmetic progressions.

Th (B + Green). If A is K-approximate subgroup of $G = \overbrace{s\text{-step}}^{\text{torsion free}}$ nilpotent group then A is controlled by a nilprogression of step s and $r = O_s(K^{c_s})$.
(upto $e^{O_s(K^{c_s})}$)

"conjecture Helfgott-Lindenstrauss".
Suppose A is a K-approximate group
then A is $c(K)$ controlled by a set of the form HL
where H is a finite subgroup of G
L normalizes H and $H \backslash HL$ is a nil progression
of rank $\leq c(K)$, step $\leq c(K)$

Another conjecture: is $c(K) \leq O(K^c)$
not known even in the abelian case (Polynomial Freiman-Ruzsa conjecture).

Rem Helfgott-Lindenstrauss conjecture $\Rightarrow$ Gromov's thm on polynomial growth.
let $\Gamma = \langle S \rangle$ with polynomial growth: $|S^n| = O(n^c)$
$\exists\, n_k$ s.t. $|S^{2n_k}| \leq K |S^{n_k}|$ for some K depending on c.
$A = S^{2n_k}$ is a K_c-approximate group $\subset \Gamma$.
HL conj $\Rightarrow$ $\langle HL \rangle$ is virtually nilpotent,
because $\langle H \backslash HL \rangle$ is a nilpotent group.
$\Rightarrow$ $A \subset$ at most $c(K)$ cosets of a v nilp subgp $\langle HL \rangle$
$\Rightarrow$ Γ is v nilpotent.