Breuillard II/1

We will prove the theorem (that there are no non-trivial approximate fields).

Lemma If A is a K approximate field, $\forall d \geq 1$ then $|Alg_d(A)| \leq K^{C(d)} |A|$.

where $C(d)$ is a constant depending on d.

Pf Induction on d. Repeated applications of the Rusza covering lemma.

Pf of Thm key observation The map $A \times A \xrightarrow{\varphi_x} F$, $(a_1, a_2) \mapsto a_1 + x a_2$ is injective iff $x \notin \frac{A-A}{A-A} \subset Alg_2(A)$

claim $Alg_2(A)$ is a subfield of F.

Pf $B = A + (Alg_2(A) \cdot Alg_2(A)) A \subset Alg_8(A)$

$C = A + (Alg_2(A) + Alg_2(A)) A \subset Alg_{16}(A)$

A is a K approximate field $\Rightarrow |B|, |C| \leq K^C |A| < |A|^2$ $(C = C(16))$ if $|A| > K^C$

$\Rightarrow Alg_2(A) \cdot Alg_2(A) \subset Alg_2(A)$.

Sum-product phenomenon (Bourgain-Katz-Tao).

Thm $\exists \varepsilon$, if $A \subset \mathbb{F}_p$ then

$\max(|AA|, |A+A|) \geq \min(|\mathbb{F}_p|, |A|^{1+\varepsilon})$

idea of the proof If A satisfies $|AA| \leq K|A|$, $|A+A| \leq K|A|$, then Katz-Tao lemma shows that there exists $A' \subset A$, $|A'| \geq K^{-C}|A|$, s.t. $Alg_2(A')$ is an K^C-approximate field

Lemma (Balog-Szemeredi-Gowers-Tao)

If $A \subset G$ = group, if $|AA| \leq K|A|$, then $\exists A' \subset A$, $|A'| \geq K^{-c}|A|$ s.t. $(A' \cup A'^{-1} \cup \{1\})^2$ is a cK^c approximate group.

(II) Rough classification

Assume $|AA| \leq K|A|$.

If $K < \frac{3}{2}$, A group (at the beginning of lecture 1).

" $\frac{3}{2} \leq K < 2 \xrightarrow{\text{(Sanders)}} A \subset$ at most $C(K)$ cosets of a finite subgroup. of $\# \leq C(K)|A|$.

$K = 2 \to$ arithmetic progressions

$A = [-N, N]$, $|A+A| \leq 2|A|$.

Question what ~~we~~ can we say about A if $K \geq 2$?

$G = \mathbb{Z}$: Th (Freiman, 60's) If $A \subset \mathbb{Z}$ is such that $|A+A| \leq K|A|$ then $A \subset P$ = "generalized arithmetic progression" of rank $O(K^c)$ and size

$$|P| \leq e^{O(K^c)}|A|.$$

A generalized arithmetic progression:

$$P = \pi\left(\prod_{i=1}^{d} [0, L_i]\right) \quad (\text{box in } \mathbb{Z}^d)$$

where $\pi : \mathbb{Z}^d \to \mathbb{Z}$ is affine. Ex ~~xxxx xxxx xxxx xxxx~~

d is called the rank of the progression.

$$P+P = \pi\left(\prod_{i=0}^{d} [0, 2L_i]\right) \Rightarrow |P+P| \leq 2^d |P|.$$

Breuillard II/2 | **Thm** (Green – Ruzsa, 90's)

$A \subset G$ abelian group, $|A+A| \leq K|A|$

$\Rightarrow$ $A \subset H + P$, H finite subgroup

P a generalized arithmetic progression.

Def (Control). A, B in G.

Say A is K-controlled by B if

- $|B| \leq K|A|$
- $A \subset BX \cap XB$ for some X, $|X| \leq K$.

Goal Classify approximate subgroups up to control.

In nilpotent groups, $\exists$ K-approximate subgroups not controlled by generalized arithmetic progressions.

Take $A = B(n) \leftarrow$ Cayley graph ball in the Heisenberg graph ball.

$|B(2n)| \leq K|B(n)|$ $\qquad |B(n)| \sim n^d$.

$|B(n)^2|$ $\qquad B(n)$ is a K approximate group.

Def (nilpotent progression) $A \subset G$, $A = \pi(B)$ where $B = B(n)$ Ball of radius n for some CC – metric on $N_{r,s}$ (r = rank, s = nilpotency class)

$N_{r,s}$ = free nilpotent group on r generators, with s steps.

Carnot – Caratheodory

$\underline{\underline{\text{Th}}}$ (B + Green). If A is K-approximate subgroup of $G = \overbrace{s\text{-step}}^{\text{torsion free}}$ nilpotent group then A is controlled by a nilprogression of step s and $r = O_s(K^{c_s})$.
(upto $e^{O_s(K^{c_s})}$)

"conjecture Helfgott-Lindenstrauss".
Suppose A is a K-approximate group
then A is $C(K)$ controlled by a set of the form HL
where H is a finite subgroup of G
L normalizes H and $H \backslash HL$ is a nil progression
of rank $\leq C(K)$, step $\leq C(K)$

Another conjecture: is $C(K) \leq O(K^c)$
not known even in the abelian case (Polynomial Freiman-Ruzsa conjecture).

$\underline{\underline{\text{Rem}}}$ Helfgott-Lindenstrauss conjecture $\Rightarrow$ Gromov's thm on polynomial growth.
let $\Gamma = \langle S \rangle$ with polynomial growth: $|S^n| \leq O(n^c)$
$\exists\, n_k$ s.t. $|S^{2n_k}| \leq K |S^{n_k}|$ for some K depending on c.
$A = S^{2n_k}$ is a K_c-approximate group $\subset \Gamma$.
HL conj $\Rightarrow \langle HL \rangle$ is virtually nilpotent,
because $\langle H \backslash HL \rangle$ is a nilpotent group.
$\Rightarrow A \subset$ at most $c(K)$ cosets of a v nilp subgp $\langle HL \rangle$
$\Rightarrow \Gamma$ is v nilpotent.