

Breuil-Laud I

Approximate groups

- I Basic yoga of approximate groups.
- II Rough classification.
- III Application to expanders.

I G a group A a finite set in G

$$AA = \{a_1 a_2, a_1, a_2 \in A\}$$

What can we say about A if $|AA| \leq 10|A|$?

Exercise If $|AA| \leq k|A|$ where $k < \frac{3}{2}$ if and only if A is contained in a normalized coset of a finite subgroup $H \subset G$, and $|H| \leq k|A|$

Rem If $A = aH = Ha$, $AA = a^2H$ so \Leftarrow is obvious.

~~to the course of the exercise is obvious.~~

Rem $\frac{3}{2}$ sharp, $A = \{0, 1\} \subset \mathbb{Z} = G$.

T. Tao: $G =$ ambient group, $k \geq 1$ a parameter

Def (Approximate group). A finite subset $A \subset G$ is called a k -approximate subgroup of G if

① $\exists X \subset G, |X| \leq k$ s.t. $AA \subset XA$

② $1 \in A, A^{-1} = A$

Rem Since $A^{-1} = A$, we also get $AA \subset AX^{-1}$.

Ex ① 1-approximate subgroup = subgroup.

② H finite subgroup, $A = H \setminus Y, |Y| \leq \sqrt{|H|}$,

~~$|Y| \leq \sqrt{|H|}$~~ $Y = Y^{-1}$ then A is a 2-approximate subgroup.

③ $A \subset Z, A = [-N, N]$

$A + A = [-2N, 2N] \subset (A + N) \cup (A - N)$

therefore A is a 2-approximate subgroup.

Rem A k -approximate group $\Rightarrow \underbrace{A \dots A}_{\text{denoted } A^k} \subset X^{k-1} A$

then $|A|^k \leq K^{k-1} |A|$

Prop If A is any subset of G , and if we suppose

$|AAA| \leq K |A|$ for some K

then $|A^n| \leq K^{2n-2} |A| \quad \forall n \geq 1$

$B = (A \cup A^{-1} \cup \{1\})^2$ is a

$C K^C$ -approximate group (with C a absolute constant).

Rem $|AA| \leq K |A|$ does not imply that $|A^n| \leq C(n, K) |A|$.

Indeed ~~we~~ take H a large finite subgroup of G

take $a \in G$ such that $H \cap a H a^{-1} = \{1\}$.

Take $A = H \cup \{a\}$. Then $AA = H \cup H a \cup a H \cup \{a^2\}$

$|AA| \leq 2|A|$

But $AAA \supset H a H$ and $|H a H| = |H|^2$

therefore $|AAA| \geq |H|^2 \gg |A|$.

Proof of the Prop Ruzsa calculus.

2 key tools ① Ruzsa triangle inequality

② Ruzsa covering lemma.

Breuil-Laud I/2

Def Ruzsa distance. $A, B \subset G$ finite subsets.

$$d(A, B) = \log \frac{|AB^{-1}|}{\sqrt{|A||B|}} \geq 0$$

because $|AB^{-1}| \geq \max(|A|, |B^{-1}|) \geq \sqrt{|A||B|}$.

Ruzsa triangle inequality: $\forall A, B, C \subset G$,

$$d(A, B) \leq d(A, C) + d(C, B).$$

Moreover $d(A, B) = 0 \Leftrightarrow A$ and B are cosets of the same finite subgroup.

Pf Triangle inequality is equivalent to $|C| |AB^{-1}| \leq |AC^{-1}| + |C^{-1}B|$.

This is true: for any $x \in A^{-1}B$ pick $a_x \in A$ and $b_x \in B$ such that $x = a_x b_x^{-1}$.

$C \times A B^{-1} \rightarrow AC^{-1} \times C^{-1}B$ is injective.

$$(c, x) \mapsto (a_x c^{-1}, c b_x)$$

Equality case left as an exercise.

Pf of (a) of the Prop: $d(A^{n-1}, A^{-2}) \leq d(A^{n-1}, A) + d(A, A^{-1}) + d(A, A^{-2})$

and therefore

$$\frac{|A^{n+1}|}{\sqrt{|A^{n-1}||A^{-2}|}} \leq \frac{|A^n|}{\sqrt{|A^{n-1}||A|}} \cdot \frac{|A^{-2}|}{\sqrt{|A^{-1}||A|}} \cdot \frac{|A^3|}{\sqrt{|A||A^{-2}|}}$$

$$\Rightarrow \frac{|A^{n+1}|}{|A|} \leq \frac{|A^n|}{|A|} \frac{|A^2|}{|A|} \frac{|A^3|}{|A|} \leq k^2 \frac{|A^n|}{|A|} \quad \square$$

Ruzsa covering lemma

If $A, B \subset G$, $|A| \leq K|B|$

then $B \subset \bigcup_{i=1}^K A^{-1}Ax$ for some subset $X \subset B$, $|X| \leq K$.

Pf look at a maximal set of disjoint translates

A, Ab_1, \dots, Ab_m such that $AB = \bigcup A b_i$

Then $m \leq K$

$\forall b \in B$, $Ab \cap A b_i \neq \emptyset$ for some i

$\Rightarrow a_1 b = a_2 b_i \Rightarrow b \in A^{-1}A b_i$, $X = \{b_1, \dots, b_m\}$.

Properties of approximate groups.

① A, B are K -approximate groups.
then $A^2 \cap B^2$ is a K^6 -approximate group

② H any subgroup, A K -approximate subgroup
 $\Rightarrow A^2 \cap H$ is a K^4 -approximate group.

③ $\pi: G \rightarrow Q$ any group homomorphism.
then $\pi(A)$ is a K -approximate group.

(if $AA \subset XA$, $\pi(A)\pi(A) \subset \pi(X)\pi(A)$)

④ Suppose G acts on a set X , A a K -approximate subgroup of G ,

(a) $\forall x \in X$, $H_x \cong \text{Stab}_G(x)$

$|A| \leq |A \cdot x| / |A^2 \cap H_x| \leq K^3 |A|$

(b) partition into orbits: if $A \cdot y_1, \dots, A \cdot y_m$ are disjoint
and $\forall x \in X$, $A \cdot x$ intersects one $A \cdot y_i$ then

$X = \bigcup A^2 y_i$ = cover with multiplicity $\leq K^5$.

Approximate fields.

F a field. Def $A \subset F$ is a K -approximate field

if $A A + A \subset (A + X) \cap A X^{-1}$ where $X^{-1} = X^{-1}$ } 0 }

Def let $\text{Alg}_d(A) = \frac{\begin{matrix} \pm A^{\pm d} & \pm A^{\pm d} & \dots & \pm A^{\pm d} \end{matrix}}{\begin{matrix} \pm A^{\pm d} & \pm A^{\pm d} & \dots & \pm A^{\pm d} \end{matrix}}$

Thm "There are no non trivial approximate fields": $\exists C$ st.
if A is a K -approximate field then

• either $|A| \leq K^C$

• or $|FA| \leq K^C |A|$ $FA = \text{subfield}$

generated by A .

