Breuillard I — Approximate groups

(I) Basic yoga of approximate groups.
(II) Rough classification.
(III) Application to expanders.

(I) G a group A a finite set in G

$$AA = \{ a_1 a_2,\ \ a_1, a_2 \in A \}$$

What can we say about A if $|AA| \leq 10|A|$?

Exercise If $|AA| \leq k|A|$ where $k < \frac{3}{2}$ if and only if ~~then~~ A is contained in a normalized coset of a finite subgroup $H \subset G$, and $|H| \leq k|A|$

~~...~~ **Rem** If $A = aH = Ha$, $AA = a^2 H$ so $\Leftarrow$ is obvious. ~~so the converse of the exercise is obvious~~.

Rem $\frac{3}{2}$ sharp, $A = \{0, 1\} \subset \mathbb{Z} = G$.

T. Tao: G = ambient group, $k \geq 1$ a parameter

Def (Approximate group). A finite subset $A \subset G$ is called a K-approximate subgroup of G if

(1) $\exists X \subset G$, $|X| \leq K$ s.t. $AA \subset XA$

(2) $1 \in A$, $A^{-1} = A$

Rem Since $A^{-1} = A$, we also get $AA \subset AX^{-1}$.

Ex (a) 1-approximate subgroup = subgroup.

(b) H finite subgroup, $A = H \setminus Y$, $|Y| \leq \sqrt{|H|}$, ~~$|Y| \leq \sqrt{|H|}$~~ $Y = Y^{-1}$ then A is a 2-approximate subgroup.

(c) $A \subset \mathbb{Z}$, $A = [-N, N]$

$A + A = [-2N, 2N] \subset (A+N) \cup (A-N)$

therefore A is a 2-approximate subgroup.

Rem A K-approximate group $\Rightarrow \underbrace{A \dots A}_{\text{denoted } A^k} \subset X^{k-1} A$

then $|A^k| \leq K^{k-1} |A|$

Prop If A is any subset of G, and if we suppose

$|AAA| \leq K |A|$ for some K

then • $|A^n| \leq K^{2n-2} |A| \quad \forall n \geq 1$

• $B = (A \cup A^{-1} \cup \{1\})^2$ is a

cK^c-approximate group (with C a absolute constant).

Rem $|AA| \leq K |A|$ does not imply that

$|A^n| \leq C(n, K) |A|$.

Indeed take H a large finite subgroup of G

take $a \in G$ such that $H \cap aHa^{-1} = \{1\}$.

Take $A = H \cup \{a\}$. Then $AA = H \cup Ha \cup aH \cup \{a^2\}$

$|AA| \leq 2|A|$

But $AAA \supset HaH$ and $|HaH| = |H|^2$

therefore $|AAA| \geq |H|^2 \gg |A|$.

Proof of the Prop Ruzsa calculus.

2 key tools (1) Ruzsa triangle inequality

(2) Ruzsa covering lemma.

Breuillard I/2

Def Ruzsa distance. $A, B \subset G$ finite subsets.

$$d(A,B) = \log \frac{|AB^{-1}|}{\sqrt{|A||B|}} \geq 0$$

because $|AB^{-1}| \geq \max(|A|, |B^{-1}|)$
$\geq \sqrt{|A||B|}$.

Ruzsa triangle inequality: $\forall A, B, C \subset G$,

$$d(A,B) \leq d(A,C) + d(C,B).$$

Moreover $d(A,B) = 0 \Leftrightarrow A$ and B are cosets of the same finite subgroup.

Pf Triangle inequality is equivalent to

$$|C||AB^{-1}| \leq |AC^{-1}| \cdot |C^{-1}B|.$$

This is true: for any $x \in A^{-1}B$ pick $a_x \in A$ and $b_x \in B$ such that $x = a_x b_x^{-1}$.

$$C \times AB^{-1} \to AC^{-1} \times C^{-1}B$$
$$(c, x) \mapsto (a_x c^{-1}, c b_x)$$

is injective.

Equality case left as an exercice.

Pf of (a) of the Prop: $d(A^{n-1}, A^{-2}) \leq d(A^{n-1}, A) + d(A, A^{-1}) + d(A, A^{-2})$

and therefore
$$\frac{|A^{n+1}|}{\sqrt{|A^{n-1}||A^{-2}|}} \leq \frac{|A^n|}{\sqrt{|A^{n-1}||A|}} \cdot \frac{|A^{-2}|}{\sqrt{|A^{-1}||A|}} \cdot \frac{|A^3|}{\sqrt{|A||A^{-2}|}}$$

$$\Rightarrow \frac{|A^{n+1}|}{|A|} \leq \frac{|A^n|}{|A|} \frac{|A^2|}{|A|} \frac{|A^3|}{|A|} \leq k^2 \frac{|A^n|}{|A|} \qquad \square$$

Rusza covering lemma

If $A, B \subset G$, $|AB| \leq K|A|$
then $B \subset A^{-1}AX$ for some subset $X \subset B$, $|X| \leq K$.

Pf look at a maximal set of disjoint translates
$A, Ab_1 \dots Ab_n$ such that $AB \supset \bigcup Ab_i$
Then $n \leq K$
$\forall b \in B$, $Ab \cap Ab_i \neq \emptyset$ for some i
$\Rightarrow a_1 b = a_2 b_i \Rightarrow b \in A^{-1}Ab_i$, $X = \{b_1 \dots b_n\}$.

Properties of approximate groups.

1. A, B are K- approximate groups.
then $A^2 \cap B^2$ is a K^6- approximate ~~group~~ subgroup
2. H any subgroup, A K- approximate group.
$\Rightarrow A^2 \cap H$ is a K^4- approximate group.
3. $\pi : G \to Q$ any group homomorphism.
K approximate group.
then $\pi(A)$ is a K- approximate group.
(if $AA \subset XA$, $\pi(A)\pi(A) \subset \pi(X)\pi(A)$)
4. Suppose G acts on a set X, A a K approximate subgroup of G,
 (a) $\forall x \in X$, $H_x \widehat{=} Stab_G(x)$
 $|A| \leq |A.x| \, |A^2 \cap H_x| \leq K^3 |A|$
 (b) partition into orbits: if $Ay_1 \dots Ay_n$ are disjoint
 and $\forall x \in X$, $A.x$ intersects one $A.y_i$ then
 $X = \bigcup A^2 y_i$ = cover with multiplicity $\leq K^5$.

Breuillard I/3

Approximate fields.

F a field. **Def** $A \subset F$ is a K-approximate field if $AA + A \subset (A + X) \cap AX^{\times}$ where $X^{\times} = X \setminus \{0\}$

Def let $\mathrm{Alg}_d(A) = \dfrac{\cancel{\pm} A^{\pm d} \pm A^{\pm d} \dots \pm A^{\pm d}}{\cancel{\pm} A^{\pm d} \pm A^{\pm d} \dots \pm A^{\pm d}}$

Thm "There are no non trivial approximate fields": $\exists C$ s.t. if A is a K-approximate field then

- either $|A| \leq K^C$
- or $|F_A| \leq K^C |A|$ F_A = subfield generated by A.