Groups with rank in $[1, 2]$

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This is a joint work with Mikaël Pichot (Tokyo).
Let L be a locally finite metric graph, denote by S the set of singular vertices of L (where we call *singular* a vertex of valency at least 3) and let

$$\Phi_L = \{\alpha : [0, \pi] \hookrightarrow L,\ \alpha(0) \in S\}$$

be the set of paths of length π in L starting from a singular vertex, where two paths are considered distinct if they have distinct images in L.
We sometimes (improperly) refer to Φ_L as the *root system* of L, and to its elements as *roots*.
For a vertex $v \in S$ denote by q_v its *order* in L, that is, $q_v = \mathrm{val}_L(v) - 1$ where val_L is the valency. For $\alpha \in \Phi_L$ we write $q_\alpha = q_{\alpha(0)}$.

Definition

We call *rank* of an element $\alpha \in \Phi_L$ the number

$$\mathrm{rk}(\alpha) = 1 + \frac{N(\alpha)}{q_\alpha}$$

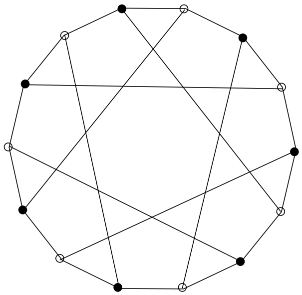
where $N(\alpha)$ is the number of path of length π in L distinct from α whose extremity coincide with that of α. Formally :

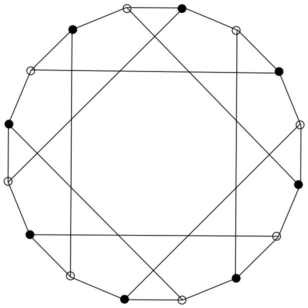
$$N(\alpha) = |\{\beta \in \Phi_L \mid \alpha \neq \beta, \alpha(0) = \beta(0), \alpha(\pi) = \beta(\pi)\}|.$$

Definition

Let X be as above and assume that $S(X, \Gamma)$ (set of orbits of singular verticies) is finite but non empty. We define the *local rank* of (X, Γ) to be the average

$$\mathrm{rk}(X, \Gamma) = \frac{1}{|S(X, \Gamma)|} \sum_{x \in S(X,\Gamma)} \mathrm{rk}(L_x).$$





Proposition

Let G be a spherical A_2 building of order q with one missing chamber. Then

$$\mathrm{rk}(G) = 2 - \frac{2}{(q+1)(q^2+q+1)-3}$$

for $q \geq 3$, while

$$\mathrm{rk}(G) = 2 - \frac{1}{8} = 1.875$$

if $q = 2$.

It is also instructive to examine the rank of this six spherical buildings :

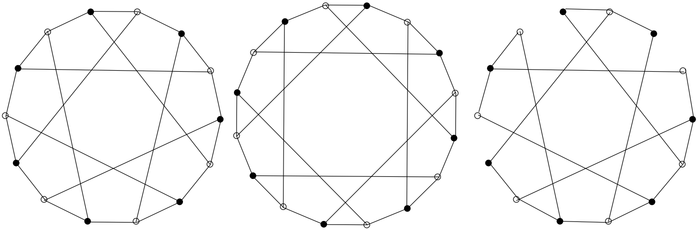
Proposition

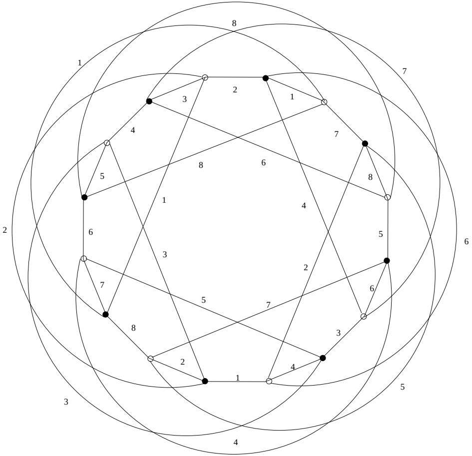
The rank of spherical A_2 buildings of order 2 with three chambers missing is given in the following table :

	G_1	G_2	G_3	G_4	G_5	G_6
$\mathrm{rk}(G_i) =$	$\frac{18}{11}$	$\frac{13}{8}$	$\frac{105}{64}$	$\frac{49}{31}$	$\frac{3}{2}$	$\frac{3}{2}$
$\mathrm{rk}(G_i) \approx$	1.636	1.625	1.640	1.58	1.5	1.5

This can be proved by a direct (but tedious) computation. In fact we have the following classification of roots :

Roots \ Buildings	G_1	G_2	G_3	G_4	G_5	G_6
rank 1	6	8	6	8	0	6
rank 3/2	36	32	34	36	60	48
rank 2	24	24	24	18	0	6
$\lvert \mathrm{Aut} \rvert$	6	2	2	2	12	6

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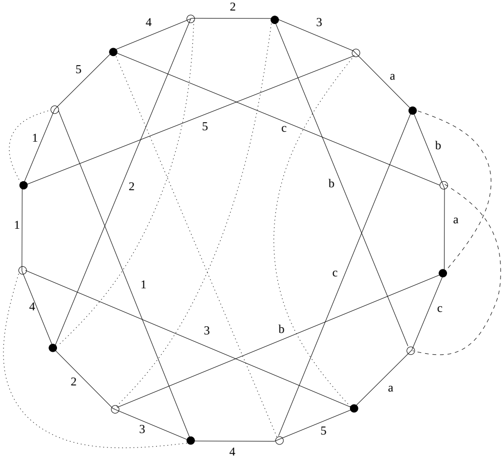
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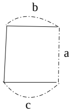
b
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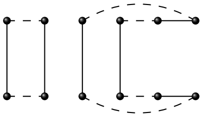
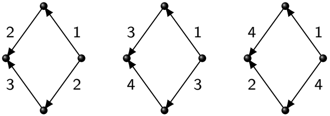
Take the Fano graph without two edges (you have 2 possibilities for that) and compute the rank of graphs obtained.

The extension invariant of the complex (X, Γ) described below is given by

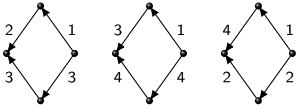


Consider the complex $\bar{V}$ defined by :

$V =$ 

This complex is hyperbolic : (building with one missing chamber)

$\dot{V}_1 =$

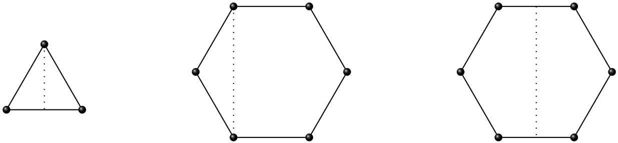


Theorem

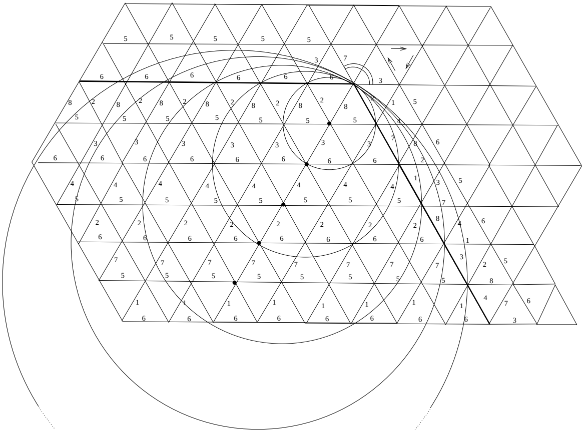
There exist two Euclidean buildings (X_0, Γ_0) and (X_1, Γ_1) of dimension 2 such that :

1. *both (X_0, Γ_0) and (X_1, Γ_1) have exactly one chamber missing;*
2. *both complexes X_i are hyperbolic, Γ_i acts freely on X_i transitively on vertices;*
3. *all links in X_0 and X_1 are isomorphic to a same graph G_6, which is a spherical building with 3 chambers missing;*
4. *there is a unique extension $(X_i, \Gamma_i) \rightsquigarrow (X_i', \Gamma_i')$ of (X_i, Γ_i) into a Euclidean building of type $\tilde{A}_2$, where the building X_i' is with a free action of a countable group Γ_i' which is transitive on vertices;*
5. *the buildings X_0' and X_1' are not isometric.*

In this case, we always have some $\mathbf{Z}^2$ in such a group.



If flats are not isolated, then there are some banchings... If a complex is not a building, but has a double system of bifurcation then it is mesoscopic. For example, B_4/Z is a mesoscopic group.



We can not answer in general, but we can give a positive answer for a class of random groups in a lattice model of triangle buildings : model of few chambers missing. [We use Cartwright-Steger lattices]
All examples with flats that we have computed, contain $\mathbf{Z}^2$...

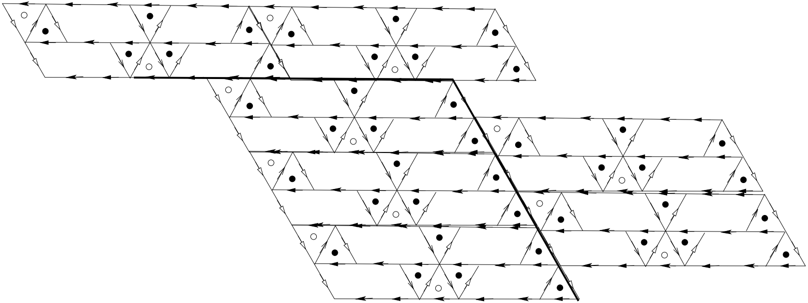
Proposition

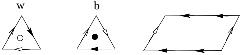
Let G be a spherical building of type A_2 and order q with a single chamber missing. Then

$$\lambda_1(G) = 1 - \frac{\sqrt{q + 1/4} + 1/2}{q + 1}.$$

In particular, $\lambda_1(G) > 1/2$ whenever $q \geq 5$.

Use the local criterion of Żuk for concluding.

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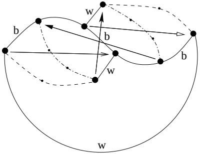
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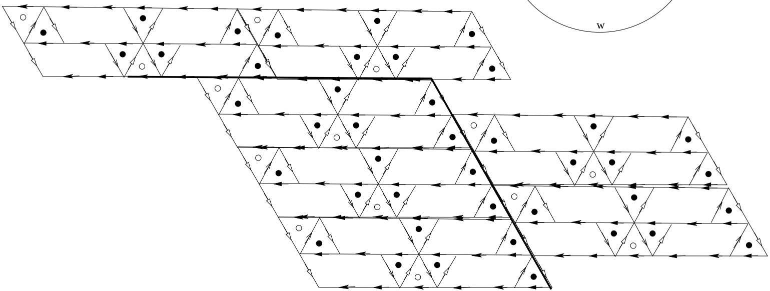

Exercise : Calculate the rank of this group. Property T, Haagerup ? Mesoscopic rank ?