Austin Irrational L^2-Betti numbers

Homology M - compact mfd CW-complex

C_p = $\mathbb{C}$-linear combination of p-balls in M

$\partial : C_p \to C_{p-1}$, $\partial\partial = 0$,

$Z_p = \ker \partial|_{C_p} \supset B_p = \mathrm{Im}\, \partial|_{C_{p+1}}$ $H_p(M, \mathbb{C}) = \frac{Z_p}{B_p}$

$\beta_p = \dim_{\mathbb{C}} H_p$. $\chi(M) = \sum_{p \geq 0} (-1)^p B_p$

Atiyah (early 70's) Γ discrete group, $\Gamma \curvearrowright M$ locally compact free proper cocompact

will virtually work with Γ-CW-complex M.

$\{p$-balls in $M\}$ is finite modulo Γ, $\forall p$

Now consider $\overline{C_p}$ = $\mathbb{C}$-linear combs of p-balls in M with square summable coefficients

Γ acts unitarily

$\partial : \overline{C_p} \to \overline{C_{p-1}}$ is a bounded operator, interwines the actions of Γ.

Take $Z_p = \ker \partial$, $B_p = \overline{\mathrm{Im}\, \partial}$, $H_p^{(2)}(M, \mathbb{C}) = Z_p / \overline{B_p} = Z_p \ominus \overline{B_p}$

unitary representation of Γ.

$\Gamma \curvearrowright \overline{C_p}$ by permuting basis

$= \rho^{\oplus n}$ for some $n \geq 1$, ρ = right regular representation.

Now $(\Gamma \curvearrowright H_p^{(2)}) \preceq \rho^{\oplus r}$, $r \geq 1$.

Murray-Von Neumann dimension: $\dim_\Gamma$ defined on subrepresentations of $\rho^{\oplus r}$, $r > 1$ such that $\dim_\Gamma 0 = 0$, $\dim_\Gamma \pi \oplus \pi' = \dim_\Gamma \pi + \dim_\Gamma \pi'$,

$\dim_\Gamma \pi = 0 \Rightarrow \pi = 0$, $\dim_\Gamma \rho = 1$.

In fact $\dim_\Gamma$ takes values in $[0, \infty[$.

Now $\beta_p^{(2)}(M, \Gamma) = \dim_\Gamma H_p^{(2)}$ L^2-Betti numbers.

N compact $\to$ $\chi(N) = \sum_{p \geq 0} (-1)^p \beta_p$.

Fact (Atiyah 78 ~~Cheeger Gromov~~). If $M = \tilde{N} \supset \pi_1(N) = \Gamma$, it is not necessary true that $\beta_p(N) = \beta_p^{(2)}(M, \Gamma)$, BUT $\chi(N) = \sum_{p \geq 0} (-1)^p \beta_p^{(2)}(M, \Gamma)$

Th If M is compact aspherical and $S^1 \curvearrowright N$ freely then $\chi(N) = 0$.

Quick sketch (Cheeger-Gromov) $\chi(N) = \sum_{p \geq 0} \beta_p^{(2)}(M, \pi_1(N))$

But the action of S^1 implies $\pi_1(N) \triangleright \mathbb{Z}$ and also $\beta_p^{(2)}(M, \pi_1(N))$ depends only on $\pi_1(N)$.

But $\pi_1(N) \triangleright \mathbb{Z}$ implies $\beta_p^{(2)}(\pi(M)) = 0 \ \forall p$.

Atiyah asked: can $\beta_p^{(2)}$ be irrational.

Th (A 09) Yes.

Other more recent results:

* For finitely generated Γ, can get any real (Pichot, Schick, Zuk)
* For fin presented Γ, can get all amenable computables number (Grabowski) ($\Leftarrow$ not known)
* For $\mathbb{Z}/m\mathbb{Z} \wr \mathbb{F}_n$ can get irrational algebraic values provided $n \geq 2$ and $m \geq 2n-1$

How about $\chi(M, \Gamma)$?

Older result: $\exists$ examples with $\beta_p^{(2)} \in \operatorname{fin}^{-1}(\Gamma) = \sum_{\substack{F \subset \Gamma \\ \text{finite}}} \mathbb{Z} \cdot \frac{1}{|F|}$ (Grigorchuk, Zuk, Dicks and Schick).

Reformulation in terms of groups / rep. theory.

$\Gamma \overset{\lambda}{\curvearrowright} \ell^2(\Gamma) \curvearrowleft \Gamma$ commute.

$p: \ell^2(\Gamma) \to \mathcal{H}_\pi$ $\quad \dim_\Gamma \pi = \langle \delta_e, P\delta_e \rangle$, $P \geq 0$. $\circledast$

If $\pi \leq \rho$

if $\pi \perp \pi'$, $\dim_\Gamma(\pi \oplus \pi') = \langle \delta_e, (p+p')\delta_e \rangle = \langle \delta_e, p\delta_e \rangle + \langle \delta_e, p'\delta_e \rangle$.

(Isom. invariance of $\dim_\Gamma$ depends on $\circledast$ being a trace).

If $\pi \leq \rho^{\oplus n}$, $P\,\ell^2(\Gamma)^{\oplus n} \to \mathcal{H}$

$p = (p_{ij})$ $\quad \dim_\Gamma \pi = \operatorname{tr}(\langle \delta_e, P_{ij}\delta_e \rangle)_{ij}$

Austin 2 Can get all such $P: \ell^2(\Gamma) \to \mathcal{H}_\pi$ from

$L\Gamma = \overline{\text{span}}^{\,wop}(\lambda(\Gamma))$.

In pratic given any $T \in (L\Gamma)_{sa}$. its spectral projections give examples.

<u>Reformulation</u> If $T \in (\mathbb{Q}\Gamma)_{sa}$ can $\dim_\Gamma \ker \lambda(T)$ be irrational?

Same question with matrices is equivalent to Atiyah's question because ∂ has coefficients in $\mathbb{Q}$ and in the other directions we can assume T has integral coefficients, and then can construct a CW- complex in several ways ---

If $F \subset \Gamma$ is a finite subgroup of Γ it is easy to get $\dim_\Gamma \ker T = 1 - \frac{1}{|F|} \in \text{fin}^{-1}(\Gamma)$: $T = \frac{1}{|F|}\left(\sum_{f \in F} [f]\right)$

$\text{Ker } \lambda(T) = \{ f \in \ell^2(\Gamma), \ \ \text{Ave}_{Fg}(f) = 0, \forall Fg \in \Gamma\backslash\Gamma \}$

Compute $\dim = 1 - \frac{1}{|F|}$

To get other values: $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^{\oplus\mathbb{Z}} \rtimes_\alpha \mathbb{Z}$.

Take $T = \frac{1}{4}\left[[(\delta_{-1}, -1)] + [0, -1] + [(\delta_1, 1)] + [(0, 1)] \right]$

Grigorchuk and Zuk show $\dim_\Gamma \ker \left(\lambda(T) - \frac{1}{2}\right) = \frac{1}{3} \in \text{fin}^{-1}(\Gamma)$

because

$\ker\left(\lambda(T) - \frac{1}{2}\right)$ is an infinite $\oplus$ of "finite subgroup examples". $\quad \overset{\shortparallel}{\sum_{n \geq 1} \frac{1}{2^{2n}}}$ ⊛

Later examples work with $\left(\overbrace{(\mathbb{Z}/2\mathbb{Z})^{\oplus G} / V}^{W}\right) \rtimes G$

where G = some group of exponential growth

V = translation invariant subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\oplus G}$.

Then the series ⊛ = an arbitrary real for non fin. pres. groups.

= any computable number for fin pres. groups.

Can re-write λ, ρ $\curvearrowright$ $L^2(\hat{W}) \otimes \ell^2(G)$
$= L^2(\hat{W} \times G)$ ---

The questions above are
Open for torsion-free groups.