

Austin Irrational L^2 Betti numbers

Homology Π -compact mfd CW-complex

$C_p = \mathbb{C}$ -linear combination of p -balls in Π

$$\partial: C_p \rightarrow C_{p-1} \quad \partial\partial = 0,$$

$$Z_p = \ker \partial|_{C_p} = B_p = \text{Im } \partial|_{C_{p+1}} \quad H_p(\Pi, \mathbb{C}) = \frac{Z_p}{B_p}$$

$$\beta_p = \dim_{\mathbb{C}} H_p \quad \chi(\Pi) = \sum_{p \geq 0} (-1)^p \beta_p$$

Atiyah (early 70's) Γ discrete group, $\Gamma \backslash \Pi$ locally compact free proper cocompact

will virtually work with Γ -CW-complex M .

$\{p\text{-balls in } M\}$ is finite modulo Γ , $\forall p$

Now consider $\bar{C}_p = \mathbb{C}$ -linear combs of p -balls in Π with square summable coefficients

Γ acts unitarily

$\partial: \bar{C}_p \rightarrow \bar{C}_{p-1}$ is a bounded operator, intertwines the actions of Γ .

Take $Z_p = \ker \partial$, $B_p = \overline{\text{Im } \partial}$, $H_p^{(2)}(\Pi, \mathbb{C}) = Z_p / \overline{B_p} = Z_p \otimes \overline{B_p}$
unitary representation of Γ .

$\Gamma \backslash \bar{C}_p$ by permuting basis

$= \rho^{\otimes n}$ for some $n \geq 1$, $\rho =$ right regular representation.

Now $(\Gamma \backslash H_p^{(2)}) \leq \rho^{\otimes n}$, $n \geq 1$.

Murray-Von Neumann dimension: \dim_{Γ} defined on subrepresentations of $\rho^{\otimes n}$, $n \geq 1$ such that $\dim_{\Gamma} 0 = 0$, $\dim_{\Gamma} \pi \oplus \pi' = \dim_{\Gamma} \pi + \dim_{\Gamma} \pi'$, $\dim_{\Gamma} \pi = 0 \Rightarrow \pi = 0$, $\dim_{\Gamma} \rho = 1$.

In fact \dim_{Γ} takes values in $[0, \infty[$.

Now $\beta_p^{(2)}(M, \Gamma) = \dim_{\Gamma} H_p^{(2)}$ L^2 -Betti numbers.

N compact $\rightarrow \chi(N) = \sum_{p \geq 0} (-1)^p \beta_p$.

Fact (Atiyah 78). If $\Pi = \tilde{N} \ni \pi_1(N) = \Gamma$, it is not necessarily true that $\beta_p(N) = \beta_p^{(2)}(M, \Gamma)$, BUT $\chi(N) = \sum_{p \geq 0} (-1)^p \beta_p^{(2)}(M, \Gamma)$

Th If Π is compact aspherical and $S^1 \curvearrowright N$ freely then $\chi(N) = 0$.

Quick sketch (Cheeger-Gromov) $\chi(N) = \sum_{p \geq 0} \beta_p^{(2)}(\Pi, \pi_1(N))$

But the action of S^1 implies $\pi_1(N) \supset \mathbb{Z}$ and also $\beta_p^{(2)}(\Pi, \pi_1(N))$ depends only on $\pi_1(N)$.

But $\pi_1(N) \supset \mathbb{Z}$ implies $\beta_p^{(2)}(\pi_1(N)) = 0 \quad \forall p$.

Atiyah asked: can $\beta_p^{(2)}$ be irrational.

Th (AOG) Yes.

Other more recent results:

* For finitely generated Γ , can get any real (Pichot, Schick, Zuk)

* For fin presented Γ , can get all computable numbers (Orbowski) (amenable)
(\Leftarrow not known)

* For $\mathbb{Z}/m\mathbb{Z} \wr \mathbb{F}_n$ can get irrational algebraic values provided $n \geq 2$ and $m \geq 2n-1$

How about $\chi(\Gamma, \Gamma)$?

Older result: \exists examples with $\beta_p^{(2)} \in \text{fin}^{-1}(\Gamma) = \sum_{F \subset \Gamma} \frac{1}{|F|}$ finite
(Grigorchuk, Zuk, Dicks and Schick).

Reformulation in terms of groups / rep. theory.

$\Gamma \curvearrowright \ell^2(\Gamma) \supset \Gamma$ commute.

If $\pi \leq \rho$ $\rho: \ell^2(\Gamma) \rightarrow \mathcal{H}_\pi$ $\dim_\Gamma \pi = \langle \delta_e, \rho \delta_e \rangle$, $\rho \geq 0$.
if $\pi \perp \pi'$, $\dim_\Gamma (\pi \oplus \pi') = \langle \delta_e, (\rho + \rho') \delta_e \rangle = \langle \delta_e, \rho \delta_e \rangle + \langle \delta_e, \rho' \delta_e \rangle$.

(Isom. invariants of \dim_Γ depends on \otimes being a trace).

If $\pi \leq \rho^{\otimes 2}$, $\rho: \ell^2(\Gamma)^{\otimes 2} \rightarrow \mathcal{H}$

$\rho = (\rho_{ij})$ $\dim_\Gamma \pi = \text{tr}(\langle \delta_e, \rho_{ij} \delta_e \rangle)_{ij}$

Austin 2] Can get all such $P: \ell^2(\mathbb{T}) \rightarrow \mathcal{H}_\pi$ from

$$LT = \overline{\text{span}}^{\text{wop}} (\lambda(\mathbb{T})).$$

In practice given any $T \in (LT)_{sa}$. its spectral projections give examples.

Reformulation If $T \in (OT)_{sa}$ can $\dim_{\mathbb{T}} \ker \lambda(T)$ be irrational?

Same question with matrices is equivalent to Atiyah's question because ∂ has coefficients in \mathbb{Q} and in the other directions we can assume T has integral coefficients, and then can construct a CW-complex in several ways ---

If $F \subset \mathbb{T}$ is a finite subgroup of \mathbb{T} it is easy to get $\dim_{\mathbb{T}} \ker T = 1 - \frac{1}{|F|} \in \text{fin}^{-1}(\mathbb{T})$: $T = \frac{1}{|F|} (\sum_{f \in F} [f])$

$$\text{Ker } \lambda(T) = \{f \in \ell^2(\mathbb{T}), \text{Ave}_{Fg} (f) = 0, \forall Fg \in \mathbb{T} \setminus \mathbb{T}\}$$

Compute $\dim = 1 - \frac{1}{|F|}$

To get other values: $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^{\otimes \mathbb{Z}} \rtimes \mathbb{Z}$.

$$\text{Take } T = \frac{1}{4} [[\delta_{-1}, -1] + [0, -1] + [(\delta_1, 1)] + [(0, 1)]]$$

Grigorchuk and Zuk show $\dim_{\mathbb{T}} \ker (\lambda(T) - \frac{1}{2}) = \frac{1}{3} \in \text{fin}^{-1}(\mathbb{T})$ because

$\ker (\lambda(T) - \frac{1}{2})$ is an infinite \oplus of "finite subgroup examples". $\sum_{n \geq 1} \frac{1}{2^{2n}} \otimes$

Later examples work with $(\overbrace{(\mathbb{Z}/2\mathbb{Z})^{\otimes G}}^W) \rtimes G$

where $G =$ some group of exponential growth

$V =$ translation invariant subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\otimes G}$.

Then the series $\otimes =$ an arbitrary real for a non fin. pres. groups.
 $=$ any computable number for fin pres. groups.

Can re-write $\lambda, \rho \int L^2(\hat{W}) \otimes L^2(G)$
 $= L^2(\hat{W} \times G) \quad \dots$

The questions above are
Open for torsion-free groups.