SKETCHES IN HIGHER CATEGORY THEORY

By

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Abstract

A sketch, in the sense of Charles Ehresmann, provides the data needed to specify a type of mathematical structure. The category of structures for a given sketch has good properties assured by the existence of reflection functors from presheaf categories. For example, a Grothendieck site is an example of a sketch and the reflection functor assigns the sheaf associated to a presheaf.

The present thesis proposes a generalisation of sketch for higher categories. The motivation for this comes from homotopy theory rather than universal algebra. For the requisite homotopy structure on a category, we introduce *vertebral categories* and *spinal categories*, rather than starting with a Quillen model category where the weak equivalences are part of the data. For us, the weak equivalences are defined from the vertebrae in much the same way as they are constructed from discs and spheres in topology. The categories of structures for our generalised sketches are categories of fibrant objects in the usual cases. Categories of stacks and spectra are examples. Moreover, our construct of the reflection functor is an extension of the small object argument of Quillen.

Finally, using our algorithm for constructing weak equivalences, we show that Grothendieck's ∞-groupoids form a spinal category. By combining all the results developed in the present thesis, future work will aim at proving that the category of ∞-groupoids admits a Quillen model structure and satisfies the *Homotopy Hypothesis* (conjectured by Grothendieck in 1983 and still unproved).

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Chapter 1

Introduction

1.1. Presentation

1.1.1. Goals of the thesis. Although there are many ways of telling a story, it is always useful to have a storyline in mind. I will therefore introduce the present work by giving several questions, which the reader may use as a breadcrumb trail to make sense of the different chapters and better appreciate the reading of this text. In this respect, the present thesis addresses the following three main questions related to homotopy theory and (higher) category theory.

1. *Is there a systematic way of producing weak equivalences for general homotopy theories? The intended description should be intuitive.*
2. *Is there a systematic way of describing the colimits of a category of models for a sketch? The intended description should allow practical computation.*
3. *Is the Homotopy Hypothesis (Grothendieck,* [**24**]*) true?*

The following sections further motivate and discuss the previously stated questions.

1.1.1.1. *Motivations of the first problem.* One of the first model categories given by D. Quillen in his monograph *Homotopical algebra* (1967, [**38**]) was that of simplicial sets. As required by the definition of model categories, Quillen gave three classes of morphisms called fibrations, trivial fibrations and weak equivalences. On the one hand, fibrations and trivial fibrations were explicitly described in the category of simplicial sets via the notion of right lifting property with respect to small sets of morphisms. On the other hand, weak equivalences were not described in the category of simplicial sets, but induced from the canonical weak equivalences of topological spaces via the underlying geometric realisation.

Such a presentation of the model category of simplicial sets shows that weak equivalences are quite difficult to describe – or in fact, not quite understood – and need external knowledge to be handled. The question that arises is the following: *is it possible to have a sound understanding of what weak equivalences look like in the category of simplicial sets without using any other language than that of simplicial sets?*

The answer is given in an article of Dugger and Isaksen [**13**] wherein a diagrammatic language is used. This language seems to have been known by various homotopy theorists (see *ibid*, [**39**, Lemma 7.5.1] or [**33**, Proposition 8]) but has never been exploited to give rise to a new point of view of abstract homotopy theory – an outlook that would be more intuitive and allow combinatorial constructions.

This is exactly one of the goals of the present thesis, namely to propose a language allowing one to handle most everyday definitions of weak equivalence in an intuitive and combinatorial way. The intuitive aspect will lie in the fact that its formulation is similar to that of the notion of bijection while the combinatorial aspect will lie in the fact that the weak equivalences will be described as morphisms that may be 'cofibrantly generated'. It is worth noting that this language will be expressed in a greater generality than in *loc. cit.* and the philosophy in which it will be utilised will fundamentally differ from these papers as no model structure will be assumed to exist from the beginning. On the contrary, our homotopy theory will systematically arise from the particular shape of our diagrams of 'cofibrations'. It is when the diagrams have nice properties that this homotopy theory will become a model category or a category of fibrant objects.

One may wonder if such a description may be used to understand the very general notion of weak equivalence attached to model categories. This is answered in the affirmative later in the introduction by using the result of [**39**] previously cited. To some extent, this reformulation of weak equivalences of (cofibrantly generated) model categories may be seen as a general statement of the Whitehead Theorem for CW-complexes.

However, note that this general Whitehead Theorem will only hold for weak equivalences between fibrant objects while the scope of the language developed herein is to allow the description of weak equivalences that are not necessarily defined between fibrant objects.

1.1.1.2. *Motivations of the second problem.* One of the most important and difficult concepts of modern algebra is the notion of colimit. The use of these objects most often requires a sound combinatorial description of them that diverges from the way that they are formally defined. In the case of a category of models for a sketch $\mathtt{S}$, say $\mathbf{Mod}(\mathtt{S})$, colimits are computed as the images of colimits in the presheaf category over $\mathtt{S}$ via the reflection functor.

$$\underset{\textit{colimit in } \mathbf{Psh}(\mathtt{S})}{\mathrm{col}\, F} \qquad \underset{\textit{reflection}}{\mathbf{Psh}(\mathtt{S}) \xrightarrow{L} \mathbf{Mod}(\mathtt{S})} \qquad \underset{\textit{colimit in } \mathbf{Mod}(\mathtt{S})}{L(\mathrm{col}\, F)}$$

This means that if one has a combinatorial description of the reflection functor, then one has a combinatorial description of the colimits of $\mathbf{Mod}(\mathtt{S})$. The remaining question is the following: *how can one describe a reflection functor in the most explicit way?*

The work of Freyd and Kelly [**19**] showed that one way of defining a reflection is to express the objects of $\mathbf{Mod}(\mathtt{S})$ as solutions of a right lifting property. More specifically, the object of $\mathbf{Mod}(\mathtt{S})$ are presheaves satisfying the property that some of the canonical functions that they induce are bijections.

$$\underset{\textit{cocone in } \mathtt{S}}{\{d_i : x_i \to x\}_i} \qquad \underset{\textit{a model}}{\xmapsto{\quad F \quad}} \qquad \underset{\lim_i F(d_i) \textit{ is a bijection}}{F(x) \xrightarrow{\cong} \lim_i F(x_i)}$$

This condition is a particular example of 'descent condition' above certain cocones of the small category $\mathtt{S}$. The right lifting property then follows from a reformulation of this particular descent condition via the Yoneda embedding.

Interestingly, this is not the only place where descent conditions are used to define constructions that looks like colimits. A notable example is the one of stacks. Stacks are a generalisation of manifolds, which may be seen as 'pseudo-gluings' of open sets coming from the Euclidian topology. Once sent to the language of stacks, the pseudo-gluings are formulated in terms of a descent condition, which intrinsically defines the concept of stack. The descent condition this time requires certain canonical morphisms to be equivalences of small categories.

$$\begin{array}{ccc} \{d_i : U_i \to U\}_i & \xmapsto{\quad F \quad} & F(U) \xrightarrow{\simeq} \lim_i F(U_i) \\ \textit{open covering of } U & \textit{a stack} & \lim_i F(d_i) \textit{ is an equivalence} \end{array}$$

Our previous discussion here suggests the following question: *May the descent condition of stacks be expressed in terms of a right lifting property so that it gives rise to a reflection functor?* This would ideally provide a way of combinatorially understanding a pseudo-colimit of stacks, which, to some extent, could be used for understanding the construction of moduli spaces – one of the other examples encompassed by stacks. The answer to such a question will be discussed in the Chapter 5, where I will show how to derive from a descent condition a combinatorial construction that turns out to be a reflection functor in practice.

Such a construction is made possible thanks to the definition of weak equivalence given herein. More specifically, in some simple cases, this definition may be expressed in terms of a right lifting property in the arrow category where the descent conditions live.

$$\begin{array}{ccc} F(U) \xrightarrow{\sim} \lim_i F(U_i) & \Leftrightarrow & F(U) \xrightarrow{\sim} \lim_i F(U_i) \\ \textit{weak equivalence} & & \textit{solution of a right lifting property} \end{array}$$

Since an abstract way of presenting a descent condition is to require the underlying arrow of the condition to be a weak equivalence (for a given homotopical context), any descent condition may be seen as the solution of a right lifting property, exactly as in the case of the paper of Freyd and Kelly.

The fact that these weak equivalences are 'cofibrantly generated' (in the sense of the Whitehead Theorem) will then enable us to use a broad generalisation of Quillen's small object argument to eventually construct what could be seen as the reflection functor. In addition to giving an inductive and objectwise presentation of the reflection, this point of view does not make use of the notions of generator, cowell-poweredness, subobject or proper factorisation system as required in the paper [**19**] of Freyd and Kelly.

Finally, the ability to express descent-like condition in terms of right lifting properties will enable us to express sheaves, spectra, models for a sketch or even flabby sheaves as fibrant objects in a certain homotopical theory. The introduced methods obviously paves the way for possible charaterisation of stacks or, more generally, (∞, n)-stacks up to generalisation of the involved structures. All this will be achieved by generalising the notion of sketch and their models in order to see all the previously listed objects as functors satisfying a general descent condition.

1.1.1.3. *Motivations of the third problem.* In 1984, A. Grothendieck published a manuscrit called *Esquisse d'un programme* [**25**] wherein is sketched a research plan gathering the main questions on which he spent most of his thinking time since 1955. The program aimed at further developing Galois Theory along Homotopy Theory. These two subjects had earlier been related in the work of C. Chevalley (1946, [**7**]), where covering spaces were classified by universal ones up to quotient by groups of permutations on the fibres. This was the Galois Theory of covering spaces. Later, in 1963, the idea of universal covering space lead J. Stasheff [**40**] to his classification theorem in which he explicitly constructed a space B such that the set of homotopy classes of continuous maps from a CW-complex X to B is naturally isomorphic to the set of fibre homotopy equivalence classes of Hurewicz fibrations over X with fixed fibre space F.

$$[X, B] \cong \mathrm{L}_F(X)$$

Interestingly, when F was a discrete group G, the space B corresponded to a $K(G,1)$ space defined by Eilenberg and MacLane in 1945 and 1950 (see [**16, 17**]). For every positive integer, the Eilenberg-MacLane spaces $K(G,n)$ are defined as CW-complexes such that their n-th homotopy groups are isomorphic to the group G. Such spaces are known to satisfy the property that the set of homotopy classes of continuous maps from a CW-complex X to $K(G,1)$ is naturally isomorphic to the n-th cohomology group of X with coeffients in G.

$$[X, K(n,G)] \cong \mathrm{H}^n(X,G)$$

In the case where n is equal to 1, the two previous isomorphisms are reminiscent of the definition of non-abelian sheaf cohomology of degree 1 introduced by Grothendieck in 1955 (see [**23**]), whose cohomology group is defined in terms of isomorphism classes of $\mathcal{G}$-torsors for a sheaf group $\mathcal{G}$.

$$\mathrm{H}^1(X,\mathcal{G}) \cong \mathrm{Tor}_{\mathcal{G}}(X)$$

It is then in the early 1960's that Grothendieck asked J. Giraud for a generalisation of sheaf cohomology to the non-abelian case in degree 2. Giraud achieved the generalisation by defining, in its manuscripts of 1966 and 1971 (see [**21, 22**]), the notions of stack and gerbe. These objects were not sheaves anymore, but pseudo-functors from a small category to the category of small groupoids. The reason as to why one leaves the world of groups to go to the world of groupoids when going to higher cohomology groups is one of the questions addressed by Grothendieck in *Esquisse d'un programme*, where he wonders what the natural notion of coefficients for higher cohomogoly groups should be. This is one of the reasons that motivated his definition of ∞-groupoid in his manuscript *À la Poursuite des Champs* [**24**]. If we leave the world of sheaf cohomology to go back to the world of singular cohomology, this suggests the idea that one could retrieve Stasheff's classification theorem in higher dimension for any ∞-groupoid G.

$$[X, B_n] \cong \mathrm{H}^n(X,G) \cong \pi\mathrm{Hom}(C_n(X), G)$$

From the point of view of Quillen's theory of model categories, the preceding isomorphism may ideally be seen as a Quillen equivalence between the homotopy category of topological spaces and the homotopy category of ∞-groupoids. Grothendieck expected that such an equivalence existed. This is sometimes called the *Homotopy Hypothesis*. A detailed proof of the existence of such an equivalence has yet to be completed.

The present thesis gives a beginning of answer by providing the category of Grothendieck's ∞-groupoids with a homotopical structure called *spinal category*. A spinal category turns out to define a model structure in the case where a certain proposition is true. Precisely, this proposition relies on a factorisation stemming from Quillen's small object argument and thereby requires to combinatorially handle a colimit of ∞-groupoids. This therefore brings us back to the previous question, which asked for an efficient way of computing colimits in a category of models for a sketch, which the category of Grothendieck's ∞-groupoids exactly is. Future work will aim at finishing the proof of the *Homotopy Hypothesis* by combining all the results developed in the present thesis. A sketch of this proof is given at the very end of the thesis.

1.1.2. Results, methods and philosophy.

1.1.2.1. *Concept of weak equivalence in higher category theory.* A general notion of weak equivalence for higher category theories, which has been around for a while, is described in a paper of J. Baez and M. Shulman [**6**]. The idea is that a weak equivalence is a morphism that is a 'surjection' in every 'dimension'. The goal of the present section is to broadly describe such a definition and use this description to slowly drift towards the version considered in this thesis.

In this respect, we need to define what we will loosely call a basic higher categorical theory. Thus, suppose to be given a category $\mathcal{C}$ whose objects X consist of sets X_n for every non-negative integer n. The elements of a set X_n will be called an *n-cells*. In addition, suppose that every n-cell in X_n is equipped with a structure of arrow $f : x \to_n y$ between two $(n-1)$-cells x and y in X_{n-1} for every $n > 1$. If one denotes by s_n and t_n the functions that map an n-cell $f : x \to_n y$ to its source x and target y, respectively, then the object X comes along with an ω-graph as follows.

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} X_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} X_3 \underset{t_3}{\overset{s_3}{\leftleftarrows}} \dots$$

To define the notion of weak equivalence, we need to introduce the notion of 'invertible n-cell' for every positive integer n. These cells have to be as weakly invertible as possible, which makes their definition quite subtle. For instance, in the paper of Metayer, Lafont and Worytkiewicz [**33**], in which they define a canonical model structure for strict ω-categories, the definition of these invertible cells is stated by co-induction. This is the type of definition that I am going to use for this first part. To do so, we will need (horizontal) compositions as well as identities at any dimension of an object in $\mathcal{C}$. Thus, suppose that any object X of $\mathcal{C}$ is equipped, for every integer $n > 1$, with

- ▷ a reflexive n-cell $\mathrm{id}_x : x \to x$ for every $(n-1)$-cell x in X;
- ▷ a composition operation $(\_\circ_n\_)$ mapping any pair of n-cells $f : y \to_n z$ and $g : x \to_n y$ in X to another one $f \circ_n g : x \to_n z$ in X;

An n-cell $f : x \to_n y$ in X is said to be *weakly invertible* if it is equipped with

1) an n-cell $f' : y \to_n x$ in X, called its *inverse*;
2) an $(n+1)$-cell $\sigma_f : f' \circ_n f \to_{n+1} \mathrm{id}_x$ that is weakly invertible;
3) an $(n+1)$-cell $\tau_f : f \circ_n f' \to_{n+1} \mathrm{id}_y$ that is weakly invertible;

Such a notion of cell comes along with a binary relation at every dimension of X as follows; for every non-negative integer n, two n-cells x and y are said to be *ω-equivalent*, which will be denoted by $x \sim y$, if there exists a weakly invertible $(n+1)$-cell from x to y. An easy co-inductive argument shows that such a binary relation is symmetric. With a little more structure on the objects of the category $\mathcal{C}$, this relation is most often reflexive – thanks to the identity cells – and transitive – thanks to the composition operations.

Along with this relation, we shall need another relation, which will stand for the notion of parallelism in higher category theory. For every non-negative integer n, two $(n+1)$-cells f and g are said to be *parallel*, which will be denoted by $f \,||\, g$, if they have the same source and target, respectively.

$$f : x \to_{n+1} y \qquad \& \qquad g : x \to_{n+1} y \qquad \Rightarrow \qquad f \,||\, g$$

For convenience, any pair of 0-cells f and g will be regarded as *parallel*, which will also be written as $f \,||\, g$.

We now have all the material to define a weak equivalence in the sense of [**6, 33**]. For consistency, a morphism $f : X \to Y$ in $\mathcal{C}$ will be supposed to be compatible with the dimensional structure of the objects of $\mathcal{C}$. Specifically, we will assume that f is equipped, for every non-negative integers n, with a function $f : X_n \to Y_n$ preserving the sources and targets of the n-cells of X to those of Y, respectively.

Definition 1.1 (Surjections)**.** A morphism $f : X \to Y$ is said to be a 0-*surjection* if for every 0-cell z in Y, there exists a 0-cell w in X for which an ω-equivalence $f(w) \sim z$ holds.

Definition 1.2 (Higher surjections)**.** A morphism $f : X \to Y$ is said to be an $(n+1)$-*surjection*, for some $n \geq 0$, if for every pair of parallel n-cells $x \,||\, y$ in X and $(n+1)$-cell $z : f(x) \to_{n+1} f(y)$ in Y, there exists an $(n+1)$-cell $w : x \to_{n+1} y$ in X for which an ω-equivalence $f(w) \sim z$ holds.

Definition 1.3 (Weak equivalences)**.** A morphism $f : X \to Y$ is said to be a *weak equivalence* if it is an n-surjection for every non-negative integer n.

Remark 1.4 (Injectivity)**.** In both papers [**6, 33**], it is noticed that the notion of weak equivalence implies that of injection with respect to the relation $\sim$. This happens in our case if the morphisms of $\mathcal{C}$ preserve the composition operations and identity cells. In this case, the idea is that if $f : X \to Y$ denotes a weak equivalence, then for every parallel pair of n-cells $x \,||\, y$ in X such that there is an invertible $(n+1)$-cell

$$z : f(x) \to_{n+1} f(y)$$

in Y, the first implication of Definition 1.2 implies that the morphism $f : X \to Y$

- lifts the cell $z : f(x) \to_{n+1} f(y)$ to a cell $w : x \to_{n+1} y$ in X;
- lifts the inverse $z' : f(y) \to_{n+1} f(x)$ to a cell $w' : y \to_{n+1} z$ in X;
- lifts the cell $\sigma_z : f(a' \circ z') \to_{n+1} \mathrm{id}_{f(x)}$ and its associated inverse to X;
- lifts the cell $\tau_z : f(z' \circ a') \to_{n+1} \mathrm{id}_{f(y)}$ and its associated inverse to X;
- etc.

eventually lifting the ω-equivalence $f(x) \sim f(y)$ in Y to another one $x \sim y$ in X.

It is not difficult to understand that the actual intuition behind the definition of weak equivalence is to recover that of bijection (up to homotopy). Let us see some examples to perceive this intuition.

Example 1.5. The category of sets, which will be denoted by **Set** throughout this text, is an example of higher category theory in which a set X is associated with the subsequent graph.

$$\begin{tikzcd} X & X \arrow[l, shift right, "\mathrm{id}_X"'] \arrow[l, shift left, "\mathrm{id}_X"] & X \arrow[l, shift right, "\mathrm{id}_X"'] \arrow[l, shift left, "\mathrm{id}_X"] & X \arrow[l, shift right, "\mathrm{id}_X"'] \arrow[l, shift left, "\mathrm{id}_X"] & \dots \arrow[l, shift right, "\mathrm{id}_X"'] \arrow[l, shift left, "\mathrm{id}_X"] \end{tikzcd}$$

This graph is equipped with natural identities $x : x \to_n x$ for every $x \in X$ and $n \geq 0$ as well as obvious compositions between identities – which are the only possible composable arrows for the previous structure.

$$x : x \to_n x \qquad \& \qquad x : x \to_n x \quad \Rightarrow \quad x \circ_n x := x$$

Since an arrow $x \to_n x$ emulates the identity relation $x = x$, it is not hard to see that an ω-equivalence in X is also given by an equality relation and a 0-surjection is exactly a surjection of sets. On the other hand, a 1-surjection is a function $f : X \to Y$ such that for every 1-cell $z : f(x) = f(y)$ in Y, there exists an 1-cell $w : x = y$ in X for which an ω-equivalence $f(w) = z$ holds. One may quickly notice that this statement is exactly the definition of an injection. However, for any positive integer $n \geq 1$, a $(n+1)$-surjection must be a function $f : X \to Y$ such that for every pair of parallel n-cells $x \,||\, x$ in X and $(n+1)$-cell $f(x) : f(x) = f(x)$ in Y, there exists an $(n+1)$-cell $x : x = x$ in X for which an ω-equivalence $f(x) = f(x)$ holds. But any function satisfy this property. In the end, a weak equivalence in **Set** is both a surjection and an injection, namely a bijection.

Example 1.6. The category of small 1-categories, which will be denoted by **Cat**(1) throughout this text, is another example of higher category theory in which a category $X : \mathrm{Mor}(X) \rightrightarrows \mathrm{Obj}(X)$ is associated the subsequent graph.

$$\begin{tikzcd} \mathrm{Obj}(X) & \mathrm{Mor}(X) \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] & \mathrm{Mor}(X) \arrow[l, shift right, "\mathrm{id}"'] \arrow[l, shift left, "\mathrm{id}"] & \mathrm{Mor}(X) \arrow[l, shift right, "\mathrm{id}"'] \arrow[l, shift left, "\mathrm{id}"] & \dots \arrow[l, shift right, "\mathrm{id}"'] \arrow[l, shift left, "\mathrm{id}"] \end{tikzcd}$$

This graph is equipped with the identities of the category X at level 1 and trivial identities such as those defined in the case of **Set** at any higher level. It is also equipped with the composition of X at level 1 while higher levels have trivial compositions such as those defined in the case of **Set**. By definition, an ω-equivalence $x \sim y$ between 0-cells is an isomorphism between objects $x \cong y$ in X while an ω-equivalence $f \sim g$ at a higher level is given by an identity relation $f = g$. It follows that a 0-surjection is exactly an essentially surjective functor of small categories; a 1-surjection is a full functor of small categories and a 2-surjection is given by a faithful functor of small categories. As in the case of **Set**, the trivial definition of cells at higher levels makes any functor into a n-surjection for $n > 2$. In the end, a weak equivalence in $\mathbf{Cat}(1)$ is an essentially surjective and fully faithful functor. This definition corresponds to the notion of weak equivalence used by A. Joyal and M. Tierney in [**31**].

Example 1.7. The category of small 2-categories, which will be denoted by $\mathbf{Cat}(2)$ throughout this text, is an example of higher category theory in which a 2-category $X : \mathrm{Cell}_2(X) \rightrightarrows \mathrm{Cell}_1(X) \rightrightarrows \mathrm{Obj}(X)$ is associated with the following graph.

$$\mathrm{Obj}(X) \underset{t_0}{\overset{s_0}{\leftleftarrows}} \mathrm{Cell}_1(X) \underset{t_1}{\overset{s_1}{\leftleftarrows}} \mathrm{Cell}_2(X) \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftleftarrows}} \mathrm{Cell}_2(X) \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftleftarrows}} \dots$$

This graph is equipped with the identities of the 2-category X at level 1 and 2 and trivial identities such as those defined in the case of **Set** at any higher level. It is also equipped with the compositions of X at level 1 and 2 while higher levels have trivial compositions such as those defined in the case of **Set**. By definition, an ω-equivalence $x \sim y$ between 0-cells is an 'adjoint equivalence' between objects $x \dashv y$ in X, an ω-equivalence $f \sim g$ between 1-cells is a 2-isomorphism $f \cong g$ between arrows in X while an ω-equivalence $f \sim g$ at a higher level is given by an identity relation $f = g$. The resulting notion of weak equivalence then corresponds to that used by S. Lack in his model structure on 2-categories (see [**32**]).

So far, I tried to present the prototypical category $\mathcal{C}$ with as little algebraic structure as possible, for its axiomatic definition could get extremely complicated in cases such as that of Grothendieck's ∞-groupoids. The present thesis is additionally interested in understanding the notion of weak equivalence not only in the case of higher category theories but also in a very general context. To this purpose, we therefore need to introduce a more tractable language, which will be that of commutative diagrams and representing objects. Below is given a reformulation of the foregoing example in terms of this diagrammatic language.

Example 1.8. Let $f : X \to Y$ be a function in **Set**. Supposing that f is a 0-surjection is equivalent to requiring that for every commutative diagram of the form given below on the left – where $\bullet$ denotes a set of one element and $\emptyset$ is the empty set – there exists an arrow $x : \bullet \to X$ making the following rightmost diagram commute.

$$\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] & X \arrow[d, "f"] \\
\bullet \arrow[r, "y"'] & Y
\end{tikzcd}
\qquad \text{“}y \in Y\text{”} \quad \Rightarrow \quad \text{“}\exists x \in X : y = f(x)\text{”} \qquad
\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] \arrow[rr, bend left, "!"] & \bullet \arrow[r, "x"] \arrow[d, equal] & X \arrow[d, "f"] \\
\bullet \arrow[r, equal] \arrow[rr, bend right, "y"'] & \bullet \arrow[r, "f(x)"] & Y
\end{tikzcd}$$

Similarly, if one denotes a set of two elements by $\bullet\,\bullet$ (double coproduct of $\bullet$), then supposing that f is a 1-surjection is equivalent to requiring that for every commutative diagram of the form given below on the left, there exists an arrow $w : \bullet \to X$ making the following right diagram commute.

$$\begin{tikzcd}
\bullet\,\bullet \arrow[r, "x \coprod y"] \arrow[d, "!"'] & X \arrow[d, "f"] \\
\bullet \arrow[r, "z"'] & Y
\end{tikzcd}
\qquad \text{“}f(x) = f(y)\text{”} \quad \Rightarrow \quad \text{“}x = y\text{”} \qquad
\begin{tikzcd}
\bullet\,\bullet \arrow[r, "!"] \arrow[d, "!"'] \arrow[rr, bend left, "x \coprod y"] & \bullet \arrow[r, "w"] \arrow[d, equal] & X \arrow[d, "f"] \\
\bullet \arrow[r, equal] \arrow[rr, bend right, "z"'] & \bullet \arrow[r, "z"] & Y
\end{tikzcd}$$

As one can quickly notice, the notion of weak equivalence is nothing but a factorisation in the arrow category of **Set**. More specifically, this factorisation is of the form displayed below on the left (see the leftmost two diagrams) where the arrow $a \Rightarrow b$ is of the form given below on the right.

$$\begin{tikzcd} a \arrow[r, Rightarrow, "\forall"] \arrow[d, Rightarrow] & f \\ b & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} a \arrow[r, Rightarrow] \arrow[d, Rightarrow] & f \\ b \arrow[ur, Rightarrow, dashed, "\exists"'] & \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \bullet \arrow[d, equal] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet\ \bullet \arrow[r, "!"] \arrow[d, "!"'] & \bullet \arrow[d, equal] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

(surjectivity) *(injectivity)*

Interestingly, the arrows forming the previous rightmost two diagrams may be organised in a unique commutative diagram of the form given below on the right, which is nothing but the following leftmost commutative diagram whose underlying pushout has been added.

$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \bullet \arrow[d, equal] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet\ \bullet \arrow[r, "!"] \arrow[d, "!"'] & \bullet \arrow[d, equal] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \bullet_1 \arrow[d] & \\ \bullet_2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \bullet_2\ \bullet_1 \arrow[r, "!"] & \bullet \end{tikzcd}$$

(surjectivity) *(injectivity)* *(bijectivity)*

In other words, the notion of injectivity in **Set** may be defined at the same level as the notion of surjectivity, that is to say at the level of the previous leftmost commutative square provided that it is equipped with a pushout. This last point will turn out to be important later on.

Example 1.9. Let $f : X \to Y$ be a functor in $\mathbf{Cat}(1)$. In the same fashion as in the category **Set**, the notions of 0-, 1- and 2-surjection may be expressed in terms of factorisations in the arrow category of $\mathbf{Cat}(1)$. More specifically, the morphism f defines a weak equivalence if every morphism $a \Rightarrow f$ in the arrow category of $\mathbf{Cat}(1)$ factorises through an arrow $a \Rightarrow b$ given by the following commutative squares (involving the obvious mappings), where $\bullet \cong \bullet$ denotes the free living isomorphism[1], $\bullet \to \bullet$ denotes the free living arrow and $\bullet \rightrightarrows \bullet$ denotes the pushout of two copies of the free living arrow along the two object category $\bullet\ \bullet$.

(1.1)
$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \bullet_1 \arrow[d] \\ \bullet_2 \arrow[r] & \bullet_2 \cong \bullet_1 \end{tikzcd} \qquad \begin{tikzcd} \bullet_2\ \bullet_1 \arrow[r] \arrow[d] & \bullet_2 \to \bullet_1 \arrow[d, equal] \\ \bullet_2 \to \bullet_1 \arrow[r, equal] & \bullet_2 \to \bullet_1 \end{tikzcd} \qquad \begin{tikzcd} \bullet_2 \rightrightarrows \bullet_1 \arrow[r] \arrow[d] & \bullet_2 \to \bullet_1 \arrow[d, equal] \\ \bullet_2 \to \bullet_1 \arrow[r, equal] & \bullet_2 \to \bullet_1 \end{tikzcd}$$

(essentially surjective) *(full)* *(faithful)*

Recall that the proof of the injectivity property noticed in Remark 1.4 consisted in using lifting properties with respect to the leftmost arrows of the previous squares.

$$\emptyset \to \{\bullet\} \qquad \{\bullet\ \bullet\} \hookrightarrow \{\bullet \to \bullet\} \qquad \{\bullet \rightrightarrows \bullet\} \to \{\bullet \to \bullet\}$$

But there is a better way of viewing the injectivity property, which has already been noticed in the case of **Set**. Specifically, the injectivity property is in fact given by lifting properties with respect to the following functors, where a lift with respect to the leftmost one gives the notion of essential injectivity while a lift with respect to the rightmost one gives the notion of faithfulness.

$$\{\bullet_2\ \bullet_1\} \hookrightarrow \{\bullet_2 \cong \bullet_1\} \qquad \{\bullet_2 \rightrightarrows \bullet_1\} \to \{\bullet_2 \to \bullet_1\}$$

These functors may be brought out in the commutative diagrams of (1.1) by forming the respective underlying pushouts of the squares. In the end, the definition of weak equivalence in $\mathbf{Cat}(1)$ only requires the following commutative diagrams, where the left one characterises

[1] i.e. the category made of two objects and an isomorphism.

the bijectivity of weak equivalences at dimension 1 (essential bijectivity) while the right one characterises the bijectivity of weak equivalences at dimension 2 (fully faithfulness).

(1.2)
$$
\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet_1 \arrow[d] & \\
\bullet_2 \arrow[r] & \bullet_2\ \bullet_1 \arrow[r, "!"] & \bullet_2 \cong \bullet_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet_2\ \bullet_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet_2 \to \bullet_1 \arrow[d] & \\
\bullet_2 \to \bullet_1 \arrow[r] & \bullet_2 \rightrightarrows \bullet_1 \arrow[r] & \bullet_2 \to \bullet_1
\end{tikzcd}
$$

Note that the pushout square of the preceding left diagram may be related to the pushout square of the right one via the $\bullet\ \bullet$. This potential pasting informs us that the preceding right diagram lives at a higher dimension than the left one. In fact, we will later see throughout this thesis that the right way of thinking of the right diagram is to regard it as a sort of dimensional tower as follows.

$$
\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet_1 \arrow[d] & & \\
\bullet_2 \arrow[r] & \bullet_2\ \bullet_1 \arrow[r, "!"] \arrow[d, "!"'] \arrow[dr, phantom, "\ulcorner", very near end] & \bullet \to \bullet \arrow[d] & \\
& \bullet \to \bullet \arrow[r] & \bullet \rightrightarrows \bullet \arrow[r, "!"] & \bullet \to \bullet
\end{tikzcd}
$$

This 'spinal structure' somehow provides the right diagram of (1.2) with a CW-complex structure by starting from nothing and gradually adding information 'dimension by dimension'. The CW-complex structure of the left diagram of (1.2) is already given by the diagram itself.

Example 1.10. The example of $\mathbf{Cat}(2)$ follows the same idea as those exposed in Example 1.8 and Example 1.9, which is to say that one may organise the diagrammatic structures characterising weak equivalences in terms of commutative diagrams as follows.

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \cdot \arrow[d] & \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \cdot \arrow[d] & & \\
\cdot \arrow[r] & \cdot & & \\
& & \ddots & \\
& & & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \cdot \arrow[d] & \\
& & & \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

The idea of the previous diagrams is that the leftmost diagram determines weak equivalences at a given dimension. This dimension is then characterised by the length of the associated 'spinal column' constructed on the right whose bottom part corresponds to the leftmost diagram.

I will finish this series of examples by briefly discussing the case of the category of strict ω-categories, which I will refer to as $\mathbf{Cat}(\omega)$. In the proof of the model structure for strict ω-groupoids [**33**], Metayer, Lafont and Worytkiewicz had to reformulate their definition of weak equivalence in terms of a factorisation in the arrow category of $\mathbf{Cat}(\omega)$, in much the same way as that shown in the previous examples. Their need for a diagrammatic language is due to the use of theorems coming from the theory of locally presentable categories – and hence sketch theory. For every non-negative integer n, their factorisations are defined with respect to the subsequent commutative square, which they call *generic n-squares*, where $\mathbf{O}^n$ is the free living n-cell strict ω-category, $\partial\mathbf{O}^n$ is empty when n is zero and otherwise equal to the strict ω-category consisting of the source and target of the n-cell in $\mathbf{O}^n$ and $\mathbf{P}^n$ is the representative object of an ω-equivalence between n-cells[2].

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r, "\mathbf{i}_n"] \arrow[d, "\mathbf{i}_n"'] & \mathbf{O}^n \arrow[d, "\mathbf{j}_n"] \\
\mathbf{O}^n \arrow[r, "\mathbf{j}'_n"'] & \mathbf{P}^n
\end{tikzcd}$$

The 'spine' resulting from the previous commutative square, for every $n \geq 0$, is of the following form, where the diagram consisting of the lowest pushout square and the arrow $\mathbf{k}^n$ that juts out of it characterises the bijectivity of the weak equivalences at dimension n.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbf{O}^0 \arrow[d] & & & \\
\mathbf{O}^0 \arrow[r] & \partial\mathbf{O}^1 & & & \\
& & \ddots & & \\
& & \partial\mathbf{O}^n \arrow[r, "\mathbf{i}_n"] \arrow[d, "\mathbf{i}_n"'] \arrow[dr, phantom, "\ulcorner"] & \mathbf{O}^n \arrow[d, "\mathbf{j}_n"] & \\
& & \mathbf{O}^n \arrow[r, "\mathbf{j}'_n"'] & \partial\mathbf{O}^{n+1} \arrow[r, "\mathbf{k}^n"] & \mathbf{P}^n
\end{tikzcd}$$

It is only when these all spines are considered together that the weak equivalences make sense at any dimension.

What the foregoing discussion showed us is that one may characterise weak equivalences locally (i.e. at any dimension) by using diagrams of the following form.

(1.3)
$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D} \arrow[d, "\delta_2"] & \\
\mathbb{D} \arrow[r, "\delta_1"'] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}'
\end{tikzcd}$$

In addition, such diagrams give rise to other diagrammatic structures that look like spinal columns. These spinal columns may be used to provide diagrams of the form (1.3) with a notion of dimension. The key idea is that these spinal columns are the central structures around which the underlying model structure organises itself, exactly as *a body would organise itself around an actual spinal column.* In the same analogy, diagram (1.3) would play the role of a vertebra for these spines as elementary structures allowing their formation. If a vertebra happens to be at the tip of a spine, then it plays a role in the determination of weak equivalences. Such a vertebra will later be referred to as the *head* of the spine.

In addition to formalising the language of higher category theories, these diagrams belong to the world of colimit sketches, which is exactly the language in which Grothendieck thought his definition of ∞-groupoid. It also notably reminds us the way how Algebraic Topology works, namely by the use of presentative objects such as spheres and disks as atomic structures to describe others up to homotopy (CW-complexes, fibrant objects and so on).

1.1.2.2. *Vertebrae and spines.* Let $\mathcal{C}$ be a category. The previous section showed how a diagram of the form (1.4), which will be called a *vertebra* in $\mathcal{C}$, could generate a notion of

[2] which may explicitly be constructed by a co-inductive small object argument.

weak equivalence consistent with the conventional intuition of bijection up to homotopy.

(1.4)
$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathbb{D} \arrow[d, "\delta_1"] \arrow[dl, phantom, "\ulcorner"] & \\ \mathbb{D} \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}' \end{tikzcd}$$

More specifically, a *weak equivalence* for the previous vertebra is a morphism $f : X \to Y$ in $\mathcal{C}$ satisfying the next conditions.

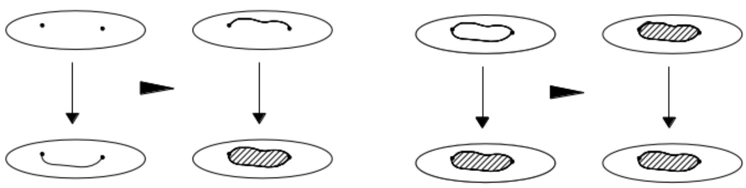
1) (*Surjectivity*) For every commutative square of the form given below on the left, there exist two arrows $x' : \mathbb{D} \to X$ and $y' : \mathbb{D}' \to Y$ making the following right diagram commute.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\ \mathbb{D} \arrow[r, "y"'] & Y \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} \mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[rr, bend left, "x"] & \mathbb{D} \arrow[r, dashed, "x'"] \arrow[d, "\beta\circ\delta_1"] & X \arrow[d, "f"] \\ \mathbb{D} \arrow[r, "\beta\circ\delta_2"] \arrow[rr, bend right, "y"'] & \mathbb{D}' \arrow[r, dashed, "y'"] & Y \end{tikzcd}$$

2) (*Injectivity*) For every commutative square of the form given below on the left, there exists an arrow $h : \mathbb{D}' \to X$ making the following right diagram commute.

$$\begin{tikzcd} \mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ \mathbb{D}' \arrow[r, "y"'] & Y \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} \mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & X \\ \mathbb{D}' \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Later on, a morphism that only satisfies item 1) with respect to diagram (1.4) will be called a *surtraction* (from latin *sur* (over or above); *traction* (a pulling or drawing)) while a morphism that only satisfies item 2) with respect to diagram (1.4) will be called an *intraction* (from latin *in* (inside); *traction*).



Surtraction for topological paths of dimension 1 *Intraction for topological paths of dimension 1*

Figure 1. Topological representation of intractions and surtractions

A morphism that is both a surtraction and an intraction for some vertebra v is a weak equivalence for v. In fact, one of the key ideas that this text aims at setting forth is that, despite the minimalistic structure of diagram (1.4), such a diagram is sufficient to make sense of a homotopy theory wherein natural notions of fibration and trivial fibration live. More specifically, my strategy is to start from scratch by considering a simple vertebra such as the one considered in (1.4) and associate it with a *zoo* of items comprising intractions, surtractions, fibrations, trivial fibrations and so on. I will then prove that one can retrieve the usual properties of standard homotopy theories (model categories, categories of fibrant objects, etc.) from the definition of these items.

It will happen that a vertebra on its own is not sufficient and requires some 'reflexive', 'reversible' or 'transitive' structure to get all the axioms of a desired homotopy theory. It will also happen that some vertebra is not sufficient on its own and needs to ally itself with other vertebrae, which will imply an interaction between the zoos of the involved vertebrae.

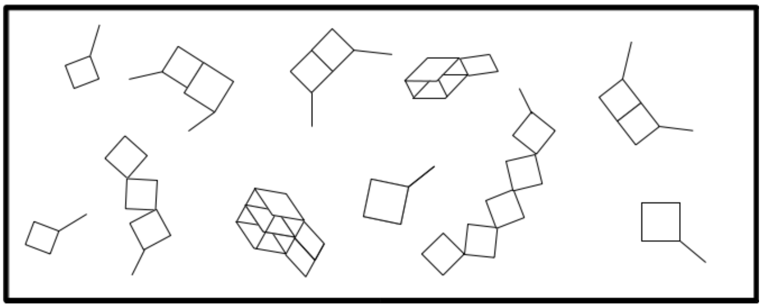


Figure 2. Some of the possible chemical interactions

It is when one combines a set of vertebrae together that the homotopy theories of all vertebrae merge to eventually give a more complex one. The goal of these thesis is therefore to give a treatment of the possible 'chemical reactions' between vertebrae in order to achieve a coherent and sound homotopy theory. The world of vertebrae would thus be a sort of chemistry where a mathematician would be free to invent their own homotopy theory provided that their set of vertebrae satisfies the necessary axioms. This said, it seems now appropriate to mention that the goal of Chapter 2, 3 and 4 is to provide an algorithm, or more specifically, a method to construct a homotopy theory rather than detect it (which is the goal of model categories).

Below is given an idea of what the chemical interactions between vertebrae look like in the case of topological spaces.

Let us start with a brief presentation of the natural set of vertebrae of the category of topological spaces and the definitions of weak equivalence resulting from it. The concerned vertebrae are given for every non-negative dimension n by diagram (1.5) where the object $\mathbb{S}^{n-1}$ denotes the topological sphere of dimension $n-1$; the object $\mathbb{D}^n$ denotes the topological disc of dimension n and the arrow $\gamma_k : \mathbb{S}^{n-1} \to \mathbb{D}^n$ denotes the canonical inclusion.

(1.5)
$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D}^n \arrow[d, "\delta_1^n"] & \\ \mathbb{D}^n \arrow[r, "\delta_2^n"'] & \mathbb{S}^n \arrow[r, "\gamma_{n+1}"'] & \mathbb{D}^{n+1} \end{tikzcd}$$

For a fixed $n > 0$, a continuous function $f : X \to Y$ between two topological spaces defines a surtraction (resp. intraction) if and only if the following morphism of homotopy groups is a surjection (resp. injection) for every point $x \in X$.

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

The fact that the previous morphism is a surjection or injection when f is an intraction or surtraction is just a consequence of what it means to be injective or surjective at the level of homotopy groups (see Figure 1). The converse is more subtle as it requires us to show that the notion of injectivity and surjectivity (up to homotopy) based on a point is the same as the notion of injectivity and surjectivity (up to homotopy) based on a path. The case $n = 1$ is easy to show. The case of surjectivity is displayed below where (1) one considers a topological 1-path between two points in Y. (2) Glueing this path with its reversed path gives a unique path based on one of the points[3]. (3) Applying the surjectivity then provides a lift as shown in the third diagram. Finally, (4) recovering the initial path then leads to the desired lift[4] to prove that $f : X \to Y$ is indeed a surtraction at dimension 1.

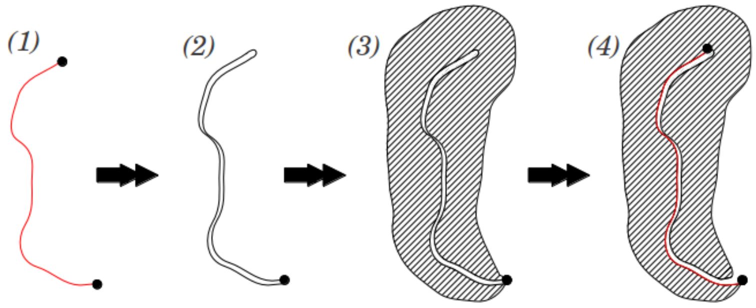


Figure 3. Homotopy surjections are surtractions

Note that the reasoning of Figure 3 uses the operation that glues two connected paths into a third one. At the level of vertebrae, such an operation requires that the 'gluing of two copies of vertebra (1.5) gives vertebra (1.5) back'. More precisely, the requisite axioms on vertebra (1.5) consist of

- a gluing of the topological discs via their connecting source and target under the form of the following pushout;

$$\begin{tikzcd} \mathbb{D}^0 \arrow[r, "\gamma_1 \circ \delta_1^0"] \arrow[d, "\gamma_1 \circ \delta_2^0"'] & \mathbb{D}^1 \arrow[d, "\varepsilon_1"] \\ \mathbb{D}^1 \arrow[r, "\varepsilon_2"'] & \mathbb{E} \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \bullet_1 \arrow[r, mapsto] \arrow[d, mapsto] & \bullet_1 - \bullet \arrow[d, mapsto] \\ \bullet - \bullet_1 \arrow[r, mapsto] & \bullet - \bullet_1 - \bullet \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

- a map $\eta : \mathbb{D}^1 \to \mathbb{E}$ composing the preceding gluing of paths into a unique topological path (with adequate compatibility between sources and targets);

$$\begin{tikzcd} \mathbb{S}^0 \arrow[dr, "\gamma_1"] & & \mathbb{D}^0 \arrow[ll, "\delta_1^0"'] \arrow[dd, "\varepsilon_2 \circ \gamma_1 \circ \delta_1^0"] \\ & \mathbb{D}^1 \arrow[dr, "\eta"] & \\ \mathbb{D}^0 \arrow[uu, "\delta_2^0"] \arrow[rr, "\varepsilon_1 \circ \gamma_1 \circ \delta_2^0"'] & & \mathbb{E} \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \bullet_0 \;\; \bullet_2 \arrow[dr, mapsto] & & \bullet_2 \arrow[ll, mapsto] \arrow[dd, mapsto] \\ & \bullet_0 - \bullet_2 \arrow[dr, mapsto, "\eta"] & \\ \bullet_0 \arrow[uu, mapsto] \arrow[rr, mapsto] & & \bullet_0 - \bullet - \bullet_2 \end{tikzcd}$$

This type of operation will later be called a *framing* and will be generalised to an operation of vertebrae taking two different vertebrae satisfying some condition of composability and giving a third vertebra. The composability condition between the two vertebrae will ask for the two input vertebrae to have an arrow in common as pictured on the left of Figure 4.

Essentially, this type of operation is used to prove that surtractions are stable under composition. For illustration, the case of vertebra (1.5) when $n = 0$ is briefly explained below.

[3]The reversed path has been shifted so that it may be seen.
[4]The same part of the lift has been shifted so that it may be seen more clearly.

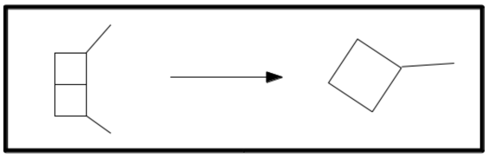


Figure 4. Chemical reaction for framings of vertebrae

Composability of surtractions. Consider two surtractions $f : Y \to Z$ and $g : X \to Y$ for vertebra (1.5) when $n = 0$. We shall denote by $h : x \sim y$ any 1-path from a point x to a point y. To show that the composite $f \circ g$ is a surtraction for vertebra (1.5) when $n = 1$, one needs to start with a point $z \in Z$ and show that there exists a point $x \in X$ such that a 1-path $h : z \sim f \circ g(x)$ holds in Z. Since z is in the codomain of $f : Y \to Z$ – which is a surtraction – there exists a point $y \in Y$ and a 1-path $h : z \sim f(y)$ in Z. Now, since y is in the codomain of $g : X \to Y$ – which is a surtraction – there exists a point $x \in X$ and a 1-path $h' : y \sim f(x)$ in Y. Applying the function f to the path h' and considering the resulting path in Z next to the path h gives a pair of connected paths, which we may compose into a unique path via the structure of framing.

$$z \underset{h}{\rightsquigarrow} f(y) \underset{f(h')}{\rightsquigarrow} f \circ g(x) \qquad \longmapsto \qquad z \underset{h''}{\rightsquigarrow} f \circ g(x)$$

In the end, the resulting composition $h'' : z \sim f \circ g(x)$ proves that $f \circ g$ is a surtraction. $\square$

Thus, one may see that a property at the level of the vertebrae will have consequences at the level of the zoo. Other compositions of importance such as interdimensional compositions will be studied later. For instance, the conjugation of a 2-path (encoded via a spine) along two 1-paths (encoded via two vertebrae) will be used to prove the cancellation of surtraction, that is to say a statement of the form *if g is a surtraction and $f \circ g$ is a surtraction, then so is f.*

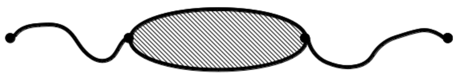


Figure 5. Conjugation of a 2-path along two 1-paths

1.1.2.3. *Model categories and vertebrae.* One may wonder if the concept of weak equivalence associated with the notion of model category falls into the theory of vertebrae. The answer is affirmative in any *tractable* model category (see [**39**]) for every weak equivalence between fibrant objects. A model category is *tractable* when it is cofibrantly generated and the generating cofibrations have cofibrant domains. The usual assumption that the model category must be cofibrantly generated with respect to a generating set is somewhat psychological and the statement may be generalised to any generating class. Below is given the explicit form of the involved vertebrae.

Recall that a *closed model category* is a category $\mathcal{C}$ equipped with three subclasses of the class of its morphisms, whose elements are respectively called *weak equivalences*, *fibrations* and *cofibrations*, that satisfy a set of four or five axioms depending on the concision of the author. The axiom of model categories that will turn out to be important for the present section is the one that states that one may factorise any morphism $f : X \to Y$ in $\mathcal{C}$ into a composite $p \circ i$ where p is a weak equivalence and i is a cofibration. The *fibrant objects* are those objects X in $\mathcal{C}$ such that any morphism of the form $A \to X$ factorises through the *acyclic cofibrations* of domain A, that is to say the cofibrations of domain A that are weak equivalences.

(1.6)
$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\forall\text{ acyclic cof. } g"'] & X \\
B &
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "g"'] & X \\
B \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}
$$

A model category will here be said to be *cofibrantly generated* if there exists a set S of cofibrations in $\mathcal{C}$ such that a morphism $f : X \to Y$ is an *acyclic fibration* (i.e. both a fibration and weak equivalence) if and only if it has the right lifting property with respect to every arrow in S.

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}
$$

Finally, a closed model category will be said to be *tractable* if it is cofibrantly generated and the domains of the arrows in the generating set S are cofibrant[5]. The vertebrae that characterise the weak equivalences in terms of intractions and surtractions are given by the *relative cylinder objects* relative to the cofibrations in S. The definition of these goes as follows: first, consider a cofibration $\gamma : U \to V$ in S and form the following left pushout square. As shown in the right-hand diagram, there is a canonical arrow $u : V \cup_U V \to V$ over this pushout generated by the identities on V.

$$
\begin{tikzcd}
U \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, "\delta_1^\gamma"] \\
V \arrow[r, "\delta_2^\gamma"'] & V \cup_U V
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
U \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, "\delta_1^\gamma"] \arrow[dr, bend left, "\mathrm{id}_V"] & \\
V \arrow[r, "\delta_2^\gamma"] \arrow[rr, bend right, "\mathrm{id}_V"'] & V \cup_U V \arrow[r, dashed, "u"] & V
\end{tikzcd}
$$

A vertebra is then given by a factorisation of the canonical arrow $u : V \cup_U V \to V$ into a weak equivalence u' and a cofibration β as follows.

(1.7)
$$
\begin{tikzcd}
U \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, "\delta_1^\gamma"] & & \\
V \arrow[r, "\delta_2^\gamma"'] & V \cup_U V \arrow[r, "\beta"] \arrow[rr, dotted, bend right, "u"'] & I(\gamma) \arrow[r, dotted, "u'"] & V
\end{tikzcd}
$$

Such a commutative diagram is usually called a *γ-relative cylinder object*. Now, there is a classical fact of the theory of model categories, whose statement looks much like that of [**39**, Lemma 7.5.1] but whose proof is exactly the same, that provides the following implication.

Proposition 1.11. *Let $\mathcal{C}$ be a tractable model category and $f : X \to Y$ be a morphism between fibrant objects satisfying the property that for every generating cofibration $\gamma : A \to B$ and commutative diagram of the form given on the left of (1.8), there exists a γ-relative cylinder object of the form (1.7) and two arrows $r : V \to X$ and $h : I(\gamma) \to Y$ making the*

[5]The *cofibrant objects* are those objects A in $\mathcal{C}$ such that any morphism of the form $A \to X$ factorises through any acyclic fibration of codomain X

right diagram of (1.8) commute.

(1.8)
$$
\begin{tikzcd}
U \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
V \arrow[r, "y"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
U \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[rr, bend left, "x"] & V \arrow[r, "r"] \arrow[d, "\beta\circ\delta_1^\gamma"] & X \arrow[d, "f"] \\
V \arrow[r, "\beta\circ\delta_2^\gamma"] \arrow[rr, bend right, "y"'] & I(\gamma) \arrow[r, "h"] & Y
\end{tikzcd}
$$

In this case, the morphism g is a weak equivalence in $\mathcal{C}$.

In other words, any morphism that is a surtraction – and *a fortiori* a weak equivalence – for every vertebra of the form (1.7), where γ runs over the set S, is a weak equivalence in $\mathcal{C}$.

Remark 1.12. Interestingly, Proposition 1.11 requires a condition more subtle than considering morphisms that are surtractions for every vertebra of the form (1.7). To see this, we shall symbolically denote by $\gamma \rightsquigarrow v$ the vertebra given in (1.7), where the symbol v refers to the vertebra itself (and stands for the first letter of the word 'vertebra') while the symbol γ refers to its seed. For any morphism $f : X \to Y$ in $\mathcal{C}$, Proposition 1.11 then states that if for every cofibration γ in S, there exists at least a vertebra of the form $\gamma \rightsquigarrow v$ for which f is a surtraction, then the morphism f is a weak equivalence. Here, we see that the notation in terms of an arrow $\gamma \rightsquigarrow v$ has some logical importance. We may somehow regard this arrow as suggestive of a logical implication of the form $\forall\, \gamma \in S$, $\exists \gamma \rightsquigarrow v$ *such that something happens.* As will be seen later, this type of writing is in fact the right logical language in which the theory of vertebrae must be expressed.

The converse of Proposition 1.11 is also the result of classical facts of the theory of model categories, which I outline below for the sake of fulfilling the curiosity of the reader.

Converse of Proposition 1.11. Suppose that $\mathcal{C}$ is equipped with an initial object $\emptyset$. First, recall that there is a equivalence relation $\sim$ defined on every hom-set $\mathcal{C}(A, X)$, called the *homotopy relation*, that states that two arrows $f, g : A \to X$ satisfy the relation $f \sim g$ if there exists a cylinder object, of the form given below on the left, whose arrow $\beta : A + A \to I(A)$ factorises the coproduct of f and g as shown on the right.

(1.9)
$$
\begin{tikzcd}
\emptyset \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "\delta_1^\gamma"] & \\
A \arrow[r, "\delta_2^\gamma"'] & A + A \arrow[r, "\beta"] & I(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
& I(A) \arrow[dr, dashed, "\exists"] & \\
A + A \arrow[ur, "\beta"] \arrow[rr, "f+g"'] & & A
\end{tikzcd}
$$

Quotienting out the hom-sets $\mathcal{C}(A, X)$ by $\sim$ for a fixed object A then provides a functor $\pi(A, \_) := \mathcal{C}(A, \_)/\sim$. It then follows from usual fact (see [**15**]) that if a morphism $f : X \to Y$ is a weak equivalence between fibrant objects, then the following function is a bijection for every cofibrant object A.

$$\pi(A, f) : \pi(A, X) \to \pi(A, Y)$$

The previous implication holds in any closed model category. The characterisation of weak equivalences in terms of surtractions and intractions will then follow from the preceding bijection when realised in the under categories $U\backslash\mathcal{C}$ for every object U in $\mathcal{C}$. Recall that $U\backslash\mathcal{C}$ denotes the category whose objects are arrows of the form $U \to X$ in $\mathcal{C}$ and whose morphisms are given by commutative triangles (under U) in $\mathcal{C}$. It is well-known that there is a closed model structure on the under categories $U\backslash\mathcal{C}$ for every object in U whose

- weak equivalences between fibrant objects are the morphisms in $\mathcal{C}$ that are weak equivalences between fibrant objects in $\mathcal{C}$;

- cofibrant objects are the cofibrations of domain U in $\mathcal{C}$.

This said, the preceding discussion implies that if $f : X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{C}$, then it induces a weak equivalence in $U\backslash\mathcal{C}$ for every commutative triangle as given below on the left. The function displayed on the right is then a bijection for every cofibration $\gamma : U \to V$.

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & U \arrow[ul, "x"] \arrow[ur, "f\circ x"'] & \end{tikzcd} \qquad \pi(\gamma, f) : \pi(\gamma, x) \to \pi(\gamma, f \circ x)$$

Also, observe that a cylinder object as given on the left of (1.9) in $U\backslash\mathcal{C}$ where A is replaced with $\gamma : U \to V$ corresponds to a γ-relative cylinder object of the form (1.7) in $\mathcal{C}$. It follows that the surjectivity of the previous right bijection may exactly be translated into the implication of (1.8). The fact that these weak equivalences are also intractions follows from the observation that the implication of (1.8) also says that for every commutative diagram of the form given below on the left, there exists an arrow $r : V \to X$ making the succeeding right-hand diagram commute.

$$\begin{tikzcd} U \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\ V \arrow[r, "y"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} U \arrow[r, "x"] \arrow[d, "\gamma"'] & X \\ V \arrow[ur, "r"'] & \end{tikzcd}$$

This therefore implies the property of being an intraction as γ covers all cofibrations, and hence the cofibrations β involved in diagram (1.7). □

Remark 1.13. The foregoing discussion in fact shows that the weak equivalences between fibrant objects are entirely characterised by surtractions for the vertebrae given by (1.7).

1.1.2.4. *Towards more general vertebrae.* Remark 1.13 of the previous section should not make the reader think that the notion of intraction is unnecessary. The identification of surtractions and weak equivalences only holds because model structures come along with an already powerful notion of homotopy. Besides, the statement only holds for weak equivalences between fibrant objects. On the contrary, one of our future goals throughout this thesis will be to understand how to construct a model category from vertebrae that are *a priori* not relative cylinder objects. For that, a key element will be the consideration of the notion of intraction, which will enable us to handle the notion of surtraction in a consistent way. Because some categories are more difficult to handle than others, the notion of intraction will force us to extend the notion of vertebra to that of 'node of vertebrae'.

(1.10)
$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathbb{D} \arrow[d, "\delta_1"] & \\ \mathbb{D} \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}' \end{tikzcd}$$

Quite often, proving that a morphism $f : X \to Y$ in some category $\mathcal{C}$ is an intraction for vertebra (1.10) will require us to start with a commutative square as given below on the left and construct, via the use of algebraic operations, a morphism $\beta' : \mathbb{S}' \to \mathbb{D}''$ making the following right diagram commute.

(1.11)
$$\begin{tikzcd} \mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ \mathbb{D}' \arrow[r, "y"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta'"'] & X \\ \mathbb{D}'' \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

In nice categories, the morphism β' turns out to be equal to β (up to possible factorisation). However, in categories with very little algebraic structure such as the category of simplical sets, the morphism β' is strictly different from β (usually involving more simplices than β in its codomain). To get around such a difficulty, we shall need to *force* the intractions to see the arrow β' as the arrow β by considering diagrams of the subsequent form, which will be called a *node of vertebrae*.

(1.12)
$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D} \arrow[d, "\delta_1"] & \mathbb{D}'' \\
\mathbb{D} \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"'] \arrow[ur, "\beta'"] & \mathbb{D}' \arrow[u, phantom, "\vdots"]
\end{tikzcd}$$

Thus, we now consider a class of vertebrae with the same pushout square but whose 'stems' are different. Somehow, putting different stems together on the same vertebra is a way of *forcing* the requisite algebraic operations to be satisfied despite the ambient category possibly not having these operations. The intractions for (1.12) are then the morphisms $f : X \to Y$ satisfying the lifting property displayed in (1.11). The notion of node of vertebrae will also turn out to be theoretically useful in Chapter 3 for the proof of the cancellation of intractions.

It is also important to mention that the concept of node of vertebrae will make sense along more general vertebrae where the associated span made of two copies of γ is turned into a pair of arrows γ and γ' as shown below (on the right is given an example of a non-symmetric vertebra in the category of simplicial sets, which will be discussed in Chapter 2).

(1.13)
$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D}_1 \arrow[d, "\delta_1"] & \mathbb{D}'' \\
\mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"'] \arrow[ur, "\beta'"] & \mathbb{D}' \arrow[u, phantom, "\vdots"]
\end{tikzcd}
\qquad
\left(\begin{tikzcd}
\partial\Delta_n \arrow[r, "\gamma_n^k"] \arrow[d, "\gamma_n"'] \arrow[dr, phantom, "\ulcorner"] & \Lambda_n^k \arrow[d] & \\
\Delta_n \arrow[r] & \partial\Delta_{n+1} \arrow[r, "\gamma_{n+1}"'] & \Delta_{n+1}
\end{tikzcd}\right)$$

This differentiation will force us to introduce some terminology. The arrows γ and γ' will be called *seed* and *coseed* (following the idea of 'generating' cofibration) while the arrows β and β' will be called *stems*. This will give rise to the study of the notion of *dual node of vertebrae* where the seed and coseed are mutually replaced with each other by flipping the square of the left vertebra of (1.13) along its diagonal from the object $\mathbb{S}$ to the object $\mathbb{S}'$.

There are particular cases where the seed and coseed are different, which include those of sheaves, stacks and other structures satisfying a descent condition. Recall that a descent condition generally requires some morphisms to be weak equivalences. As noticed throughout various passages of this introduction, a weak equivalence for a vertebra of the form (1.10) is a morphism $f : X \to Y$ that satisfies the kind of right lifting property used to describe fibrant objects in section 1.1.2.3, namely implication (1.6). Specifically, this right lifting property (see subsequent right diagram) is defined with respect to the arrow $\gamma \Rightarrow \beta \circ \delta_1$ encoded by the following left commutative square in the arrow category of $\mathcal{C}$.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathbb{D} \arrow[d, "\beta\circ\delta_1"] \\
\mathbb{D} \arrow[r, "\beta\circ\delta_1"'] & \mathbb{D}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \arrow[rr, "x"] \arrow[dr, "\gamma"] \arrow[dd, "\gamma"'] & & X \arrow[dr, "f"] & \\
& \mathbb{D} \arrow[rr, "y"] \arrow[dd, "\beta\circ\delta_2"] & & Y \\
\mathbb{D} \arrow[uurr, dashed] \arrow[dr, "\beta\circ\delta_1"'] & & & \\
& \mathbb{D}' \arrow[uurr, dashed, "\exists"'] & &
\end{tikzcd}$$

It will later be seen that, in the language of vertebrae, the fibrant objects for vertebra (1.10) will correspond to those objects that satisfy implication (1.6) where the acyclic cofibration is replaced with the arrow $\beta \circ \delta_1 : \mathbb{D}_1 \to \mathbb{D}'$ – this arrow will later be referred to as the *trivial stem* of vertebra (1.10). The arising question therefore asks whether there is a vertebra in the ambient arrow category whose trivial stem is given by the previous left commutative square. This would allow us to see weak equivalences as fibrant objects for this vertebra. Obviously, it should preferably be constructed by only using the data available from the initial vertebra in $\mathcal{C}$. Chapter 5 will answer such a question in the affirmative by using the other morphisms provided by vertebra (1.10). Explained in somewhat broad terms, the idea will be to consider the subsequent left 3-dimensional vertebra.

(1.14)
$$\begin{tikzcd}
 & \cdot \arrow[rr, "\gamma"] \arrow[dd, "\gamma"] & & \cdot \arrow[dd, "\delta_1"] & & \\
\cdot \arrow[ur, equal] \arrow[rr, equal] \arrow[dd, "\gamma"'] & & \cdot \arrow[ur, "\gamma"] \arrow[dd, "\gamma"] & & & \\
 & \cdot \arrow[rr, "\delta_1"] & & \cdot \arrow[rr, "\beta"] & & \cdot \\
\cdot \arrow[ur, equal] \arrow[rr, equal] & & \cdot \arrow[ur, "\delta_1"] \arrow[rr, equal] & & \cdot \arrow[ur, "\beta\circ\delta_1"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\binom{\emptyset}{\emptyset} \arrow[r] \arrow[d] & \binom{\emptyset}{\bullet} \arrow[d] \arrow[dl, phantom, "\ulcorner"] & \\
\binom{\bullet}{\bullet} \arrow[r] & \binom{\bullet}{\bullet\ \bullet} \arrow[r] & \binom{\bullet}{\bullet \cong \bullet}
\end{tikzcd}$$

Notice that the seed and coseed of this vertebra are different. On the above right is given a translation of this 3-dimensional vertebra in the case of the left vertebra of (1.2) in $\mathbf{Cat}(1)$ where a symbol $\binom{a}{b}$ represents the obvious arrow $a \to b$.

To see what this vertebra does in the case of sheaves in categories, consider a small category D, two functors $F : D \to \mathbf{Cat}(1)$ and $G : D \to \mathbf{Cat}(1)$ and a natural transformation $\theta : F \Rightarrow G$. For every cone in D of the form given below on the left, the induced commutative square given below on the right is a weak equivalence for the right vertebra of (1.14) if the statement beneath the two following diagrams holds.

$$\begin{tikzcd}
 & U \arrow[dl, "t_i"'] \arrow[dr, "t_j"] & \\
U_i \arrow[rr, "d_{i,j}"'] & & U_j
\end{tikzcd}
\qquad
\begin{tikzcd}
F(U) \arrow[r, "\lim_i F(t_i)"] \arrow[d, "\theta_U"'] & \lim_i F(U_i) \arrow[d, "\lim_i \theta_{U_i}"] \\
G(U) \arrow[r, "\lim_i G(t_i)"] & \lim_i G(U_i)
\end{tikzcd}$$

For every object x in $G(U)$, there exists an object y in $\lim_i F(U_i)$
such that x is isomorphic to y when sent to $\lim_i G(U_i)$.

Such a property is usually referred to as *local essential surjectivity* (see [**37**]). In other words, the difference between the seed and the coseed encodes the information needed to express the idea of locality. But this same difference will prevent us from obtaining some of the important axioms needed to generate a sound homotopy theory. A first remedy to this problem will be to expand the definition of these vertebrae from an arrow category to the category of sequences of arrows.

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

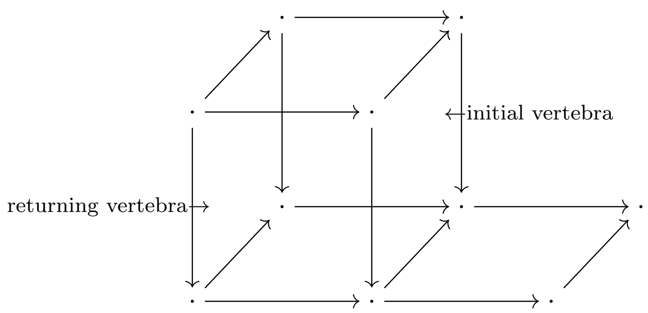
A chain of arrows as above will somehow encode different levels of locality. The descent condition of a sheaf will then be sent to a chain of arrows as above by replacing the arrow $X_0 \to X_1$ with the canonical arrow $F(U) \to \lim_i F(U_i)$ while the other arrows will be replaced with identities. These identities will somehow force the local specification of already-local objects to simply be local (*local* + *local* = *local*).

In general, the consideration of nodes of vertebrae with different seeds and coseeds does not suffice to handle all the algebraic operations. For instance, in the world of stacks, a composition as pictured in Figure 5 will involve two vertebrae on each side of a spine that are going to reduce the level of locality on the seed and coseeds of the output spines. The following pictures try[6] to illustrate this phenomenon for a conjugation of a spine of dimension 1 along two vertebrae of the same dimension.

$$\begin{pmatrix} & & \bullet \\ \bullet & \cong & \bullet \\ \bullet & \cong & \bullet \end{pmatrix} \otimes \underbrace{\begin{pmatrix} \bullet & & \\ \bullet & \cong & \bullet \\ \bullet & \cong & \bullet \end{pmatrix}}_{\text{initial spine}} \otimes \begin{pmatrix} \bullet & & \\ \bullet & \cong & \bullet \end{pmatrix} \mapsto \begin{pmatrix} \bullet & & \\ \bullet & \cong & \bullet \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \bullet & & \\ \bullet & \cong & \bullet \\ \bullet & \cong & \bullet \end{pmatrix}}_{\text{dual}} \otimes \begin{pmatrix} \bullet & & \\ \bullet & \cong & \bullet \end{pmatrix} \otimes \underbrace{\begin{pmatrix} & & \bullet \\ \bullet & \cong & \bullet \end{pmatrix}}_{\text{dual}} \mapsto \underbrace{\begin{pmatrix} \bullet & & \\ \bullet & & \bullet \\ \bullet & \cong & \bullet \end{pmatrix}}_{\text{returning spine}}$$

Composing a second time (as shown in the previous second line) along the two dual vertebrae will not bring us back to the initial vertebra. It will bring us back to a vertebra that will look like the initial vertebra but with a shift in the local parameters, so that there exists a canonical morphism of diagrams between the two vertebrae, as pictured below.



Such a structure will be called an *alliance of vertebrae* and will be denoted as an arrow $v \rightsquigarrow v_*$ where v denotes the 'initial vertebra' while v_* denotes the 'returning vertebra'. This type of structure will turn out to be the right structure to handle the concept of vertebra. For instance, we will see that the notation $\gamma \rightsquigarrow v$ needed in Remark 1.12 will naturally arise from the notation $v \rightsquigarrow v_*$. The notion of zoo will also be adapted to the notion of alliance of nodes of vertebrae, so that the axioms for intractions and surtractions will use commutative squares instead of arrows.

1.1.2.5. *Homotopy Hypothesis.* It is with the goal of understanding the ultimate language of cohomology theories and, more specifically, the natural context in which the proper base change theorems and Lefschetz hyperplane theorems live that Grothendieck introduced his definition of ∞-groupoid (see [**10**, page 5, 1st par.] and [**24**, page 39 of scan., 2nd par.]). The point was to parameterise the ∞-groupoids by the open sets of a topological space – thus replacing sheaves with ∞-*stacks* – to generalise the Galois-Poincaré theory developed in [**26**] to higher order homotopy and eventually retrieve the results of [**27**] at this level. In particular, one of the ideas behind ∞-groupoids was to recover the viewpoint of complexes of sheaves, injective resolutions and derived categories without 'resolving' the coefficients (see [**10**, page 5, last three par.] and [**24**, page 39 of scan., 2nd par.]). The reading of [**24**] also gives the feeling that the point of Grothendieck was to make his definition

i) *algebraic*, by defining an ∞-groupoid as the model for a certain sketch;
ii) *algorithmic*, by exposing an explicit and inductive construction of the operations and coherences encoded by the sketch (see [**24**, p. 20 of scan., l. 2-4]);

[6] The case that is presented there could be recovered by the notion of nodes of vertebrae. Alliances become really needed when considering framings of spines of dimension 2 along vertebrae of dimension 1.

iii) *intuitive*, by reproducing the inherent dynamic of higher topological paths via a set of abstract discs forming the objects of the sketch (see [**24**, p. 19 of scan., l. 21-28]).

Specifically, the formation of operations and coherences is handled via the notion of parallelism. By analogy with topological paths, two parallel paths are algebraically related when they form the border of another path. The main task is thus to define the class of parallel paths that may be the borders of higher paths. The definition of this class first requires the notion of *globular sum*, which encompasses all the possible gluings of discs of the sketch. The definition of the sketch then states that any parallel pair of arrows (f, g) from an n-disc $\mathbb{D}_n$ to a globular sum $\mathbb{B}$ is the border of an $(n+1)$-disc living in $\mathbb{B}$ (see next diagram).

(1.15)
$$\begin{tikzcd} \mathbb{D}_{n+1} \arrow[rrd, dashed, "\exists"] & & \\ \mathbb{D}_n \arrow[u, shift left, "s_n"] \arrow[u, shift right, "t_n"'] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & \mathbb{B} \end{tikzcd}$$

Interestingly, if one authorises the formation of all pushouts in the sketch, the notion of parallelism between f and g may be expressed in terms of a commutative diagram as given below on the left-hand side, where $\mathbb{S}_{n-1}$ is a pushout of discs that may be identified with an $(n-1)$-sphere. Because the sketch has now all the pushouts, one may also form the succeeding middle one. Factorisation (1.15) is then equivalent to saying that the dashed arrow $\lceil f, g \rceil$ of the succeeding middle diagram may be factorised as shown on the right.

$$\begin{tikzcd} \mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[d, "g"] \\ \mathbb{D}_n \arrow[r, "f"'] & \mathbb{B} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[d, "\delta_1^n"] \arrow[rd, bend left, "g"] & \\ \mathbb{D}_n \arrow[r, "\delta_2^n"'] \arrow[rr, bend right, "f"'] & \mathbb{S}_n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\lceil f,g \rceil"] & \mathbb{B} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & \mathbb{D}_{n+1} \arrow[rd, dashed, "\exists"] & \\ \mathbb{S}_n \arrow[ru, "\gamma_{n+1}"] \arrow[rr, "\lceil f,g \rceil"] & & \mathbb{B} \end{tikzcd}$$

In other words, the operations and coherences of ∞-groupoids come from the filling of a given commutative square of parellel arrows by a vertebra as follows.

$$\begin{tikzcd} \mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[d, "\delta_1^n"] \arrow[rrd, bend left, "g"] & & \\ \mathbb{D}_n \arrow[r, "\delta_2^n"'] \arrow[rrr, bend right, "f"'] & \mathbb{S}_n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\gamma_{n+1}"] & \mathbb{D}_{n+1} \arrow[r, dotted] & \mathbb{B} \end{tikzcd}$$

On the other hand, all the algebraic operations defined in this thesis to recover the properties of model categories may be expressed in terms of the existence of a certain vertebra filling a certain given commutative square of parallel arrows going to a certain globular sum. This last point will therefore allow us, in Chapter 6, to provide ∞-groupoids with a homotopy theory called *spinal category*. A full model structure may result from such a structure when some condition on a pushout holds (see Chapter 5). The computation of this pushout in the category of ∞-groupoids may be achieved via the reflection functor that will be constructed in Chapter 5. Future work will aim at using the results of the present thesis to provide ∞-groupoids with a closed model structure and prove that the Homotopy Hypothesis holds.

1.1.3. Chapter 2. This chapter develops the theory of alliances of nodes of vertebrae. The propositions given there may be restricted to the world of vertebrae by taking 'identity alliances of vertebrae'. However, important theories such as that of stacks do require the notion of alliance. As explained at the end of section 1.1.2.4, alliances are in this case needed when considering framings of spines of dimension 2 along vertebrae of dimension 1. This chapter is fairly formal and the astute reader can recover the proofs of the propositions without too much difficulty.

1.1.4. Chapter 3. This chapter finishes the proof of the two-out-of-three property for the notion of weak equivalence introduced in Chapter 2. The level of this chapter requires some effort on the part of the readers to digest the different notions required by the proof. The chapter therefore includes a long introduction discussing the case of topological spaces. I warn the reader that the difficulty of this chapter is not due to the 3-dimensional structure of alliances, but the iterative nature of the notion of coherence in higher category theory. On the other hand, the notion of alliance makes many concepts easier to handle by allowing the relaxation of constraints that would have been inevitable in the case of vertebrae.

1.1.5. Chapter 4. This chapter organises the different objects defined in Chapters 2 & 3 into mathematical structures whose properties are very close to those of model categories and categories of fibrant objects. This chapter presents no actual difficulty other than the need for a long – but inevitable – glossary of elementary structures. The different notions are rather uninteresting from a general and theoritical point of view and their abstract formulations always come along with examples illustrating their essence from the point of view of vertebrae.

1.1.6. Chapter 5. This chapter extends the construct initiated by Chapter 4 in order to include theories such as those of sheaves and spectra. The chapter only shows the way towards the construction of homotopy theories without entering the details, which are left to the reader because of the various interests by which they might be motivated. Instead, it focuses on the technical constructions, such as those of fibrant replacements, (weak) factorisation systems or reflection functors. These constructions are independent of the previous chapters.

1.1.7. Chapter 6. This chapter uses the formalism attached to the theory of vertebrae to equip Grothendieck's ∞-groupoids with a spinal category. The difficulty of this chapters only lies in the information carried along the proof as well as the fact that the reader is required to master the notions of Chapter 3. Other than that, no new idea is introduced and the constructions only aims at a direct application of the concepts developed in Chapters 2 & 3.

1.2. Conventions and usual theory

The goal of this section is to recall the notions that will be considered known by the reader throughout this thesis. Most of them may be found in [**34**].

1.2.1. Set theory and category theory.

1.2.1.1. *Foundation.* In the literature, one can come across three main theories axiomatising the notion of set. The first one is called Zermelo-Fraenkel set theory with axiom of choice (abbr. ZFC) and focuses on sets only. The second one is called von Neumann-Bernays-Gödel set theory (abbr. NBG) and focuses on sets and classes. The third one is called Kelley-Morse set theory (abbr. KM) and also focuses on sets and classes. The second one is a conservative extension[7] of the first one while the third one is stronger than the first and second one. The transition from ZFC to NBG solves the Russell's Paradox when considering the 'set of sets' in ZFC. The passage from NBG to KM solves the problem of defining a product of classes over a class for example. More generally, it solves the problem of defining large limits and colimits. However, KM is not a conservative extension of ZFC, which would be a desirable property. Thoughout the present thesis, in order to avoid the Russell's Paradox and be able to define large constructions, we will consider a conservative extension of ZFC, called NBG_ω, by considering the axioms of NBG up to a transfinite induction. The domain of discourse

[7] A theory T' is said to be a *conservative extension* of a theory T if the language of T' extends that of T, every theorem of T holds in T' and any theorem of T' holds in T if written in the language of T.

of NBG_ω consists of *k-classes* for any non-negative integer k. The term 0-class will later be replaced with the term *set*. The set theory NBG_ω is then defined as follows:

for all non-negative integer k, k-classes and $(k+1)$-classes satisfy NBG by taking the role of sets and classes, respectively.

Later on, the term 1-class will be shortened to *class* while any type of class will be referred to as *higher class*. The restriction of NBG_ω to sets is exactly ZFC while its restriction to sets and classes shows more structure than NBG at the level of classes. The difference between KM amd NBG_ω is that the former could be seen as including what could stand for *ω-classes* in its theory; that is to say a type of classes including all the others and thus creating an impredicative axiomatisation. By not doing so, NBG_ω defines a conservative extension of ZFC.

1.2.1.2. *Metamathematics and mathematics.* In the spirit of [**34**], the prefix *meta*- will be utilised to refer to the size of the considered objects, mostly because they are not sets. In this sense, a *metafunction* will refer to any 'function' between any type of class while a *function* will refer to any function between sets. This terminology is motivated by the use of mixed structures such as the notion of *category*, which, herein, refers to the usual notion of locally small category whose collection of objects is a class. When higher classes are involved, the category will be called a *metacategory*.

1.2.1.3. *Convention on the notation of compositions.* In the sequel, we will meet various structures equipped with a notion of 'hom-set' – or 'hom-class' – on which a notion of composition will take a pair of arrows $f : a \to b$ and $g : b \to c$ to carry out an arrow $h : a \to c$. Such an operation will be denoted as an arrow of the form $\mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ while the image of a pair (g, f) throught this operation will be denoted by a functional notation $f \circ g$.

1.2.1.4. *Category of sets.* The category whose objects are sets and whose morphisms are functions will be denoted by **Set**.

1.2.1.5. *Categories of presheaves.* For any small category D, the category[8] whose objects are presheaves $D^{\mathrm{op}} \to \mathbf{Set}$ and whose morphisms are natural transformations in **Set** between the presheaves in question will be denoted by $\mathbf{Psh}(D)$.

1.2.1.6. *Categories of simplicial sets.* Let $\boldsymbol{\Delta}$ denote the *simplex category* whose objects are non-zero finite ordinals and whose morphisms are order-preserving maps. Then, the category $\mathbf{Psh}(\boldsymbol{\Delta})$ will be called the *category of simplical sets* and denoted by **sSet**.

1.2.1.7. *Categories of functors.* Let $\mathcal{C}$ be a category. For any fixed small category D, the category whose objects are functors $D \to \mathcal{C}$ and whose morphisms are the natural transformations in $\mathcal{C}$ between the functors in question will be denoted by $\mathcal{C}^D$.

1.2.1.8. *Metacategory of functors.* Let $\mathcal{C}$ be a metacategory. For any fixed category $\mathcal{D}$, the metacategory whose objects are functors $\mathcal{D} \to \mathcal{C}$ and whose morphisms are the natural transformations in $\mathcal{C}$ between the functors in question will be denoted by $[\mathcal{D}, \mathcal{C}]$.

1.2.1.9. *Category of topological spaces.* The category whose objects are topological spaces and whose morphisms are continuous functions will be denoted by **Top**.

1.2.1.10. *Category of nonnegatively graded chain complexes.* Let R be a ring. The category whose objects are nonnegatively graded chain complexes of left R-modules and whose morphisms are (nonnegatively graded) sequences of homomorphisms of left R-modules preserving the boundary maps will be denoted by $\mathbf{Ch}_R$.

1.2.1.11. *Category of small n-categories.* Let n denote a natural number. The category whose objects are small (strict) n-categories and whose morphisms are (strict) n-functors will be

[8] The term *category* is justified by the axiom of limitation of size in NBG.

denoted by $\mathbf{Cat}(n)$. An explicit definition of the category of small n-categories may be found in [**36**].

1.2.1.12. *Ordinals.* Recall that in ZFC, an *ordinal* is the isomorphism class of well-ordered[9] sets. It follows from this definition that the proper class of ordinals is also well-ordered. As a result, ordinals as well as the class of all ordinals may be seen as ordered categories. In the sequel, the category whose objects are ordinals and whose morphisms $\alpha \to \beta$ correspond to the order relations $\alpha < \beta$ will be denoted by $\mathbf{O}$. Then, for every ordinal number n in $\mathbf{O}$, the full subcategory of $\mathbf{O}$ whose objects are those ordinals less than n will be denoted by $\mathbf{O}(n)$. Note that for any ordinal n, the category $\mathbf{O}(n)$ is small.

1.2.1.13. *Representation of ordinals.* Recall that ordinals are associated with a class of representatives (introduced by von Neumann in 1920) whose elements are constructed by only using the empty set and the usual structures given by ZFC. For instance, the finite ordinal 2 is coded by a sequence containing two empty sets.

$$2 = (\emptyset, \emptyset) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$$

This provides the class of ordinals with a nice arithmetic where the strict order relation is given by the binary relation $\in$ (*being a member of*). If the union operator of ZFC is denoted by $\cup$, this representation characterises the ordinals of $\mathbf{O}$ as follows.

Lemma 1.14. *Let n be the representative of an ordinal in $\mathbf{O}$. The ordinal n*

1) *is either 0, i.e. the empty set;*
2) *or a limit ordinal, i.e. n satisfies the equation $n = \cup n$;*
3) *or a successor ordinal, i.e. there exists an ordinal m such that $n = m \cup \{m\}$.*

In the sequel, a successor ordinal of the form $m \cup \{m\}$ will be denoted by $m + 1$. The ordinal m will also be said to be the *predecessor* of $m+1$. Because the ordinal 0 satisfies the equation $0 = \cup 0$, it will later be regarded as a limit ordinal.

1.2.1.14. *Omega.* The least infinite ordinal will be denoted by ω. Recall that its set of elements is in bijection with the set of non-negative integers $\mathbb{N}$. By definition, the ordinal ω is a limit ordinal.

1.2.1.15. *Categorical arithmetic.* The notation $\mathbf{O}(\_)$ for ordinal categories will mainly serve to make our structure clearly determined; e.g. the notation $\mathbf{O}(\kappa + 1)$ will be used to avoid any ambiguity caused by a notation of the form $\kappa + 1$. However, some obvious categories will regularly be used and denoted using special notations. We will thus denote by

- $\mathbf{1}$ a terminal category.
- $\mathbf{2}$ a category consisting of two objects $\{0, 1\}$ and an arrow $c : 0 \to 1$;
- $\mathbf{3}$ a category consisting of three objects $\{0, 1, 2\}$ and two arrows $a : 0 \to 1$ and $b : 1 \to 2$ together with their composition $b \circ a : 0 \to 2$;
- $\emptyset$ an initial category (i.e. empty category);
- $\mathbf{2} + \mathbf{2}$ a category consisting of four objects and two arrows $a : 0 \to 1$ and $b : 2 \to 3$;
- $\mathbf{sq}$ the product of $\mathbf{2}$ with itself, that is to say a square category of four objects and four arrows.

[9] Recall that a set A is said to be *well-ordered* if it is equipped with a well-order, where a *well-order on A* is a total order on A satisfying the property that every non-empty subset of A has a least element for the order.

Remark 1.15. There is a commutative diagram of functors of the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbf{2}+\mathbf{2} \arrow[d] \\ \mathbf{2} \arrow[r] & \mathbf{3} \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} (\quad) \arrow[r, mapsto] \arrow[d, mapsto] & (\bullet \to \bullet \quad \bullet \to \bullet) \arrow[d, mapsto] \\ (\bullet \to \bullet) \arrow[r, mapsto] & (\bullet \to \bullet \to \bullet) \end{tikzcd}$$

where the bottom functor sends the arrow $c : 0 \to 1$ to the composite $b \circ a : 0 \to 2$ and the vertical right functor is the obvious functor sending $1 \mapsto 1$, $2 \mapsto 1$ and $3 \mapsto 2$ with respect to the notations given above.

1.2.1.16. *Commutative squares.* A *commutative square* in some category $\mathcal{C}$ is a diagram of the form $(\mathbf{sq}, \alpha)$. More explicitly, it is encoded by a 4-tuple of four morphisms $\gamma : A \to B$, $\gamma' : A \to B$, $u : A \to A'$ and $v : B \to B'$ such that the relation $v \circ \gamma = u \circ \gamma'$ holds. This data will usually be denoted as a diagram of the form given below on the left.

$$\begin{tikzcd} A \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & A' \arrow[d, "u"] \\ B \arrow[r, "v"] & B', \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] & B \arrow[d, "v"] \\ A' \arrow[r, "u"] & B', \end{tikzcd}$$

For short, such a diagram will be denoted as either $\gamma' : \gamma \Rightarrow u$ or $v : \gamma \Rightarrow u$, exactly as if it were seen from above or below. The underlying order on its arrows allows one to define a notion of *dual* for the preceding left commutative square, namely the right commutative square.

1.2.1.17. *Arrow categories.* Let $\mathcal{C}$ be a category. The functor category $\mathcal{C}^{\mathbf{2}}$ will later be referred to as the *arrow category* of $\mathcal{C}$. The objects of $\mathcal{C}^{\mathbf{2}}$ are arrows in $\mathcal{C}$ while the morphisms are commutative squares. A morphism will generally be denoted as an arrows $x : a \Rightarrow b$ when it is encoded by a commutative squares of the following form (as if it were seen from above).

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d, "a"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "x'"'] & \cdot \end{tikzcd}$$

For convenience, the functor categories $\mathcal{C}^{\mathbf{2}}$, $\mathcal{C}^{\mathbf{3}}$ and $\mathcal{C}^{\mathbf{O}(\omega)}$ will later be denoted by the ordinal notations as follows: $\mathcal{C}^2$, $\mathcal{C}^3$ and $\mathcal{C}^\omega$.

1.2.1.18. *Domain and codomain functors.* Let $\mathcal{C}$ be a category. In the sequel, the term *domain functor of* $\mathcal{C}$ will be used to name the obvious functor $\mathcal{C}^2 \to \mathcal{C}$ mapping an arrow $f : X \to Y$ of $\mathcal{C}$ to the object X, which will be denoted by $\mathrm{dom}_{\mathcal{C}}$ or dom if the category $\mathcal{C}$ is obvious. Similarly, the term *codomain functor* of $\mathcal{C}$ will be sued to name the obvious functor $\mathcal{C}^2 \to \mathcal{C}$ mapping an arrow $f : X \to Y$ of $\mathcal{C}$ to the object Y, which will be denoted by $\mathrm{cod}_{\mathcal{C}}$ or cod if the category $\mathcal{C}$ is obvious.

Remark 1.16. The domain functor of $\mathcal{C}^2$, which will sometimes be denoted by $\mathrm{top} : \mathcal{C}^{\mathbf{sq}} \to \mathcal{C}^2$, truncates a commutative square at the level of its top arrow.

1.2.1.19. *Natural transformation as functors.* This section points out a common fact about natural transformation that will repeatedly be used in this text. Let $\mathcal{C}$ be a category, D be a small category and consider two functors A and B of type $D \to \mathcal{C}$. In the sequel, any natural transformation $\eta : A \Rightarrow B$ will equivalently be regarded as a functor $\eta : D \to \mathcal{C}^2$. In this case, the functor A is identified with the composite $\mathrm{dom}_{\mathcal{C}} \circ \eta$ while the functor B is identified with the composite $\mathrm{cod}_{\mathcal{C}} \circ \eta$.

1.2.1.20. *Functor of functor categories.* For any functor $M : \mathcal{C} \to \mathcal{B}$, any small category D induces an obvious post-composition functor $\mathcal{C}^D \to \mathcal{B}^D$. This functor will be denoted by the same letter M.

1.2.1.21. *Subfunctors.* Let D be a small category and $F : D \to \mathbf{Set}$ be a functor. A *subfunctor of F* is a functor $G : D \to \mathbf{Set}$ such that

1) for every object d in D, the inclusion $G(d) \subseteq F(d)$ holds;
2) for every morphism $t : d \to d'$ in D, the function $G(t) : F(d) \to F(d')$ is the restriction of $F(t)$ along the respective inclusion of the domains and codomains.

1.2.1.22. *Fibres.* For every functor $F : \mathcal{K} \to \mathcal{D}$ and object d in $\mathcal{D}$, the *fibre of F above d* will refer to the subcategory of $\mathcal{K}$ made of the objects x satisfying the equation $H(x) = d$ and the arrows whose images via the functor F are identities on d. Such a category will be denoted by $F^{-1}(d)$ or F_d when specified so.

1.2.1.23. *Cones.* Let D be a small category and $\mathcal{C}$ be a category. Denote by Δ_D the functor $\mathcal{C} \to \mathcal{C}^D$ that maps every object X of $\mathcal{C}$ to its associated constant functor $D \to \mathcal{C}$ mapping any object and arrow in D to the object X and identity on X, respectively. We shall speak of a *cone in $\mathcal{C}$ over D* to refer to a natural transformation of the form $\Delta_D(X) \Rightarrow F$ where X is an object in $\mathcal{C}$, which will be referred to as the *peak*, and F is a functor from D to $\mathcal{C}$.

1.2.1.24. *Cocones.* Let D be a small category, $\mathcal{C}$ be a category and denote by Δ_D the functor $\mathcal{C} \to \mathcal{C}^D$ defined in section 1.2.1.23. We shall speak of a *cocone in $\mathcal{C}$ over D* to refer to a natural transformation of the form $F \Rightarrow \Delta_D(X)$ where X is an object in $\mathcal{C}$, which will be referred to as the *peak*, and F is a functor from D to $\mathcal{C}$.

1.2.1.25. *Diagrams.* Let $\mathcal{C}$ be a category. A *diagram in $\mathcal{C}$* consists of a small category I and a functor $\alpha : I \to \mathcal{C}$. Such a structure will usually be denoted as a pair (I, α). If I is a terminal category, the diagram (I, α) will instead be denoted as $(\mathbf{1}, a)$ where a is the object picked out by α in $\mathcal{C}$.

Remark 1.17. Later on, we shall speak of a cone in $\mathcal{C}$ *defined over a diagram* (I, α) to refer to any cone of the form $\Delta_I(X) \Rightarrow \alpha$. Similarly, we shall speak of a cocone in $\mathcal{C}$ *defined over a diagram* (I, α) to refer to any cocone of the form $\Delta_I(X) \Rightarrow \alpha$.

1.2.1.26. *Limits.* Let D be a small category, $\mathcal{C}$ be a category and denote by Δ_D the functor $\mathcal{C} \to \mathcal{C}^D$ defined in section 1.2.1.23. For every functor $F : D \to \mathcal{C}$, a *limit of F in $\mathcal{C}$* is an object X in $\mathcal{C}$ equipped with a cone $\alpha : \Delta_D(X) \Rightarrow F$ such that, for every other object Y in $\mathcal{C}$ equipped with a cone $\beta : \Delta_D(Y) \Rightarrow F$, there exists a unique morphism $f : Y \to X$ in $\mathcal{C}$ for which the equation $\beta = \alpha \circ \Delta_D(f)$ holds. A cone such as α will later be said to be *universal* over its diagram (D, F).

1.2.1.27. *Colimits.* Let D be a small category, $\mathcal{C}$ be a category and denote by Δ_D the functor $\mathcal{C} \to \mathcal{C}^D$ defined in section 1.2.1.23. Similarly, for every functor $F : D \to \mathcal{C}$, a *colimit of F in $\mathcal{C}$* is an object X in $\mathcal{C}$ equipped with a cocone $\alpha : F \Rightarrow \Delta_D(X)$ such that, for every other object Y in $\mathcal{C}$ equipped with a cocone $\beta : F \Rightarrow \Delta_D(Y)$, there exists a unique morphism $f : X \to Y$ in $\mathcal{C}$ for which the equation $\beta = \Delta_D(f) \circ \alpha$ holds. A cocone such as α will later be said to be *universal* over its diagram (D, F).

1.2.1.28. *Yoneda lemma.* Recall that for every small category D, the *Yoneda Lemma* (see [**34**]) states that there are two natural isomorphisms in d and F of the following form.

$$[D^{\mathrm{op}}, \mathbf{Set}](D(\_, d), F) \cong F(d) \qquad [D, \mathbf{Set}](D(d, \_), F) \cong F(d)$$

It is well-known that such isomorphisms imply that the functors $D(\_, d) : D \to \mathbf{Psh}(D)$ and $D(d, \_) : D^{\mathrm{op}} \to \mathbf{Set}^D$ are fully faithful. These functors are usually referred to as the *Yoneda embeddings*.

1.2.1.29. *Colimit sketches.* A *colimit sketch* is a small category $\mathtt{S}$ equipped with a subset Q of its universal cocones (see section 1.2.1.27) such that, for every object x in $\mathtt{S}$, there exists a unique cocone in Q defined over a diagram of the form $(\mathbf{1}, x)$. The cocones in Q as well as their associated diagrams and colimits (i.e. their peaks) will be said to be *chosen.* A *model for a colimit sketch* $\mathtt{S}$ *in a category* $\mathcal{C}$ is a functor $\mathtt{S} \to \mathcal{C}$ that sends the chosen cocones of $\mathtt{S}$ to universal cocones in $\mathcal{C}$. The models of a colimit sketch $\mathtt{S}$ in $\mathcal{C}$ define the objects of a category, denoted by $\mathbf{Mod}_{\mathcal{C}}(\mathtt{S})$, whose morphisms are natural transformations in $\mathcal{C}$ over $\mathtt{S}$. For any colimit sketch $\mathtt{S}$, the category of models for $\mathtt{S}$ in $\mathbf{Set}$ will be denoted by $\mathbf{Mod}(\mathtt{S})$.

1.2.1.30. *Limit sketches.* A *limit sketch* is a small category $\mathtt{S}$ equipped with a subset Q of its universal cones (see section 1.2.1.26) such that, for every object x in $\mathtt{S}$, there exists a unique cone in Q defined over a diagram of the form $(\mathbf{1}, x)$. The cones in Q as well as their associated diagrams and limits (i.e. their peaks) will be said to be *chosen.* A *model for a limit sketch* $\mathtt{S}$ *in a category* $\mathcal{C}$ is a functor $\mathtt{S} \to \mathcal{C}$ that sends the chosen cones of $\mathtt{S}$ to universal cones in $\mathcal{C}$. The models of a limit sketch $\mathtt{S}$ in $\mathcal{C}$ define the objects of a category, denoted by $\mathbf{Mod}_{\mathcal{C}}(\mathtt{S})$, whose morphisms are natural transformations in $\mathcal{C}$ over $\mathtt{S}$. For any limit sketch $\mathtt{S}$, the category of models for $\mathtt{S}$ in $\mathbf{Set}$ will be denoted by $\mathbf{Mod}(\mathtt{S})$.

Later on, the obvious limit sketch obtained from inverting the arrows of a colimit sketch $\mathtt{S}$ will be denoted by $\mathtt{S}^{\mathrm{op}}$. The reader interested to know more about the theory of sketches may refer to [**1**].

Proposition 1.18. *Let* $\mathtt{A}$ *be a colimit sketch. The hom-bifunctor – or Yoneda embedding – given by* $\mathtt{A}(\_,\_) : \mathtt{A} \to \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$ *is a model of* $\mathtt{A}$ *in* $\mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$.

Proof. Let $x(\_) : I \to \mathtt{A}$ be a chosen diagram of $\mathtt{A}$ whose colimit will be denoted as $\lim_i x_i$. The statement follows from the following series of natural isomorphisms.

$$\begin{aligned}
\mathbf{Mod}(\mathtt{A}^{\mathrm{op}})(\mathrm{col}_i\, \mathtt{A}(_, x_i), X(_)) &\cong \lim_i \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})(\mathtt{A}(_, x_i), X(_)) \\
&\cong \lim_i X(x_i) \\
&\cong X(\lim_i x_i) \\
&\cong X(\mathrm{col}_i\, x_i) \\
&\cong \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})(\mathtt{A}(_, \mathrm{col}_i\, x_i), X(_))
\end{aligned}$$

By the Yoneda Lemma, it follows from the previous natural isomorphisms that an isomorphism $\mathrm{col}_i \mathtt{A}(\_, x_i) \cong \mathtt{A}(\_, \mathrm{col}_i x_i)$ holds if $\mathrm{col}_i x_i$ denotes a chosen colimit of $\mathtt{A}$. $\square$

1.2.1.31. *Inclusion of sketches.* Any functor from a colimit (resp. limit) sketch $\mathtt{A}$ to a colimit (resp. limit) sketch $\mathtt{A}'$ that is faithful, injective on object and is a model of $\mathtt{A}$ in $\mathtt{A}'$ (i.e. preserves the chosen colimits (resp. limits) of $\mathtt{A}$ to those of $\mathtt{A}'$) will be called an *inclusion of sketches.* A colimit (resp. limit) sketch $\mathtt{A}$ will be said to be a *subsketch* of a colimit (resp. limit) sketch $\mathtt{A}'$ if it is equipped with an inclusion of sketches $\mathtt{A} \to \mathtt{A}'$.

1.2.1.32. *Free completion and cocompletion.* Let $\mathtt{A}$ be a colimit (resp. limit) sketch. A *free cocompletion (resp. completion)* of $\mathtt{A}$ consists of a colimit (resp. limit) sketch $\mathtt{A}^{\triangledown}$ together with an inclusion of sketches $j : \mathtt{A} \to \mathtt{A}^{\triangledown}$ such that for every cocomplete (resp. complete) category $\mathcal{C}$ and model $F : \mathtt{A} \to \mathcal{C}$ in $\mathbf{Mod}_{\mathcal{C}}(\mathtt{A})$, there exists a unique model $F^{\triangledown} : \mathtt{A}^{\triangledown} \to \mathcal{C}$ in $\mathbf{Mod}_{\mathcal{C}}(\mathtt{A})$ up to natural isomorphism making the following diagram commute.

$$\begin{tikzcd}
\mathtt{A}^{\triangledown} \arrow[dr, "F^{\triangledown}"] & \\
\mathtt{A} \arrow[u, "j"] \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}$$

The functor $F^{\triangledown}$ will later be referred to as the *free extension* of F.

Remark 1.19. Any free completion of the underlying category of $\mathtt{A}$ provides a free cocompletion (resp. completion) at the level of the sketch. This means that $\mathtt{A}^{\triangledown}$ is equivalent to $\mathbf{Psh}(A)$ up to chosen colimits (resp. limits).

Example 1.20. Let $\mathtt{A}$ be a colimit sketch. For any free cocompletion $j : \mathtt{A} \to \mathtt{A}^{\triangledown}$, the free extension of the model $\mathtt{A}(\_,\_) : \mathtt{A} \to \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$ given by Proposition 1.18 produces a model $\mathtt{A}[\_] : \mathtt{A}^{\triangledown} \to \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$.

1.2.1.33. *Overcategories.* Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. The *category over* X is the category whose objects are the arrows of $\mathcal{C}$ going to X and whose morphisms are given by commutative triangle as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

Such a category will be called an *overcategory* and denoted by $\mathcal{C}/X$.

1.2.1.34. *Covering families.* Let D be a small category and d be an object in D. A *covering family* on d is a collection $S := \{u_i : d_i \to d\}_{i\in A}$ of arrows in D. In this case, it makes sense to define, for every morphism $f : c \to d$ in D, the *pullback of S along f* as the collection of arrows $f^*S := \{v_i : c_i \to c\}_{i\in A}$ where the arrow v_i is the pullback of u_i along f. Note that every morphism $g : d \to c$ gives rise to a family $g \circ S := \{g \circ u_i\}_{i\in A}$. This last operation is used to define a more complex operation on S as follows. For every $i \in A$, take a covering family T_i on d_i. We will denote by $S \circ \{T_i\}_{i\in A}$ the covering family on d obtained by the disjoint union of families $u_i \circ T_i$ for every $i \in A$.

1.2.1.35. *Grothendieck pretopologies.* Let D be a small category. A *Grothendieck pretopology* on D consists, for every object d in D, of a collection J_d of covering families S on d such that

1) (Stability) for every arrow $f : c \to d$ in D, the pullback f^*S exists in J_c;
2) (Locality) for every $i \in A$ and T_i in J_{d_i}, the covering family $S \circ \{T_i\}_{i\in A}$ is in J_d;
3) (Identity) for every object d in D, the singleton $\{\mathrm{id}_d : d \to d\}$ is in J_d.

Such a collection will usually be denoted by J. A category D equipped with a Grothendieck pretopology J on D will be called a *site*.

Remark 1.21. Every covering family $S = \{u_i : d_i \to d\}_{i\in A}$ on an object d in J_d may be seen as a functor $A \to D/d$ if A is seen as a discrete category. It follows from the stability and locality axioms that this functor extends to a product-preserving functor $A' \to D/d$ where A' is the product completion of A. This functor will be called the *stabilisation of S*.

1.2.1.36. *Adjunctions.* Recall that an *adjunction* between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ and two natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow U \circ F$ (the *unit*) and $\varepsilon : F \circ U \Rightarrow \mathrm{id}_{\mathcal{D}}$ (the *counit*) such that the composites

$$F \xRightarrow{F\eta} FUF \xRightarrow{\varepsilon F} F \quad \text{and} \quad U \xRightarrow{\eta U} UFU \xRightarrow{U\varepsilon} U \tag{1.16}$$

are the identity transformations on F and U, respectively. In the previous situation, the functor F is said to be the *left adjoint of* U while the functor U is said to be the *right adjoint of* F. This will be written as a pair $(F \dashv U : \mathcal{D} \to \mathcal{C}, \eta, \varepsilon)$ or as a diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

and for short denoted as $F \dashv U$. It follows from the definition of an adjunction that the subsequent functions are inverses of each other for every object $X \in \mathrm{Obj}(\mathcal{C})$ and $Y \in \mathrm{Obj}(\mathcal{D})$.

$$\left[\begin{array}{ccc} \mathcal{C}(X,U(Y)) & \to & \mathcal{D}(F(X),Y) \\ a & \mapsto & \varepsilon_Y \circ Fa \end{array}\right] \qquad \left[\begin{array}{ccc} \mathcal{D}(F(X),Y) & \to & \mathcal{C}(X,U(Y)) \\ b & \mapsto & Ub \circ \eta_X \end{array}\right]$$

It is not hard to check, using naturality and functoriality, that the previous functions are natural in X and Y. Conversely, it is well-known that any binatural bijection of the form $\mathcal{D}(F(X),Y) \cong \mathcal{C}(X,U(Y))$ induces an adjunction $F \dashv U$.

Example 1.22 (Kan extensions)**.** A particular case of the notion of adjunction is the notion of left (resp. right) Kan extension. Recall that the *left Kan extension* of a functor $F : \mathcal{D} \to \mathcal{C}$ along a functor $i : \mathcal{D} \to \mathcal{E}$ consists of a functor $\mathrm{Lan}_i F : \mathcal{E} \to \mathcal{C}$ and a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_i F \circ i$ such that for any other functor $G : \mathcal{E} \to \mathcal{C}$ and natural transformation $\theta : F \Rightarrow G \circ i$, there exists a unique natural transformation $\tau : \mathrm{Lan}_i F \Rightarrow G$ making the following diagram commute.

$$\begin{tikzcd} F \arrow[rr, Rightarrow, "\theta"] \arrow[dr, Rightarrow, "\eta"'] & & G \circ i \\ & \mathrm{Lan}_i F \circ i \arrow[ur, Rightarrow, "\tau i"'] & \end{tikzcd}$$

Dually, there is a notion of *right Kan extension*, which will be omitted as it is not required in the sequel.

Let now D be a small category, $\mathcal{C}$ be a category and $\mathbf{1}$ denote a terminal category. For every object d of D, denote by d the functor $\mathbf{1} \to D$ picking out the object d in D. It is well-known (see [**34**]) that when the extension $\mathrm{Lan}_d X : D \to \mathcal{C}$ exists for every functor $X : \mathbf{1} \to \mathcal{C}$ picking out an object in X, then the mapping $\mathrm{Lan}_d(\_)$ extends to a functor $\mathcal{C} \to \mathcal{C}^D$ that is left adjoint to the restriction functor $\nabla_d : F \mapsto F \circ d = F(d)$.

$$\begin{tikzcd} \mathcal{C}^D \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "\mathrm{Lan}_d"'] \arrow[from=1-1, bend right, "\nabla_d"'] \end{tikzcd}$$

Similarly, limits and colimits over a small category D may be defined as right and left Kan extension along the canonical functor $! : D \to \mathbf{1}$, respectively.

$$\begin{tikzcd} \mathcal{C}^D \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "\mathrm{col}_D"'] \arrow[from=1-1, bend right, "\Delta_D"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, phantom, "\perp"] & \mathcal{C}^D \arrow[l, bend right, "\Delta_D"'] \arrow[from=1-1, bend right, "\lim_D"'] \end{tikzcd}$$

Here, the functor Δ_D is regarded as the pre-composition functor $\mathcal{C} \to \mathcal{C}^D$ mapping an object $X : \mathbf{1} \to \mathcal{C}$ to the following constant functor.

$$D \xrightarrow{\;!\;} \mathbf{1} \xrightarrow{\;X\;} \mathcal{C}$$

An interesting relationship between the preceding three adjunctions is that for every object d in D, the equation $\nabla_d \circ \Delta_D = \mathrm{id}_{\mathcal{C}}$ holds. Later on, this equation will turn out to be useful.

1.2.1.37. *Limits and colimits as adjoints.* Let D be a small category and $\mathcal{C}$ be a category. The category $\mathcal{C}$ will be said to *admit limits over* D if the functor $\Delta_D : \mathcal{C} \to \mathcal{C}^D$ has a right adjoint, which will be denoted by $\lim_D : \mathcal{C}^D \to \mathcal{C}$. The latter functor is called the *limit functor over* D. Similarly, the category $\mathcal{C}$ will be said to *admit colimits over* D if the functor $\Delta_D : \mathcal{C} \to \mathcal{C}^D$ has a left adjoint, which will be denoted by $\mathrm{col}_D : \mathcal{C}^D \to \mathcal{C}$. This other functor is called the *colimit functor over* D.

1.2.1.38. *Limits and Yoneda embedding.* This section only focuses on limits since a dual version of the subsequent arguments leads to similar facts for colimits. Let D be a small category and $\mathcal{C}$ be a category that admits limits over D. It is quite straightforward to see that the following isomorphism holds.

$$\begin{aligned}
\lim_D \mathcal{C}(X, F(_)) &\cong \mathbf{Set}(\mathbf{1}, \lim_D \mathcal{C}(X, F(_))) && (\mathbf{1} \text{ is a generator of } \mathbf{Set}) \\
&\cong \mathbf{Set}^D(\Delta_D(\mathbf{1}), \mathcal{C}(X, F(_))) && \text{(adjointness)} \\
&\cong \mathcal{C}^D(\Delta_D(X), F(_)) && \text{(reformulation)} \\
&\cong \mathcal{C}(X, \lim_D F(_)) && \text{(adjointness)}
\end{aligned}$$

In particular, this shows that limits commute with the hom-sets of $\mathcal{C}$. Similarly, it follows from the definition of $\Delta_D : \mathcal{C} \to \mathcal{C}^D$ and $\Delta_D : \mathbf{Set} \to \mathbf{Set}^D$ that the following equation holds.

$$\Delta_D(\mathcal{C}(X, Y))(\_) \cong \mathcal{C}(X, \Delta_D(Y)(\_))$$

In fact, not only the hom-sets commute with $\lim_D$ and Δ_D, but they also commute with the unit $\eta : \mathrm{id}_D \Rightarrow \lim_D \Delta_D$ and counit $\varepsilon : \Delta_D \lim_D \Rightarrow \mathrm{id}_D$ of the adjunction $\Delta_D \dashv \lim_D$ in $\mathbf{Set}$.

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "="] \arrow[d, "\eta_{\mathcal{C}(X,Y)}"'] & \mathcal{C}(X,Y) \arrow[d, "\mathcal{C}(X,\eta_Y)"] \\
\lim_D \Delta_D \mathcal{C}(X,Y) \arrow[r, "\cong"'] & \mathcal{C}(X, \lim_D \Delta_D(Y))
\end{tikzcd}$$

1.2.1.39. *Wedges and ends.* This section only recalls the notion of end as the dual notion of coend may be recovered in much the way. Let $\mathcal{C}$ be a category and $\mathtt{T}$ be a small category. Consider a functor $F : \mathtt{T} \times \mathtt{T}^{\mathrm{op}} \to \mathcal{C}$. A *wedge over* F is given by an object e in $\mathcal{C}$ and, for every object t in T, an arrow $K_t : e \to F(t,t)$ such that the subsequent left diagram commutes for every arrow $f : t \to t'$ in $\mathtt{T}$.

$$\begin{tikzcd}
e \arrow[r, "K_t"] \arrow[d, "K_{t'}"'] & F(t,t) \arrow[d, "F(t,f)"] \\
F(t',t') \arrow[r, "F(f,t')"'] & F(t,t')
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[r, "g"] \arrow[rr, bend left, "K_t"] \arrow[dr, bend right, "K_{t'}"'] & e^* \arrow[r, "K_t^*"] \arrow[d, "K_{t'}^*"] & F(t,t) \arrow[d, "F(t,f)"] \\
& F(t',t') \arrow[r, "F(f,t')"'] & F(t,t')
\end{tikzcd}$$

A *morphism of wedges* from (e, K) to (e^*, K^*) is given by an arrow $g : e \to e'$ making the above-displayed right diagram commute. An *end* for the functor $F : \mathtt{T} \times \mathtt{T}^{\mathrm{op}} \to \mathcal{C}$ is a terminal object, say (e, K), in the category of wedges over F. The object e is usually denoted by an integral as follows.

$$\int_{t \in \mathtt{T}} F(t,t) \tag{1.17}$$

Example 1.23. For any functors $A : \mathtt{T} \to \mathcal{C}$ and $B : \mathtt{T} \to \mathcal{C}$, it is well-known that the set $\mathcal{C}^{\mathtt{T}}(A, B)$ is given by (or isomorphic to) the end $\int_{t \in \mathtt{T}} \mathcal{C}(A(t), B(t))$ in $\mathbf{Set}$.

Even if there only appear the terms $F(t,t)$ in the expression (1.17), the end of F depends on the object $F(t,t')$ for every arrow $f : t \to t'$ in $\mathtt{T}$. In fact, this may be made more explicit in terms of limits. Denote by $\mathbf{F3}(\mathtt{T})$ the category whose objects are arrows $f : t \to t'$ and whose morphisms from $f : t \to t'$ to $g : s \to s'$ are given by commutative diagrams of the following form.

(1.18)
$$\begin{tikzcd} t \arrow[d, "f"'] & s \arrow[l, "a"'] \arrow[d, "g"] \\ t' \arrow[r, "b"'] & s' \end{tikzcd}$$

The functor $F : \mathtt{T} \times \mathtt{T}^{\mathrm{op}} \to \mathcal{C}$ then induces a functor $\tilde{F} : \mathbf{F3}(\mathtt{T}) \to \mathcal{C}$ mapping an object $f : t \to t'$ of $\mathbf{F3}(\mathtt{T})$ to the object $F(t,t')$ and a commutative square of the form (1.18) to the morphism $F(a,b) : F(t,t') \to F(s,s')$. The next proposition is a very well-known result about ends.

Proposition 1.24. *If $\mathcal{C}$ admits limits over $\mathbf{F3}(\mathtt{T})$, then the end $\int_{t\in\mathtt{T}} F(t,t)$ is isomorphic to the limit $\lim_{\mathbf{F3}(\mathtt{T})}\tilde{F}$.*

Proof. This is discussed in [**34**]. The idea is that the universal wedge provides a universal cone

$$\Delta_{\mathbf{F3}(\mathtt{T})}\Big(\int_{t\in\mathtt{T}} F(t,t)\Big) \Rightarrow \tilde{F}$$

whose component at $f : t \to t'$ is given by the following arrow.

$$\textstyle\int_{t\in\mathtt{T}} F(t,t) \xrightarrow{K_t} F(t,t) \xrightarrow{F(t,f)} F(t,t')$$

It is then not hard to show that the previous arrows are compatible with the morphisms of the form $F(a,b) : F(t,t') \to F(s,s')$. □

Remark 1.25. Somehow, Proposition 1.24 shows that a better notation for $\int_{t\in\mathtt{T}} F(t,t)$ could be the following one.

$$\int_{f:s\to t\in\mathtt{T}} F(s,t)$$

Proposition 1.24 might also help the reader to see that, for every pair of functors $F : \mathtt{T}\times\mathtt{T}^{\mathrm{op}} \to \mathcal{C}$ and $G : \mathtt{T}\times\mathtt{T}^{\mathrm{op}} \to \mathcal{C}$, an isomorphism will hold between $\int_{t\in\mathtt{T}} F(t,t)$ and $\int_{t\in\mathtt{T}} G(t,t)$ if there is a natural isomorphism $F(s,t) \cong G(s,t)$ in the variables s and t.

1.2.1.40. *Complete and cocomplete categories.* A category $\mathcal{C}$ will be said to be *complete* if it admits limits over all small categories D. This therefore requires the limit functor $\lim_D : \mathcal{C}^D \to \mathcal{C}$ to exist for every small category D. Similarly, a category $\mathcal{C}$ will be said to be *cocomplete* if it admits colimits over all small categories D.

Example 1.26. For any ordinal κ in $\mathbf{O}$, the subcategory $\mathbf{O}(\kappa)$ is cocomplete. The colimit of a functor $F : D \to \mathbf{O}(\kappa)$ for some small category D is the union (or supremum) of the set of the ordinals in the image of F.

$$\lim_D F = \cup_{d\in\mathrm{Obj}(d)} F(d)$$

Example 1.27. The category **Set** is complete and cocomplete. The limit $\lim_D F$ of a functor $F : D \to \mathbf{Set}$ for some small category D is given by the set

(1.19) $$\{(x_d)_{d\in\mathrm{Obj}(D)} \mid x_d \in F(d) \text{ and for any } t : d \to d' \text{ in } D :\ F(t)(x_d) = x_{d'}\}$$

while the colimit $\mathrm{col}_D F$ of a functor $F : D \to \mathbf{Set}$ for some small category D is given by the quotient set

$$\{(d,x) \mid d \in \mathrm{Obj}(D);\ x \in F(d)\}/\sim$$

where $\sim$ denotes the binary relation whose relations $(d,x) \sim (d',x')$ are defined when there exists an object e and two arrow $t : d \to e$ and $t' : d' \to e$ in D such that the equation $F(t)(x) = F(t')(x')$ holds. Note that in the case where D is a preordered category $\mathbf{O}(\kappa)$ for some ordinal κ, the binary relation $\sim$ is an equivalence relation.

The next proposition recalls a very well-known fact.

Proposition 1.28. *If a category $\mathcal{C}$ is complete (resp. cocomplete), then so is $\mathcal{C}^D$ for any small category D where the limits (resp. colimits) are defined objectwise in $\mathcal{C}$.*

Proof. Straightforward. □

Remark 1.29. It follows from classical results (see [**34**]) that if a category is complete, then it admits right Kan extensions along any functor. Similarly, if a category is cocomplete, then it admits left Kan extensions along any functor.

1.2.1.41. *Cardinality.* Let A be an object in **Set**. The *cardinality of* A is the least ordinal κ such that there is a bijection between A and κ. In ZFC, the axiom of choice ensures that the cardinality of a set A always exists, which will be denoted by $|A|$. A notion of cardinality for small categories follows from that for sets. Consider a small category D. The *cardinality of* D is the cardinality of the set

$$\mathrm{Ar}(D) := \coprod_{a,b\in\mathrm{Obj}(D)} D(a,b)$$

where $\mathrm{Obj}(D)$ is the set of objects of D. The cardinality of D will be denoted by $|D|$. Below is given a well-known result on the commutativity of limits and colimits.

Proposition 1.30. *For every small category D and limit ordinal $\kappa \geq |D|$, the following canonical natural transformation valued in* **Set** *over* $\mathbf{Set}^{\mathbf{O}(\kappa)\times D}$ *is an isomorphism.*

$$\mathrm{col}_{\mathbf{O}(\kappa)} \lim_D(\_) \Rightarrow \lim_D \mathrm{col}_{\mathbf{O}(\kappa)}(\_)$$

Proof. A proof may be found in [**2**, Corollaire 9.8]. For the sake of self-containedness, the proof is recalled below. We essentially keep the same notations as in Example 1.27. Let $F_{\_}(\_) : \mathbf{O}(\kappa) \times D \to \mathbf{Set}$ be a functor. An equivalence class for the equivalence relation $\sim$ will be denoted using brackets, i.e. $[(k,x)]$. The notation

$$(x_d)^F_{d\in\mathrm{Obj}(D)}$$

will be used to mean that the collection $(x_d)_{d\in\mathrm{Obj}(D)}$ is compatible with the action of the functor F in the appropriate way (see equation (1.19)). By definition, the following equations hold.

$$\mathrm{col}_{\mathbf{O}(\kappa)}\mathrm{lim}_D F = \{[k,(x_d)^F_{d\in\mathrm{Obj}(D)}] \mid (x_d)^F_{d\in\mathrm{Obj}(D)} \in \mathrm{lim}_D F_k(\_)\}$$
$$\mathrm{lim}_D\mathrm{col}_{\mathbf{O}(\kappa)} F = \{([k_d,x_d])^F_{d\in\mathrm{Obj}(D)} \mid [k_d,x_d] \in \mathrm{col}_{\mathbf{O}(\kappa)} F_{\_}(d)\}$$

The natural transformation $\mathrm{col}_{\mathbf{O}(\kappa)} \lim_D \Rightarrow \lim_D \mathrm{col}_{\mathbf{O}(\kappa)}(\_)$ is given by the following mapping.

$$[k,(x_d)^F_{d\in\mathrm{Obj}(D)}] \mapsto ([k,x_d])^F_{d\in\mathrm{Obj}(D)}$$

Let us prove its surjectiveness. Consider an element in $\mathrm{lim}_D\mathrm{col}_{\mathbf{O}(\kappa)}F$ of the following form.

$$([k_d,x_d])^F_{d\in\mathrm{Obj}(D)}$$

By definition of the compatibility with the action of F, for any arrow $t : d \to d'$ in D, there exist arrows $s_d : k_d \to e_t$ and $s_{d'} : k_{d'} \to e_t$ in $\mathbf{O}(\kappa)$ such that the next equation holds.

$$F_{s_d}(d) \circ F_{k_d}(t)(x_d) = F_{s_{d'}}(d')(x_{d'}) \tag{1.20}$$

Since κ is a limit ordinal greater than or equal to $|D|$, we may define the following supremum in $\mathbf{O}(\kappa)$.

$$\begin{tikzcd} & & \cup_{t\in\mathrm{Ar}(D)} e_t & & \\ e_{t_0} \arrow[urr, "g_{t_0}"] & e_{t_1} \arrow[ur, "g_{t_1}"] & e_{t_2} \arrow[u, "g_{t_2}"] & \dots & e_t \arrow[ull, "g_t"'] \end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{cardinality given by } |D|}$$

Denote the supremum $\cup_{t\in \mathrm{Ar}(D)} e_t$ by e. Note that for any pair of arrows $t : d \to d'$ and $t' : d'' \to d$ in D, the arrows $g_t \circ s_d : k_d \to e$ and $g_{t'} \circ s_d : k_d \to e$ are equal in $\mathbf{O}(\kappa)$. The family made of the elements $F_{g_t \circ s_d}(d)(x_d)$ for every object d in D is then compatible with the action of F, since, for any arrow $t : d \to d'$ in D, the following equation holds from equation (1.20).

$$F_e(t) \circ F_{g_t \circ s_d}(d)(x_d) = F_{g_t \circ s_d}(d) \circ F_{k_d}(t)(x_d) = F_{g_t}(d) \circ F_{s_{d'}}(d')(x_{d'})$$

In addition, it is not hard to check that the mapping rule of the natural transformation $\mathrm{col}_{\mathbf{O}(\kappa)} \lim_D(\_) \Rightarrow \lim_D \mathrm{col}_{\mathbf{O}(\kappa)}(\_)$ includes the rule

$$[e, (F_{g_t \circ s_d}(d)(x_d))^F_{d\in \mathrm{Obj}(D)}] \mapsto ([k_d, x_d])^F_{d\in \mathrm{Obj}(D)}$$

since $(k_d, x_d) \sim (e, F_{g_t \circ s_d}(d)(x_d))$. Let us now prove its injectiveness. Note that any equality $([k, x_d])^F_{d\in \mathrm{Obj}(D)} = ([k', x'_d])^F_{d\in \mathrm{Obj}(D)}$ implies the existence of cospans

$$\begin{tikzcd} & e_d & \\ k \arrow[ur, "s_d"] & & k' \arrow[ul, "s'_d"'] \end{tikzcd}$$

such that the identity $F_{s_d}(d)(x_d) = F_{s'_d}(d)(x'_d)$ holds for every object d in D. Now, define the following supremum, which will be denoted by e'.

$$\begin{tikzcd} & & \cup_{d\in \mathrm{Obj}(D)} e_d & & \\ e_{d_0} \arrow[urr, "g_0"] & e_{d_1} \arrow[ur, "g_1"] & e_{d_2} \arrow[u, "g_2"] & \dots & e_d \arrow[ull, "g_d"'] \end{tikzcd}$$

cardinality below $|D|$

For every object d in D, the arrows $g_d \circ s_d : k \to e'$ are equal in $\mathbf{O}(\kappa)$. The same is true for $g_d \circ s'_d : k' \to e'$. It follows that the equation

$$\lim_d F_{g_d \circ s_d}(d)((x_d)^F_d) = \lim_d F_{g_d \circ s'_d}(d)((x'_d)^F_d)$$

holds, which implies the identity $[k, (x_d)^F_{d\in \mathrm{Obj}(D)}] = [k', (x'_d)^F_{d\in \mathrm{Obj}(D)}]$. $\square$

In general, when a category $\mathcal{C}$ is cocomplete, the functor $\Delta_D : \mathcal{C} \to \mathcal{C}^D$ commutes with every colimit for any small category D (see Proposition 1.28). It follows from Proposition 1.30 that the unit of the adjunction $\Delta_D \vdash \lim_D$ commutes with colimits in **Set** as shown in the next proposition.

Proposition 1.31. *For every small category D and limit ordinal $\kappa \geq |D|$, denote by the letter η the units of the two adjunctions $\Delta_D \vdash \lim_D$ in* **Set** *and* $\mathbf{Set}^{\mathbf{O}(\kappa)}$. *The following diagram of canonical arrows in* **Set** *commutes for any functor $F : \mathbf{O}(\kappa) \to \mathbf{Set}$.*

$$\begin{tikzcd} \mathrm{col}_{\mathbf{O}(\kappa)} F(\_) \arrow[r, "\mathrm{col}_{\mathbf{O}(\kappa)} \eta_{F(\_)}"] \arrow[d, equal] & \mathrm{col}_{\mathbf{O}(\kappa)} \lim_D \Delta_D F(\_) \arrow[d, "\cong"] \\ \mathrm{col}_{\mathbf{O}(\kappa)} F(\_) \arrow[r, "\eta_{\mathrm{col}_{\mathbf{O}(\kappa)} F(\_)}"'] & \lim_D \Delta_D(\mathrm{col}_{\mathbf{O}(\kappa)} F(\_)) \end{tikzcd}$$

Proof. We keep the convention set in the proof of Proposition 1.30. We only need to check that the diagram of the statement commutes. For any set X, the unit $\eta_X : X \to \lim_D \Delta_D(X)$ maps an element of $x \in X$ to the constant collection $(x)_{d\in \mathrm{Obj}(D)}$. Similarly, for any functor $X : \mathbf{O}(\kappa) \to \mathbf{Set}$, the unit $\eta_{X(\_)} : X(\_) \to \lim_D \Delta_D(X(\_))$ maps an element of $x \in X(k)$ to the constant collection $(x)_{d\in\mathrm{Obj}(D)}$ in $\lim_D\Delta_D(X(k))$. The diagram of the statement is therefore encoded by the following mapping rules.

$$\begin{tikzcd}
{[(k,x)]} \arrow[r, mapsto, "\mathrm{col}_{\mathbf{O}(\kappa)}\eta_{F(_)}"] \arrow[d, equal] & {([k,(x)_{d\in\mathrm{Obj}(D)}])} \arrow[d, mapsto, "\cong"] \\
{[(k,x)]} \arrow[r, mapsto, "\eta_{\mathrm{col}_{\mathbf{O}(\kappa)}F(_)}"'] & {([(k,x)])_{d\in\mathrm{Obj}(D)}}
\end{tikzcd}$$

In particular, this shows that the diagram commutes. □

1.2.2. Abstract homotopy theory.

1.2.2.1. *Lifting properties.* Let $\mathcal{C}$ be a category and $f : X \to Y$ and $g : A \to B$ be two morphisms in $\mathcal{C}$. A diagram as given below on the left will be said to *admit* a lift if there exists an arrow $h : B \to X$ making the following right diagram commute.

(1.21)
$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
B \arrow[r, "v"] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
B \arrow[r, "v"] \arrow[ur, "h"] & Y
\end{tikzcd}$$

If every diagram as given on the left of (1.21) admits a lift then

- f will be said to *have the right lifting porperty* (abbrev. rlp) with respect to g;
- g will be said to *have the left lifting porperty* (abbrev. llp) with respect to f;

The class of morphisms in $\mathcal{C}$ that have the rlp with respect to a certain class $\mathcal{A}$ of morphisms in $\mathcal{C}$ will be denoted by $\mathbf{rlp}(\mathcal{A})$. Similarly, the class of morphisms in $\mathcal{C}$ that have the llp with respect to a certain class $\mathcal{A}$ of morphisms in $\mathcal{C}$ will be denoted by $\mathbf{llp}(\mathcal{A})$. The following properties are classical results.

Proposition 1.32. *Let $f : A \to B$ be an arrow that has the rlp with respect to an arrow $g : A \to B$ and suppose to be given a pullback as follows.*

$$\begin{tikzcd}
P \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Z \arrow[r, "f'"'] & Y
\end{tikzcd}$$

The arrow $p : P \to Z$ has the rlp with respect to $g : A \to B$.

Proposition 1.33. *Let $\gamma : A \to B$ be an arrow that has the llp with respect to an arrow $f : X \to Y$. Suppose to be given a pushout as follows.*

$$\begin{tikzcd}
A \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & A' \arrow[d, "p'"] \\
B \arrow[r, "p"'] \arrow[ur, phantom, "\ulcorner", very near end] & B'
\end{tikzcd}$$

The arrow $p' : A' \to B'$ has the left lifting property with respect to $f : X \to Y$.

Proposition 1.34. *If a morphism has the rlp with respect to two arrows $\delta : A \to B$ and $\beta : B \to C$, then it has the rlp with respect to $\beta \circ \delta : A \to C$.*

Proposition 1.35. *If a morphism has the llp with respect to two arrows $f : A \to B$ and $g : B \to C$, then it has the llp with respect to $g \circ f : A \to C$.*

Remark 1.36. Every isomorphism has the right and left lifting property with respect to any other arrows.

1.2.2.2. *Retracts.* Let $\mathcal{C}$ be a category and $f : X \to A$ and $g : Y \to B$ be two morphisms in $\mathcal{C}$. The morphism f is said to be a *retract of* g if there exist morphisms $i : X \to Y$, $j : A \to B$, $r : Y \to X$ and $s : B \to A$ satisfying the identities $r \circ i = \mathrm{id}_X$ and $s \circ j = \mathrm{id}_{X'}$ and such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ A \arrow[r, "j"] & B \arrow[r, "s"] & A. \end{tikzcd}$$

commutes in $\mathcal{C}$. It follows from the definitions that if the morphism $g : Y \to B$

- has the rlp with respect to certain morphism, then so does f;

- has the llp with respect to certain morphism, then so does f;

1.2.3. Abstract homotopy theory.

1.2.3.1. *Category-classes.* Let $\mathcal{C}$ be a category. We shall call *$\mathcal{C}$-class* a class of morphisms in $\mathcal{C}$ that contains all identity morphisms of $\mathcal{C}$ and is closed under composition. Thus, a $\mathcal{C}$-class may be seen as a subcategory of $\mathcal{C}$ whose objects are those of $\mathcal{C}$. A $\mathcal{C}$-class will be said to be *coherent* if it also contains every isomorphism of $\mathcal{C}$.

1.2.3.2. *Homotopy categories.* Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a $\mathcal{C}$-class of morphisms, called the *weak equivalences*. The goal of this section is to present a way of formally inverting the arrows of $\mathcal{C}$ that belongs to the class $\mathcal{W}$. Categorically, this is done in terms of a metacategory $\mathcal{C}[\mathcal{W}^{-1}]$ and a functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, called the *localisation functor*, sending arrows in $\mathcal{W}$ to isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$. The metacategory $\mathcal{C}[\mathcal{W}^{-1}]$ is generally not locally small and cannot thus be called a category. More specifically, the objects of $\mathcal{C}[\mathcal{W}^{-1}]$ are those of $\mathcal{C}$ and its arrows from an object A to an object B are equivalence classes of finite zigzags of arrows in $\mathcal{C}$ of the form

$$A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow \quad \longrightarrow B,$$

where the non-oriented arrows between objects may be either left or right and all left-oriented arrows are in $\mathcal{W}$, such that two zigzags are equivalent if and only if they are equal after

i) composing some arrows in the same direction;
ii) or replacing two adjacent copies of the same arrow in $\mathcal{W}$ of opposite directions with an identity.

The previous relation only becomes an equivalence relation after completion under transitivity and symmetry. With such a definition of $\mathcal{C}[\mathcal{W}^{-1}]$, we see that it could be useful to be able to permute the alternating arrows belonging to either $\mathcal{W}$ or $\mathcal{C}$ in order to get a minimal representation of the form

$$A \xleftarrow{\in\mathcal{W}} X \longrightarrow B. \tag{1.22}$$

In fact, it turns out that, in practice, a reasonable form to consider is the following:

$$A \xleftarrow{\in\mathcal{W}} X \longrightarrow Y \xleftarrow{\in\mathcal{W}} B.$$

These representations are one of reasons (among others) for most of the axioms of homotopy theories that will later follow. They will usually be expressed in terms of 'square' and 'factorisation' properties. On the other hand, the axioms called *two-out-of-three* or *two-out-of-six property* rather allow one to prove *saturation* results, namely every arrow of $\mathcal{C}$ that is an isomorphism in the image of the localisation functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is actually in $\mathcal{W}$. All these ideas are more deeply discussed in [**14**] via the notion of homotopical category and three-arrow calculus (see section 1.2.3.6 below). In the sequel and everywhere in the literature, the metacategory $\mathcal{C}[\mathcal{W}^{-1}]$ is called the *homotopy category* of $(\mathcal{C}, \mathcal{W})$ and denoted by $\mathrm{Ho}(\mathcal{C})$. It turns out to be equivalent to an actual category in the case of *model categories* (see section 1.2.3.3 below). This is possible due to the 'lifting' and 'factorisation' properties allowing one to construct approximated inverses of the arrows of $\mathcal{W}$ in the zigzags of $\mathrm{Ho}(\mathcal{C})$ up to isomorphisms in $\mathrm{Ho}(\mathcal{C})$.

1.2.3.3. *Closed model categories.* A *closed model*[10] *category* (originally defined in [**38**]) consists of a cocomplete and finitely complete category $\mathcal{C}$ endowed with three $\mathcal{C}$-classes of morphisms, whose elements will be called *weak equivalences*, *fibrations* and *cofibrations*, and, agreeing that a morphism being both a fibration (resp. cofibration) and a weak equivalence is called *acyclic fibration* (resp. *acyclic cofibration*), for which the following axioms are true:

M1 Let f and g be morphisms in $\mathcal{C}$ such that $f \circ g$ exists. If two of the three morphisms f, g and $f \circ g$ are weak equivalences, then so is the third;

[10] The term *model* refers to the idea of providing a model for the *homotopy theory of topological spaces*.

M2 Let f and g be morphisms in $\mathcal{C}$. If f is a retract of g and g is either a weak equivalence, or a fibration or a cofibration, then so is f, respectively;

M3 Every cofibration has the llp with respect to acyclic fibrations and every fibration has the rlp with respect to acyclic cofibrations;

M4 Every morphism in $\mathcal{C}$ may be factorised in two ways as a composite $f \circ g$ where, in one case, g is a cofibration and f is a acyclic fibration and, in the other case, g is an acyclic cofibration and f is a fibration.

In the literature, the first axiom M1 is called the *two-out-of-three property.* We will later see that it may be refined into a *two-out-of-six property.*

Example 1.37 (Topological spaces)**.** The category **Top** is a closed model category (see [**38**]) whose weak equivalences, fibrations and cofibrations are weak homotopy equivalences, Serre fibrations and morphisms that have the left lifting property with respect to the trivial fibrations, respectively.

Example 1.38 (1-Categories)**.** The category **Cat**(1) is a closed model category whose structure was first published in [**31**]. The weak equivalences are the equivalences between small categories. This structure is usually called the *canonical (or folk) model structure* in the literature.

Example 1.39 (2-Categories)**.** The category **Cat**(2) also has a canonical model structure, published in [**32**]. The weak equivalences are biequivalences between small categories.

Example 1.40 (n-Categories)**.** More generally, it is proven in [**33**] that the category **Cat**(n) may be provided with a canonical model structure for every $n \geq 1$, generalising the result of [**32**].

Example 1.41 (Chain complexes)**.** For a fixed ring R, the category $\mathbf{Ch}_R$ is a closed model category for which the class of weak equivalences exactly contains the morphisms whose image via the k-th homology group functor are isomorphisms, for every $k \geq 0$. The class of fibrations, for its part, consists of the componentwise epimorphisms.

Example 1.42 (Presheaves)**.** A result of [**8**] shows that the category $\mathbf{Psh}(D)$ has a model structure when D is a *test category.* In particular, it may be shown that the simplex category $\mathbf{\Delta}$ is a test category, providing **sSet** with a natural model structure, which was originally published in [**38**].

1.2.3.4. *Category of fibrant objects.* A *category of fibrant objects* (originally defined in [**12**]) is a category $\mathcal{C}$ with finite products endowed with two coherent $\mathcal{C}$-classes of morphisms, whose elements are called *weak equivalences* and *fibrations*, and, agreeing that a morphism being both a fibration and a weak equivalence is called *acyclic fibration*, for which the following axioms are true:

F1 Let f and g be morphisms in $\mathcal{C}$ such that $f \circ g$ exists. If two of the three morphisms f, g and $f \circ g$ are weak equivalences, then so is the third;

F2 Fibrations and acyclic fibrations are *preserved under pullbacks*, meaning that if f is a fibration (resp. an acyclic fibration), then for every pullback of the form

$$\begin{tikzcd} A' \arrow[r, "\delta_2"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "f'"'] & B, \end{tikzcd}$$

the morphism δ_1 is a fibration (resp. an acyclic fibration);

F3 Every morphism to the terminal object is a fibration;

F4 Every morphism in $\mathcal{C}$ may be factorised as a composite $f \circ g$ where g is a weak equivalence and f is a fibration.

First of all, axiom F1 may obviously be identified with axiom M1 in the definition of closed model categories. Axiom F4 is stated in a stronger way in [**12**], but as shown in *ibid*, our version and that of *ibid* are equivalent. Interestingly, axiom F2 is always true in a closed model structure thanks to axiom M3 and Proposition 1.32. Notice that axiom M4 and Proposition 1.33 imply that the cofibrations of any closed model structure are *preserved under pushouts*, namely if g is a cofibration, then for every pushout of the form

$$\begin{tikzcd} A' \arrow[r, "g'"] \arrow[d, "g"'] & A \arrow[d, "\delta_1"] \\ B' \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & B, \end{tikzcd}$$

the morphism δ_1 is a cofibration. An important point in the definition of categories of fibrant objects is that there is no notion of cofibrations in $\mathcal{C}$. This does not however mean that there does not exist any suitable notion of cofibrations associated with categories of fibrant object in practice.

Example 1.43 (Sheaves)**.** The need of considering categories of fibrant objects rather than model categories was motivated by the question of finding the right setting for the homotopy theory of sheaves of spectra and, in particular, recovering the Leray spectral sequence (see [**12**]).

1.2.3.5. *Calculus of fractions.* A *right calculus of fractions* (originally defined in [**20**]) consists of a category $\mathcal{C}$ equipped with a $\mathcal{C}$-class of morphisms, whose elements are called *weak equivalences* satisfying the following conditions hold:

C1 For every weak equivalence $v : X \to Y$ and arrow $f : Z \to Y$ in $\mathcal{C}$, there exists a weak equivalence $v' : P \to Z$ and an arrow $f' : P \to X$ in $\mathcal{C}$ such that the subsequent diagram commutes;

(1.23)
$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "v'"'] & X \arrow[d, "v"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

C2 For every weak equivalence $v : Y \to Z$ and pair of morphisms $f, g : X \to Y$ such that $v \circ f = v \circ g$, there exists a weak equivalence $v' : A \to X$ such that $f \circ v' = g \circ v'$.

Notice that axiom C1 is an abstraction of axiom F2 for categories of fibrant objects. Axiom C2 is usually used to handle compositions in the homotopy category and allows one to give some representative for the equivalence classes of zigzags in $\mathrm{Ho}(\mathcal{C})$. The representative zigzags then have the form shown in diagram (1.22).

Example 1.44 (Categories of fibrant objects)**.** When quotiented with respect to a certain equivalence relation, a category of fibrant objects $\mathcal{C}$ – or rather its quotient $\pi\mathcal{C}$ – admits a structure of calculus of fractions, which is used to describe its homotopy category is terms of a localisation $\pi\mathcal{C}[\mathcal{W}^{-1}]$ (see [**12**] for more details).

1.2.3.6. *Homotopical categories and three-arrows calculus.* A *homotopical category* is a category $\mathcal{C}$ equipped with a $\mathcal{C}$-class of morphisms, whose elements are called *weak equivalences* satisfying the *two-out-of-six* property:

H0 Let f, g and h be morphisms in $\mathcal{C}$ such that $f \circ g \circ h$ exists. If both composite $f \circ g$ and $g \circ h$ are weak equivalences, then so are f, g, h and $f \circ g \circ h$;

A *three-arrow calculus* (originally defined in [**14**]) consists of a homotopical category $\mathcal{C}$ equipped with two subclasses of its class of equivalences, whose elements will be called *acyclic fibrations* and *acyclic cofibrations*, for which the following axioms hold:

H1 For every acyclic fibration $v : X \to Y$ and arrow $f : Z \to Y$ in $\mathcal{C}$, there exists a acyclic fibration $v' : P \to Z$ and an arrow $f' : P \to X$ in $\mathcal{C}$ such that diagram (1.23) commutes;

H2 For every acyclic cofibration $v' : P \to Z$ and arrow $f' : P \to X$ in $\mathcal{C}$, there exists a acyclic cofibration $v : X \to Y$ and an arrow $f : Z \to Y$ in $\mathcal{C}$ such that diagram (1.23) commutes;

H3 Every weak equivalence may be functorially factorised as a composite $f \circ g$ where g is a acyclic cofibration and f is a acyclic fibration.

Notice that axioms H1 and H2 are similar to axiom C1 and axiom F2. In axiom H3, functorial means that the factorisation may be seen as a functor $\mathcal{C}^2 \to \mathcal{C}^3$ of the form $f \circ g \mapsto (f, g)$. Finally, axiom H0 may be shown to imply the two-out-of-three property required for model categories (see axiom M1).

Example 1.45 (Model categories)**.** Every model category equipped with a functorial factorisation is a three-arrow calculus when endowed with its weak equivalences, acyclic fibrations and acyclic cofibrations (see [**14**] for more details).

Example 1.46 (Categories of fibrant objects)**.** Every category of fibrant objects equipped with a functorial factorisation is a three-arrow calculus when endowed with its weak equivalences, acyclic fibrations and all its morphisms as acyclic cofibrations.

1.2.3.7. *Quillen functors.* This section recalls the usual theorems on functors and adjunctions between models categories. To start with, consider the following adjunction where $\mathcal{C}$ and $\mathcal{D}$ are model categories.

(1.24)
$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

Theorem 1.47 (see [**38, 15**])**.** *If one of the following equivalent conditions is true:*

1) *F preserves cofibrations and acyclic cofibrations;*
2) *G preserves fibrations and acyclic fibrations;*
3) *F preserves cofibrations and G preserves fibrations;*
4) *F preserves acyclic cofibrations and G preserves acyclic fibrations;*

then it is possible to derive from (1.24) an adjunction of the form

(1.25)
$$\begin{tikzcd} \mathrm{Ho}(\mathcal{C}) \arrow[r, shift left, "\mathbf{L}F"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{D}). \arrow[l, shift left, "\mathbf{R}G"] \end{tikzcd}$$

where the functors $\mathbf{L}F$ and $\mathbf{R}G$ are derived from F and G, respectively. The adjunction (1.24) is then called a Quillen adjunction.

One of the achievements of [**38**] was to show that an adjunction of the from (1.25) may give rise to an equivalence of categories under certain conditions on the counit of the adjunction. In the next proposition, the symbol ε denotes the counit of adjunction (1.24).

Proposition 1.48 (see [**38, 15**]). *If the canonical mapping $f \mapsto \varepsilon_Y \circ F(f)$ defined over $\mathcal{C}(X, G(Y))$ for every object X in $\mathcal{C}$ and Y in $\mathcal{D}$ (see section 1.2.1.36), preserves and reflects weak equivalences, then the adjunction (1.25) is an equivalence of categories.*

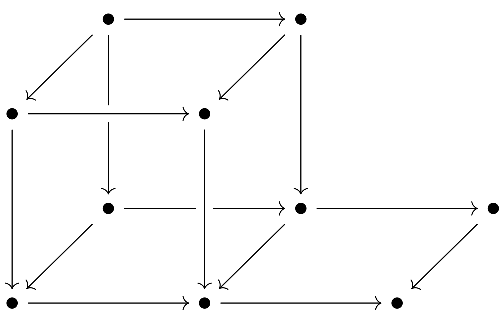
Chapter 2

Vertebrae

2.1. Introduction

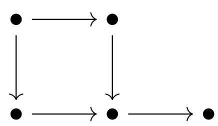
This second chapter aims at defining the basic concepts of the theory of vertebrae. Most of the properties provided in this chapter are those intrinsic to the world of vertebrae while Chapter 3 will focus on properties relative to the world of spines. At the end of the chapter is given a brief account of how to use and combine all the proven results to build a homotopy theory. This last point will be fully discussed in Chapters 4 & 5.

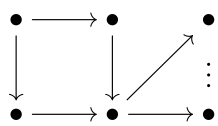
This chapter starts with a preparatory section (see section 2.2) meant to define all the factorisation properties that will be used to define the *zoo* associated with an *alliance of nodes of vertebrae*. Alliances define a 3-dimensional generalisation of the 2-dimensional notion of *node of vertebrae*. For their part, nodes of vertebrae generalise the notion of vertebra to a class of vertebrae with a common base.



alliance between the front and back vertebrae

vertebra





node of vertebrae

The main properties discussed in section 2.2.1 are the following.

▷ *Relative right lifting propery*: makes sense of the notion of fibration and trivial fibration for an alliances of nodes of vertebrae;

▷ *Simplicity*: allows the definition of the notion of injectivity associated with a weak equivalence for an alliances of nodes of vertebrae;

▷ *Divisibility*: allows the definition of the notion of surjectivity associated with a weak equivalence for an alliances of nodes of vertebrae;

The next section (section 2.3) concerns the theory of vertebrae itself. The section organises itself so that the definition of alliance of nodes of vertebrae is gradually given through intermediate structures, which will play substantial roles throughout the entire thesis. These structures, which will also be associated with a notion of zoo, are the following.

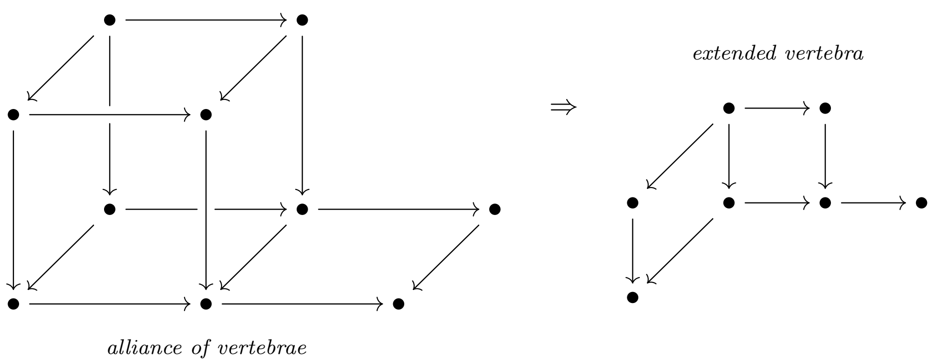
Alliance of nodes of vertebrae	*Extended node of vertebrae*	*Communication*
its zoo is made of weak equivalences; intractions; surtractions; fibrations; trivial fibrations and pseudofibrations.	this structure has less information than an alliance and will only allow the definition of surtractions and pseudofibrations.	this structure has even less information than an extended node of vertebrae and will allow the definition of pseudofibrations

The foregoing structures will interact in a fundamental way. The idea is that there is an right action of alliances of nodes of vertebrae on extended nodes of vertebrae while there is a left action of communication on extended nodes of vertebrae.

$$[\mathit{Communications}] \times [\mathit{Extended\ nodes}] \to [\mathit{Extended\ nodes}]$$

$$[\mathit{Extended\ nodes}] \times [\mathit{Alliances}] \to [\mathit{Extended\ nodes}]$$

These actions come from the fact that extended nodes of vertebrae may be seen from two points of view. On the one hand, they may be seen as 'truncated' alliances of nodes of vertebrae (see next picture), which allows us to compose them with alliances. On the other hand, they give a generalisation of the notion of node of vertebrae. From this latest point of view, the notion of extended node of vertebrae may be regarded as the stabilisation of node of vertebrae under the left action of communications.



These actions will give rise to various notions of module in Chapter 4, which will be used to build up various homotopical structures somewhat close to model categories and categories of fibrant objects.

Section 2.3 then continues by giving properties attached to the zoos of the different structures. For instance, some properties require a unique alliance of nodes of vertebrae. Other properties start with items from the zoos of an alliances $\mathfrak{a}$ and an extended node of vertebrae $\mathfrak{n}$ and deduce some properties regarding the zoo of the left action of the alliance on the extended node of vertebrae $\mathfrak{a} \circledcirc \mathfrak{n}$. These actions will therefore enable us to achieve many properties on the zoos.

The previous structures may sometimes turn out to be not enough to obtain properties that are characteristic of homotopy theories. We will then need to introduce the notion of reflexive vertebra to obtain properties ensuring that 'being strict implies being weak'. For instance, this type of structure will be used to show that deformation retracts as well as isomorphisms are weak equivalences.

Then will arrive a point where the notion of reflexivity is no longer sufficient to show properties such as the stability of weak equivalences under composition. The structure that will be required is called *framing* and may be seen as a 'horizontal composition of vertebrae'. Somehow, the notion of vertebra may be seen as a notion of cell, arrow or homotopy path in the ambient category.

$$\begin{tikzcd}
\textit{source} + \textit{target} \arrow[r, squiggly] \arrow[d, squiggly] & \textit{target arrow} \arrow[d, squiggly] & \\
\textit{source arrow} \arrow[r, squiggly] & \textit{parallel arrows} \arrow[r, squiggly] & \text{cell}
\end{tikzcd}
\qquad
\begin{tikzcd}
\textit{source} \arrow[d, bend right=60, "\textit{source arrow}"'] \arrow[d, bend left=60, "\textit{target arrow}"] \arrow[d, phantom, "\overset{\text{cell}}{\Rightarrow}"] \\
\textit{target}
\end{tikzcd}$$

A framing of vertebrae should then be regarded as a composition of the involved arrows or paths. Such an operation will take two vertebrae satisfying some condition of composability as input and will give a third vertebra, which will be said to *frame* the two previous ones.

$$\begin{tikzcd}
\textit{source} \arrow[d, bend right=60, "\gamma"'] \arrow[d, bend left=60, "\gamma'"] \arrow[d, phantom, "\overset{\text{cell}}{\Rightarrow}"] \\
\textit{target}
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\textit{source} \arrow[d, bend right=60, "\gamma'"'] \arrow[d, bend left=60, "\gamma''"] \arrow[d, phantom, "\overset{\text{cell}}{\Rightarrow}"] \\
\textit{target}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\textit{source} \arrow[d, bend right=60, "\gamma"'] \arrow[d, bend left=60, "\gamma''"] \arrow[d, phantom, "\overset{\text{cell}}{\Rightarrow}"] \\
\textit{target}
\end{tikzcd}$$

For illustration, the process of framing two vertebrae will look like as follows. We will first consider two vertebrae whose commutative squares from the *source* + *target* to the *cell*, later called *diskad*, will be pasted as shown below on the right.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & \cdot \arrow[d, dotted] & {} \arrow[from=1-2, to=2-3, bend left] \\
\cdot \arrow[r, dotted] \arrow[rr, bend right] & \cdot \arrow[r, dotted] & \cdot
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \cdot \arrow[d, dotted] & {} \arrow[from=1-2, to=2-3, bend left] \\
\cdot \arrow[r, dotted] \arrow[rr, bend right] & \cdot \arrow[r, dotted] & \cdot
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\cdot \arrow[rr] \arrow[dr] \arrow[dd] & & \cdot \arrow[d] \arrow[dl, phantom, "(b)"] \\
& \cdot \arrow[r] \arrow[d] \arrow[dl, phantom, "(a)"] & \cdot \\
\cdot \arrow[r] & \cdot &
\end{tikzcd}$$

diskad of (a) as input 1 *diskad of (b) as input 2*

The preceding rightmost diagram then offers a span which may be used to form a pushout as shown in the following leftmost diagram. This pushout square produces the commutative square (c) given by the outer square. The diskad of the third vertebra then appears after factorising (c) via a morphism η, which will be called *cooperadic transition*, making the following rightmost diagram commute. The commutative square (d) defines the diskad of the output vertebra, which appears after forming a pushout in (d).

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[dr] \arrow[dd] & & \cdot \arrow[d] \arrow[dl, phantom, "(b)"] \\
& \cdot \arrow[r] \arrow[d] \arrow[dl, phantom, "(a)"] \arrow[dr, phantom, "\ulcorner"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\cdot \arrow[rr] \arrow[dd] & & \cdot \arrow[d] \\
& (c) & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\cdot \arrow[rr] \arrow[dd] \arrow[dr, phantom, "(d)"] & & \cdot \arrow[d] \arrow[dl, dotted] \\
& \cdot \arrow[dr, dashed, "\eta"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[ur, dotted] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

The name *framing* comes from the fact that the square (c) literally frames both squares (a) and (b). Later on, a framing will be defined for any 'communicating' pair of extended nodes of vertebrae, that is to say a pair of extended nodes of vertebrae whose components are composable up to action of a communication.

One of the subtleties associated with the framing operation is that it is only defined for the notion of extended node of vertebrae (and its degenerated cases). Framings of alliances do not exist because their structure is to rich. This is the main reason for which we have to juggle the notions of alliances, nodes of vertebrae and extended nodes of vertebrae. Other type of framing operations will be considered in Chapter 3 where two extended vertebrae and a spine will be taken as input to produce a second spine that frames the whole structure.

Section 2.3 will discuss the case of reversible vertebrae, which are vertebrae having the property that their zoos correspond to those of their duals.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] & \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] & \mathbb{D}_2 \arrow[d, "\delta_2"] & \\ \mathbb{D}_1 \arrow[r, "\delta_1"'] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}' \end{tikzcd}$$

a vertebra *the dual vertebra*

We will finish the section by discussing the construction of a mini-homotopy theory based on a fairly tractable alliance of nodes of vertebrae. The case of general alliances will be fully discussed in Chapter 4 as they require modular structures.

Section 2.4 will finally provide a non-exhaustive list of examples of vertebrae together with a description of their zoos. Some of them have already been discussed in Chapter 1. Other examples will be discussed in more details in Chapter 6.

2.2. Preparation

2.2.1. Factorisation properties.

2.2.1.1. *Relative lifting properties.* Let $\mathcal{C}$ be a category and $f : X \to Y$ be a morphism in $\mathcal{C}$. For any commutative square of the form

(2.1)
$$\begin{tikzcd} A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[d, "\gamma"] \\ B' \arrow[r, "\theta'"'] & B \end{tikzcd}$$

the morphism $f : X \to Y$ will be said to *have the right lifting property (abbrev. rlp) with respect to diagram (2.1)* if for every commutative square of the form given on the left of diagram (2.2), there exists a morphism $h : B' \to X$ making the right diagram of (2.2) commute. In this case, the arrow h will be called a *lift* for the left-hand commutative diagram.

(2.2)
$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[r, "x"] & X \arrow[d, "f"] \\ B' \arrow[r, "\theta'"'] \arrow[urr, dashed, "h"] & B \arrow[r, "y"'] & Y \end{tikzcd}$$

For convenience, because diagram (2.1) also encodes a morphism $\theta : \gamma' \Rightarrow \gamma$ in $\mathcal{C}^2$, the morphism f will often be said to *have the rlp with respect to the arrow* $\theta : \gamma' \Rightarrow \gamma$. Note that if this morphism is an identity of the form $\mathrm{id}_\gamma : \gamma \Rightarrow \gamma$ in $\mathcal{C}^2$, then the previous rlp corresponds to the rlp with respect to the arrow γ as defined in section 1.2.2.1. To better acquaint the reader with this type of lifting property, below is given an easy example in which it appears.

Example 2.1. It is well-known that the image of a functor does not necessarily have a category structure. It turns out that the preceding right lifting property characterises the functors whose images are categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and denote by $F(\mathcal{C})$ the subgraph of $\mathcal{D}$ whose objects are the objects of $\mathcal{D}$ that are images of objects of $\mathcal{C}$ via F and whose morphisms are morphisms of $\mathcal{D}$ that are images of morphisms of $\mathcal{C}$ via F. Recall that we defined the following commutative diagram in Remark 1.15.

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \mathbf{2}+\mathbf{2} \arrow[d] \\
\mathbf{2} \arrow[r] & \mathbf{3}
\end{tikzcd}
\quad\Leftrightarrow\quad
\begin{tikzcd}
(\quad) \arrow[r, mapsto] \arrow[d, mapsto] & (\bullet_0 \to \bullet \quad \bullet \to \bullet_1) \arrow[d, mapsto] \\
(\bullet_0 \to \bullet_1) \arrow[r, mapsto] & (\bullet_0 \to \bullet \to \bullet_1)
\end{tikzcd}
\tag{2.3}
$$

Claim: For any functor $F : \mathcal{C} \to \mathcal{D}$, the subgraph $F(\mathcal{C})$ of $\mathcal{D}$ forms a category for the composition of $\mathcal{D}$ if and only if F has the right lifting property with respect to diagram (2.3). Suppose that F has the rlp with respect to diagram (2.3) and let us show that $F(\mathcal{C})$ is a category for the composition of $\mathcal{D}$. Consider two composable arrows $F(f) : F(y) \to F(z)$ and $F(g) : F(w) \to F(x)$ in $F(\mathcal{C})$ where the arrows $f : y \to z$ and $g : w \to x$ belong to $\mathcal{C}$. Because $F(f)$ and $F(g)$ are composable, the equation $F(z) = F(w)$ must hold.

$$
\begin{tikzcd}
(\bullet_0 \to \bullet \quad \bullet \to \bullet_1) \arrow[r, mapsto] \arrow[d, mapsto] & y \to z \quad w \to x \arrow[d, mapsto, "F"] \\
(\bullet_0 \to \bullet \to \bullet_1) \arrow[r, mapsto] & F(y) \to F(x)
\end{tikzcd}
$$

The right lifting property with respect to diagram (2.3) then says that there must exist $h : w' \to z'$ such that the equation $F(f) \circ F(g) = F(h)$ holds, which means that the composition of $F(f)$ and $F(g)$ in $\mathcal{D}$ belongs to $F(\mathcal{C})$ and proves that $F(\mathcal{C})$ is a subcategory of $\mathcal{D}$. Conversely, if $F(\mathcal{C})$ is a category, then for every composable arrows $F(f) : F(y) \to F(z)$ and $F(g) : F(w) \to F(x)$ in $F(\mathcal{C})$, where $f : y \to z$ and $g : w \to x$ are arrows in $\mathcal{C}$, there exists an arrow $h : w' \to z'$ in $\mathcal{C}$ for which the equation $F(f) \circ F(g) = F(h)$ holds. This directly implies the rlp with respect to diagram (2.3).

Remark 2.2. When the ambient category $\mathcal{C}$ admits all pushouts, the rlp with respect to a commutative square may be reduced to (i.e. is equivalent to) the rlp with respect to a commutative square whose top face is an identity, which is to say a commutative triangle or, in other words, an object of $\mathcal{C}^3$ (see rightmost square of the following right diagram).

$$
\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[d, "\gamma"] \\
B' \arrow[r, "\theta'"'] & B
\end{tikzcd}
\quad\Leftrightarrow\quad
\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[r, equal] \arrow[d, dashed] & A \arrow[d, "\gamma"] \\
B' \arrow[r, dashed] \arrow[rr, bend right, "\theta'"'] & P \arrow[r, dashed] \arrow[ul, phantom, "\ulcorner" very near start] & B
\end{tikzcd}
$$

This point may help the reader understand why the assumptions of Proprosition 2.5 (see below) are natural. However, it turns out, in practice, that the square form is the best suited in terms of representatives. For instance, even if Example 2.1 is relatively simple, it already appears easier to present it in terms of the objects $\mathbf{2}$ and $\mathbf{3}$ and $\mathbf{2}+\mathbf{2}$, than in terms of a coproduct $\mathbf{2}+(\mathbf{2}+\mathbf{2})$ and an arrow $\mathbf{2}+(\mathbf{2}+\mathbf{2}) \to \mathbf{3}$. The square form also naturally arises from the squarelike shape of a vertebra, which is not necessarily the case of the triangle form. Finally, the square form is by far the best shape that gets along with the other properties that will be introduced in section 2.2.1.2 and section 2.2.1.3.

Proposition 2.3. *Let $f : X \to Y$ be an arrow that has the rlp with respect to diagram (2.1) and suppose to be given a pullback as follows.*

$$
\begin{tikzcd}
P \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d, "f"] \\
Z \arrow[r, "f'"'] & Y
\end{tikzcd}
$$

The arrow $p : P \to Z$ has the rlp with respect to diagram (2.1) .

Proof. The proof resembles standard proofs for usual lifting properties. According to implication (2.2), the proof needs to start with a diagram of the form given below on the left. The implication then describes the pasting of this commutative square with the pullback square of the statement and diagram (2.1).

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\gamma"'] & P \arrow[d, "p"] \\
B \arrow[r, "y"'] & Z
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[r, "x"] \arrow[d, "\gamma"'] & P \arrow[r, "p'"] \arrow[d, "p"] & X \arrow[d, "f"] \\
B' \arrow[r, "\theta'"'] & B \arrow[r, "y"'] & Z \arrow[r, "f'"'] & Y
\end{tikzcd}$$

Because the morphism $f : X \to Y$ has the rlp with respect to diagram (2.1), the outer commutative rectangle of the previous pasting produces a lift $h : B' \to X$. The cone formed by $h : B' \to X$ and the composite $y \circ \theta' : B' \to Z$ then induces a canonical arrow $h : B' \to P$ above the pullback P as shown by the following implication.

$$\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[r, "x"] & P \arrow[r, "p'"] & X \arrow[d, "f"] \\
B' \arrow[r, "\theta'"'] \arrow[rrru, dashed, "h"] & B \arrow[r, "y"'] & Z \arrow[r, "f'"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[r, "x"] & P \arrow[d, "p"] \\
B' \arrow[r, "\theta'"'] \arrow[rru, dashed, "h'"] & B \arrow[r, "y"'] & Z
\end{tikzcd}$$

The previous right diagram finally produces a lift for the initial commutative square. □

Remark 2.4. If a morphism in $\mathcal{C}$ has the rlp with respect to the right commutative square of diagram (2.4), then it has the rlp with respect to the left commutative square of diagram (2.4).

(2.4)
$$\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[d, "\gamma"] \\
B' \arrow[r, "\theta'"'] & B
\end{tikzcd}
\quad \Leftarrow \quad
\begin{tikzcd}
A' \arrow[r, equal] \arrow[d, "\gamma'"'] & A' \arrow[d, "\gamma\circ\theta"] \\
B' \arrow[r, "\theta'"'] & B
\end{tikzcd}$$

In this case, the rlp with respect to a triangle is stronger than the rlp with respect to a square. In the next proposition, a commutative square as displayed on the left of (2.4) will be called the *proper square* of the commutative square given on the right. Conversely, a commutative square as displayed on the right of (2.4) will be called the *biased square* of the commutative square given on the left.

Proposition 2.5. *If a morphism $f : X \to Y$ in $\mathcal{C}$ has the rlp with respect to the following two leftmost commutative squares, then it has the rlp with respect to the vertical pasting of the underlying proper squares as given on the right.*

$$\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\delta'"'] & A \arrow[d, "\delta"] \\
B' \arrow[r, "\theta_*"'] & B
\end{tikzcd}
\quad
\begin{tikzcd}
B' \arrow[r, equal] \arrow[d, "\beta'"'] & B' \arrow[d, "\beta\circ\theta_*"] \\
C' \arrow[r, "\theta_\dagger"'] & C
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\beta'\circ\delta'"'] & A \arrow[d, "\beta\circ\delta"] \\
C' \arrow[r, "\theta_\dagger"'] & C
\end{tikzcd}$$

Proof. To prove the proposition, let us start with the leftmost commutative diagram bellow, which, after some rearrangement, leads to the commutative diagram in the middle. By assumption on $f : X \to Y$, this commutative diagram produces a lift $h : B' \to X$ as shown in the succeeding rightmost diagram.

(2.5)
$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\beta\circ\delta"'] & X \arrow[d, "f"] \\
C \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\delta"'] & X \arrow[d, "f"] \\
B \arrow[r, "y\circ\beta"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\delta'"'] & A \arrow[r, "x"] & X \arrow[d, "f"] \\
B' \arrow[r, "\theta_*"'] \arrow[urr, dashed, "h"] & B \arrow[r, "y\circ\beta"'] & Y
\end{tikzcd}
$$

In particular, the bottom right corner of the latest commutative diagram gives the following leftmost commutative diagram. By using the assumptions on $f : X \to Y$, we may produce a lift $h' : C' \to X$ as shown on the right.

(2.6)
$$
\begin{tikzcd}
B' \arrow[r, "h"] \arrow[d, "\beta\circ\theta_*"'] & X \arrow[d, "f"] \\
C \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
B' \arrow[r, equal] \arrow[d, "\beta'"'] & B' \arrow[r, "h"] & X \arrow[d, "f"] \\
C' \arrow[r, "\theta_\dagger"'] \arrow[urr, dashed, "h'"] & C \arrow[r, "y"'] & Y
\end{tikzcd}
$$

Finally, vertically pasting the top left corner of the rightmost commutative diagram of (2.5) with the right commutative diagram of (2.6) along the arrow h provides a lift $h' : C' \to X$ for the initial commutative diagram of (2.5). $\square$

Proposition 2.6. *If a morphism $g : X \to Y$ has the rlp with respect to the leftmost commutative square below and a morphism $f : Y \to Z$ has the rlp with respect to the middle commutative square, then the composite $f \circ g : X \to Y$ has the rlp with respect to the horizontal pasting of both squares as shown on the right.*

$$
\begin{tikzcd}
A_\dagger \arrow[r, "\theta_*"] \arrow[d, "\gamma_\dagger"'] & A_* \arrow[d, "\gamma_*"] \\
B_\dagger \arrow[r, "\theta'_*"'] & B_*
\end{tikzcd}
\qquad
\begin{tikzcd}
A_* \arrow[r, "\theta"] \arrow[d, "\gamma_*"'] & A \arrow[d, "\gamma"] \\
B_* \arrow[r, "\theta'"'] & B
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A_\dagger \arrow[r, "\theta_*"] \arrow[d, "\gamma_\dagger"'] & A_* \arrow[r, "\theta"] \arrow[d, "\gamma_*"] & A \arrow[d, "\gamma"] \\
B_\dagger \arrow[r, "\theta'_*"'] & B_* \arrow[r, "\theta'"'] & B
\end{tikzcd}
$$

Proof. To prove the statement, start with the leftmost commutative diagram below, which, after some arrangement, gives the commutative diagram in the middle. By assumption on $f : X \to Y$, this commutative diagram produces a lift $h : B_* \to Z$ as shown in the corresponding rightmost diagram.

(2.7)
$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f\circ g"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A \arrow[r, "g\circ x"] \arrow[d, "\gamma"'] & Z \arrow[d, "f"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A_* \arrow[r, "\theta"] \arrow[d, "\gamma_*"'] & A \arrow[r, "g\circ x"] & Z \arrow[d, "f"] \\
B_* \arrow[r, "\theta'"'] \arrow[urr, dashed, "h"] & B \arrow[r, "y"'] & Y
\end{tikzcd}
$$

In particular, the top left corner of the latest commutative diagram gives the following leftmost commutative diagram. By using the assumptions on $g : X \to Z$, we may produce a lift $h' : B_\dagger \to X$ as shown in the middle.

$$
\begin{tikzcd}
A_* \arrow[r, "x\circ\theta"] \arrow[d, "\gamma_*"'] & X \arrow[d, "g"] \\
B_* \arrow[r, "h"'] & Z
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A_\dagger \arrow[r, "\theta_*"] \arrow[d, "\gamma_\dagger"'] & A_* \arrow[r, "x\circ\theta"] & X \arrow[d, "g"] \\
B_\dagger \arrow[r, "\theta'_*"'] \arrow[urr, dashed, "h'"] & B_* \arrow[r, "h"'] & Z
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A_\dagger \arrow[r, "\theta\circ\theta_*"] \arrow[d, "\gamma_\dagger"'] & A_* \arrow[r, "x"] & X \arrow[d, "f\circ g"] \\
B_\dagger \arrow[r, "\theta'\circ\theta'_*"'] \arrow[urr, dashed, "h'"] & B_* \arrow[r, "y"'] & Y
\end{tikzcd}
$$

As shown by the earlier implication, vertically pasting the previous middle commutative diagram with the bottom right corner of the rightmost commutative diagram of (2.7) after pre-composing it with the arrow θ' provides the desired lift $h' : B_\dagger \to X$ for the initial commutative diagram of (2.7). $\square$

The previous two propositions are generalisations of Proposition 1.34 and Proposition 1.35 that were given in Chapter 1.

2.2.1.2. *Scales and simplicity.* Let $\mathcal{C}$ be a category. A *scale* in $\mathcal{C}$ consists of an arrow $\varkappa : A' \to A$ in $\mathcal{C}$ as well as two classes Ω and Ω' of morphisms in $\mathcal{C}$ whose domains are equal to A and A', respectively.

$$\begin{tikzcd}
 & A' \arrow[rr, "\varkappa"] \arrow[dl, "\beta'_*"'] \arrow[d, "\beta'"] & & A \arrow[d, "\beta_*"'] \arrow[dr, "\beta"] & \\
B'_* \arrow[r, phantom, "\cdots"] & B' & & B \arrow[r, phantom, "\cdots"] & B_*
\end{tikzcd}$$
$$\underbrace{\hphantom{B'_* \quad \cdots \quad B'}}_{\in\Omega'} \qquad\qquad \underbrace{\hphantom{B \quad \cdots \quad B_*}}_{\in\Omega}$$

A morphism $f : X \to Y$ in $\mathcal{C}$ will be said to be *simple with respect to a scale* $(\Omega, \varkappa, \Omega')$ if for every arrow $\beta : A \to B$ in Ω and commutative square of the form given on the left of diagram (2.8), there exist an arrow $\beta' : A' \to B'$ in Ω' and a morphism $h : B' \to X$ in $\mathcal{C}$ such that the succeeding right diagram commutes.

(2.8)
$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A' \arrow[r, "\varkappa"] \arrow[d, "\beta'"'] & A \arrow[r, "x"] & X \\
B' \arrow[urr, dashed, "h"'] & &
\end{tikzcd}$$

The morphism $h : B' \to X$ (thought of as coming equipped with β') will often be called a *lift* (or *semi-lift* if ambiguous).

Proposition 2.7. *If a morphism $f : Z \to Y$ is simple with respect to a scale $(S, \varkappa, S_*)$ and a morphism $g : X \to Z$ is simple with respect to a scale $(S_*, \varkappa_*, S_\dagger)$, then $f \circ g : X \to Y$ is simple with respect to $(S, \varkappa \circ \varkappa_*, S_\dagger)$.*

Proof. To prove the statement, start with the leftmost commutative diagram below, which, after some arrangement, gives the commutative diagram in the middle. By assumption on $f : X \to Y$, this commutative diagram produces a lift $h : B_* \to Z$ as shown in the following rightmost diagram, where β_* belongs to Ω_*.

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f\circ g"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A \arrow[r, "g\circ x"] \arrow[d, "\beta"'] & Z \arrow[d, "f"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A_* \arrow[r, "\varkappa"] \arrow[d, "\beta_*"'] & A \arrow[r, "g\circ x"] & Z \\
B_* \arrow[urr, dashed, "h"'] & &
\end{tikzcd}$$

In particular, the top left corner of the latest commutative diagram gives the following leftmost commutative diagram. By using the assumption on $g : X \to Z$, we may produce a lift $h' : B_\dagger \to X$ as shown on the right.

$$\begin{tikzcd}
A_* \arrow[r, "x\circ\varkappa"] \arrow[d, "\beta_*"'] & X \arrow[d, "g"] \\
B_* \arrow[r, "h"'] & Z
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A_\dagger \arrow[r, "\varkappa_*"] \arrow[d, "\beta_\dagger"'] & A \arrow[r, "x\circ\varkappa"] & Z \\
B_\dagger \arrow[urr, dashed, "h'"'] & &
\end{tikzcd}$$

This last diagram proves the statement. $\square$

A scale $(\Omega, \varkappa, \Omega')$ in $\mathcal{C}$ will be said to be *oriented along* a metafunction $\varphi : \Omega \to \Omega'$ if, for every arrow $\beta : A \to B$ in Ω, it is equipped with a commutative square of the following form in $\mathcal{C}$.

$$\begin{tikzcd} A' \arrow[r, "\varkappa"] \arrow[d, "\varphi(\beta)"'] & A \arrow[d, "\beta"] \\ B' \arrow[r, "u_\beta"'] & B \end{tikzcd}$$

The previous commutative square will be called the *orientation of* $(\Omega, \varkappa, \Omega')$ *at* β.

Proposition 2.8. *Let* $(\Omega, \varkappa, \Omega')$ *be a scale in* $\mathcal{C}$ *that is oriented along a metafunction* $\varphi : \Omega \to \Omega'$*. Every morphism that has the rlp with respect to the orientation of* $(\Omega, \varkappa, \Omega')$ *at every* $\beta \in \Omega$ *is simple with respect to* $(\Omega, \varkappa, \Omega')$*.*

Proof. If a morphism $f : X \to Y$ has the rlp with respect to the orientation of $(\Omega, \varkappa, \Omega')$ at every $\beta \in \Omega$, then the next series of implication holds.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A' \arrow[r, "\varkappa"] \arrow[d, "\varphi(\beta)"'] & A \arrow[r, "x"] & X \arrow[d, "f"] \\ B' \arrow[r, "u_\beta"'] \arrow[urr, dashed, "h"] & B \arrow[r, "y"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A' \arrow[r, "\varkappa"] \arrow[d, "\varphi(\beta)"'] & A \arrow[r, "x"] & X \\ B' \arrow[urr, dashed, "h"'] & & \end{tikzcd}$$

This therefore proves that $f : X \to Y$ is simple with respect to $(\Omega, \varkappa, \Omega')$. $\square$

In the sequel, being simple with respect to a scale of the form $(\Omega, \mathrm{id}_A, \Omega)$ will be shortened as being *simple with respect to the class* Ω and being simple with respect to a singleton set $\{\beta : A \to B\}$ will be shortened as being *simple with respect to the morphism* $\beta : A \to B$. In this last case, a morphism $f : X \to Y$ in $\mathcal{C}$ is simple with respect to $\beta : A \to B$ if for every commutative square of the form given below on the left, there exist a morphism $h : B \to X$ in $\mathcal{C}$ such that the succeeding right diagram commutes.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "\beta"'] & X \\ B \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

This means that the morphism $x : A \to X$ may be factorised through $\beta : A \to B$.

2.2.1.3. *Besoms and division.* The following notion is a mix between the notion of simplicity and that of factorisation in an arrow category. Let $\mathcal{C}$ be a category. A *besom* in $\mathcal{C}$ consists of two commutative squares in $\mathcal{C}$ as follows together with a class Ω of morphisms in $\mathcal{C}$ whose domains are all equal to the object B''.

$$\begin{tikzcd} A' \arrow[r, "\theta"] \arrow[d, "\overline{\gamma}"'] & A \arrow[d, "\gamma"] \\ B' \arrow[r, "\theta'"'] & B \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "\gamma'"] \arrow[d, "\overline{\gamma}"'] & A'' \arrow[d, "\delta_1"] \\ B' \arrow[r, "\delta_2"'] & B'' \end{tikzcd} \tag{2.9}$$

A morphism $f : X \to Y$ in $\mathcal{C}$ will be said to be *divisible by a besom as above* if for every commutative diagram of the form given below on the left, there exist an arrow $\beta : B'' \to D$ in Ω and two morphisms $x' : A' \to X$ and $y' : B' \to Y$ in $\mathcal{C}$ such that the second diagram commutes in $\mathcal{C}$.

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] & Y
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
A' \arrow[r, "\gamma'"'] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "x\circ\theta"] & A'' \arrow[r, dashed, "x'"'] \arrow[d, "\beta\circ\delta_1"] & X \arrow[d, "f"] \\
B' \arrow[r, "\beta\circ\delta_2"] \arrow[rr, bend right, "y\circ\theta'"'] & D \arrow[r, dashed, "y'"] & Y
\end{tikzcd}
$$

Proposition 2.9. *In the case of the previous definition, the morphism $f : X \to Y$ is simple with respect to the singleton scale $(\{\gamma\}, \theta, \{\gamma'\})$.*

Proof. The semi-lift is given by the arrow $x' : A'' \to X$. □

It is possible to use a more compact language for besoms. First notice that the left commutative square of (2.9) defines a morphism of the form $\theta : \overline{\gamma} \Rightarrow \gamma$ in $\mathcal{C}^2$. The post-composition of the right square diagram of (2.9) with an arrow $\beta : B'' \to D$ in Ω also provides a morphism $\mathbf{d}(\beta) : \overline{\gamma} \Rightarrow \beta \circ \delta_1$ in $\mathcal{C}^2$. In the end, this leads to the following picture where θ may be seen as the pole of a besom[1] whose twigs are given by the arrows $\mathbf{d}(\beta) : \overline{\gamma} \Rightarrow \beta \circ \delta_1$ for every $\beta \in \Omega$.

$$
\begin{tikzcd}
(\beta, \dots, \beta_* \in \Omega) & \beta_* \circ \delta_1 & & \\
& \vdots & & \\
& \beta \circ \delta_1 & \overline{\gamma} \arrow[l, Rightarrow, "\mathbf{d}(\beta)"] \arrow[uul, Rightarrow, "\mathbf{d}(\beta_*)"'] \arrow[r, Rightarrow, "\theta"] & \gamma
\end{tikzcd}
$$

The previous picture motivates the notation $(\Omega, \mathbf{d}, \theta)$ for such a structure. In this language, a morphism $f : X \to Y$ in $\mathcal{C}$ is divisible by a besom $(\Omega, \mathbf{d}, \theta)$ as above if for every morphism $x : \gamma \Rightarrow f$ in $\mathcal{C}^2$, there exist an arrow $\beta \in \Omega$ and a morphism $x' : \beta \circ \delta_1 \Rightarrow f$ in $\mathcal{C}^2$ such that the following diagram commutes.

$$
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\mathbf{d}(\beta)"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\beta \circ \delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f
\end{tikzcd}
$$

This kind of reformulation simplifies the proof of the following propositions.

Proposition 2.10. *Let $(\Omega, \mathbf{d}, \theta)$ be a besom in $\mathcal{C}$ with $\theta : \overline{\gamma} \Rightarrow \gamma$. If a morphism $f : X \to Y$ is divisible by $(\Omega, \mathbf{d}, \theta)$, then it is divisible by $(\Omega, \mathbf{d}, \theta_* \circ \theta)$ for any commutative square $\theta_* : \gamma \Rightarrow \gamma_*$.*

Proof. To prove the proposition, start with the left diagram below. Rearranging this diagram and applying the assumption of the proposition then leads to the existence of some $\beta \in \Omega$ making the middle diagram commute.

$$
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\theta"] & \gamma \arrow[r, Rightarrow, "\theta_*"] & \gamma_* \arrow[d, Rightarrow, "x"] \\
& & f
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\mathbf{d}(\beta)"'] & \gamma \arrow[d, Rightarrow, "x\circ\theta"] \\
\beta \circ \delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\mathbf{d}(\beta)"'] & \gamma \arrow[r, Rightarrow, "\theta_*"] & \gamma_* \arrow[d, Rightarrow, "x"] \\
\beta \circ \delta_1 \arrow[rr, Rightarrow, dashed, "x'"'] & & f
\end{tikzcd}
$$

Finally, putting the diagram back in its original form (leftmost diagram) provides the desired factorisation. □

[1] 'Besom' is another word for 'broom', in particular, a broom made of actual twigs.

A *couple of besoms* consists of two besoms $(\Omega, \mathbf{d}, \theta)$ and $(\Omega_*, \mathbf{d}_*, \theta_*)$ equipped with a metafunction $\phi : \Omega \to \Omega_*$ and, for every β in Ω, a commutative diagram as follows in $\mathcal{C}^2$.

$$\begin{tikzcd}
\gamma_* \arrow[r, Rightarrow, "\theta_*"] \arrow[d, Rightarrow, "\mathbf{d}_*(\phi(\beta))"'] & \overline{\gamma} \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\mathbf{d}(\beta)"'] & \gamma \\
\phi(\beta) \circ \delta_1^* \arrow[r, Rightarrow, "u_\beta"'] & \beta \circ \delta_1 &
\end{tikzcd}$$

Note that such a structure also provides a third besom $(\Omega_*, \mathbf{d}_*, \theta \circ \theta_*)$.

Proposition 2.11. *Suppose to be given a couple of besoms as above. If a morphism $f : X \to Y$ is divisible by $(\Omega, \mathbf{d}, \theta)$, then it is divisible by $(\Omega_*, \mathbf{d}_*, \theta \circ \theta_*)$.*

Proof. To prove the statement, start with the left diagram below. Because $f : X \to Y$ is divisible by the besom $(\Omega, \mathbf{d}, \theta)$, there exists $\beta \in \Omega$ for which this diagram admits a factorisation as shown in next right commutative diagram.

$$\begin{tikzcd}
\gamma_* \arrow[r, Rightarrow, "\theta_*"] & \overline{\gamma} \arrow[r, Rightarrow, "\theta"] & \gamma \arrow[d, Rightarrow, "x"] \\
 & & f
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\gamma_* \arrow[r, Rightarrow, "\theta_*"] & \overline{\gamma} \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\mathbf{d}(\beta)"'] & \gamma \arrow[d, Rightarrow, "x"] \\
 & \beta \circ \delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f
\end{tikzcd}$$

Using the structure of couple between the two besoms $(\Omega, \mathbf{d}, \theta)$ and $(\Omega_*, \mathbf{d}_*, \theta_*)$, the previous factorisation may be extended to the following left factorisation. A pasting of the squares provides the commutative diagram on the right.

$$\begin{tikzcd}
\gamma_* \arrow[r, Rightarrow, "\theta_*"] \arrow[d, Rightarrow, "\mathbf{d}_*(\beta)"'] & \overline{\gamma} \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\mathbf{d}(\beta)"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\phi(\beta) \circ \delta_1^* \arrow[r, Rightarrow, "\varkappa"'] & \beta \circ \delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\gamma_* \arrow[r, Rightarrow, "\theta_*"] \arrow[d, Rightarrow, "\mathbf{d}_*(\beta)"'] & \overline{\gamma} \arrow[r, Rightarrow, "\theta"] & \gamma \arrow[d, Rightarrow, "x"] \\
\phi(\beta) \circ \delta_1^* \arrow[rr, Rightarrow, "x' \circ \varkappa"'] & & f
\end{tikzcd}$$

Finally, since $\phi(\beta)$ belongs to the class Ω_*, the last diagram shows that f is divisible by the besom $(\Omega_*, \mathbf{d}_*, \theta \circ \theta_*)$. □

2.2.1.4. *Stability under retracts.* Let $\mathcal{C}$ be a category and $f : X \to A$ and $g : Y \to B$ be two morphisms in $\mathcal{C}$. Recall the morphism f is said to be a retract of g if there exist morphisms $i : X \to Y$, $j : A \to B$, $r : Y \to X$ and $s : B \to A$ satisfying the identities $r \circ i = \mathrm{id}_X$ and $s \circ j = \mathrm{id}_{X'}$ and such that the following diagram commutes in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\
A \arrow[r, "j"] & B \arrow[r, "s"] & A.
\end{tikzcd}$$

It directly follows from the definitions that if the morphism $g : Y \to B$

- has the rlp with respect to a certain commutative square, then so does f;
- is simple with respect to a certain scale $(\Omega, \varkappa, \Omega')$, then so is f;
- is divisible by a certain besom $(\Omega, \gamma', \theta)$, then so is f.

2.3. Theory of vertebrae

2.3.1. Vertebrae. The goal of this section is to define the vocabulary necessary to the discussions of the present and next chapters. The idea is to gradually introduce all the requisite structure that we will require via intermediate structures, all of which playing significant roles in the development of the theory. The next table gives the conventional notations of the arrows that will form the structures in question.

	(Object)	Extended	Alliance of
Prevertebra	γ , γ', δ_1, δ_2	$\varkappa$, ϱ	$\varkappa$, ϱ, ϱ', $\varkappa'$
Vertebra	β		u
Node of vertebrae	Ω		$\{u_\beta \mid \beta \in \Omega\}$

Because each of these arrows play an important role in the definition of the 'zoos' of these structures, it turned out to be necessary to give them names. To prepare the reader to the variety of different names, below is given a table summarising most of them, where the abbreviation *tr.* stands for *transition.*

γ	γ'	δ_1	δ_2	β
seed, preseed	*coseed*	*antiseed*	*anticoseed*	*stem*
$\beta \circ \delta_1$	$\varkappa$	ϱ	ϱ'	$\varkappa'$
trivial stem	*spherical tr.*	*discal tr.*	*codiscal tr.*	*cospherical tr.*

Thoughout the paper, we will usually keep the previous notations conventional. If two structures are involved, we will distinguish one from the other by introducing different indexing notations for both structures (e.g. γ_*, $\overline{\gamma}$ and $\gamma_\dagger$ for the seeds).

2.3.1.1. *Prevertebrae.* A *prevertebra* in a category $\mathcal{C}$ is a structure consisting of four objects $\mathbb{S}$, $\mathbb{D}_1$, $\mathbb{D}_2$ and $\mathbb{S}'$ and a commutative square of four morphisms producing a pushout in $\mathcal{C}$ as shown in diagram (2.10).

(2.10)
$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}' \end{tikzcd}$$

The morphisms γ, γ', δ_1 and δ_2 will be called *seed*, *coseed*, *antiseed* and *anticoseed* of diagram (2.10). In the sequel, a prevertebra as above will be denoted by the symbols $\|\gamma, \gamma' : \mathbb{S}'\|$ or simply $\|\gamma, \gamma'\|$ when the pushout is obvious. The *dual* of a prevertebra $\|\gamma, \gamma'\|$ will be the dual commutative square encoding the prevertebra $\|\gamma', \gamma\|$. We shall denote by p^{rv} the dual of any prevertebra p.

Remark 2.12. A prevertebra defines a model of a colimit sketch `Prev` in $\mathcal{C}$. The colimit sketch `Prev` consists of a four arrows making a pushout, which is part of in the chosen colimits.

2.3.1.2. *Domain and codomain.* Let $\mathcal{C}$ be a category and p be a prevertebra in $\mathcal{C}$ as displayed in diagram (2.10). Such a prevertebra will be written $p : \mathbb{S} \multimap \mathbb{S}'$, where the objects $\mathbb{S}$ and $\mathbb{S}'$ will be called *domain* and *codomain* of p, respectively. In the case where $\mathbb{S}'$ is thought of as the universal cocone (δ_1, δ_2), it will be quite natural to write $\|\gamma, \gamma'\| : \mathbb{S} \multimap (\delta_1, \delta_2)$, thereby exposing every morphism contained in a vertrebra.

2.3.1.3. *Alliances of prevertebrae.* Let $\mathcal{C}$ be a category. An *alliance of prevertebrae* in $\mathcal{C}$ consists of two prevertebrae p and $\overline{p}$ equipped with a morphism of models $\overline{p} \Rightarrow p$ in the category $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Prev})$. Such an alliance will be said to *go from* p *to* $\overline{p}$ and, for this reason, will be regarded as an arrow $\mathtt{p} : p \rightsquigarrow \overline{p}$ in the opposite category $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Prev})^{\mathrm{op}}$. The vertebra $\overline{p}$ will be said to be *allied* to p.

Remark 2.13 (Notation)**.** Let us explain the reversed notation $\mathfrak{p} : p \rightsquigarrow \overline{p}$. A prevertebra may be seen as a colimit sketch by itself in $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Prev})$ and so do alliances of prevertebrae in $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Prev})^2$. As is often the case for colimit sketches, the language of alliances is better behaved when seen in the opposite ambient category. For instance, such need arises when 'homing' the sketch in question via the Yoneda embedding on the contravariant variable. The homing operation will extensively be used in Chapter 3 and Chapter 4.

To resume, an alliance of prevertebrae $\mathfrak{p}$ from a prevertebra $\|\gamma, \gamma'\| : \mathbb{S} \multimap (\delta_1, \delta_2)$ to a prevertebra $\|\overline{\gamma}, \overline{\gamma}'\| : \overline{\mathbb{S}} \multimap (\overline{\delta}_1, \overline{\delta}_2)$ consists of three morphisms $\varkappa : \overline{\mathbb{S}} \to \mathbb{S}$, $\varrho : \overline{\mathbb{D}}_2 \to \mathbb{D}_2$ and $\varrho' : \overline{\mathbb{D}}_1 \to \mathbb{D}_1$ in $\mathcal{C}$ making the following diagram commutes.

(2.11)
$$\begin{tikzcd}
\overline{\mathbb{S}} \arrow[rr, "\overline{\gamma}'"] \arrow[dr, "\varkappa"'] \arrow[dd, "\overline{\gamma}"'] & & \overline{\mathbb{D}}_1 \arrow[dr, "\varrho'"] \arrow[dd, "\overline{\delta}_1", near start] & \\
& \mathbb{S} \arrow[rr, "\gamma'", near start] \arrow[dd, "\gamma"', near start] & & \mathbb{D}_1 \arrow[dd, "\delta_1"] \\
\overline{\mathbb{D}}_2 \arrow[rr, "\overline{\delta}_2"', near end] \arrow[dr, "\varrho"'] & & \overline{\mathbb{S}}' \arrow[dr, dashed, "\varkappa'"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& \mathbb{D}_2 \arrow[rr, "\delta_2"'] & & \mathbb{S}'
\end{tikzcd}$$

Although the fourth morphism $\varkappa' : \overline{\mathbb{S}}' \to \mathbb{S}'$ is induced by universality over the pushout $\overline{\mathbb{S}}'$, it will come in handy to identify the alliance $\mathfrak{p}$ with the 4-tuple $(\varkappa, \varrho, \varrho', \varkappa')$. The arrows $\varkappa$, ϱ, ϱ' and $\varkappa'$ will be called *spherical, discal, codiscal and cospherical transitions*, respectively. Because the commutative squares encoding the left and right face of diagram (2.11) will play substantial roles in the sequel, they will be denoted by $\mathbf{seed}(\mathfrak{p}) : \overline{\gamma} \Rightarrow \gamma$ and $\mathbf{asee}(\mathfrak{p}) : \overline{\delta}_1 \Rightarrow \delta_1$ in $\mathcal{C}^2$, respectively. The next proposition is a generalisation of Proposition 1.33.

Proposition 2.14. *If a morphism $f : X \to Y$ in $\mathcal{C}$ has the rlp with respect to the commutative square* $\mathbf{seed}(\mathfrak{p})$*, then it has the rlp with respect to the commutative square* $\mathbf{asee}(\mathfrak{p})$*.*

Proof. To prove the statement, start with the following leftmost commutative square. By pre-composing this diagram with the prevertebra $\|\gamma, \gamma'\|$, we obtain the middle and, in fact, right commutative diagrams.

$$\begin{tikzcd}
\mathbb{D}_1 \arrow[r, "x"] \arrow[d, "\delta_1"'] & X \arrow[d, "f"] \\
\mathbb{S} \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[r, "x"] \arrow[d, "\delta_1"'] & X \arrow[d, "f"] \\
\mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S} \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "x\circ\gamma'"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
\mathbb{D}_2 \arrow[r, "y\circ\delta_2"'] & Y
\end{tikzcd}$$

The right lifting property with respect to the commutative square $\mathbf{seed}(\mathfrak{p}) : \overline{\gamma} \Rightarrow \gamma$ then implies that there exists a lift $h : \overline{\mathbb{D}}_2 \to X$ making the leftmost diagram, below, commute. By using the pushout of the allied prevertebra $\|\overline{\gamma}, \overline{\gamma}'\|$, we may show that there exists a canonical arrow $h' : \overline{\mathbb{S}}' \to X$ making the following right diagram commute.

(2.12)
$$\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "x\circ\varrho'\circ\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] & X \arrow[d, "f"] \\
\overline{\mathbb{D}}_2 \arrow[r, "y\circ\delta_2\circ\varrho"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] & \overline{\mathbb{D}}_1 \arrow[r, "x\circ\varrho'"] \arrow[d, "\overline{\delta}_1"] & X \arrow[d, "f"] \\
\overline{\mathbb{D}}_2 \arrow[r, "\overline{\delta}_2"] \arrow[urr, bend right, "h"'] \arrow[rr, bend right=40, "y\circ\delta_2\circ\varrho"'] & \overline{\mathbb{S}}' \arrow[ur, dashed, "h'"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

This latest diagram provides the equation $f \circ h' \circ \overline{\delta}_2 = y \circ \delta_2 \circ \varrho$, whose last term is also equal to $y \circ \varkappa' \circ \overline{\delta}_2$ (see diagram (2.11)). Similarly, it provides the equation $f \circ h' \circ \overline{\delta}_1 = f \circ x \circ \varrho'$, whose right-hand term is equal to $y \circ \delta_1 \circ \varrho'$, and, in fact, $y \circ \varkappa' \circ \overline{\delta}_1$ (see diagram (2.11)). In other words, the following left equations hold, which, by universality of $\overline{\mathbb{S}}'$, imply the equation $f \circ h' = y \circ \varkappa'$. This last equation together with the rightmost top corner of the right diagram of (2.12) provides the following right commutative diagram.

$$\left\{\begin{array}{c} f \circ h' \circ \overline{\delta}_2 = y \circ \varkappa' \circ \overline{\delta}_2 \\ f \circ h' \circ \overline{\delta}_1 = y \circ \varkappa' \circ \overline{\delta}_1 \\ \Downarrow \\ f \circ h' = y \circ \varkappa' \end{array}\right\} \quad \Rightarrow \quad \begin{tikzcd} \overline{\mathbb{D}}_1 \arrow[r, "x\circ\varrho'"] \arrow[d, "\overline{\delta}_1"'] & X \arrow[d, "f"] \\ \overline{\mathbb{S}}' \arrow[r, "y\circ\varkappa'"'] \arrow[ur, dashed, "h'"] & Y \end{tikzcd}$$

This diagram provides a lift for the very first commutative square considered at the beginning of the proof. □

Remark 2.15. Proposition 2.14 holds even when the front face of diagram 2.11 is not a pushout square.

Similarly, it is not hard to prove the following result.

Proposition 2.16. *If a morphism $f : X \to Y$ in $\mathcal{C}$ is simple with respect to $(\{\gamma\}, \varkappa, \{\overline{\gamma}\})$, then it is simple with respect to $(\{\delta_1\}, \varrho', \{\overline{\delta}_1\})$.*

Proof. Mimic the proof of Proposition 2.14 by focusing on the top parts of the various commutative diagrams. However, the discussion after (2.12) is not needed. □

Later on, the *dual* of an alliance of prevertebrae $(\varkappa, \varrho, \varrho', \varkappa') : p \rightsquigarrow \overline{p}$ will be the alliance of prevertebrae $p^{\mathrm{rv}} \rightsquigarrow \overline{p}^{\mathrm{rv}}$ encoded by the 4-tuple $(\varkappa, \varrho', \varrho, \varkappa')$.

2.3.1.4. *Extended prevertebrae.* Let $\mathcal{C}$ be a category. An *extended prevertebra*[2] in $\mathcal{C}$ consists of an arrow $\gamma : \mathbb{S} \to \mathbb{D}_2$, a prevertebra $\|\overline{\gamma}, \overline{\gamma}'\| : \overline{\mathbb{S}} \multimap (\overline{\delta}_1, \overline{\delta}_2)$ and two morphisms $\varkappa : \overline{\mathbb{S}} \to \mathbb{S}$ and $\varrho : \overline{\mathbb{D}}_2 \to \mathbb{D}_2$ making the following left diagram commute. On the right is given the global structure in brackets.

$$(2.13) \qquad \begin{tikzcd} \overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[d, "\overline{\gamma}"'] & \mathbb{S} \arrow[d, "\gamma"] \\ \overline{\mathbb{D}}_2 \arrow[r, "\varrho"'] & \mathbb{D}_2 \end{tikzcd} \qquad \left(\begin{tikzcd} \mathbb{S} \arrow[d, "\gamma"'] & \overline{\mathbb{S}} \arrow[l, "\varkappa"'] \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] & \overline{\mathbb{D}}_1 \arrow[d, "\overline{\delta}_1"] \\ \mathbb{D}_2 & \overline{\mathbb{D}}_2 \arrow[l, "\varrho"] \arrow[r, "\overline{\delta}_2"'] \arrow[ur, phantom, "\ulcorner"] & \overline{\mathbb{S}}' \end{tikzcd}\right)$$

The arrows γ, $\varkappa$ and ϱ will be called *preseed, spherical transition* and *discal transition.* As in the case of an alliance of prevertebrae, the structure defined by $(\varkappa, \varrho)$ will be denoted as an arrow $\mathfrak{p} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \|\overline{\gamma}, \overline{\gamma}'\|$ where $\mathfrak{p}$ will often be replaced with the pair $(\varkappa, \varrho)$. This notation is motivated by the following implication.

Proposition 2.17. *Every alliance of prevertebrae $(\varkappa, \varrho, \varrho', \varkappa) : \|\gamma, \gamma'\| \rightsquigarrow \|\overline{\gamma}, \overline{\gamma}'\|$ involves an extended prevertebra $(\varkappa, \varrho) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \|\overline{\gamma}, \overline{\gamma}'\|$.*

Proof. Removing the front and right faces of diagram (2.11) provides the rightmost commutative diagram of (2.13) in brackets. □

In the spirit of the analogy made by Proposition 2.17, for any extended prevertebra $\mathfrak{p} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \|\overline{\gamma}, \overline{\gamma}'\|$ as defined above, the left commutative square of diagram (2.13) will be denoted as an arrow $\mathbf{seed}(\mathfrak{p}) : \overline{\gamma} \Rightarrow \gamma$ in $\mathcal{C}^2$.

[2] Here, 'extended' refers to the picture of a prevertebra whose seed has been *extended* to a square.

2.3.1.5. *Vertebrae.* Let $\mathcal{C}$ be a category. A *vertebra* in $\mathcal{C}$ consists of a prevertebra $\|\gamma,\gamma' : \mathbb{S}'\| : \mathbb{S} \multimap (\delta_1,\delta_2)$ and a further morphism $\beta : \mathbb{S}' \to \mathbb{D}'$ in $\mathcal{C}$, called the *stem* of the vertebra (see the left diagram of (2.14)). The right commutative square of (2.14) resulting from the composition of β with the prevertebra $\|\gamma,\gamma'\|$ will be called the *diskad* of the vertebra while the term *codiskad* will be used to denominate its dual commutative square.

(2.14)
$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \arrow[dl, phantom, "\ulcorner"] & \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}' \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\beta\circ\delta_1"] \\ \mathbb{D}_2 \arrow[r, "\beta\circ\delta_2"'] & \mathbb{D}' \end{tikzcd}$$

The composite morphisms $\beta \circ \delta_1$ and $\beta \circ \delta_2$ will be called the *trivial stem* and the *trivial costem*, respectively. In the sequel, a vertebra as given above will be denoted by the symbols $\|\gamma,\gamma'\| \cdot \beta$.

The *dual* of a vertebra $\|\gamma,\gamma'\| \cdot \beta$ will be the vertebra $\|\gamma',\gamma\| \cdot \beta$, whose diskad is the codiskad of $\|\gamma,\gamma'\| \cdot \beta$. The prevertebra of a vertebra will sometimes be called the *base* of the vertebra. If the prevertebra $\|\gamma,\gamma'\|$ is denoted by p, then the vertebra defined by $\|\gamma,\gamma'\| \cdot \beta$ will be shortened as $p \cdot \beta$. The dual of the latter is then given by $p^{\mathrm{rv}} \cdot \beta$. For any vertebra $v := \|\gamma,\gamma'\| \cdot \beta$ as defined above, the diskad of the vertebra v will be denoted as an arrow $\mathbf{disk}(v) : \gamma \Rightarrow \beta \circ \delta_1$ in $\mathcal{C}^2$. Finally, the *domain* and *codomain* of a vertebra are the respective domain and codomain of its base.

Remark 2.18. As in the case of prevertebrae, a vertebra is the model of a colimit sketch `Vert` in $\mathcal{C}$. The sketch `Vert` consists of the colimit sketch `Prev` augmented by an additional arrow whose domain is the chosen pushout of `Prev`.

2.3.1.6. *Alliances of vertebrae.* Let $\mathcal{C}$ a category. An *alliance of vertebrae* in $\mathcal{C}$ consists of two vertebrae $p \cdot \beta$ and $\overline{p} \cdot \overline{\beta}$ equipped with an arrow from the former to the latter in the opposite category of $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Vert})$. In other words, an alliance from $p \cdot \beta$ to $\overline{p} \cdot \overline{\beta}$ consists of an alliance of prevertebrae $(\varkappa,\varrho,\varrho',\varkappa') : p \rightsquigarrow \overline{p}$ and a commutative square as follows.

$$\begin{tikzcd} \overline{\mathbb{S}}' \arrow[r, "\varkappa'"] \arrow[d, "\overline{\beta}"'] & \mathbb{S}' \arrow[d, "\beta"] \\ \overline{\mathbb{D}}' \arrow[r, "u"] & \mathbb{D}' \end{tikzcd}$$

When seen as a morphism in the opposite category of $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Vert})$, it will make sense to denote the previous alliance $(\varkappa,\varrho,\varrho',\varkappa',u)$ as an arrow $\mathfrak{a} : p \cdot \beta \rightsquigarrow \overline{p} \cdot \overline{\beta}$ where $\mathfrak{a}$ will often be replaced with the 5-tuple $(\varkappa,\varrho,\varrho',\varkappa',u)$. The previous commutative square will later be denoted as an arrow $\mathbf{stem}(\mathfrak{a}) : \overline{\beta} \Rightarrow \beta$ in $\mathcal{C}^2$ while the arrows $\mathbf{seed}(\mathfrak{p})$ and $\mathbf{asee}(\mathfrak{p})$ associated with the alliance of prevertebrae $\mathfrak{p} := (\varkappa,\varrho,\varrho',\varkappa')$ will be replaced with the notations $\mathbf{seed}(\mathfrak{a})$ and $\mathbf{asee}(\mathfrak{a})$. We will also need the biased square (see Remark 2.4) of $\mathbf{stem}(\mathfrak{a})$, which will be denoted as an arrow $\mathbf{bste}(\mathfrak{a}) : \overline{\beta} \Rightarrow \beta \circ \varkappa'$ in $\mathcal{C}^2$. If the commutative square $\mathbf{asee}(\mathfrak{a})$ is of the form $\overline{\delta}_1 \Rightarrow \delta_1$ in $\mathcal{C}^2$, the vertical pasting of $\mathbf{asee}(\mathfrak{a})$ with $\mathbf{stem}(\mathfrak{a}) : \overline{\beta} \Rightarrow \beta$ (see below) will be denoted as $\mathbf{triv}(\mathfrak{a}) : \overline{\beta} \circ \overline{\delta}_1 \Rightarrow \beta \circ \delta_1$.

$$\begin{tikzcd} \overline{\mathbb{D}}_1 \arrow[r, "\varrho'"] \arrow[d, "\overline{\delta}_1"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \\ \overline{\mathbb{S}}' \arrow[r, "\varkappa'"'] & \mathbb{S}' \end{tikzcd} \quad \begin{tikzcd} \overline{\mathbb{S}}' \arrow[r, "\varkappa'"] \arrow[d, "\overline{\beta}"'] & \mathbb{S}' \arrow[d, "\beta"] \\ \overline{\mathbb{D}}' \arrow[r, "u"'] & \mathbb{D}' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \overline{\mathbb{D}}_1 \arrow[r, "\varrho'"] \arrow[d, "\overline{\beta}\circ\overline{\delta}_1"'] & \mathbb{D}_1 \arrow[d, "\beta\circ\delta_1"] \\ \overline{\mathbb{D}}' \arrow[r, "u"'] & \mathbb{D}' \end{tikzcd}$$

The *dual* of an alliance $(\varkappa,\varrho,\varrho',\varkappa',u) : p \cdot \beta \rightsquigarrow \overline{p} \cdot \overline{\beta}$ is the alliance of vertebrae $p^{\mathrm{rv}} \cdot \beta \rightsquigarrow \overline{p}^{\mathrm{rv}} \cdot \overline{\beta}$ encoded by the 4-tuple $(\varkappa,\varrho',\varrho,\varkappa',u)$.

2.3.1.7. *Extended vertebrae.* Let $\mathcal{C}$ be a category. An *extended vertebra* in $\mathcal{C}$ consists of a vertebra $\overline{p}\cdot\overline{\beta}$ in $\mathcal{C}$ endowed with an extended prevertebra $\mathfrak{p} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{p}$. The whole data consists of a diagram as follows.

$$\begin{tikzcd}
\mathbb{S} \arrow[d, "\gamma"'] & \overline{\mathbb{S}} \arrow[l, "\varkappa"'] \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] & \overline{\mathbb{D}}_1 \arrow[d, "\overline{\delta}_1"] \arrow[dl, phantom, "\ulcorner", very near end] & \\
\mathbb{D}_2 & \overline{\mathbb{D}}_2 \arrow[l, "\varrho"] \arrow[r, "\overline{\delta}_2"'] & \overline{\mathbb{S}}' \arrow[r, "\overline{\beta}"] & \overline{\mathbb{D}}'
\end{tikzcd}$$

Such a structure will be denoted as an arrow $\mathfrak{v} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{p}\cdot\overline{\beta}$. For convenience, the associated arrows $\mathbf{seed}(\mathfrak{p})$ and $\mathbf{disk}(\overline{p}\cdot\overline{\beta})$ in $\mathcal{C}^2$ will be denoted as $\mathbf{seed}(\mathfrak{v})$ and $\mathbf{disk}(\mathfrak{v})$, respectivey. The extended vertebra $\mathfrak{v}$ will sometimes be denoted by $\mathfrak{p}\cdot\overline{\beta}$.

Remark 2.19 (Notation)**.** As in the case of extended prevertebrae, an alliance of vertebrae of the form $(\varkappa,\varrho,\varrho',\varkappa',u) : p\cdot\beta \rightsquigarrow \overline{p}\cdot\overline{\beta}$ always involves an extended vertebra $(\varkappa,\varrho) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{p}\cdot\overline{\beta}$, but the converse is generally false. In fact, extended vertebrae should more be thought of as a generalised version of a vertebra than the truncation of some alliance of vertebrae. This point of view is enhanced by the notation $\mathfrak{p}\cdot\overline{\beta}$, or, in the present case, $(\varkappa,\varrho)\cdot\overline{\beta}$, thereby copying the notation of vertebrae.

2.3.1.8. *Nodes of vertebrae.* A *node of vertebrae* in a category $\mathcal{C}$ is equivalently

1) a class of vertebrae in $\mathcal{C}$ whose bases are equal;
2) a prevertebra p endowed with a class Ω of morphisms in $\mathcal{C}$ such that the domain of every element in Ω is the codomain of the prevertebra p.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \arrow[dl, phantom, "\ulcorner", very near end] & \mathbb{D}'_* \\
& & \vdots \\
\mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"] \arrow[uur, "\beta_*"] & \mathbb{D}'
\end{tikzcd}$$
$$\underbrace{\qquad\qquad\qquad}_{\Omega}$$

The prevertebra p will be called the *base* of the node of vertebrae while the elements of Ω will be called the *stems* of the node. Later on, a node of vertebrae such as that given above will be denoted by the symbols $p\cdot\Omega$. The *dual* of $p\cdot\Omega$ will be the node of vertebrae $p^{\mathrm{rv}}\cdot\Omega$. Finally, the *domain* and *codomain* of a node of vertebrae are the respective domain and codomain of its base.

2.3.1.9. *Alliances of nodes of vertebrae.* Let $\mathcal{C}$ a category. An *alliance of nodes of vertebrae* in $\mathcal{C}$ consists of two nodes of vertebrae $p\cdot\Omega$ and $\overline{p}\cdot\overline{\Omega}$ equipped with

1) an alliance of prevertebrae $(\varkappa,\varrho,\varrho',\varkappa') : p \rightsquigarrow \overline{p}$;
2) a metafunction $\phi : \Omega \to \overline{\Omega}$ and an alliance of vertebrae $(\varkappa,\varrho,\varrho',\varkappa',u_\beta) : p\cdot\beta \rightsquigarrow \overline{p}\cdot\phi(\beta)$ for every stem $\beta\in\Omega$.

Later on, the collection of arrows u_β indexed by $\beta\in\Omega$ will symbolically be represented by its associate letter, namely u. The set of data $(\varkappa,\varrho,\varrho',\varkappa',\phi,u)$ will then be called the *structure of alliance* and denoted as an arrow $p\cdot\Omega \rightsquigarrow \overline{p}\cdot\overline{\Omega}$. When the structure of alliance $(\varkappa,\varrho,\varrho',\varkappa',\phi,u)$ is given a name, say $\mathfrak{a}$, the alliance of vertebrae $(\varkappa,\varrho,\varrho',\varkappa',u_\beta)$, for some $\beta\in\Omega$, will be denoted by $\mathfrak{a}_\beta$ and referred to as to *the component of* $\mathfrak{a}$ *at* β. Because the commutative square $\mathbf{seed}(\mathfrak{a}_\beta)$ does not depend on the stem $\beta\in\Omega$, it will later be denoted as $\mathbf{seed}(\mathfrak{a})$. The *dual* of an alliance of nodes of vertebrae such as above is the alliance defined by $(\varkappa,\varrho',\varrho,\varkappa',\phi,u) : p^{\mathrm{rv}}\cdot\Omega \rightsquigarrow \overline{p}^{\mathrm{rv}}\cdot\overline{\Omega}$, which will later be denoted as $\mathfrak{a}^{\mathrm{rv}}$.

Remark 2.20. The triple $(\Omega, \varkappa, \overline{\Omega})$ defines a scale in $\mathcal{C}$, which is oriented along the meta-function $\phi : \Omega \to \overline{\Omega}$. This scale will later be referred to as the *underlying scale* of the alliance of nodes of vertebrae $\mathfrak{a}$.

Remark 2.21. The triple $(\overline{\Omega}, \mathbf{disk}(\overline{p} \cdot \_), \mathbf{seed}(\mathfrak{a}))$ defines a besom in $\mathcal{C}$ (see diagram below). This besom will later be referred to as the *underlying besom* of the alliance of nodes of vertebrae $\mathfrak{a}$.

$$\begin{tikzcd}
\overline{\beta}_* \circ \delta_1 & & \\
\vdots & & \\
\overline{\beta} \circ \delta_1 & \overline{\gamma} \arrow[l, Rightarrow, "\mathbf{disk}(\overline{p}\cdot\overline{\beta})"'] \arrow[luu, Rightarrow, "\mathbf{disk}(\overline{p}\cdot\overline{\beta}_*)"'] \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a})"] & \gamma
\end{tikzcd}
\qquad (\overline{\beta}, \overline{\beta}_* \in \overline{\Omega})$$

2.3.1.10. *Extended nodes of vertebrae.* Let $\mathcal{C}$ be a category. An *extended node of vertebrae* in $\mathcal{C}$ consists of a class of extended vertebrae in $\mathcal{C}$ whose extended prevertebrae are equal (see next diagram). This is also equivalent to considering an extended prevertebra $(\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{p}$ endowed with a node of vertebrae $\overline{p} \cdot \overline{\Omega}$ in $\mathcal{C}$.

$$\begin{tikzcd}
\mathbb{S} \arrow[d, "\gamma"'] & \overline{\mathbb{S}} \arrow[l, "\varkappa"'] \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] & \overline{\mathbb{D}}_1 \arrow[d, "\overline{\delta}_1"] & \overline{\mathbb{D}}'_* \\
 & & & \vdots \\
\mathbb{D}_2 & \overline{\mathbb{D}}_2 \arrow[l, "\varrho"] \arrow[r, "\overline{\delta}_2"'] \arrow[ur, phantom, "\ulcorner"] & \overline{\mathbb{S}}' \arrow[r, "\overline{\beta}"] \arrow[ruu, "\overline{\beta}_*"] & \overline{\mathbb{D}}'
\end{tikzcd}$$

Such a structure will be denoted as an arrow $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{p} \cdot \overline{\Omega}$ where the notation $\mathfrak{n}$ will often be replaced with the pair $(\varkappa, \varrho)$. For convenience, the extended vertebrae $\gamma \overset{\text{ex}}{\rightsquigarrow} \overline{p}\cdot\overline{\beta}$ encoded by this same pair $(\varkappa, \varrho)$, for some $\beta \in \overline{\Omega}$, will be denoted by $\mathfrak{n}_\beta$ and referred to as *the component of* $\mathfrak{n}$ *at* β. Also, because the commutative square $\mathbf{seed}(\mathfrak{n}_\beta)$ does not depend on the stem β, for every $\beta \in \overline{\Omega}$, it will later be denoted as $\mathbf{seed}(\mathfrak{n})$. As in the case of extended vertebrae, an extended node of vertebrae $\mathfrak{n}$ whose extended prevertebrae is denoted by $\mathfrak{p} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{p}$ will sometimes be denoted as $\mathfrak{p} \cdot \overline{\Omega}$. Notice that every alliance of nodes of vertebrae $(\varkappa, \varrho, \varrho', \varkappa', \phi, u) : \nu \rightsquigarrow \overline{\nu}$ gives rise to an extended node of vertebrae $(\varkappa, \varrho) : \gamma \rightsquigarrow \overline{\nu}$ where γ denotes the seed of ν. This extended node of vertebrae will be referred to as the *underlying extended node of vertebrae* of the structure of alliance $(\varkappa, \varrho, \varrho', \varkappa', \phi, u)$ and denoted by $\mathbf{ext}(\mathfrak{a}) : \gamma \rightsquigarrow \overline{\nu}$.

Remark 2.22. The triple $(\overline{\Omega}, \mathbf{disk}(\overline{p} \cdot \_), \mathbf{seed}(\mathfrak{n}))$ defines a besom in $\mathcal{C}$ (see next diagram). This besom will later be referred to as the *underlying besom* of the extended nodes of vertebrae $\mathfrak{n}$.

$$\begin{tikzcd}
\overline{\beta}_* \circ \delta_1 & & \\
\vdots & & \\
\overline{\beta} \circ \delta_1 & \overline{\gamma} \arrow[l, Rightarrow, "\mathbf{disk}(\overline{p}\cdot\overline{\beta})"'] \arrow[luu, Rightarrow, "\mathbf{disk}(\overline{p}\cdot\overline{\beta}_*)"'] \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] & \gamma
\end{tikzcd}
\qquad (\overline{\beta}, \overline{\beta}_* \in \overline{\Omega})$$

2.3.1.11. *Communications.* In the sequel, it will be necessary to make extended nodes of vertebrae 'communicate' between their preseeds, seeds and coseeds. This section introduce the language that will enable us to handle such 'communications'. Let $\mathcal{C}$ be a category and denote by $\mathbf{Com}(\mathcal{C})$ the opposite category $(\mathcal{C}^2)^{\text{op}}$. The arrows of the category $\mathbf{Com}(\mathcal{C})$ will be called *communications* and denoted with the symbol $\rightsquigarrow$. Any arrow $\gamma_* \Rightarrow \gamma$ in $\mathcal{C}^2$ encoded by a diagram as given below will define a communication $(\varkappa, \varrho) : \gamma \rightsquigarrow \gamma_*$ in $\mathbf{Com}(\mathcal{C})$.

$$\begin{tikzcd}
\mathbb{S}_* \arrow[r, "\varkappa"] \arrow[d, "\gamma_*"'] & \mathbb{S} \arrow[d, "\gamma"] \\
\mathbb{D}_* \arrow[r, "\varrho"'] & \mathbb{D}
\end{tikzcd}$$

The category $\mathbf{Com}(\mathcal{C})$ will be called the *category of communications of* $\mathcal{C}$ and its composition operation will be denoted by the symbol $\odot$. Note that every extended node of vertebrae $\mathfrak{n} := (\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ gives rise to a communication $(\varkappa, \varrho) : \gamma \rightsquigarrow \overline{\gamma}$ where $\overline{\gamma}$ denotes the seed of $\overline{\nu}$. This communication will be referred to as the *underlying communication* of $\mathfrak{n}$ and denoted as an arrow $\mathbf{com}(\mathfrak{n}) : \gamma \rightsquigarrow \overline{\gamma}$. In the spirit of this analogy, the commutative square in $\mathcal{C}^2$ encoding a communication $(\varkappa, \varrho) : \gamma \rightsquigarrow \overline{\gamma}$ will be denoted as $\mathbf{seed}(t) : \gamma \Rightarrow \overline{\gamma}$. It directly follows from such a definition that for any pair of composable communications $t : \gamma \rightsquigarrow \gamma_*$ and $t_* : \gamma_* \rightsquigarrow \gamma_\flat$, the following relation holds.

$$\mathbf{seed}(t) \circ \mathbf{seed}(t_*) = \mathbf{seed}(t_* \odot t) \tag{2.15}$$

Finally, for the sake of convenience, the communication $\mathbf{com}(\mathbf{ext}(\mathfrak{a}))$ associated with an alliance of node of vertebrae $\mathfrak{a}$ will later be denoted as $\mathbf{com}(\mathfrak{a})$.

2.3.1.12. *Convention on notations.* Throughout the present and next chapters, the indexing notations of a node of vertebrae will be coherent with the indexing notations of its objects, e.g. the base p_* of a node of vertebrae ν_* shall be encoded as $\|\gamma_*, \gamma'_* : \mathbb{S}'_*\| : \mathbb{S}_* \multimap (\delta_1^*, \delta_2^*)$. This convention will be extended to any additional structure with which the node of vertebrae is equipped. Similarly, the indexing notations for the structure of an alliance will be coherent with the indexing notations of the nodes of vertebrae for which it is defined. The letters $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{v}$ and $\mathfrak{n}$ will (usually) be reserved for alliances, extended prevertebrae, extended vertebrae and extended nodes of vertebrae, respectively. For their part, communications will usually be denoted by letters t and be coherent with the indexing notations of theirs components.

2.3.2. Actions and compositions.

2.3.2.1. *Composition of alliances of nodes of vertebrae.* Let $\mathcal{C}$ be a category. It is not difficult to see that the structures of alliance in $\mathcal{C}$ induce a metacategory whose objects are the nodes of vertebrae in $\mathcal{C}$ and whose morphisms are alliances of nodes of vertebrae. The composition of two alliances of nodes of vertebrae, say of the form $(\varkappa_0, \varrho_0, \varrho'_0, \varkappa'_0, \phi_0, u^0) : p_0 \cdot \Omega_0 \rightsquigarrow p_1 \cdot \Omega_1$ and $(\varkappa_1, \varrho_1, \varrho'_1, \varkappa'_1, \phi_1, u^1) : p_1 \cdot \Omega_1 \rightsquigarrow p_2 \cdot \Omega_2$, is defined by the structure of alliance

$$(\varkappa_0 \circ \varkappa_1, \varrho_0 \circ \varrho_1, \varrho'_0 \circ \varrho'_1, \varkappa'_0 \circ \varkappa'_1, \phi_0 \circ \phi_1, u^0 \circ u^1) : p_0 \cdot \Omega_0 \rightsquigarrow p_2 \cdot \Omega_2 \tag{2.16}$$

where $(u^0 \circ u^1)_\beta := u^0_\beta \circ u^1_{\phi_0(\beta)}$ for every stem $\beta \in \Omega_0$. This metacategory will be denoted by $\mathbf{Ally}(\mathcal{C})$ and its composition will be written with the symbol $\circledcirc$. Seeing alliances of vertebrae as particular alliances of nodes of vertebrae in $\mathbf{Ally}(\mathcal{C})$, the operations $\mathbf{stem}(\_)$, $\mathbf{seed}(\_)$, $\mathbf{asee}(\_)$ and $\mathbf{triv}(\_)$ appears to be functorial from $\mathbf{Ally}(\mathcal{C})$ to $\mathbf{Com}(\mathcal{C})$ in the sense that for any pair of composable alliances of vertebrae $\mathfrak{a} : v \rightsquigarrow v_*$ and $\mathfrak{a}_* : v_* \rightsquigarrow v_\flat$, the relation

$$\mathbf{oper}(\mathfrak{a}) \circ \mathbf{oper}(\mathfrak{a}_*) = \mathbf{oper}(\mathfrak{a}_* \circledcirc \mathfrak{a})$$

hold when replacing $\mathbf{oper}(\_)$ with any of the operators previously listed. The only exception to the foregoing identity is for the biased operator $\mathbf{bste}(\_)$. It follows from formula (2.16) that these relations generalise to alliances of nodes of vertebrae as follows: for any pair of alliances of nodes of vertebrae $\mathfrak{a} : p \cdot \Omega \rightsquigarrow p_* \cdot \Omega_*$ and $\mathfrak{a}^* : p_* \cdot \Omega_* \rightsquigarrow p_\flat \cdot \Omega_\flat$, the following relation holds for every stem $\beta \in \Omega$, where $\phi : \Omega \to \Omega_*$ denotes the metafunction associated with $\mathfrak{a}$.

$$\mathbf{oper}(\mathfrak{a}_\beta) \circ \mathbf{oper}(\mathfrak{a}^*_{\phi(\beta)}) = \mathbf{oper}(\mathfrak{a}^*_{\phi(\beta)} \circledcirc \mathfrak{a}_\beta) = \mathbf{oper}((\mathfrak{a}_* \circledcirc \mathfrak{a})_\beta) \tag{2.17}$$

Note that in the case of **seed**(_), the previous relation does not depend on β. In the same fashion, the operation **com**(_) is functorial from $\mathbf{Com}(\mathcal{C})$ to $\mathbf{Ally}(\mathcal{C})$, in the sense that the following equation is satisfied for any pair of composable alliances of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ and $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$.

$$\mathbf{com}(\mathfrak{a}_*) \odot \mathbf{com}(\mathfrak{a}) = \mathbf{com}(\mathfrak{a}_* \circledcirc \mathfrak{a}) \tag{2.18}$$

Remark 2.23. Any pair of composable alliances of nodes of vertebrae $\mathfrak{a} : p \cdot \Omega \rightsquigarrow p_* \cdot \Omega_*$ and $\mathfrak{a}^* : p_* \cdot \Omega_* \rightsquigarrow p_\flat \cdot \Omega_\flat$ induces a couple structure (see section 2.2.1.3) between the underlying besoms of $\mathfrak{a}$ and $\mathfrak{a}_*$. The couple is then given by the metafunction $\phi_* : \Omega_* \to \Omega_\flat$ associated with $\mathfrak{a}^*$ and the following commutative diagram for every stem β_* in Ω_*.

$$\begin{tikzcd}
\gamma_\flat \arrow[rr, Rightarrow, "\mathbf{seed}(\mathfrak{a}^*)"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\phi_*(\beta_*))"'] & & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma \\
\phi_*(\beta_*) \circ \delta_1^\flat \arrow[rr, Rightarrow, "\mathbf{triv}(\mathfrak{a}^*_{\beta_*})"'] & & \beta_* \circ \delta_1^* &
\end{tikzcd}$$

2.3.2.2. *Action of alliances on extended nodes of vertebrae.* Let $\mathcal{C}$ be a category. This section defines a right action of $\mathbf{Ally}(\mathcal{C})$ on the extended nodes of vertebrae of $\mathcal{C}$. Consider an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ in $\mathcal{C}$, encoded by a pair $(\varkappa, \varrho)$, and an alliance of nodes of vertebrae $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ in $\mathcal{C}$, encoded by a 5-tuple $(\varkappa_*, \varrho_*, \varrho'_*, \varkappa'_*, \phi_*, u^*)$. We will later denote by $\mathfrak{a}_* \circledcirc \mathfrak{n}$ the extended node of vertebrae $\gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\flat$ encoded by the pair $(\varkappa \circ \varkappa_*, \varrho \circ \varrho_*)$ or, more diagrammatically, by the following diagram.

$$\begin{tikzcd}
\mathbb{S} \arrow[d, "\gamma"'] & \mathbb{S} \arrow[l, "\varkappa"'] \arrow[d, dotted] & \mathbb{S}_\flat \arrow[l, "\varkappa_*"'] \arrow[r, "\gamma'_\flat"] \arrow[d, "\gamma_\flat"'] & \mathbb{D}_1^\flat \arrow[d, "\delta_1^\flat"] & \mathbb{D}'_\dagger \\
 & & & & \vdots \\
\mathbb{D}_2 & \mathbb{D}_2 \arrow[l, "\varrho"] & \mathbb{D}_2^\flat \arrow[l, "\varrho_*"] \arrow[r, "\delta_2^\flat"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}'_\flat \arrow[r, "\beta_\flat"] \arrow[uur, "\beta_\dagger"] & \mathbb{D}'_\flat
\end{tikzcd}$$

$$\underbrace{\hphantom{\mathbb{D}_2 \longleftarrow \mathbb{D}_2}}_{\mathfrak{n}}\underbrace{\hphantom{\longleftarrow \mathbb{D}_2^\flat \longrightarrow \mathbb{S}'_\flat \longrightarrow \mathbb{D}'_\flat}}_{\mathfrak{a}_*}$$

This operation defines a right action in the sense that it is associative and neutral with respect to the identity alliance. It is straightforward to check that this action is compatible with the operation **ext**(_) in the sense that for any pair of composable alliances of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ and $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, the following identity holds.

$$\mathfrak{a}_* \circledcirc \mathbf{ext}(\mathfrak{a}) = \mathbf{ext}(\mathfrak{a}_* \circledcirc \mathfrak{a}) \tag{2.19}$$

Similarly, the action $\circledcirc$ is compatible with the operation **seed**(_) in the sense that for any extended node of vertebrae $\mathfrak{n} : \gamma \rightsquigarrow \nu_*$ and alliance of nodes of vertebrae $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, the following equation holds.

$$\mathbf{seed}(\mathfrak{n}) \circ \mathbf{seed}(\mathfrak{a}_*) = \mathbf{seed}(\mathfrak{a}_* \circledcirc \mathfrak{n}) \tag{2.20}$$

Remark 2.24. Any extended node of vertebrae $\mathfrak{n} : p \cdot \Omega \rightsquigarrow p_* \cdot \Omega_*$ and alliance $\mathfrak{a}^* : p_* \cdot \Omega_* \rightsquigarrow p_\flat \cdot \Omega_\flat$ in $\mathbf{Ally}(\mathcal{C})$ induce a couple structure between the underlying besoms of $\mathfrak{n}$ and $\mathfrak{a}^*$. The couple is given by the metafunction $\phi_* : \Omega_* \to \Omega_\flat$ associated with $\mathfrak{a}^*$ and the following commutative diagram for every stem β_* in Ω_*.

$$\begin{tikzcd}
\gamma_\flat \arrow[rr, Rightarrow, "\mathbf{seed}(\mathfrak{a}^*)"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\phi_*(\beta_*))"'] & & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma \\
\phi_*(\beta_*) \circ \delta_1^\flat \arrow[rr, Rightarrow, "\mathbf{triv}(\mathfrak{a}^*_{\beta_*})"'] & & \beta_* \circ \delta_1^* &
\end{tikzcd}$$

2.3.2.3. *Action of communications on extended nodes of vertebrae.* Let $\mathcal{C}$ be a category. This section defines a left action of $\mathbf{Com}(\mathcal{C})$ on the extended nodes of vertebrae of $\mathcal{C}$. Consider an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ in $\mathcal{C}$, encoded by a pair $(\varkappa, \varrho)$, and a communication $t_\flat : \gamma_\flat \rightsquigarrow \gamma$ in $\mathbf{Com}(\mathcal{C})$, encoded by a pair $(\varkappa_\flat, \varrho_\flat)$. We will later denote by $\mathfrak{n} \odot t_\flat$ the extended node of vertebrae $\gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_*$ encoded by the pair $(\varkappa \circ \varkappa_\flat, \varrho \circ \varrho_\flat)$ or, more diagrammatically, by the following diagram.

$$\begin{tikzcd}
\mathbb{S}^\flat \arrow[d, "\gamma_\flat"'] & \mathbb{S} \arrow[l, "\varkappa_\flat"'] \arrow[d, dotted] & \mathbb{S}_* \arrow[l, "\varkappa"'] \arrow[r, "\gamma'_*"] \arrow[d, "\gamma_*"'] & \mathbb{D}_1^* \arrow[d, "\delta_1^*"] & \mathbb{D}'_\dagger \arrow[d, phantom, "\vdots"] \\
\mathbb{D}_2^\flat & \mathbb{D}_2 \arrow[l, "\varrho_\flat"] & \mathbb{D}_2^* \arrow[l, "\varrho"] \arrow[r, "\delta_2^*"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}'_* \arrow[r, "\beta_*"] \arrow[ur, "\beta_\dagger"] & \mathbb{D}'_* \\
\arrow[r, phantom, "\underbrace{\qquad\qquad}_{t_\flat}"] & \arrow[rrr, phantom, "\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\mathfrak{n}}"] & & &
\end{tikzcd}$$

This operation defines a left action in the sense that it is associative and neutral with respect to the identity communication. This action is compatible with the operations $\mathbf{seed}(\_)$ and $\mathbf{com}(\_)$ in the sense that for any extended nodes of vertebrae $\mathfrak{n}_* : \gamma_* \rightsquigarrow \nu_*$ and communication $t : \gamma \rightsquigarrow \gamma_*$, the following identities hold.

$$\mathbf{seed}(t) \circ \mathbf{seed}(\mathfrak{n}_*) = \mathbf{seed}(\mathfrak{n}_* \odot t) \qquad \mathbf{com}(\mathfrak{n}_* \odot t) = \mathbf{com}(\mathfrak{n}_*) \odot t \tag{2.21}$$

Similarly, for any pair of composable alliances of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ and $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, it is not hard to check that the following equation is satisfied.

$$\mathbf{ext}(\mathfrak{a}_*) \odot \mathbf{com}(\mathfrak{a}) = \mathbf{ext}(\mathfrak{a}_* \circledcirc \mathfrak{a}) \tag{2.22}$$

2.3.3. Zoo for an elementary homotopy theory.

2.3.3.1. *Zoo of an alliance of nodes of vertebrae.* Let $\mathcal{C}$ be a category and consider an alliance of nodes of vertebrae $\mathfrak{a} : p \cdot \Omega \rightsquigarrow \overline{p} \cdot \overline{\Omega}$. A morphism $f : X \to Y$ in $\mathcal{C}$ will respectively be said to be a *i) fibration; ii) trivial fibration; iii) pseudofibration; iv) intraction; v) surtraction for the alliance* $\mathfrak{a}$ if it

i) has the rlp with respect to $\mathbf{triv}(\mathfrak{a}_\beta)$ for every stem $\beta \in \Omega$;

ii) has the rlp with respect to $\mathbf{seed}(\mathfrak{a})$ and $\mathbf{bste}(\mathfrak{a}_\beta)$ for every stem $\beta \in \Omega$;

iii) has the rlp with respect to $\mathbf{seed}(\mathfrak{a})$;

iv) is simple with respect to the scale $(\Omega, \varkappa, \overline{\Omega})$;

v) is divisible by the besom $(\overline{\Omega}, \mathbf{disk}(\overline{p} \cdot \_), \mathbf{seed}(\mathfrak{a}))$;

In addition, the morphism $f : X \to Y$ will be called a *weak equivalence* if it is both an intraction and a surtraction. This set of terminology will be referred to as the *zoo of the alliance* $\mathfrak{a} : p \cdot \Omega \rightsquigarrow \overline{p} \cdot \overline{\Omega}$.

Remark 2.25. When the alliance $\mathfrak{a}$ is an identity on a node of vertebrae $\|\gamma, \gamma'\| \cdot \Omega$ in $\mathbf{Ally}(\mathcal{C})$, the right lifting properties given in i), ii) and iii) are usual right lifting properties. In this case, the biased square $\mathbf{bste}(\mathfrak{a}_\beta)$ is equal to its proper square $\mathbf{stem}(\mathfrak{a}_\beta)$. The right lifting properties are then defined with respect to the arrows of the vertebra $\|\gamma, \gamma'\| \cdot \Omega$ having the same names i.e. trivial stems $(\beta \circ \delta_1)$, seed (γ) and stems (β). Finally, in the case where the alliance $\mathfrak{a}$ is an identity on a vertebra $v := \|\gamma, \gamma'\| \cdot \beta$, property iv) says that $f : X \to Y$ is simple with respect to the arrow β. For its part, property v) amounts to saying that any arrow $\gamma \Rightarrow f$ in $\mathcal{C}^2$ factorises through the diskad $\mathbf{disk}(v) : \gamma \Rightarrow \beta \circ \delta_1$.

Notice that the notions of pseudofibration and surtraction do not require the entire structure of an alliance and may be defined with respect to its underlying extended node of vertebrae (see Proposition 2.26). Later on, it will actually be useful to have such notions defined with respect to general extended nodes of vertebrae, which motivates the next section.

2.3.3.2. *Zoo of an extended node of vertebrae.* Let $\mathcal{C}$ be a category and consider an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{p} \cdot \overline{\Omega}$. A morphism $f : X \to Y$ in $\mathcal{C}$ will respectively be said to be a *i) pseudofibration; ii) surtraction for* $\mathfrak{n}$ if it

- i) has the rlp with respect to $\mathbf{seed}(\mathfrak{n})$;
- ii) is divisible by the underlying besom $(\overline{\Omega}, \mathbf{disk}(\overline{p} \cdot \_), \mathbf{seed}(\mathfrak{n}))$.

This set of terminology will be referred to as the *zoo of the extended nodes of vertebrae* $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{p} \cdot \overline{\Omega}$.

Proposition 2.26. *A morphism is a pseudofibration (resp. surtraction) for an alliance of nodes of vertebrae if and only if it is a pseudofibration (resp. surtraction) for its underlying extended nodes of vertebrae.*

Proof. Straightforward. □

In fact, one can again reduce the structure for which a pseudofibration is defined, which only requires the structure of the underlying communication of $\mathfrak{n}$ (see Proposition 2.27).

2.3.3.3. *Zoo of a communication.* Let $\mathcal{C}$ be a category and consider a communication $t : \gamma \rightsquigarrow \overline{\gamma}$. A morphism $f : X \to Y$ in $\mathcal{C}$ will be said to be a *pseudofibration for* t if it has the rlp with respect to the commutative square $\mathbf{seed}(t) : \overline{\gamma} \Rightarrow \gamma$.

Proposition 2.27. *A morphism is a pseudofibration for an extended node of vertebrae if and only if it is a pseudofibration for its underlying communication.*

Proof. Straightforward. □

2.3.4. Properties associated with zoos.

2.3.4.1. *Direct properties.* Let $\mathcal{C}$ be a category. The next propositions for which no alliance of nodes of vertebrae, extended nodes of vertebrae or communication is specified involve a *unique* such structure as defined in section 2.3.3. In this case, Proposition 2.26 and Proposition 2.27 ensure that not mentioning the type of structure along which the zoo is defined does not matter.

Proposition 2.28. *Every isomorphism in $\mathcal{C}$ is a pseudofibration, fibration and trivial fibration.*

Proof. All isomorphisms have the rlp with respect to any commutative square in the ambient category $\mathcal{C}$. □

Proposition 2.29. *Pseudofibrations, fibrations and trivial fibrations are preserved under pullbacks.*

Proof. Follows from Proposition 2.3. □

Proposition 2.30. *The classes of pseudofibrations, fibrations, trivial fibrations, surtractions and intractions are stable under retracts.*

Proof. See the properties listed in section 2.2.1.4. □

The next three propositions show that the class of trivial fibrations is at the intersection of three different other classes of the zoo.

Proposition 2.31. *Every trivial fibration is an pseudofibration.*

Proof. By definition. □

Proposition 2.32. *Every trivial fibration is a fibration.*

Proof. By Proposition 2.14, a trivial fibration f for an alliance of nodes of vertebrae $\mathfrak{a}$ must have the rlp with respect to $\mathbf{asee}(\mathfrak{a})$. Proposition 2.5 then implies that since f has the rlp with respect to $\mathbf{asee}(\mathfrak{a})$ and the biased square $\mathbf{bste}(\mathfrak{a})$, it also has the rlp with respect to their vertical composition $\mathbf{triv}(\mathfrak{a})$. □

Proposition 2.33. *Every trivial fibration is an intraction.*

Proof. Remark 2.4 implies that a trivial fibration f for an alliance of nodes of vertebrae $\mathfrak{a}$ has the rlp with respect to the proper square $\mathbf{stem}(\mathfrak{a})$. Proposition 2.8 then implies that f is simple with respect to the underlying oriented scale of $\mathfrak{a}$ defined in Remark 2.20. □

Proposition 2.34. *Let f and g be two morphisms such that $f \circ g$ exists. If $f \circ g$ is an intraction, then so is g.*

Proof. Straightforward (see definition of simplicity with respect to a scale). □

Proposition 2.35. *Every isomorphism in $\mathcal{C}$ is an intraction.*

Proof. Follows from Proposition 2.28 and Proposition 2.33. □

Proposition 2.36. *If two morphisms $f : Y \to Z$ and $g : X \to Y$ are trivial fibrations for an identity alliance id_ν, then so is the composite $f \circ g$.*

Proof. Follows from Remark 2.25 and Proposition 1.35. □

2.3.4.2. *Properties relative to actions and compositions.* Let $\mathcal{C}$ be a category. The notations for the alliances of nodes of vertebrae, extended nodes of vertebrae and communications that are given in this section follow the conventions of section 2.3.1.12. The other morphisms of $\mathcal{C}$ will be denoted by the letters f and g.

Proposition 2.37. *If $f : Y \to Z$ is a pseudofibration for $t : \gamma \rightsquigarrow \gamma_*$ and $g : X \to Y$ is a pseudofibration for $t_* : \gamma_* \rightsquigarrow \gamma_\flat$, then $f \circ g$ is a pseudofibration for the composite arrow $t_* \odot t : \gamma \rightsquigarrow \gamma_\flat$.*

Proof. Follows from Proposition 2.6 and formula (2.15). □

Proposition 2.38. *If $f : Y \to Z$ is a pseudofibration for $t : \gamma \rightsquigarrow \gamma_*$ and $g : X \to Y$ is a pseudofibration for $\mathfrak{n}_* : \gamma_* \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$, then $f \circ g$ is a pseudofibration for the composite $\mathfrak{n}_* \odot t : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$.*

Proof. By Proposition 2.27, the morphism $g : X \to Y$ is also a pseudofibration for $\mathbf{com}(\mathfrak{n}_*) : \gamma_* \rightsquigarrow \gamma_\flat$. It follows from Proposition 2.37 that $f \circ g$ is a pseudofibration for $\mathbf{com}(\mathfrak{n}_*) \odot t$. The right equation of (2.21) then shows that $f \circ g$ is a pseudofibration for $\mathbf{com}(\mathfrak{n}_* \odot t)$, which, by Proposition 2.27, implies the statement. □

Proposition 2.39. *If $f : Y \to Z$ is a pseudofibration for $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ and $g : X \to Y$ is a pseudofibration for $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, then $f \circ g$ is a pseudofibration for the composite $\mathfrak{a}_* \odot \mathfrak{a} : \nu \rightsquigarrow \nu_\flat$.*

Proof. By Proposition 2.26 and Proposition 2.27, the morphism f is a pseudofibration for the underlying extended vertebrae $\mathbf{ext}(\mathfrak{a})$ while the morphism g is a pseudofibration for the underlying communication $\mathbf{com}(\mathfrak{a})$. It follows from Proposition 2.38 that the composite $f \circ g$ is a pseudofibration for $\mathbf{ext}(\mathfrak{a}) \odot \mathbf{com}(\mathfrak{a}_\flat)$, which, by formula (2.22), implies that $f \circ g$ is a pseudofibration for $\mathbf{ext}(\mathfrak{a} \circledcirc \mathfrak{a}_\flat)$. The statement then follows from Proposition 2.26. □

Proposition 2.40. *If $f : Y \to Z$ is a fibration for $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ and $g : X \to Y$ is a fibration for $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, then $f \circ g$ is a fibration for the composite $\mathfrak{a}_* \circledcirc \mathfrak{a}$.*

Proof. Let $p \cdot \Omega$ and $p_* \cdot \Omega_*$ denote the nodes of vertebrae ν and ν_*, respectively. Consider a stem $\beta : \mathbb{S}' \to \mathbb{D}'$ in Ω. Since f is a fibration for $\mathfrak{a}$, it has the rlp with respect to $\mathbf{triv}(\mathfrak{a}_\beta) : \phi(\beta) \circ \delta_1^* \Rightarrow \beta \circ \delta_1$. By assumption, the stem $\phi(\beta)$ belongs to Ω_* and since g is a fibration for $\mathfrak{a}_* : p \cdot \Omega_* \rightsquigarrow \nu_\flat$, it has the rlp with respect to the commutative square

$$\mathbf{triv}(\mathfrak{a}^*_{\phi(\beta)}) : \phi_*(\phi(\beta)) \circ \delta_1^\flat \Rightarrow \phi(\beta) \circ \delta_1^*$$

where $\mathfrak{a}^*_{\phi(\beta)}$ denotes the component of $\mathfrak{a}_*$ at $\phi(\beta)$. By Proposition 2.6, it follows that $f \circ g$ has the rlp with respect to $\mathbf{triv}(\mathfrak{a}_\beta) \circ \mathbf{triv}(\mathfrak{a}^*_{\phi(\beta)})$. By formula (2.17), this means that $f \circ g$ has the rlp with respect to $\mathbf{triv}((\mathfrak{a}_* \circledcirc \mathfrak{a})_\beta)$ and is hence a fibration for $\mathfrak{a}_* \circledcirc \mathfrak{a} : \nu \rightsquigarrow \nu_\flat$. ☐

Proposition 2.41. *Let f and g be two morphisms such that $f \circ g$ exists. If f is an intraction for $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ and g is an intraction for $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, then $f \circ g$ is an intraction for the composite $\mathfrak{a}_* \circledcirc \mathfrak{a}$.*

Proof. Follows from Proposition 2.7. ☐

Proposition 2.42. *If a morphism is a fibration (resp. intraction) for an alliance $\mathfrak{a}$, then so is it for any composite of the form $\mathfrak{a}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{a}_\flat$.*

Proof. Suppose that a morphism $f : X \to Y$ is a fibration for $\mathfrak{a}$. Proposition 2.28 shows that the identities id_X and id_Y are also fibrations for $\mathfrak{a}_*$ and $\mathfrak{a}_\flat$, respectively. Applying Propoposition 2.40 twice shows that $f = \mathrm{id}_Y \circ f \circ \mathrm{id}_X$ is a fibration for $\mathfrak{a}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{a}_\flat$. A similar reasoning that uses Proposition 2.41 and Proposition 2.35 shows that the statement also holds for intractions. ☐

Proposition 2.43. *If a morphism is a pseudofibration for a communication t, then so is it for any composite of the form $\mathfrak{n}_* \circledcirc t \odot t_\flat$.*

Proof. Suppose that a morphism $f : X \to Y$ is a fibration for $\mathfrak{a}$. Proposition 2.28 shows that the identities id_X and id_Y are also pseudofibrations for $t_\flat$ and $\mathfrak{n}_*$, respectively. Applying Proposition 2.38 and Propoposition 2.37 then shows that $f = \mathrm{id}_Y \circ f \circ \mathrm{id}_X$ is a fibration for $\mathfrak{n}_* \circledcirc t \circledcirc t_\flat$. ☐

Proposition 2.44. *If a morphism is a pseudofibration for a extended $\mathfrak{n}$, then so is it for any composite of the form $\mathfrak{a}_* \odot \mathfrak{n} \odot t_\flat$.*

Proof. If a morphism f is a pseudofibration for $\mathfrak{n}$, then so is it for the underlying extended node of vertebrae $\mathbf{com}(\mathfrak{n})$ by Proposition 2.27. It follows from Proposition 2.43 that f is a pseudofibration for $\mathbf{ext}(\mathfrak{a}_*) \odot \mathbf{com}(\mathfrak{n}) \odot \mathbf{com}(t_\flat)$. By formula (2.18) and formula (2.22), this means that f is a pseudofibration for $\mathbf{ext}(\mathfrak{a}_* \circledcirc \mathfrak{n} \circledcirc t_\flat)$, which is equal to the extended node of vertebrae $\mathfrak{a}_* \circledcirc \mathfrak{n} \circledcirc t_\flat$ itself. ☐

Proposition 2.45. *If a morphism is a pseudofibration for an alliance $\mathfrak{a}$, then so is it for any composite of the form $\mathfrak{a}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{a}_\flat$.*

Proof. If a morphism f is a pseudofibration for $\mathfrak{a}$, then so is it for the underlying communication $\mathbf{ext}(\mathfrak{a})$ by Proposition 2.26. It follows from Proposition 2.43 that f is a pseudofibration for $\mathfrak{a}_* \odot \mathbf{ext}(\mathfrak{a}) \odot \mathbf{com}(\mathfrak{a}_\flat)$, which, by formula (2.22) and formula (2.19), implies that f is a pseudofibration for $\mathbf{ext}(\mathfrak{a}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{a}_\flat)$. The statement then follows from Proposition 2.26. ☐

Proposition 2.46. *If a morphism is a surtraction for an extended node of vertebrae $\mathfrak{n}$, then so is it for any composite of the form $\mathfrak{a}_* \circledcirc \mathfrak{n} \odot t_\flat$.*

Proof. Denote $t : \gamma_\flat \rightsquigarrow \gamma$, $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and $\mathfrak{a} : \nu_* \rightsquigarrow \nu_\dagger$ and suppose to be given a surtraction f for $\mathfrak{n}$. By definition, the morphism f is divisible by the besom $(\Omega_*, \mathbf{disk}(p_* \cdot \_), \mathbf{seed}(\mathfrak{n}))$ associated with $\mathfrak{n}$ while the statement requires to prove that f is divisible by the besom

$(\Omega_\dagger, \mathbf{disk}(p_\dagger \cdot \_), \mathbf{seed}(\mathfrak{a}_* \circledcirc \mathfrak{n} \circledcirc t_\flat))$. Since $\mathbf{seed}(\mathfrak{n}) : \gamma_* \Rightarrow \gamma$ and $\mathbf{seed}(t_\flat) : \gamma \Rightarrow \gamma_\flat$ are composable arrows in $\mathcal{C}^2$, it follows from Proposition 2.10 that the morphism f must be divisible by the following besom.

$$(\Omega_*, \mathbf{disk}(p_* \cdot \_), \mathbf{seed}(t_\flat) \circ \mathbf{seed}(\mathfrak{n}))$$

By the leftmost formula of (2.21), the previous besom is also the underlying besom of the extended node of vertebrae $\mathfrak{n}_* \odot t$. Now, as noticed in Remark 2.24, the fact that $\mathfrak{a}_*$ and $\mathfrak{n}_* \odot t$ are composable implies that their underlying besoms form a couple of besoms. Proposition 2.11 then implies that f must be divisible by the following besom.

$$(\Omega_\dagger, \mathbf{disk}(p_\dagger \cdot \_), \mathbf{seed}(\mathfrak{n} \odot t_\flat) \circ \mathbf{seed}(\mathfrak{a}_*))$$

It follows from formula (2.20) that this is the besom of the composite $\mathfrak{a}_* \circledcirc \mathfrak{n} \odot t_\flat$, which proves the statement. $\square$

Proposition 2.47. *If a morphism is a surtraction for an alliance $\mathfrak{a}$, then so is it for any composite of the form $\mathfrak{a}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{a}_\flat$.*

Proof. If a morphism f is a surtraction for $\mathfrak{a}$, then so is it for the underlying extended nodes of vertebrae $\mathbf{ext}(\mathfrak{a})$ by Proposition 2.26. It follows from Proposition 2.46 that f is a pseudofibration for $\mathfrak{a}_* \circledcirc \mathbf{ext}(\mathfrak{a}) \odot \mathbf{com}(\mathfrak{a}_\flat)$, which, by formula (2.22) and formula (2.19), means that f is a pseudofibration for $\mathbf{ext}(\mathfrak{a}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{a}_\flat)$. The statement then follows from Proposition 2.26. $\square$

Proposition 2.48. *If a morphism is a surtraction for $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and a fibration for $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, then it is a pseudofibration for the composite $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$.*

Proof. The notations of the nodes of vertebrae ν, ν_* and $\nu_\flat$ follow the usual conventions. The goal is to show that any morphism $f : X \to Y$ that is a surtraction for $\mathfrak{n}$ and a fibration for $\mathfrak{a}_*$ has the rlp with respect to $\mathbf{seed}(\mathfrak{a}_* \circledcirc \mathfrak{n}) : \gamma_\flat \Rightarrow \gamma$. In this respect, consider the leftmost commutative square, below. Because f is a surtraction for $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \Omega_*$, there exists a stem $\beta_* \in \Omega_*$ and two arrows $x' : \mathbb{D}_1^* \to X$ and $y' : \mathbb{D}'_* \to Y$ making the following rightmost diagram commute (the middle diagram is its translation in $\mathcal{C}^2$).

(2.23)
$$\begin{tikzcd} \mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\ \mathbb{D}_2 \arrow[r, "y"] & Y \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_* \cdot \beta_*)"'] & \gamma \arrow[d, Rightarrow, "x"] \\ \beta_* \circ \delta_1^* \arrow[r, Rightarrow, dashed, "x'"'] & f \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \mathbb{S}_* \arrow[r, "\gamma'_*"'] \arrow[rr, bend left, "x \circ \varkappa"] \arrow[d, "\gamma_*"'] & \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, "\beta_* \circ \delta_1^*"] & X \arrow[d, "f"] \\ \mathbb{D}_2^* \arrow[r, "\beta_* \circ \delta_2^*"] \arrow[rr, bend right, "y \circ \varrho"'] & \mathbb{D}'_* \arrow[r, "y'"] & Y \end{tikzcd}$$

The rightmost commutative diagram of (2.23) provides the following left commutative square. Now, since f is a fibration for $\mathfrak{a}_* : p_* \cdot \Omega_* \rightsquigarrow \nu_\flat$, there exists a lift $h : \mathbb{D}'_* \to X$ making the corresponding right diagram commute.

(2.24)
$$\begin{tikzcd} \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, "\beta_* \circ \delta_1^*"'] & X \arrow[d, "f"] \\ \mathbb{D}'_* \arrow[r, "y'"'] & Y \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \mathbb{D}_1^\flat \arrow[r, "x' \circ \varrho'_*"] \arrow[d, "\phi_*(\beta_*) \circ \delta_1^\flat"'] & X \arrow[d, "f"] \\ \mathbb{D}'_\flat \arrow[r, "y' \circ u^*_{\beta_*}"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

Because $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ are composable, Remark 2.24 shows that the underlying besoms of $\mathfrak{n}$ and $\mathfrak{a}_*$ form a couple. In particular, Remark 2.24 produces the following left commutative diagram. Pasting this diagram with the middle commutative diagram of (2.23) then provides the corresponding right commutative diagram.

$$\begin{tikzcd}
\gamma_\flat \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a}_*)"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\phi_*(\beta_*))"'] & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma & \Rightarrow & \gamma_\flat \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a}_*)"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\phi_*(\beta_*))"'] & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\phi_*(\beta_*)\circ\delta_1^* \arrow[r, Rightarrow, "\mathbf{triv}(\mathfrak{a}^*_{\beta_*})"'] & \beta_*\circ\delta_1 & & & \phi_*(\beta_*)\circ\delta_1^* \arrow[r, Rightarrow, "\mathbf{triv}(\mathfrak{a}^*_{\beta_*})"'] & \beta_*\circ\delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f
\end{tikzcd}$$

The outer commutative square of the latest diagram may be rewritten in the form of the leftmost diagram, below. Merging this diagram with the lifting obtained in the rightmost commutative diagram of (2.24) then provides the commutative diagram on the right, below.

$$\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"'] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}_1^\flat \arrow[r, "x'\circ\varrho'_*"] \arrow[d, "\phi_*(\beta_*)\circ\delta_1^\flat"] & X \arrow[d, "f"] & \Rightarrow & \mathbb{S}_\flat \arrow[r, "x\circ\varkappa\circ\varkappa_*"] \arrow[d, "\gamma_\flat"'] & X \arrow[d, "f"] \\
\mathbb{D}_2^\flat \arrow[r, "\phi_*(\beta_*)\circ\delta_2^\flat"] \arrow[rr, bend right, "y\circ\varrho\circ\varrho'_*"'] & \mathbb{D}'_\flat \arrow[r, "y'\circ u^*_{\beta_*}"] & Y & & \mathbb{D}_2^\flat \arrow[r, "y\circ\varrho\circ\varrho'_*"'] \arrow[ur, "h\circ\phi_*(\beta_*)\circ\delta_2^\flat"] & Y
\end{tikzcd}$$

This shows that the leftmost commutative diagram of (2.23) admits a lift along the commutative square $\mathbf{seed}(\mathfrak{a}_* \circledcirc \mathfrak{n}) : \gamma_\flat \Rightarrow \gamma$. $\square$

Proposition 2.49. *Let f and g be two morphisms such that $f \circ g$ exists. If $f \circ g$ is a surtraction for $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ and f is an intraction for $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, then g is a surtraction for the composite $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\flat$.*

Proof. We are going to prove that g is divisible by the underlying besom of $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\flat$. To do so, denote $g : X \to Y$, $f : Y \to Z$ and consider the leftmost commutative diagram, below. Post-composing this diagram with $f : Y \to Z$ leads to the middle commutative square. Because $f \circ g$ is divisible by the underlying besom of $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$, there exists $\beta_* \in \Omega_*$ and two morphisms $x' : \mathbb{D}_1^* \to X$ and $y' : \mathbb{D}'_* \to Y$ making the corresponding rightmost diagram commute.

(2.25)
$$\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "g"] & \Rightarrow & \mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f\circ g"] & \Rightarrow & \mathbb{S}_* \arrow[r, "\gamma'_*"'] \arrow[d, "\gamma_*"'] \arrow[rr, bend left, "x\circ\varkappa"] & \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, "\beta_*\circ\delta_1^*"] & X \arrow[d, "f\circ g"] \\
\mathbb{D}_2 \arrow[r, "y"'] & Y & & \mathbb{D}_2 \arrow[r, "f\circ y"'] & Z & & \mathbb{D}_2^* \arrow[r, "\beta_*\circ\delta_2^*"] \arrow[rr, bend right, "f\circ y\circ\varrho"'] & \mathbb{D}'_* \arrow[r, "y'"] & Z
\end{tikzcd}$$

For now, note that the preceding rightmost diagram gives a factorisation of the composite $x \circ \varkappa$ in terms of the morphisms x' with γ'_*. Pre-composing the leftmost diagram of (2.25) with the commutative square $\mathbf{seed}(\mathfrak{n}) : \gamma \Rightarrow \gamma_*$ and using the previous expression of $x \circ \varkappa$ provides the middle commutative diagram, below. Then, using the span made by γ_* and γ'_* and forming the pushout associated with the base $\|\gamma_*, \gamma'_*\|$ of ν_* provides a canonical morphisms $h : \mathbb{S}'_* \to Y$ making the rightmost diagram commute.

(2.26)
$$\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "g"] & \Rightarrow & \mathbb{S}^* \arrow[r, "\varkappa"] \arrow[d, "\gamma^*"'] \arrow[rr, bend left, dashed, "x'\circ\gamma'_*"] & \mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "g"] & \Rightarrow & \mathbb{S}_* \arrow[r, "\gamma'_*"] \arrow[d, "\gamma_*"'] \arrow[rr, bend left, "x\circ\varkappa"] & \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, "\delta_1^*"] & X \arrow[d, "g"] \\
\mathbb{D}_2 \arrow[r, "y"'] & Y & & \mathbb{D}_2^* \arrow[r, "\varrho"'] & \mathbb{D}_2 \arrow[r, "y"'] & Y & & \mathbb{D}_2^* \arrow[r, "\delta_2^*"] \arrow[rr, bend right, "y\circ\varrho"'] & \mathbb{S}'_* \arrow[r, dashed, "h"] & Y
\end{tikzcd}$$

Post-composing the earlier rightmost diagram with $f : Y \to Z$, we see that the composite $f \circ h : \mathbb{S}'_* \to Z$ defines a solution for the leftmost commutative diagram of (2.27). Similarly, the rightmost diagram of (2.25) exposes the composite $y' \circ \beta_* : \mathbb{S}'_* \to Z$ as a solution for the same problem. If follows from the universality of the pushout $\mathbb{S}'_*$ that the two arrows are equal, which provides the succeeding middle commutative diagram. Now, because f is an intraction for $\mathfrak{a}_* : p_* \cdot \Omega_* \rightsquigarrow p_\flat \cdot \Omega_\flat$, there exists a stem $\beta_\flat \in \Omega_\flat$ and a morphism $h' : \mathbb{D}'_\flat \to Y$ factorising the composite $h \circ \varkappa'_*$ as shown in the following rightmost diagram.

(2.27)
$$
\begin{tikzcd}
\mathbb{S}_* \arrow[r, "\gamma'_*"] \arrow[d, "\gamma_*"'] & \mathbb{D}_1^* \arrow[d, "\delta_1^*"] \arrow[dr, bend left, "f\circ g\circ x'"] & \\
\mathbb{D}_2^* \arrow[r, "\delta_2^*"] \arrow[rr, bend right, "f\circ y\circ \varrho"'] & \mathbb{S}'_* \arrow[r, dashed] & Y
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\mathbb{S}'_* \arrow[r, "h"] \arrow[d, "\beta_*"] & Y \arrow[d, "f"] \\
\mathbb{D}'_* \arrow[r, "y'"] & Z
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\mathbb{S}'_\flat \arrow[r, "\varkappa'_*"] \arrow[d, "\beta_\flat"] & \mathbb{S}'_* \arrow[r, "h"] & Y \\
\mathbb{D}'_\flat \arrow[urr, "h'"'] & &
\end{tikzcd}
$$

Expressing the rightmost diagram of (2.26) in $\mathcal{C}^2$ gives the following leftmost commutative diagram where the prevertebra p_* of ν_* is seen as an arrow $\gamma_* \Rightarrow \delta_1^*$ in $\mathcal{C}^2$. Pasting this diagram with the commutative cuboid encoding the underlying alliance of prevertebrae $p_* \rightsquigarrow p_\flat$ of $\mathfrak{a}_*$ gives the middle commutative diagram. The outer commutative square of this diagram in $\mathcal{C}$ is then given below on the right.

$$
\begin{tikzcd}
\gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "p_*"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\delta_1^* \arrow[r, Rightarrow, dashed, "h"'] & g
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\gamma_\flat \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a}_*)"] \arrow[d, Rightarrow, "p_\flat"'] & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "p_*"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\delta_1^\flat \arrow[r, Rightarrow, "\mathbf{asee}(\mathfrak{a}_*)"'] & \delta_1^* \arrow[r, Rightarrow, dashed, "h"'] & g
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}_1^\flat \arrow[r, "x'\circ\varrho'"] \arrow[d, "\delta_1^\flat"] & X \arrow[d, "g"] \\
\mathbb{D}_2^\flat \arrow[r, "\delta_2^\flat"] \arrow[rr, bend right, "y\circ\varrho\circ\varrho_*"'] & \mathbb{S}'_\flat \arrow[r, "h\circ\varkappa'_*"] & Y
\end{tikzcd}
$$

Finally, merging the latest rightmost commutative diagram with the rightmost commutative diagram of (2.27) provides the following left commutative diagram.

$$
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}_1^\flat \arrow[r, "x'\circ\varrho'"] \arrow[d, "\beta_\flat\circ\delta_1^\flat"] & X \arrow[d, "g"] \\
\mathbb{D}_2^\flat \arrow[r, "\beta_\flat\circ\delta_2^\flat"] \arrow[rr, bend right, "y\circ\varrho\circ\varrho_*"'] & \mathbb{D}'_\flat \arrow[r, "h'"] & Y
\end{tikzcd}
\Leftrightarrow
\begin{tikzcd}
\gamma^\flat \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a}_*\circledcirc\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\beta_\flat)"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\beta_\flat\circ\delta_2^\flat \arrow[r, Rightarrow, dashed, "x'\circ\varrho'"'] & g
\end{tikzcd}
$$

The preceding right diagram is a reformulation of the leftmost one in $\mathcal{C}^2$, thereby showing that g is divisible by the besom of $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$. $\square$

2.3.4.3. *Towards more properties.* Unfortunately, the type of statement provided by Proposition 2.28 does not hold for surtractions in general. In order to include isomorphisms in the class of surtractions, we need a bit more structure on the underlying nodes of vertebrae. This leads us to the next section, which introduces the notion of reflexive vertebra (see section 2.3.5). Similarly, the type of statement provided by Proposition 2.41 does not hold for surtractions in general. Such a property may only be achieved in the case of surtractions when restricting to extended nodes of vertebrae as the structure of alliance of nodes of vertebra is too rich for such a property. The idea will be to define a new type of composition on extended node of vertebrae, called a *framing of extended nodes of vertebrae* (see section 2.3.6). This composition will have some similarity with the usual composition of cospans (i.e. using a pushout ; see [**34**]).

2.3.5. Reflexive vertebrae.

2.3.5.1. *Reflexive prevertebrae.* Let $\mathcal{C}$ be a category. A prevertebra $\|\gamma,\gamma'\|$ in $\mathcal{C}$ will be said to be *reflexive* if it is endowed with a morphism $\lambda : \mathbb{D}_1 \to \mathbb{D}_2$ such that the following left diagram commutes.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[dl, "\lambda"] \\
\mathbb{D}_2 &
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \arrow[dr, bend left, "\lambda"] & \\
\mathbb{D}_2 \arrow[r, "\delta_2"] \arrow[rr, equal, bend right] & \mathbb{S}' \arrow[r, dashed] & \mathbb{D}_2
\end{tikzcd}
$$

The universal solution $\mathbb{S}' \to \mathbb{D}_2$ induced by this diagram over the pushout $\mathbb{S}'$ (see previous right diagram) will be called *boundary contraction* of $\|\gamma,\gamma'\|$. The morphism λ will be referred to as the *reflexive transition* of the prevertebra.

2.3.5.2. *Reflexive vertebrae.* Consider a reflexive prevertebra p as in section 2.3.5.1. A vertebra $p \cdot \beta$ will be said to be *reflexive above* p if it is equipped with a morphism $\alpha : \mathbb{D}' \to \mathbb{D}_2$ such that the boundary contraction of p is equal to the composite

$$\mathbb{S}' \xrightarrow{\ \beta\ } \mathbb{D}' \xrightarrow{\ \alpha\ } \mathbb{D}_2.$$

The morphism α will be called the *homotopy contraction* of $p \cdot \beta$.

2.3.5.3. *Reflexive nodes of vertebrae.* Let p be a reflexive prevertebra as defined in section 2.3.5.1. A node of vertebrae $p \cdot \Omega$ will be said to be *reflexive above* p if one of its vertebrae is reflexive above p.

2.3.5.4. *Reflexive extended nodes of vertebrae.* Let $\mathcal{C}$ be a category. An extended node of vertebrae will be said to be *reflexive* in $\mathcal{C}$ if its associated node of vertebrae is reflexive in $\mathcal{C}$. The next propositions consider a reflexive extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in the category $\mathcal{C}$. The notations for $\mathfrak{n}$ shall follow the conventions of section 2.3.1.12.

Proposition 2.50. *Every pseudofibration for* $\mathfrak{n}$ *is a surtraction for* $\mathfrak{n}$.

Proof. Let $f : X \to Y$ be a pseudofibration for $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$. We are going to show that f is divisible by the underlying besom of $\mathfrak{n}$. Start with the following leftmost commutative square. Because f is a pseudofibration, it has the rlp with respect to $\mathbf{seed}(\mathfrak{n}) : \overline{\gamma} \Rightarrow \gamma$, which means that there exists a lift $h : \mathbb{D}_2 \to X$ making the succeeding middle diagram commute. Using the reflexive structure associated with the base of the node of vertebrae $\overline{\nu}$ turns this diagram into the rightmost one.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
\mathbb{D}_2 \arrow[r, "y"'] & Y
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "x\circ\varkappa"] \arrow[d, "\overline{\gamma}"'] & X \arrow[d, "f"] \\
\overline{\mathbb{D}}_2 \arrow[r, "y\circ\varrho"'] \arrow[ur, "h"] & Y
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] \arrow[rrr, bend left, "x\circ\varkappa"] & \overline{\mathbb{D}}_1 \arrow[r, "\overline{\lambda}"] & \overline{\mathbb{D}}_2 \arrow[r, "h"] \arrow[dr, "y\circ\varrho"'] \arrow[dll, equal] & X \arrow[d, "f"] \\
\overline{\mathbb{D}}_2 \arrow[rrr, "y\circ\varrho"'] & & & Y
\end{tikzcd}
\tag{2.28}
$$

By assumption on $\mathfrak{n}$, there exists a reflexive vertebra $\overline{p} \cdot \overline{\beta}$ in the node of vertebrae $\overline{\nu}$. The reflexive structure associated with this vertebra allows us to factorise the rightmost commutative diagram (2.28) into the following left commutative diagram (by making the homotopy contraction appear in the left-top corner).

$$
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[rrr, bend left, "x\circ\varkappa"] \arrow[dd, "\overline{\gamma}"'] & \overline{\mathbb{D}}_1 \arrow[r, "\overline{\lambda}"] \arrow[d, "\overline{\delta}_1"] & \overline{\mathbb{D}}_2 \arrow[r, "h"] \arrow[d, equal] & X \arrow[dd, "f"] \\
 & \overline{\mathbb{S}}' \arrow[r, "\overline{\alpha}\circ\overline{\beta}"] & \overline{\mathbb{D}}_2 \arrow[dr, "y\circ\varrho"] & \\
\overline{\mathbb{D}}_2 \arrow[ur, "\overline{\delta}_2"] \arrow[urr, equal] \arrow[rrr, "y\circ\varrho"'] & & & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(\overline{p}\circ\overline{\beta})"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\overline{\beta}\circ\overline{\delta}_1 \arrow[r, Rightarrow, dashed, "x'"'] & f
\end{tikzcd}
$$

This diagram implies the preceding right commutative square in $\mathcal{C}^2$, where the arrow $x' : \overline{\beta}\circ\overline{\delta}_1 \Rightarrow f$ is encoded by the arrows h and $y\circ\varrho\circ\overline{\alpha}$. This last diagram shows that f is divisible by the underlying besom of $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$. □

We easily deduce that

Proposition 2.51. *Every trivial fibration for $\mathfrak{n}$ is a fibration and a weak equivalence for $\mathfrak{n}$.*

Proof. It follows from Proposition 2.32 and Proposition 2.33 that a trivial fibration for $\mathfrak{n}$ is a fibration and an intraction. It follows from Proposition 2.31 and Proposition 2.50 that a trivial fibration is a surtraction, which proves the statement. □

2.3.5.5. *Coreflexive alliances.* Let $\mathcal{C}$ be a category. An alliance of nodes of vertebrae will be said to be *coreflexive* in $\mathcal{C}$ if its domain is reflexive. The next few results involve a coreflexive alliance $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ in $\mathbf{Ally}(\mathcal{C})$, which implies that the domain ν contains at least a reflexive vertebra.

Lemma 2.52. *Let $i : X \to Y$ be a morphism in $\mathcal{C}$. If there exists an intraction $r : Y \to X$ for $\mathfrak{n}$ such that the equality $r\circ i = \mathrm{id}_X$ holds, then i is a weak equivalence for $\mathfrak{n}$.*

Proof. The fact that the morphism i is an intraction is a consequence of Proposition 2.34 and Proposition 2.35. Let us now prove that i is a surtraction for $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$. Start with a commutative diagram as given below on the left. Post-composing this commutative square with the arrow $r : Y \to X$ and using the relation $r\circ i = \mathrm{id}_X$ implies the equation $u = r\circ i\circ u = r\circ v\circ\gamma$. This equation leads to the commutativity of the following right diagram.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "u"] \arrow[d, "\gamma"'] & X \arrow[d, "i"] \\
\mathbb{D}_2 \arrow[r, "v"] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[rr, bend left, "u"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[r, "r\circ v\circ\lambda"] & X \arrow[d, "i"] \\
\mathbb{D}_2 \arrow[rr, "v"] & & Y
\end{tikzcd}
\tag{2.29}
$$

Denote by $h : \mathbb{S}' \to Y$ the universal solution induced after formation of the pushout $\mathbb{S}'$ in the right diagram of (2.29) over the span of the prevertebra $\|\gamma, \gamma'\|$ (see left diagram of (2.30)). Because the equation $r\circ i = \mathrm{id}_X$ holds, post-composing the right diagram of (2.29) with r implies that $r\circ h : \mathbb{S}' \to X$ defines a solution for the following right commutative diagram over the pushout $\mathbb{S}'$.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[rr, bend left, "u"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D}_1 \arrow[r, "r\circ v\circ\lambda"] \arrow[d, "\delta_1"] & X \arrow[d, "i"] \\
\mathbb{D}_2 \arrow[r, "\delta_2"] \arrow[rr, bend right, "v"'] & \mathbb{S}' \arrow[r, dashed, "h"] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D}_1 \arrow[d, "\delta_1"] \arrow[dr, bend left, "r\circ v\circ\lambda"] & \\
\mathbb{D}_2 \arrow[r, "\delta_2"] \arrow[rr, bend right, "r\circ v"'] & \mathbb{S}' \arrow[r, dashed] & X
\end{tikzcd}
\tag{2.30}
$$

If $p \cdot \beta$ denotes the reflexive vertebra of ν, the reflexive structure of $p \cdot \beta$ implies that the composite $r \circ v \circ \alpha \circ \beta : \mathbb{S}' \to X$ is also a solution of the right commutative diagram of (2.30). By the universal property of a pushout, this implies that the following leftmost diagram must commute. Because r is an intraction for $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$, it is simple for the underlying scale $(\overline{\Omega}, \varkappa, \Omega)$ of $\mathfrak{a}$. This means that there exist a stem $\overline{\beta} : \overline{\mathbb{S}}' \to \overline{\mathbb{D}}'$ in $\overline{\Omega}$ and a morphism $h' : \overline{\mathbb{D}}' \to Y$ such that the following right triangle commutes.

(2.31)
$$
\begin{tikzcd}
\mathbb{S}' \arrow[r, "h"] \arrow[d, "\beta"'] & Y \arrow[d, "r"] \\
\mathbb{D}' \arrow[r, "r\circ v\circ\alpha"'] & X
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overline{\mathbb{S}}' \arrow[r, "\varkappa'"] \arrow[d, "\overline{\beta}"'] & \mathbb{S}' \arrow[r, "h"] & Y \\
\overline{\mathbb{D}}' \arrow[urr, dashed, "h'"'] & &
\end{tikzcd}
$$

Now, expressing the left diagram of (2.30) in $\mathcal{C}^2$ gives the next left commutative diagram where the prevertebra p of ν is seen as an arrow $\gamma \Rightarrow \delta_1$ in $\mathcal{C}^2$. Pasting this diagram with the commutative cuboid encoding the underlying alliance of prevertebrae $p \rightsquigarrow \overline{p}$ of $\mathfrak{a}$ gives the middle commutative diagram. The outer commutative square of this diagram in $\mathcal{C}$ is then given below on the right, where $r' : \overline{\mathbb{D}}_1 \to X$ stands for the composite $r \circ v \circ \lambda \circ \varrho'$.

$$
\begin{tikzcd}
\gamma \arrow[r, Rightarrow, "\mathrm{id}_\gamma"] \arrow[d, Rightarrow, "p"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\delta_1 \arrow[r, Rightarrow, dashed, "r\circ v\circ\lambda"'] & i
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a})"] \arrow[d, Rightarrow, "\overline{p}"'] & \gamma \arrow[r, equal, "\mathrm{id}_\gamma"] \arrow[d, Rightarrow, "p"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\overline{\delta}_1 \arrow[r, Rightarrow, "\mathbf{asee}(\mathfrak{a})"'] & \delta_1 \arrow[r, Rightarrow, dashed, "r\circ v\circ\lambda"'] & i
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "x\circ\varkappa"] & \overline{\mathbb{D}}_1 \arrow[r, "r'"] \arrow[d, "\overline{\delta}_1"] & X \arrow[d, "i"] \\
\overline{\mathbb{D}}_2 \arrow[r, "\overline{\delta}_2"] \arrow[rr, bend right, "y\circ\varrho"'] & \overline{\mathbb{S}}' \arrow[r, "h\circ\varkappa'"] & Y
\end{tikzcd}
$$

Finally, merging the latest right commutative diagram with the right commutative diagram of (2.31) provides the following left commutative diagram.

$$
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "x\circ\varkappa"] & \overline{\mathbb{D}}_1 \arrow[r, "r'"] \arrow[d, "\overline{\beta}\circ\overline{\delta}_1"] & X \arrow[d, "g"] \\
\overline{\mathbb{D}}_2 \arrow[r, "\overline{\beta}\circ\overline{\delta}_2"] \arrow[rr, bend right, "y\circ\varrho"'] & \overline{\mathbb{D}}' \arrow[r, "h'"] & Y
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\overline{\gamma} \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a})"] \arrow[d, Rightarrow, "\mathbf{disk}(\overline{p}\cdot\overline{\beta})"'] & \gamma \arrow[d, Rightarrow, "x"] \\
\overline{\beta}\circ\overline{\delta}_1 \arrow[r, Rightarrow, dashed, "r'"'] & g
\end{tikzcd}
$$

The corresponding right diagram is a reformulation of the left one in $\mathcal{C}^2$, thereby showing that i is divisible by the besom of $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$. $\square$

The previous lemma implies – and is even equivalent to – the next proposition.

Proposition 2.53. *Every isomorphism in $\mathcal{C}$ is a surtraction for $\mathfrak{n}$.*

Proof. Let $f : X \to Y$ be an isomorphism in $\mathcal{C}$. By definition, its inverse $f' : Y \to X$ implies the relation $f' \circ f = \mathrm{id}_X$ in $\mathcal{C}$. By Proposition 2.52, the statement is proven if the isomorphism f' is an intraction. But this is precisely the statement of Proposition 2.35. $\square$

2.3.6. Communications and framings.

2.3.6.1. *Communication of extended prevertebrae.* The notion of communication of extended prevertebrae will mostly be used together with the notions of pseudofibration and surtraction. Let $\mathcal{C}$ be a category. Two extended prevertebrae $\mathfrak{p} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_*$ and $\mathfrak{p}_\flat : \gamma_\flat \overset{\mathrm{ex}}{\rightsquigarrow} p_\dagger$ will be said to *communicate* if the coseed of $\mathfrak{p}$ (given by the coseed of p_*) is equal to the preseed of $\mathfrak{p}_\flat$ (see section 2.3.1.4). In other words, the prevertebra p_* is of the form $\|\gamma_*, \gamma_\flat\|$. More generally, two extended prevertebrae $\mathfrak{p} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_*$ and $\mathfrak{p}_\flat : \gamma_\flat \overset{\mathrm{ex}}{\rightsquigarrow} p_\dagger$ will be said to *communicate via a* *communication* $t : \gamma'_* \rightsquigarrow \gamma_\flat$ in $\mathbf{Com}(\mathcal{C})$ if the arrow γ'_* denotes the coseed of p_*. In this case, the two extended prevertebrae $\mathfrak{p}$ and $\mathfrak{p}_\flat \odot t : \gamma'_* \overset{\text{ex}}{\rightsquigarrow} \overline{p}_\flat$ communicate.

2.3.6.2. *Framing of extended prevertebrae.* First, the reader might want to go back to section 2.1 to remind themselves about the intuition behind the concept of framing (of vertebrae). Let now $\mathcal{C}$ be a category and consider two extended prevertebrae $(\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} \|\gamma_*, \gamma'_*\|$ and $(\varkappa_*, \varrho_*) : \gamma'_* \overset{\text{ex}}{\rightsquigarrow} \|\gamma_\flat, \gamma'_\flat\|$ that communicate in $\mathcal{C}$. This type of data implies that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\varkappa_*"] \arrow[d, "\gamma_* \circ \varkappa_*"'] & \mathbb{S}_* \arrow[r, "\varkappa"] \arrow[d, "\gamma_*"] & \mathbb{S} \arrow[d, "\gamma"] \\
\mathbb{D}_2^* \arrow[r, equal] & \mathbb{D}_2^* \arrow[r, "\varrho"'] & \mathbb{D}_2
\end{tikzcd}$$

An extended prevertebra $\mathfrak{p}_\bullet$ will be said to *frame* the previous pair of extended prevertebrae if it is of the form $(\varkappa \circ \varkappa_*, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} \|\gamma_* \circ \varkappa_*, \gamma'_\flat\|$ up to a choice of pushout object in $\mathcal{C}$. In this case, the arrow $\mathbf{seed}(\mathfrak{p}_\bullet)$ is given by the outer commutative square of the foregoing diagram.

Remark 2.54. The framing of two extended prevertebrae that both lift to alliances of prevertebrae (see the dotted arrows of the lefmost diagram, below) do not necessarily provides an extended prevertebra that lifts to an alliance of prevertebrae in $\mathcal{C}$ (see right diagram, below). This fact mainly explains our interest in studying extended vertebrae along with alliances of vertebrae.

$$\begin{tikzcd}[column sep=small]
 & \mathbb{S}_\flat \arrow[rr, dotted] & & \cdot & \mathbb{S}_* \arrow[rr, dotted] & & \cdot \\
\mathbb{D}_1^\flat \arrow[ur] & & \mathbb{D}_1^* \arrow[ul] \arrow[r, "\varrho_*"] & \mathbb{D}_2^* \arrow[u, dotted] \arrow[ur] & & \mathbb{D}_2^* \arrow[ul] \arrow[r, "\varrho"] & \mathbb{D}_2 \arrow[u, dotted] \\
 & \mathbb{S}_\flat \arrow[ul, "\gamma'_\flat"] \arrow[ur, "\gamma_\flat"'] \arrow[rrr, "\varkappa_*"'] & & & \mathbb{S}_* \arrow[ul, "\gamma'_*"] \arrow[ur, "\gamma_*"'] \arrow[rr, "\varkappa"'] & & \mathbb{S} \arrow[u, "\gamma"']
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}[column sep=small]
 & \mathbb{S}_\bullet \arrow[rr, dotted, "?"] & & \cdot \\
\mathbb{D}_1^\flat \arrow[ur] & & \mathbb{D}_2^* \arrow[ul] \arrow[r, "\varrho"] & \mathbb{D}_2 \arrow[u, dotted, "?"'] \\
 & \mathbb{S}_\flat \arrow[ul, "\gamma'_\flat"] \arrow[ur, "\gamma_* \circ \varkappa_*"'] \arrow[rr, "\varkappa \circ \varkappa_*"'] & & \mathbb{S} \arrow[u, "\gamma"']
\end{tikzcd}$$

Remark 2.55. If the two extended prevertebrae $\gamma \overset{\text{ex}}{\rightsquigarrow} \|\gamma_*, \gamma'_*\|$ and $\gamma'_* \overset{\text{ex}}{\rightsquigarrow} \|\gamma_\flat, \gamma'_\flat\|$ are reflexive, then so is $\mathfrak{p}_\bullet$. The reflexive transition associated with the prevertebra $\|\gamma_* \circ \varkappa_*, \gamma'_\flat\|$ is given by the following composite arrow, where λ_* and $\lambda_\flat$ stand for the respective reflexive transitions of the prevertebrae $\|\gamma_*, \gamma'_*\|$ and $\|\gamma_\flat, \gamma'_\flat\|$.

$$\mathbb{D}_1^\flat \xrightarrow{\lambda_\flat} \mathbb{D}_2^\flat \xrightarrow{\varrho_*} \mathbb{D}_1^* \xrightarrow{\lambda_*} \mathbb{D}_2^*$$

2.3.6.3. *Framing of extended vertebrae.* Let $\mathcal{C}$ be a category and $(\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} p_*$ and $(\varkappa_*, \varrho_*) : \gamma'_* \overset{\text{ex}}{\rightsquigarrow} p_\flat$ be two communicating extended prevertebrae admitting a framing extended prevertebra $\mathfrak{p}_\bullet : \gamma \overset{\text{ex}}{\rightsquigarrow} p_\bullet$ in $\mathcal{C}$. An extended vertebra $\mathfrak{n}_\bullet$ of the form $\mathfrak{p}_\bullet \cdot \beta_\bullet$ will be said to *frame* two extended vertebrae of the form $(\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} p_* \cdot \beta_*$ and $(\varkappa_*, \varrho_*) : \gamma'_* \overset{\text{ex}}{\rightsquigarrow} p_\flat \cdot \beta_\flat$ if it is equipped with a pushout of the form given below on the left as well as a morphism $\eta : \mathbb{D}'_\bullet \to \mathbb{E}$, called *cooperadic transition*, making the following right-hand diagram commute.

$$\begin{tikzcd}
\mathbb{D}_2^\flat \arrow[r, "\beta_* \circ \delta_1^* \circ \varrho_*"] \arrow[d, "\beta_\flat \circ \delta_2^\flat"'] & \mathbb{D}'_* \arrow[d, "\varepsilon_1"] \\
\mathbb{D}'_\flat \arrow[r, "\varepsilon_2"'] & \mathbb{E} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}'_\bullet \arrow[dr, "\eta \circ \beta_\bullet"] & \mathbb{D}_1^\flat \arrow[l, "\delta_1^\bullet"'] \arrow[d, "\varepsilon_2 \circ \beta_\flat \circ \delta_1^\flat"] \\
\mathbb{D}_2^* \arrow[u, "\delta_2^\bullet"] \arrow[r, "\varepsilon_1 \circ \beta_* \circ \delta_2^*"'] & \mathbb{E}
\end{tikzcd}
\tag{2.32}$$

The next proposition extends Remark 2.55 to the notion of extended vertebrae.

Proposition 2.56. *If both extended vertebrae $(\varkappa, \varrho) \cdot \beta$ and $(\varkappa_*, \varrho_*) \cdot \beta_*$ are reflexive, then so is the framing $\mathfrak{p}_\bullet \cdot \beta_\bullet$.*

Proof. Let λ_* and $\lambda_\flat$ denote the respective reflexive transitions of p_* and $p_\flat$. We are going to show that the boundary contraction induced by the reflexive transition $\lambda_* \circ \varrho_* \circ \lambda_\flat : \mathbb{D}_1^\flat \to \mathbb{D}_2^*$ defined in Remark 2.55 factorises through $\beta_\bullet : \mathbb{S}'_\bullet \to \mathbb{D}'_\bullet$. It follows from the equations $\lambda_* = \alpha_* \circ \beta_* \circ \delta_1^*$ and $\mathrm{id}_{\mathbb{D}_2^\flat} = \alpha_\flat \circ \beta_\flat \circ \delta_2^\flat$ (coming from the reflexivity of p_* and $p_\flat$) that the following left diagram commutes. The outer commutative square of this diagram then implies the existence of a canonical arrow over the pushout square of (2.32) as shown below on the right.

$$
\begin{tikzcd}
\mathbb{D}_2^\flat \arrow[r, "\beta_*\circ\delta_1^*\circ\varrho_*"] \arrow[d, "\beta_\flat\circ\delta_2^\flat"'] \arrow[dr, "\lambda_*\circ\varrho_*"] & \mathbb{D}' \arrow[d, "\alpha_*"] \\
\mathbb{D}'_\flat \arrow[r, "\lambda_*\circ\varrho_*\circ\alpha_\flat"'] & \mathbb{D}_2^*
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{D}_2^\flat \arrow[r, "\beta_*\circ\delta_1^*\circ\varrho_*"] \arrow[d, "\beta_*\circ\delta_2^*"'] & \mathbb{D}'_* \arrow[d, "\varepsilon_1"] \arrow[dr, bend left, "\alpha_*"] & \\
\mathbb{D}'_\flat \arrow[r, "\varepsilon_2"] \arrow[rr, bend right, "\lambda_*\circ\varrho_*\circ\alpha_\flat"'] \arrow[ur, phantom, "\ulcorner", near end] & \mathbb{E} \arrow[r, dashed, "\alpha'"] & \mathbb{D}_2^*
\end{tikzcd}
$$

Now, composing the right-hand diagram of (2.32) with the arrow $\alpha' : \mathbb{E} \to \mathbb{D}_2^*$ and using the relations provided by the preceding right commutative diagram together with the equations $\lambda_\flat = \alpha_\flat \circ \beta_\flat \circ \delta_1^\flat$ and $\mathrm{id}_{\mathbb{D}_2^*} = \alpha_* \circ \beta_* \circ \delta_2^*$ leads to the following left commutative diagram.

$$
\begin{tikzcd}
\mathbb{S}'_\bullet \arrow[dr, "\eta\circ\beta_\bullet"] & \mathbb{D}_1^\flat \arrow[l, "\delta_1^\bullet"'] \arrow[d, "\varepsilon_2\circ\beta_\flat\circ\delta_1^\flat"] \arrow[dr, bend left, "\lambda_*\circ\varrho_*\circ\lambda_\flat"] & \\
\mathbb{D}_2^* \arrow[u, "\delta_2^\bullet"] \arrow[r, "\varepsilon_1\circ\beta_*\circ\delta_2^*"] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{D}_2^*}"'] & \mathbb{E} \arrow[r, "\alpha'"] & \mathbb{D}_2^*
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_\bullet \arrow[r, "\gamma'_\bullet"] \arrow[d, "\gamma_\bullet"'] & \mathbb{D}_1^\flat \arrow[d, "\delta_1^\bullet"] \arrow[dr, bend left, "\lambda_*\circ\varrho_*\circ\lambda_\flat"] & \\
\mathbb{D}_2^* \arrow[r, "\delta_1^\bullet"] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{D}_2^*}"'] \arrow[ur, phantom, "\ulcorner", near end] & \mathbb{S}'_\bullet \arrow[r, dashed] & \mathbb{D}_2^*
\end{tikzcd}
$$

By universality, the composite arrow $\alpha' \circ \eta \circ \beta_\bullet : \mathbb{S}'_\bullet \to \mathbb{D}_2^*$ must be a solution of the problem given on the above right and hence equal to the boundary contraction of the prevertebra $\mathfrak{p}_\bullet := \|\gamma_\bullet, \gamma'_\bullet : \mathbb{S}'_\bullet\|$. This exposes a factorisation of the boundary contraction by the stem $\beta_\bullet$ and thus provides $\mathfrak{p}_\bullet \cdot \beta_\bullet$ with a reflexive structure where the homotopy contraction is given by the composite arrow $\alpha' \circ \eta : \mathbb{D}'_\bullet \to \mathbb{D}_2^*$. $\square$

Remark 2.57. Other notions of framing will arise in Chapter 3. These will be defined with respect to extended vertebrae whose spherical transitions are identities. In Chapter 4 will be explained how to transform any extended vertebra into these.

2.3.6.4. *Framing for extended nodes of vertebrae.* Let $\mathcal{C}$ be a category and $\mathfrak{p} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_*$ and $\mathfrak{p}_* : \gamma'_* \overset{\mathrm{ex}}{\rightsquigarrow} p_\flat$ be two extended prevertebrae framed by a third one $\mathfrak{p}_\bullet : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_\bullet$ in $\mathcal{C}$. An extended node of vertebrae $\mathfrak{n}_\bullet$ of the form $\mathfrak{p}_\bullet \cdot \Omega_\bullet$ will be said to *frame* two extended nodes of vertebrae of the form $\mathfrak{n} := \mathfrak{p} \cdot \Omega_*$ and $\mathfrak{n}_* := \mathfrak{p}_* \cdot \Omega_\flat$ if every pair of extended vertebrae $\mathfrak{v} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \beta_*$ and $\mathfrak{v}_* : \gamma'_* \overset{\mathrm{ex}}{\rightsquigarrow} p_\flat \cdot \beta_\flat$, taken in $\mathfrak{n}$ and $\mathfrak{n}_*$, is framed by one in $\mathfrak{n}_\bullet$. Endowed with such a structure, the next proposition follows.

Proposition 2.58. *Let f and g be two morphisms such that $f \circ g$ exists. If f is a surtraction for $\mathfrak{n}$ and g is a surtraction for $\mathfrak{n}_*$, then $f \circ g$ is a surtraction for $n_\bullet$.*

Proof. Our goal is to prove that the composite $f \circ g$ is divisible by the underlying besom of $\mathfrak{n}_\bullet$. By definition, the base $\mathfrak{p}_\bullet$ must be of the form $\gamma \overset{\mathrm{ex}}{\rightsquigarrow} \|\gamma_* \circ \varkappa_*, \gamma'_\flat\|$. Denote $g : X \to Y$ and $f : Y \to Z$ and consider a commutative square as given below on the left. Rearranging this diagram a little gives the one in the middle. Now, because f is a surtraction for the extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \Omega_*$, there exists $\beta_* \in \Omega_*$ and two morphisms $x' : \mathbb{D}_1^* \to X$ and

$z' : \mathbb{D}'_* \to Z$ making the following rightmost diagram commute.

(2.33)
$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f\circ g"] \\
\mathbb{D}_2 \arrow[r, "z"'] & Z
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "g\circ x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
\mathbb{D}_2 \arrow[r, "z"'] & Z
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_* \arrow[r, "\gamma'_*"] \arrow[d, "\gamma_*"'] \arrow[rr, bend left, "g\circ x\circ\varkappa"] & \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, dashed, "\beta_*\circ\delta_1^*"] & Y \arrow[d, "f"] \\
\mathbb{D}_2^* \arrow[r, "\beta_*\circ\delta_2^*"] \arrow[rr, bend right, "z\circ\varrho"'] & \mathbb{D}'_* \arrow[r, "z'"] & Z
\end{tikzcd}
$$

Note that the top part of the last commutative diagram provides the next left commutative square. Because g is a surtraction for the extended node of vertebrae $\mathfrak{n} : \gamma'_* \overset{\text{ex}}{\rightsquigarrow} p_\flat \cdot \Omega_\flat$, there exists $\beta_\flat \in \Omega_\flat$ and two morphisms $x'' : \mathbb{D}_1^\flat \to X$ and $y : \mathbb{D}'_\flat \to Y$ making the right-hand diagram in (2.34) commute.

(2.34)
$$
\begin{tikzcd}
\mathbb{S}_* \arrow[r, "x\circ\varkappa"] \arrow[d, "\gamma'_*"'] & X \arrow[d, "g"] \\
\mathbb{D}_2^* \arrow[r, "x'"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}_1^\flat \arrow[r, "x''"] \arrow[d, dashed, "\beta_\flat\circ\delta_1^\flat"] & Y \arrow[d, "g"] \\
\mathbb{D}_2^\flat \arrow[r, "\beta_\flat\circ\delta_2^\flat"] \arrow[rr, bend right, "x'\circ\varrho_*"'] & \mathbb{D}'_\flat \arrow[r, "y"] & Y
\end{tikzcd}
$$

As illustrated below, pre-composing the rightmost diagram of (2.33) with $\varkappa_*$ and using the diagrammatic relation involved in the arrow $\mathbf{seed}(\mathfrak{n}_*) : \gamma_\flat \Rightarrow \gamma_*$ (precisely, in the top part of the resulting diagram) gives rise to the right-hand commutative diagram, below.

$$
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\varkappa^*"] & \mathbb{S}_* \arrow[r, "\gamma'_*"] \arrow[d, "\gamma_*"'] \arrow[rr, bend left, "g\circ x\circ\varkappa"] & \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, dashed, "\beta_*\circ\delta_1^*"] & Y \arrow[d, "f"] \\
& \mathbb{D}_2^* \arrow[r, "\beta_*\circ\delta_2^*"] \arrow[rr, bend right, "z\circ\varrho"'] & \mathbb{D}'_* \arrow[r, "z'"] & Z
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\varkappa_*"] \arrow[d, "\gamma_\flat"'] & \mathbb{S}_* \arrow[d, "\gamma'_*"'] \\
\mathbb{D}_2^\flat \arrow[r, "\varrho_*"'] & \mathbb{D}_2^*
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"] \arrow[d, dashed, "\gamma_*\circ\varkappa_*"'] \arrow[rr, bend left, "g\circ x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}_1^* \arrow[r, "x'\circ\varrho_*"] \arrow[d, dashed, "\beta_*\circ\delta_1^*\circ\varrho_*"] & X \arrow[d, "f"] \\
\mathbb{D}_2^* \arrow[r, "\beta_*\circ\delta_2^*"] \arrow[rr, bend right, "z\circ\varrho"'] & \mathbb{D}'_* \arrow[r, "z'"] & Z
\end{tikzcd}
$$

Now, inserting the diagramatic relation of the right commutative diagram of (2.34) into the top part of the previous rightmost commutative diagram provides the left commutative diagram of (2.35). By assumption, there exists an extended vertebra $\mathfrak{p}_\bullet \cdot \beta_\bullet$ in $\mathfrak{p}_\bullet \cdot \Omega_\bullet$ that frames the pair of extended vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} p_* \cdot \beta_*$ and $\mathfrak{n}_* : \gamma'_* \overset{\text{ex}}{\rightsquigarrow} p_\flat \cdot \beta_\flat$, say via a cooperadic transition $\eta : \mathbb{D}' \to \mathbb{E}$. In particular, one may form the associated pushout $\mathbb{E}$ in the bottom right rectangle of the following left diagram (see pushout square of (2.32)). The universal property of $\mathbb{E}$, provides a canonical morphism $h : \mathbb{E} \to Z$ making the following right diagram commute.

(2.35)
$$
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"] \arrow[rrr, bend left, "x\circ\varkappa\circ\varkappa_*"] \arrow[dr, "\gamma_\flat"'] \arrow[dd, "\gamma_*\circ\varkappa_*"'] & \mathbb{D}_1^\flat \arrow[rr, "x''"] \arrow[dr, dashed, "\beta_\flat\circ\delta_1^\flat"] & & X \arrow[d, "g"] \\
& \mathbb{D}_2^\flat \arrow[r, "\beta_\flat\circ\delta_2^\flat"] \arrow[rr, bend right, "x'\circ\varrho_*"'] \arrow[d, dashed, "\beta^*\circ\delta_1^*\circ\varrho_*"'] & \mathbb{D}'_\flat \arrow[r, "y"] & Y \arrow[d, "f"] \\
\mathbb{D}_2^* \arrow[r, "\beta_*\circ\delta_2^*"] \arrow[rrr, bend right, "z\circ\varrho"'] & \mathbb{D}'_* \arrow[rr, "z'"] & & Z
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, dotted] \arrow[dr, dotted] \arrow[dd, dotted] & \mathbb{D}_1^\flat \arrow[rr, dotted] \arrow[dr, dotted] & & X \arrow[d, dotted] \\
& \mathbb{D}_2^\flat \arrow[r, "\beta_\flat\circ\delta_2^\flat"] \arrow[d, dashed, "\beta^*\circ\delta_1^*\circ\varrho_*"'] & \mathbb{D}'_\flat \arrow[r, "y"] \arrow[d, dashed, "\varepsilon_1"] & Y \arrow[d, "f"] \\
\mathbb{D}_2^* \arrow[r, dotted] & \mathbb{D}'_* \arrow[r, dashed, "\varepsilon_1"] \arrow[rr, bend right, "y'"'] & \mathbb{E} \arrow[r, dashed, "h"] \arrow[ul, phantom, "\ulcorner", very near start] & Z
\end{tikzcd}
$$

After inserting the previous right diagram into the left one and using the relations involved in the right diagram of (2.32), one deduces that the next diagram commutes. The left trapezoid of the next diagram is obtained from the prevertebra associated with the extended vertebra $\mathfrak{n}_\bullet$, which must be of the form $\|\gamma_* \circ \varkappa_*, \gamma'_\flat\|$.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'_\flat"] \arrow[rrr, bend left, "x \circ \varkappa \circ \varkappa_*"] \arrow[dd, "\gamma_* \circ \varkappa_*"'] & \mathbb{D}_1^\flat \arrow[rr, "x''"] \arrow[d, "\delta_1^\bullet"] & & X \arrow[d, "g"] \\
 & \mathbb{S}'_\bullet \arrow[r, "\eta \circ \beta_\bullet"] & \mathbb{D}'_\bullet \arrow[dr, "h"] & Y \arrow[d, "f"] \\
\mathbb{D}_2^* \arrow[ur, "\delta_2^\bullet"] \arrow[r, "\beta \circ \delta_2"] \arrow[rrr, bend right, "z \circ \varrho"'] & \mathbb{D}' \arrow[rr, "z'"] & & Z
\end{tikzcd}$$

This last diagram exactly states that the composite $f \circ g$ is divisible by the underlying besom of the extended node of vertebrae $n_\bullet : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \|\gamma_* \circ \varkappa_*, \gamma'_\flat\| \cdot \Omega_\bullet$. □

We also deduce from the previous section the following proposition.

Proposition 2.59. *If both extended nodes of vertebrae $\mathfrak{n}$ and $\mathfrak{n}_*$ are reflexive, then so is the framing $\mathfrak{n}_\bullet$.*

Proof. Follows from Proposition 2.56. □

2.3.7. Reflections and reversions.

2.3.7.1. *Reflection of nodes of vertebrae.* Let $\mathcal{C}$ be a category and ν and ν_* be two nodes of vertebrae in $\mathcal{C}$. The node of vertebrae ν will be said to be *reflected* by the node of vertebrae ν_* if it is equipped with an alliance of nodes of vertebrae $\nu \rightsquigarrow \nu_*^{\mathrm{rv}}$.

2.3.7.2. *Reversible alliances of nodes of vertebrae.* Let $\mathcal{C}$ be a category and $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ be an alliance of nodes of vertebrae in $\mathcal{C}$. The alliance $\mathfrak{a}$ will be said to be *reversible* if it is equipped with two reflections $\mathfrak{r} : \nu \rightsquigarrow \nu^{\mathrm{rv}}$ and $\mathfrak{r}_* : \nu_* \rightsquigarrow \nu_*^{\mathrm{rv}}$ such that $\mathfrak{a}^{\mathrm{rv}} = \mathfrak{r}_* \circledcirc \mathfrak{a} \circledcirc \mathfrak{r}^{\mathrm{rv}}$. In this case, the dual $\mathfrak{a}^{\mathrm{rv}}$ is also reversible as the previous equation implies the relation $\mathfrak{a} = \mathfrak{r}_*^{\mathrm{rv}} \circledcirc \mathfrak{a}^{\mathrm{rv}} \circledcirc \mathfrak{r}$.

Proposition 2.60. *Suppose that $\mathfrak{a}$ is a reversible alliance of nodes of vertebrae. If a morphism is a fibration (resp. trivial fibration; pseudofibration; intraction; surtraction) for $\mathfrak{a}$, then so is it for the dual $\mathfrak{a}^{\mathrm{rv}}$.*

Proof. Follows from Proposition 2.42, Proposition 2.47 and Proposition 2.45. □

The next property shows that *right properness* – i.e. the property that the pullback of a weak equivalence along a fibration is still a weak equivalence – naturally lies in a dual context.

Proposition 2.61. *If $f : X \to Y$ is a surtraction for $\mathfrak{n} : \gamma \rightsquigarrow \nu_*$ and $g : Z \to Y$ is a fibration for a dual alliance $\mathfrak{a}_*^{\mathrm{rv}} : \nu_*^{\mathrm{rv}} \rightsquigarrow \nu_\flat^{\mathrm{rv}}$, then every pullback of f along g is a surtraction for $\mathfrak{a}_* \odot \mathfrak{n} : \gamma \rightsquigarrow \nu_\flat$.*

Proof. To show the statement, start with a pullback of $f : X \to Y$ along $g : Z \to Y$ as the one given below on the left. We are going to show that $f^* : P \to Z$ is divisible by the besom of $\mathfrak{a}_* \odot \mathfrak{n} : \gamma \rightsquigarrow \nu_\flat$. To do so, start with the middle commutative square, below. Pasting the leftmost commutative diagram with the middle one along f^* and using the fact that f is divisible by the besom of $\mathfrak{n} : \gamma \rightsquigarrow \nu_*$ allows us to show that there exists a vertebra $\|\gamma_*, \gamma'_*\| \cdot \beta_*$ in ν_* and two arrows $x' : \mathbb{D}_1^* \to X$ and $y' : \mathbb{D}'_* \to Y$ making the following rightmost diagram commute.

(2.36)
$$\begin{tikzcd} P \arrow[r, "g^*"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & P \arrow[d, "f^*"] \\ \mathbb{D}_2 \arrow[r, "y"'] & Z \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S}_* \arrow[r, "\gamma'_*"'] \arrow[d, "\gamma_*"'] \arrow[rr, bend left, "g^*\circ x\circ\varkappa"] & \mathbb{D}_1^* \arrow[r, "x'"] \arrow[d, "\beta_*\circ\delta_1^*"] & X \arrow[d, "f"] \\ \mathbb{D}_2^* \arrow[r, "\beta_*\circ\delta_2^*"] \arrow[rr, bend right, "g\circ y\circ\varrho"'] & \mathbb{D}'_* \arrow[r, "y'"] & Y \end{tikzcd}$$

In particular, the lower part of the earlier diagram provides the following left commutative diagram. Now, because $g : Z \to Y$ is a fibration for $\mathfrak{a}_*^{\mathrm{rv}} : \nu_*^{\mathrm{rv}} \rightsquigarrow \nu_\flat^{\mathrm{rv}}$, it has the rlp with respect to the commutative square $\mathbf{triv}(\mathfrak{a}_*^{\mathrm{rv}}) : \phi_*(\beta) \circ \delta_2^\flat \Rrightarrow \beta_* \circ \delta_2^*$. In other words, there exists a lift $h : \mathbb{D}'_\flat \to Z$ making the following right diagram commute.

(2.37)
$$\begin{tikzcd} \mathbb{D}_2^* \arrow[r, "v\circ\varrho"] \arrow[d, "\beta_*\circ\delta_2^*"'] & Z \arrow[d, "g"] \\ \mathbb{D}'_* \arrow[r, "y'"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{D}_2^\flat \arrow[r, "\varrho_*"] \arrow[d, "\phi_*(\beta_*)\circ\delta_2^\flat"'] & \mathbb{D}_2^* \arrow[r, "v\circ\varrho"] & Z \arrow[d, "g"] \\ \mathbb{D}'_\flat \arrow[r, "u^*_{\beta_*}"'] \arrow[urr, dashed, "h"] & \mathbb{D}'_* \arrow[r, "y'"'] & Y \end{tikzcd}$$

Now, because $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ are composable, Remark 2.24 shows that the underlying besoms of $\mathfrak{n}$ and $\mathfrak{a}_*$ form a couple structure and provides the following commutative diagram on the left. On the other hand, a reformulation of the rightmost commutative diagram of (2.36) in the arrow category $\mathcal{C}^2$ provides the commutative diagram given below on the right.

$$\begin{tikzcd} \gamma_\flat \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a}_*)"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\phi_*(\beta_*))"'] & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma \\ \phi_*(\beta_*) \circ \delta_1^* \arrow[r, Rightarrow, "\mathbf{triv}(\mathfrak{a}^*_{\beta_*})"'] & \beta_* \circ \delta_1 \end{tikzcd} \qquad \begin{tikzcd} \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma \arrow[d, Rightarrow, "g^*\circ x"] \\ \beta_* \circ \delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f \end{tikzcd}$$

Pasting the previous two commutative diagrams in the obvious way leads to the left following commutative diagram in $\mathcal{C}^2$. The outer commutative square of this diagram may be rewritten in the form of the following rightmost diagram in $\mathcal{C}$.

$$\begin{tikzcd} \gamma_\flat \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{a}_*)"] \arrow[d, Rightarrow, "\mathbf{disk}(p_\flat\cdot\phi_*(\beta_*))"'] & \gamma_* \arrow[r, Rightarrow, "\mathbf{seed}(\mathfrak{n})"] \arrow[d, Rightarrow, "\mathbf{disk}(p_*\cdot\beta_*)"'] & \gamma \arrow[d, Rightarrow, "g^*\circ x"'] \\ \phi_*(\beta_*) \circ \delta_1^* \arrow[r, Rightarrow, "\mathbf{triv}(\mathfrak{a}^*_{\beta_*})"'] & \beta_* \circ \delta_1 \arrow[r, Rightarrow, dashed, "x'"'] & f \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S}_\flat \arrow[r, "\gamma'_\flat"'] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "g^*\circ x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}_1^\flat \arrow[r, "x'\circ\varrho'_*"] \arrow[d, "\phi_*(\beta_*)\circ\delta_1^\flat"] & X \arrow[d, "f"] \\ \mathbb{D}_2^\flat \arrow[r, "\phi_*(\beta_*)\circ\delta_2^\flat"] \arrow[rr, bend right, "y\circ\varrho\circ\varrho'_*"'] & \mathbb{D}'_\flat \arrow[r, "y'\circ u^*_{\beta_*}"] & Y \end{tikzcd}$$

Merging the bottom part of the earlier right diagram with the lifting obtained in the right commutative diagram of (2.37) provides the commutative diagram given below on the left. Pulling back $f : X \to Y$ along $g : Z \to Y$ in this diagram provides the existence of a canonical morphism $h' : \mathbb{D}^\flat_1 \to P$ making the following right diagram commute.

$$\begin{tikzcd}
\mathbb{S}_\flat \arrow[r, "\gamma'_\flat"'] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "g^*\circ x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}^\flat_1 \arrow[r, "x'"] \arrow[d, "\phi_*(\beta_*)\circ\delta^\flat_1"] & X \arrow[d, "f"] & \Rightarrow & \mathbb{S}_\flat \arrow[r, "\gamma'_\flat"'] \arrow[d, "\gamma_\flat"'] \arrow[rr, bend left, "x\circ\varkappa\circ\varkappa_*"] & \mathbb{D}^\flat_1 \arrow[r, dashed, "h'"'] \arrow[d, "\phi_*(\beta_*)\circ\delta^\flat_1"] & P \arrow[d, "f^*"] \\
\mathbb{D}^\flat_2 \arrow[r, "\phi_*(\beta_*)\circ\delta^\flat_2"] \arrow[dr, "y\circ\varrho\circ\varrho_*"'] & \mathbb{D}'_\flat \arrow[r, "y'"] \arrow[d, "h"] & Y & & \mathbb{D}^\flat_2 \arrow[r, "\phi_*(\beta_*)\circ\delta^\flat_2"] \arrow[rr, bend right, "y\circ\varrho\circ\varrho_*"'] & \mathbb{D}'_\flat \arrow[r, "h"] & Z \\
& Z \arrow[ur, "g"'] & & & & &
\end{tikzcd}$$

This last diagram finishes the proof of the statement. □

The previous statement does not have any version in terms of intractions. Right properness mainly relies on the transfer of surtractions. The transfer of intractions along pullbacks is instead subtly done under the transfer of surtractions. This is possible when surtractions happens to emcompass all the intractions. Such a phenomenon may already be noticed in the statement of Proposition 2.9. Proving the stability of intractions under pullbacks in such a way would also require one to prove that if a morphism is simple with respect to the scale $(\{\gamma\}, \varkappa \circ \varkappa_*, \{\gamma'_\flat\})$, then it is simple with respect to any scale $(\Omega, \varkappa', \overline{\Omega})$. A result such as Proposition 2.16 would therefore allow one to conclude. This will be studied further in Chapter 4.

2.3.8. From vertebrae to homotopy theories.

2.3.8.1. *Towards homotopy theories.* This section resumes and combines the results of sections 2.3.1, 2.3.5, 2.3.6 and 2.3.7 with the goal of retrieving the usual properties generally satisfied by homotopy theories. To do so, we will suppose given a certain node of vertebrae $\nu := p \cdot \Omega$ in some category $\mathcal{C}$ and try to retrieve classical statements of the theory of model categories for the zoo of an identity alliance $\mathrm{id}_\nu : \nu \rightsquigarrow \nu$ in $\mathbf{Ally}(\mathcal{C})$. Thus, all the terms given in this section such as 'weak equivalence', 'fibrations' and 'cofibrations' should be read as being 'for id_ν' when the associated alliance is not specified. In addition, we will assume that

1) the node of vertebrae ν is reflexive;
2) the underlying extended node of vertebrae of id_ν frames two copies of itself, which forces its coseed to be equal to its seed.

First of all, it is worth noting that the classes of fibrations, trivial fibrations, pseudofibrations and weak equivalences are coherent $\mathcal{C}$-classes (see Proposition 2.28 and Proposition 2.35 for the isomorphisms and Proposition 2.39, Proposition 2.40 and Proposition 2.41 for stability under composition). In fact, the auto-framing structure associated with id_ν is almost sufficient to prove the entire two-out-of-six property and hence the two-out-of-three property.

Theorem 2.62. *Let f, g and h be morphisms such that the composite $f \circ g \circ h$ exists. If $f \circ g$ and $g \circ h$ are weak equivalences for* id_ν*, then g is an intraction and both h and $f \circ g \circ h$ are weak equivalences for* id_ν*.*

Proof. First, notice that the relation $\mathrm{id}_\nu = \mathrm{id}_\nu \circledcirc \mathrm{id}_\nu$ holds. Now, if the composite arrows $f \circ g$ and $g \circ h$ are intractions, then so are g, h and $f \circ g \circ h$ by Proposition 2.34 and Proposition 2.41. If $g \circ h$ is also a surtraction, then h is a surtraction by Proposition 2.49. Finally, if, in addition $f \circ g$ is a surtraction, then $f \circ g \circ h$ is a surtraction by Proposition 2.58. □

The proof of the two-out-of-six property will be finished in Chapter 3 by using a much more general notion of framing. On the other hand, the fact that $p \cdot \Omega$ is reflexive almost proves axiom M3 of section 1.2.3.3.

Theorem 2.63. *A trivial fibration is both a fibration and a weak equivalence. Conversely, a fibration that is a weak equivalence is a pseudofibration.*

Proof. Follows from Proposition 2.48, Proposition 2.51 and the fact that relation $\mathrm{id}_\nu = \mathrm{id}_\nu \circledcirc \mathrm{id}_\nu$ holds. $\square$

The previous result exactly corresponds to the first part of axiom M3 of section 1.2.3.3 when every pseudofibration is a trivial fibration, which almost always happens in practice. In the case of the identity alliance $\mathrm{id}_\nu : \nu \rightsquigarrow \nu$, this is equivalent to saying that if a morphism has the rlp with respect to the seed of p, then it has the rlp with respect to every stem $\beta \in \Omega$. In general, this phenomenon will be a consequence of Proposition 2.14 when combined with a generalised notion of 'saturation'. This last statement will be made clearer in the Chapter 4. Finally, some examples may not follow the previous idea and rely on the following lemma.

Lemma 2.64. *Suppose that there exists an arrow $\iota : \mathbb{S}' \to \mathbb{D}_1$ through which all the stems of Ω factorise (i.e. $\beta = h \circ \iota$). If the pushout of any stem β along ι is the composite $\beta \circ \delta_1$, then a fibration that is a weak equivalence is a trivial fibration.*

Proof. By Proposition 2.48, it only suffices to prove the right lifting property with respect to every stem $\beta \in \Omega$. Let $f : X \to Y$ be both a fibration and an intraction for id_ν and consider the commutative square given below on the left. Because f is an intraction for id_ν, the arrow $u : \mathbb{S}' \to \mathbb{D}'$ factorises through a stem $\beta' \in \Omega$. But, since the stem β' factorises through the arrow $\iota : \mathbb{S}' \to \mathbb{D}_1$, so does the arrow $u : \mathbb{S}' \to \mathbb{D}'$. This therefore provides the following right commutative diagram.

(2.38)
$$
\begin{tikzcd}
\mathbb{S}' \arrow[r, "u"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r, "v"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "\iota"] \arrow[rr, bend left, "u"] \arrow[d, "\beta"'] & \mathbb{D}_1 \arrow[r, "u'"] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[rr, "v"'] & & Y
\end{tikzcd}
$$

By assumption, the pushout of β along ι is the composite $\beta \circ \delta_1$ (see the left-hand diagram below). As $f : X \to Y$ is a fibration for id_ν, it has the rlp with respect to the trivial stem $\beta \circ \delta_1$, which leads to the existence of a lift $h : \mathbb{D}' \to X$ making the following right diagram commute.

$$
\begin{tikzcd}
\mathbb{S}' \arrow[r, "\iota"] \arrow[rr, bend left, "u"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D}_1 \arrow[r, "u'"] \arrow[d, "\beta\circ\delta_1"] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r] \arrow[rr, bend right, "v"'] & \mathbb{D}' \arrow[r] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "\iota"] \arrow[rr, bend left, "u"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{D}_1 \arrow[r, "u'"] \arrow[d, "\beta\circ\delta_1"] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r] \arrow[rr, bend right, "v"'] & \mathbb{D}' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

This last diagram provides a lift for the left-hand diagram (2.38) and shows that f is a trivial fibration for id_ν. $\square$

The use of the previous lemma will, most of the time, involve weak equivalences that look like isomorphisms (see the examples of section 2.4).

2.4. Examples of everyday vertebrae

2.4.1. Set and higher category theory.

2.4.1.1. *Sets.* The category **Set** has an obvious but fundamental vertebra given by diagram 2.39. In the sequel, many other examples will copy the shape of this vertebra by replacing the object **1** with a 'generator' object.

(2.39)
$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, "\delta_1"] \arrow[dl, phantom, "\ulcorner"] & \\ \mathbf{1} \arrow[r, "\delta_2"'] & \mathbf{1}+\mathbf{1} \arrow[r, "\beta"] & \mathbf{1}, \end{tikzcd}$$

Intractions and surtractions for (2.39) are exactly injections and surjections in **Set**, respectively. A weak equivalence is thus a bijection in **Set**. Trivial fibrations turn out to correspond to weak equivalences while any function defines a fibration.

Remark 2.65 (Properties)**.** Vertebra (2.39) is reflexive via the trivial reflexive transition $\mathrm{id}_{\mathbf{1}} : \mathbf{1} \to \mathbf{1}$ and frames two copies of itself via the trivial cooperadic transition $\mathrm{id}_{\mathbf{1}} : \mathbf{1} \to \mathbf{1}$. In addition, it is not hard to see that it satisfies Lemma 2.64 by taking ι to be β since the arrow β is an epimorphism of sets.

2.4.1.2. *Small categories.* The category of small categories $\mathbf{Cat}(1)$ is equipped with two natural vertebrae that extend that of **Set**. Recall that $\emptyset$, **1** and **2** denote the initial category, the terminal category and the category generated by two objects and one arrow, respectively. The symbol **iso** will denote the free living isomorphism[3]. The first vertebra is as follows where the arrow $\beta_0 : \mathbf{1}+\mathbf{1} \to \mathbf{iso}$ stands for the inclusion of the source and target into **iso**.

(2.40)
$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, "\delta_1^0"] \arrow[dl, phantom, "\ulcorner"] & \\ \mathbf{1} \arrow[r, "\delta_2^0"'] & \mathbf{1}+\mathbf{1} \arrow[r, "\beta_0"] & \mathbf{iso} \end{tikzcd}$$

An intraction for the preceding vertebra is an essentially injective functor while a surtraction is an essentially sujective functor. A weak equivalence is therefore an essential bijection on objects.

Remark 2.66 (Properties)**.** Vertebra (2.40) is reflexive via the trivial reflexive transition $\mathrm{id}_{\mathbf{1}} : \mathbf{1} \to \mathbf{1}$. The homotopy contraction is given by the functor $\mathbf{iso} \to \mathbf{1}$ mapping the isomorphism to the identity. Vertebra (2.40) frames two copies of itself via the cooperadic transition $\mathbf{iso} \to \mathbf{iso} \oplus_{\mathbf{1}} \mathbf{iso}$ mapping the isomorphism to the composite of the two isomorphisms of each component of the pushout $\mathbf{iso} \to \mathbf{iso} \oplus_{\mathbf{1}} \mathbf{iso}$

$$\bullet_0 \cong \bullet_2 \longmapsto \bullet_0 \cong \bullet_1 \cong \bullet_2$$

The other vertebra is given by the next commutative diagram wherein $\mathbf{2} \oplus \mathbf{2}$ denotes the category of 2 parallel arrows and the arrow $\beta_0 : \mathbf{2} \oplus \mathbf{2} \to \mathbf{2}$ is the functor contracting the two parallel arrows of $\mathbf{2} \oplus \mathbf{2}$ into the unique arrow in **2**.

(2.41)
$$\begin{tikzcd} \mathbf{1}+\mathbf{1} \arrow[r, "\gamma_1"] \arrow[d, "\gamma_1"'] & \mathbf{2} \arrow[d, "\delta_1^1"] \arrow[dl, phantom, "\ulcorner"] & \\ \mathbf{2} \arrow[r, "\delta_2^1"'] & \mathbf{2} \oplus \mathbf{2} \arrow[r, "\beta_1"] & \mathbf{2}, \end{tikzcd}$$

An intraction for the preceding vertebra is a faithful functor while a surtraction is a full functor. A weak equivalence is therefore a fully faithful functor.

[3] i.e. the category made of two objects and an isomorphism.

Remark 2.67 (Properties)**.** Vertebra (2.41) is reflexive via the trivial reflexive transition $\mathrm{id}_{\mathbf{2}} : \mathbf{2} \to \mathbf{2}$. It also frames two copies of itself via the trivial cooperadic transition $\mathrm{id}_{\mathbf{2}} : \mathbf{2} \to \mathbf{2}$. Besides, vertebra (2.41) satisfies Lemma 2.64 by taking ι to be β_1 since the arrow β_1 is an epimorphism of small categories.

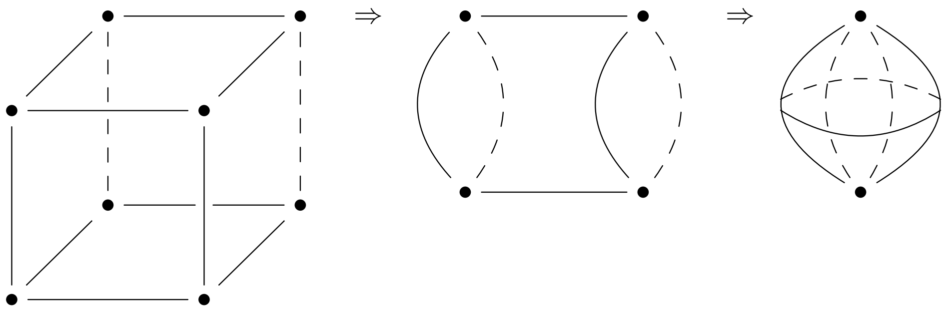
2.4.1.3. *Higher categories.* The case of 2-categories as well as that of strict ω-categories are the natural extensions of that of **Set** and $\mathbf{Cat}(1)$ and have already been discussed in the introduction. Chapter 6 gives a detailed treatment of the case of weak ω-groupoids.

2.4.2. Algebraic topology.

2.4.2.1. *Topological spaces (1).* The category of topological spaces **Top** contains the archetype of the concept of vertebra. The involved vertebrae are those defined by using topological spheres and discs as follows

$$V_n := \begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}^n \arrow[d, "\delta_1^n"] & \\ \mathbb{D}^n \arrow[r, "\delta_2^n"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}^n \arrow[r, "\gamma_{n+1}"'] & \mathbb{D}^{n+1} \end{tikzcd} \tag{2.42}$$

where, for every integer $n \in \mathbb{N}$, the object $\mathbb{S}^{n-1}$ denotes the topological sphere of dimension $n-1$; the object $\mathbb{D}^n$ denotes the topological disc of dimension n and the arrow $\gamma_n : \mathbb{S}^{n-1} \to \mathbb{D}^n$ denotes the canonical inclusion. Recall that topological spheres and discs are different from the Euclidean spheres and discs. The latter are given by semi-algebraic equations while the former are successive quotients of cubical constructions as shown below for the disc of dimension 3.



More specifically, the successive quotients are given by functions $\mathrm{pr}_{k+1}^n : [0,1]^n \to [0,1]^n$ defined as follows for every $0 \leq k \leq n-2$.

$$\begin{array}{ccccl} [0,1]^n & \to & [0,1]^n & \\ (x_0,\dots,x_k,b,x_{k+2},\dots,x_{n-1}) & \mapsto & (1/2,\dots,1/2,b,x_{k+2},\dots,x_{n-1}) & \text{if } b = 0,1 \\ (x_i)_{0\leq i\leq n-1} & \mapsto & (x_i)_{0\leq i\leq n-1} & \text{otherwise} \end{array}$$

If one denotes the composite $\mathrm{d}_n = \mathrm{pr}_{n-1}^n \circ \cdots \circ \mathrm{pr}_1^n$, then the topological spheres and discs are given by the following quotients for every non-negative integer n (where $\partial[0,1]^n$ stands for the topological boundary of $[0,1]^n$).

$$\mathbb{D}^n = [0,1]^n/d_n \qquad \mathbb{S}^{n-1} = (\partial[0,1]^n)/d_n \qquad \text{where} \quad \mathbb{S}^{-1} = \emptyset$$

The pushout of the vertebra is then given by the rightmost commutative diagram below, whose commutativity follows from the equations $d_{n+1}(1,x) = d_{n+1}(1,y) = d_{n+1}(0,x) =$

$d_{n+1}(0,y)$ satisfied by any pair x and y in $[0,1]^n$ for which the equation $d_n(x) = d_n(y)$ holds.

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d] \\ \mathbb{D}^n \arrow[r] & \mathbb{S}^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad := \quad \begin{tikzcd} (\partial[0,1]^n)/d_n \arrow[r, "\subset"] \arrow[d, "\subset"'] & {[0,1]^n/d_n} \arrow[d, "\{0\}\times(\_)"] \\ {[0,1]^n/d_n} \arrow[r, "\{1\}\times(\_)"'] & (\partial[0,1]^{n+1})/d_{n+1} \end{tikzcd}$$

The previous definitions provide topological spheres and discs with nice properties inherited from cubical shapes. In particular, since the pasting of two cubes gives another one up to reparametrisation, the pasting of two discs will give a disc up to reparametrisation and adequate combination of quotients.

$$\begin{tikzcd} \bullet_{1,0} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\sim"] & \bullet_{1,1} \arrow[d, dash] \\ \bullet_{0,0} \arrow[r, dash] & \bullet_{0,1} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet_{1,0} \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "\sim"] & \cdot \arrow[r, dash] \arrow[d, dash, dashed] \arrow[dr, phantom, "\sim"] & \bullet_{1,2} \arrow[d, dash] \\ \bullet_{0,0} \arrow[r, dash] & \cdot \arrow[r, dash] & \bullet_{0,2} \end{tikzcd}$$

It follows from this that vertebra V_n is reflexive and frames two copies of itself. The reflexive transition is induced by the obvious contraction of cubes $[0,1]^{n+1} \to \mathbf{1} \times [0,1]^n$ while the cooperadic transition is induced by the composition of cubes as shown above for dimension 2. More details regarding the properties of V_n are given in Chapter 6.

A weak equivalence (resp. intraction; surtraction) for V_n is a morphism $f : X \to Y$ for which the following left morphism is a bijection of sets (resp. injection; surjection) if $n = 0$ and for which the corresponding right morphism is a bijection of sets (resp. injection; surjection) if $n > 0$ for every point $x \in X$.

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y) \qquad\qquad \pi_n(f,x) : \pi_n(X,x) \to \pi_n(Y, f(x))$$

Similarly, a morphism that is a fibration for every vertebra V_n, where n runs over $\mathbb{N}$, corresponds to a Serre fibration.

2.4.2.2. *Topological spaces (2).* The category **Top** has many other types of vertebrae. For instance, tensoring the 0-dimensional version of vertebra (2.42) with any topological space U gives the next vertebra.

(2.43)
$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & U \arrow[d, "\delta_1"] & \\ U \arrow[r, "\delta_2"'] & U + U \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\beta"] & U \times [0;1]. \end{tikzcd}$$

A fibration for the preceding vertebra is a Hurewicz fibration.

Remark 2.68 (Properties)**.** Vertebra (2.43) is reflexive via the reflexive transition induced by the canonical map $[0,1] \to \mathbf{1}$ and frames two copies of itself via the cooperadic transition induced by the composition of intervals $[0,1] \to [0,2]$.

2.4.2.3. *Pointed topological spaces.* The category of pointed topological spaces inherits the vertebrae of section 2.4.2.1 up to a quotient of their first sphere as follows for every $n \geq 1$.

$$V_n/\partial V_n := \quad \begin{tikzcd} \mathbb{S}^{n-1}/\mathbb{S}^{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}^n/\mathbb{S}^{n-1} \arrow[d, "\delta_1^n"] & \\ \mathbb{D}^n/\mathbb{S}^{n-1} \arrow[r, "\delta_2^n"'] & \mathbb{S}^n/\mathbb{S}^{n-1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\gamma_{n+1}"'] & \mathbb{D}^{n+1}/\mathbb{S}^{n-1} \end{tikzcd}$$

The underlying points of the spheres and discs is then given by the image of the point $\mathbb{S}^{n-1}/\mathbb{S}^{n-1} \cong \mathbf{1}$ through the different maps. A weak equivalence (resp. intraction; surtraction) for the vertebra $V_n/\partial V_n$ is a morphism $f : (X, x) \to (Y, y)$ for which the following morphism is a bijection (resp. injection; surjection).

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, y)$$

Remark 2.69 (Properties)**.** The vertebra V_n is reflexive via the quotient of the reflexive transition defined in section 2.4.2.1 while it frames two copies of itself via the quotient of the cooperadic transition defined in section 2.4.2.1.

2.4.2.4. *Simplicial sets.* The example of the category of simplical sets is an example where object have little algebraic structure, thereby making the definition of vertebrae quite large and complex. Their construction is achieved by forcing the algebraic relation by means of the notion of node of vertebrae. First, the category **sSet** has an 'intuitive' set of vertebrae, which we are going to use to generate all the others. Recall that **sSet** is the presheaf category $\mathbf{Psh}(\mathbf{\Delta})$ where $\mathbf{\Delta}$ is the category of non-zero finite ordinals and order-preserving maps between them. For every non-zero ordinal $n \in \omega$, denote by Δ_n the object $\Delta(\_, n) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ in **sSet**. Similarly, denote by $\partial\Delta_n$ the subfunctor of Δ_n removing all split epimorphisms of $\mathbf{\Delta}$ in the hom-sets $\Delta(k, n)$.

$$\begin{tikzcd} & k \arrow[dr, "\textit{split epi.}"] & \\ n \arrow[ur] \arrow[rr, equal] & & n \end{tikzcd} \qquad \begin{tikzcd} & k \arrow[dr, "u"] & \\ n \arrow[ur] \arrow[rr, "f_k"'] & & n+1 \end{tikzcd}$$

This defines a functor as if a composite $f \circ g$ is a split epimorphism, then so is f. Finally, write as $f_k : n \to n + 1$ the injective order-presering map that does not contain $k \in n + 1$ in its image. For every relation $k \in n + 1$, denote by Λ_n^k the subfunctor of $\partial\Delta_{n+1}$ removing all arrows $k \to n + 1$ in $\Delta(k, n + 1)$ that satisfy the previous right factorisation of f_k in Δ_n. In the end, it takes a few line of calculations to see that these objects define the following vertebra in **sSet** (made of inclusions of subfunctors).

(2.44)
$$\begin{tikzcd} \partial\Delta_n \arrow[r, "\gamma_n^k"] \arrow[d, "\gamma_n"'] & \Lambda_n^k \arrow[d, "\delta_1^k"] & \\ \Delta_n \arrow[r, "\delta_2^k"'] \arrow[ur, phantom, "\ulcorner"] & \partial\Delta_{n+1} \arrow[r, "\gamma_{n+1}"] & \Delta_{n+1}. \end{tikzcd}$$

Even if this vertebra is reflexive, it does not permit any kind of framing, even up to communication $\gamma_n^k \overset{\mathrm{ex}}{\rightsquigarrow} \gamma_n$, which does not exist. We are therefore required to define more general vertebrae, which will include those defined above. First, recall that it is possible to factorise any morphism $f : X \to Y$ in **sSet** into two arrows $f = p \circ i$ where p has the rlp with respect to the set of arrows

$$\Gamma := \{\gamma_n : \partial\Delta_n \to \Delta_n\}_{n\in\omega}$$

while the arrow i is in the class $\mathbf{llp}(\mathbf{rlp}(\Gamma))$. This type of factorisation will be referred to as Γ-*factorisation.* First, consider the canonical arrow given below, on the left, from the initial object to the terminal object in **sSet**. Then, Γ-factorise this arrow as indicated above into $a_0 \circ g_0$ and similarly, Γ-factorise g_0 into $l_0 \circ g_0'$ as shown below in the middle. The pair of arrows g_0 and g_0' may be equipped with a structure of prevertebra as displayed on the right.

This prevertebra is reflexive when considered with the reflexive transition $l_0 : \mathbb{D}_1^0 \to \mathbb{D}_2^0$.

$$
\emptyset \xrightarrow{!} \mathbf{1} \quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{D}_1^0 \arrow[r, "l_0"] & \mathbb{D}_2^0 \arrow[dr, "a_0"] & \\
\emptyset \arrow[u, "g_0'"] \arrow[ur, "g_0"'] \arrow[rr, "!"'] & & \mathbf{1}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\emptyset \arrow[r, "g_0'"] \arrow[d, "g_0"'] & \mathbb{D}_1^0 \arrow[d, "\delta_0^1"] \arrow[dr, "l_0"] & \\
\mathbb{D}_2^0 \arrow[r, "\delta_0^2"] \arrow[rr, bend right, equal] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_0 \arrow[r, dashed, "b_0"] & \mathbb{D}_2^0
\end{tikzcd}
$$

Now, we may generate a node of vertebra of base $\|g_0, g_0'\|$ by considering the arrows β stemming from all the possible Γ-factorisations of the arrow b_0 as shown in the left diagram, below. This obviously gives the following right node of vertebrae in **sSet** whose vertebrae are all reflexive.

$$
\begin{tikzcd}
& \mathbb{D} \arrow[dr, "\alpha_\beta"] & \\
\mathbb{S}_0 \arrow[ur, "\beta"'] \arrow[rr, "b_0"'] & & \mathbb{D}_2^0
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\emptyset \arrow[r, "g_0'"] \arrow[d, "g_0"'] & \mathbb{D}_1^0 \arrow[d, "\delta_0^1"] & \mathbb{D}' \\
\mathbb{D}_2^0 \arrow[r, "\delta_0^2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_0 \arrow[r, "\beta"] \arrow[ur, "\beta'"] & \mathbb{D} \arrow[u, phantom, "\vdots"]
\end{tikzcd}
$$

Now, may now repeat the previous operation by replacing the canonical arrow $\emptyset \to \mathbf{1}$ with $b_0 : \mathbb{S}_0 \to \mathbb{D}_2^0$ and so on, so that we obtain a reflexive node of vertebrae of the form given below on the right for every non-zero $n \in \omega$.

$$
\mathbb{S}_n \xrightarrow[b_n]{} \mathbb{D}_2^n \quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{D}_1^{n+1} \arrow[r, "l_{n+1}"] & \mathbb{D}_2^{n+1} \arrow[d, "a_{n+1}"] \\
\mathbb{S}_n \arrow[u, "g_{n+1}'"] \arrow[ur, "g_{n+1}"] \arrow[r, "b_n"'] & \mathbb{D}_2^n
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_n \arrow[r, "g_{n+1}'"] \arrow[d, "g_{n+1}"'] & \mathbb{D}_1^{n+1} \arrow[d, "\delta_{n+1}^1"] & \mathbb{D}' \\
\mathbb{D}_2^{n+1} \arrow[r, "\delta_{n+1}^2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_{n+1} \arrow[r, "\beta"] \arrow[ur, "\beta'"] & \mathbb{D} \arrow[u, phantom, "\vdots"]
\end{tikzcd}
$$

In the end, we obtain a whole class of nodes of vertebrae $\mathcal{E}$ for all possible choices of Γ-factorisations in **sSet**. In addition, this class contains vertebra (2.44) for every non-zero $n \in \omega$. The point of the previous construction is that framings exist between any communicating pairs of nodes of vertebrae in $\mathcal{E}$. To see this, consider two communicating pairs $\|g, g'\| \cdot \Omega$ and $\|g_*, g_*'\| \cdot \Omega_*$ where $g' = g_*$ in $\mathcal{E}$. For every stem $\beta \in \Omega$ and $\beta_* \in \Omega$, it is always possible to associate this pair with the left-hand pushout, below. Then, pre-composing this commutative square with $g' : \mathbb{S} \to \mathbb{D}_1$ and using some diagrammatic rearrangement leads to the following right commutative diagram where $\mathbb{S}_\bullet'$ denotes the pushout of g and g_*'

$$
\begin{tikzcd}
\mathbb{D}_1 \arrow[r, "\beta \circ \delta_1"] \arrow[d, "\beta_* \circ \delta_2^*"'] & \mathbb{D}' \arrow[d, "\varepsilon_1"] \\
\mathbb{D}_*' \arrow[r, "\varepsilon_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{E}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_\bullet' \arrow[dr, dashed, "e"] & \mathbb{D}_1^* \arrow[l, "\delta_1^\bullet"'] \arrow[d, "\varepsilon_2 \circ \beta_* \circ \delta_1^*"] \\
\mathbb{D}_2 \arrow[u, "\delta_2^\bullet"] \arrow[r, "\varepsilon_1 \circ \beta \circ \delta_2"'] & \mathbb{E}
\end{tikzcd}
$$

Using a Γ-factorisation on the canonical arrow $e : \mathbb{S}_\bullet' \to \mathbb{E}$, say of the form $\eta \circ \beta_\bullet$, then shows that the pair of vertebrae $\|g, g'\| \cdot \beta$ and $\|g_*, g_*'\| \cdot \beta_*$ is framed by the vertebra $\|g, g_*'\| \cdot \beta_\bullet$. In the end, this shows that the pair of nodes of vertebrae $\|g, g'\| \cdot \Omega$ and $\|g_*, g_*'\| \cdot \Omega_*$ is framed by the canonical node of vertebrae of the form $\|g, g_*'\| \cdot \Omega_\bullet$ in $\mathcal{E}$.

This node is indeed in $\mathcal{E}$ as the reflexive structures of $\|g, g'\| \cdot \Omega$ and $\|g_*, g_*'\| \cdot \Omega_*$ induce a reflexive structure for $\|g, g_*'\| \cdot \Omega_\bullet$ by Remark 2.55. This therefore provides the factorisation, below – where $b : \mathbb{S} \to \mathbb{D}$ denotes the arrow b_n associated with the construction of g in the previous algorithm – which forces $\|g, g_*'\| \cdot \Omega_\bullet$ to be in $\mathcal{E}$ since the composite $\lambda \circ \lambda_*$ is in $\mathbf{rlp}(\Gamma)$

by Proposition 1.35.

$$\begin{tikzcd} \mathbb{D}_1^* \arrow[r, "\lambda\circ\lambda_*"] & \mathbb{D}_2 \arrow[dr, "a"] & \\ \mathbb{S} \arrow[u, "g'_*"] \arrow[ur, "g"'] \arrow[rr, "b"'] & & \mathbb{D} \end{tikzcd}$$

Remark 2.70. In fact, the definition of the nodes of vertebrae $\mathcal{E}$ permit any 'algebraic' operation that resembles a framing. The idea of defining stems, seeds and coseeds using Γ-factorisations is the key idea of many other constructions such as the homotopy theory of weak ω-categories and that of weak ω-groupoids discussed in Chapter 5.

To conclude, a morphism $f : X \to Y$ that is a trivial fibration for every node of vertebrae in $\mathcal{E}$ is a morphism that has the rlp with respect to the arrows of Γ and the seeds in $\mathcal{E}$. Because the seeds are also in $\mathbf{llp}(\mathbf{rlp}(\Gamma))$, a trivial fibration $f : X \to Y$ is equivalently a morphism that has the rlp with respect to the arrow $\gamma_n : \partial\Delta_n \to \Delta_n$ for every non-zero $n \in \omega$. This exactly correspond to the usual definition of acyclic fibration for **sSet**. Similarly, by Proposition 1.33 and Propsoition 1.34, the trivial stems $\beta \circ \delta_n^1 : \mathbb{D}_n^1 \to \mathbb{D}'$ in $\mathcal{E}$ must be in $\mathbf{llp}(\mathbf{rlp}(\Gamma))$. In fact, they are also weak equivalences in the usual sense. This follows from the two-out-out three property satisfied by the usual weak equivalences and the fact that the canonical arrows $\mathbb{D}_n^1 \to \mathbf{1}$ and $\mathbb{D}' \to \mathbf{1}$ are acyclic fibrations in the usual sense[4]. All this indicates that the usual homotopy theory for **sSet** is at least a *Bousfield localization* (see [**28**]) of the homotopy theory resulting from the nodes of vertebrae in $\mathcal{E}$.

2.4.3. Universal algebra.

2.4.3.1. *Rings.* As in the case of **Set**, the category of rings **Rng** has an initial object $\mathbb{Z}$ and a generator object $\mathbb{Z}[X]$, that may be used to define the following vertebra, where δ_1 and δ_2 are the obvious inclusions and β maps x and y to z.

(2.45)
$$\begin{tikzcd} \mathbb{Z} \arrow[r, "!"] \arrow[d, "!"'] & \mathbb{Z}[y] \arrow[d, "\delta_1"] & \\ \mathbb{Z}[x] \arrow[r, "\delta_2"'] & \mathbb{Z}[x] \oplus_{\mathbb{Z}} \mathbb{Z}[y] \arrow[r, "\beta"] & \mathbb{Z}[z] \end{tikzcd}$$

An intraction for (2.45) is a monomorphism of rings while a surtraction is an epimorphism of rings. A weak equivalence thus turns out to be an isomorphism of rings. Also, note that a trivial fibration is exactly a weak equivalence while any morphism is a fibration.

Remark 2.71 (Properties)**.** Vertebra (2.45) is reflexive via the trivial reflexive transition $\mathrm{id}_{\mathbb{Z}[x]} : \mathbb{Z}[x] \to \mathbb{Z}[x]$ and frames two copies of itself via the trivial cooperadic transition $\mathrm{id}_{\mathbb{Z}[x]} : \mathbb{Z}[x] \to \mathbb{Z}[x]$. In addition, it is not hard to see that it satisfies Lemma 2.64 by taking ι to be β since the arrow β is an epimorphism of rings.

2.4.3.2. *Modules (1).* Let R be a ring and denote by $\mathbf{Mod}_R$ the category of left R-modules. As in the case of **Set**, the category $\mathbf{Mod}_R$ has an initial object 0 and a generator object R that may be used to define the following vertebra, where δ_1 maps an element b to the pair $(0, b)$, δ_2 maps an element a to the pair $(a, 0)$ and β maps a pair (a, b) to the sum $a + b$.

(2.46)
$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & R \arrow[d, "\delta_1"] & \\ R \arrow[r, "\delta_2"'] & R \oplus R \arrow[r, "\beta"] & R \end{tikzcd}$$

[4]See the algorithm generating the vertebrae, wherein the objects $\mathbb{D}_n^1$ and $\mathbb{D}'$ may be related to the terminal object by finite compositions of arrows in $\mathbf{rlp}(\Gamma)$

An intraction for (2.46) is a monomorphism of modules while a surtraction is an epimorphism of modules. A weak equivalence thus turns out to be a bijection of modules. A trivial fibration is exactly a weak equivalence while any morphism is a fibration.

Remark 2.72 (Properties)**.** Vertebra (2.46) is reflexive via the trivial reflexive transition $\mathrm{id}_{D_1} : D_1 \to D_1$ and frames two copies of itself via the trivial cooperadic transition $\mathrm{id}_{D_1} : D_1 \to D_1$. In addition, it is easy to check that it satisfies Lemma 2.64 by taking ι to be β since the arrow β is an epimorphism of R-modules.

2.4.3.3. *Modules (2).* Let R be a ring. There is another type of vertebrae for $\mathbf{Mod}_R$ for which the notion of fibration is more interesting. Denote by $I \hookrightarrow J$ an inclusion of ideals of R. The ideals I and J may be seen as left R-modules in a trivial way so that the inclusion $I \hookrightarrow J$ defines a morphism of left R-modules. The inclusion $I \hookrightarrow J$ induces a coproduct morphism $I \oplus I \to J$ mapping a pair (a, b) to the sum $a + b$. This arrow may be used to define the next vertebra where δ_1^I maps an element b to the pair $(0, b)$ and δ_2^I maps an element a to the pair $(a, 0)$.

(2.47)
$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & I \arrow[d, "\delta_1^I"] & \\ I \arrow[r, "\delta_2^I"'] \arrow[ur, phantom, "\ulcorner"] & I \oplus I \arrow[r, "\beta_{I,J}"] & J \end{tikzcd}$$

If the ideal I is principal (i.e. of the form $I = (x)$), then an intraction for (2.47) is a monomorphism of modules while a surtraction is an epimorphism of modules. In this case, a weak equivalence is a bijection of modules. Also, a trivial fibration is exactly a weak equivalence when I is principal. In the case of a general ideal I, a fibration is a morphism that has the rlp with respect to the inclusion $I \hookrightarrow J$. It follows from Baer's criterion and Zorn's Lemma[5] that a morphism is a fibration for every vertebra (2.47), where $I \hookrightarrow J$ runs over all the inclusions of ideals of R, if and only if it is an epimorphism whose kernel is injective in $\mathbf{Mod}_R$.

Remark 2.73 (Properties)**.** Vertebra (2.47) is reflexive when $J = I$. The reflexive transition is then given by the identity on I. Regarding the framing structure, Vertebra (2.47) relative to an inclusion $I \hookrightarrow J$ communicates with any other vertebra relative to some inclusion $I \hookrightarrow J'$. The vertebra relative to the inclusion $I \hookrightarrow J + J'$ (where $J + J'$ denotes the sum of J and J') then frames the previous communication via the cooperadic transition $J + J' \cong J \oplus_I J'$ in the case where $J \cap J' = I$ (where $J \oplus_I J'$ denotes the pushout of J and J' over I in $\mathbf{Mod}_R$). Finally, vertebra (2.47) satisfies Lemma 2.64 when the equality $I = J$ holds.

The fact that not every vertebra of the form (2.47) is reflexive suggests that vertebra (2.47) would be better behaved when considering a node of vertebrae instead of a vertebra. In addition, the fact that an equality of type $J \cap J' = I$ restricts the framings suggests that vertebra (2.47) should be defined for more general inclusions. Specifically, denote by $\mathcal{S}(I)$ the class of modules made of pushouts of the following form for every collection of inclusions of ideals $\{I \hookrightarrow J_k\}_{k \in A}$.

$$\bigoplus_{k \in A}{}_I \, J_k$$

Every object $B \in \mathcal{S}(I)$ is equipped with an inclusion $I \hookrightarrow B$, which induces coproduct morphisms of the form $I \oplus I \to B$. These arrows may be used to define the following node

[5] Use Zorn's Lemma to split the morphism and show it is a split epimorphism. The splitting and Baer's criterion then allow an easy characterisation of the kernel as an injective module.

of vertebrae.

$$
\begin{tikzcd}
0 \arrow[r, "!"] \arrow[d, "!"'] & I \arrow[d, "\delta_1^I"] & & B \arrow[d, phantom, "\vdots"] \\
I \arrow[r, "\delta_2^I"'] & I \oplus I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rr, "\beta'"'] \arrow[urr, "\beta"] & & B'
\end{tikzcd}
$$

By Remark 2.73, the previous node of vertebrae is reflexive. It also frames two copies of itself via trivial cooperadic transitions of the form $B \oplus_I B' \to B \oplus_I B'$. A morphism that is a weak equivalence for every node of vertebrae of the preceding form is a bijection of modules. A morphism that is a fibration for every node of vertebrae of the previous form is an epimorphism whose kernel is injective in $\mathbf{Mod}_R$.

2.4.3.4. *Chain complexes.* Let R be a ring. This section aims at describing the vertebrae that recovers the projective model structure on the category of non-negatively graded chain complexes over R, which will be denoted by $\mathbf{Ch}_R$.

Recall that the objects of $\mathbf{Ch}_R$ are sequences of left R-modules of the form given below on the left, where the equation $d_k \circ d_{k+1} = 0$ holds for every $k \in \mathbb{N}$, while morphisms are given by sequences $(f_k : M_k \to M'_k)_{k \geq 0}$ of morphisms of left R-modules making the succeeding right square commute for every $k \in \mathbb{N}$.

$$
\begin{tikzcd}
M_0 & M_1 \arrow[l, "d_0"'] & \dots \arrow[l, "d_1"'] & M_k \arrow[l, "d_{k-1}"'] & \dots \arrow[l, "d_k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M_k \arrow[d, "f_k"'] & M_{k+1} \arrow[l, "d_k"'] \arrow[d, "f_{k+1}"] \\
M'_k & M'_{k+1} \arrow[l, "d_k"]
\end{tikzcd}
$$

In the sequel, we will denote by $R(\delta)$ the module equal to $R^{\oplus 2}$ when $\delta = 1$ and 0 when $\delta = 0$. We will also denote by $\mu_\delta : R(\delta) \to R$ the addition of R when $\delta = 1$ and the canonical map $0 \to R$ when $\delta = 0$. Now, for every integer $n \geq 0$, define the following chain complexes in $\mathbf{Ch}_R$ for the obvious morphisms of R-modules.

$$
\begin{cases}
D_n : 0 \xleftarrow[d_0]{} 0 \xleftarrow[d_1]{} \dots \longleftarrow 0 \xleftarrow[d_{n-1}]{} R \xleftarrow[d_n]{} 0 \xleftarrow[d_{n+1}]{} 0 \longleftarrow \dots \\
D_n(\delta) : 0 \xleftarrow[d_0]{} 0 \xleftarrow[d_1]{} \dots \longleftarrow 0 \xleftarrow[d_{n-1}]{} R \xleftarrow[\mu_\delta]{} R(\delta) \xleftarrow[d_{n+1}]{} 0 \longleftarrow \dots \\
S_n : 0 \xleftarrow[d_0]{} 0 \xleftarrow[d_1]{} \dots \longleftarrow 0 \xleftarrow[d_{n-1}]{} R^{\oplus 2} \xleftarrow[d_n]{} 0 \xleftarrow[d_{n+1}]{} 0 \longleftarrow \dots
\end{cases}
$$

It is not hard to see that that there exist obvious vertebrae of the form given on the left of (2.48) for every $n \in \mathbb{N}$. The maps in degree n and $n+1$ are given below on the right.

(2.48)

$$
\begin{tikzcd}
0 \arrow[r, "!"] \arrow[d, "!"'] & D_n \arrow[d] & \\
D_n \arrow[r] & S_n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "\beta_n(\delta)"] & D_n(\delta)
\end{tikzcd}
\qquad
\begin{tikzcd}
& 0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] & \\
0 \arrow[rr] \arrow[dd] & & R \arrow[dd] & & \\
& 0 \arrow[rr] \arrow[dl] & & 0 \arrow[r] \arrow[dl] & R(\delta) \arrow[dl, "\mu_\delta"] \\
R \arrow[rr] & & R^{\oplus 2} \arrow[r, "\mu_1"] & R &
\end{tikzcd}
$$

The subtlety lying in the preceding vertebra is the existence of a non-trival term in degree $n+1$ when $\delta = 1$. The fact that this term is a power of R is required by a framing structure, but does not have any influence on the definition of weak equivalences. Note that the front face of the preceding right diagram always corresponds to vertebra (2.46) defined in the case of left R-modules. This implies that a weak equivalence for the version of vertebra (2.48) when $\delta = 0$ is a bijection of left R-modules in degree n. On the other hand, when $\delta = 1$, an intraction for vertebra (2.48) is a morphism $f : M \to N$ for which the following property holds.

$$\textit{If } f(x) = d_n z \textit{ in } N_n, \textit{ then } \exists\, z' \textit{ in } M_{n+1} \textit{ such that } x = d_n z'$$

Similarly, one may show that, when $\delta = 1$, a surtraction for vertebra (2.48) is a morphism $f : M \to N$ for which the following property holds.

$$\textit{For every } y \in N_n, \textit{ then } \exists\, x \in N_n \textit{ and } \exists\, z \in N_{n+1} \textit{ such that } f(x) + y = d_n z$$

It is thus not hard to satisfy that a weak equivalence for vertebra (2.48) is the morphism $f : M \to N$ for which the induced morphism of homology groups $\mathrm{H}_n(f) : \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ is an isomorphism.

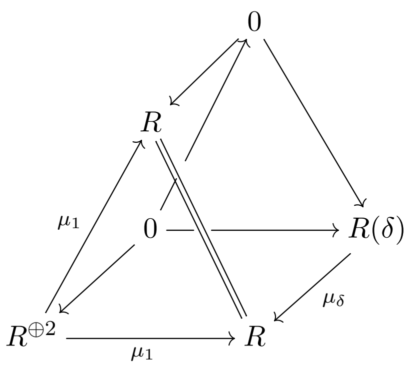
Remark 2.74 (Properties)**.** The version of vertebra (2.48) when $\delta = 0$ is reflexive via a trivial reflexive transition. Vertebra (2.48) frames two copies of itself via the following lefthand cooperadic transition when $\delta = 1$ and via the right one when $\delta = 0$.

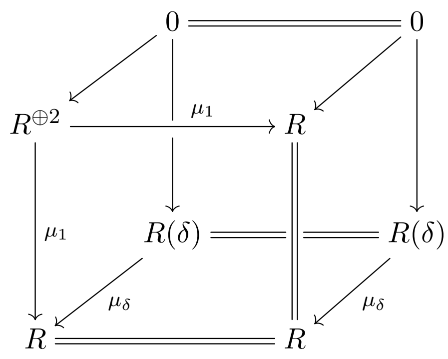
$$\begin{tikzcd} R \oplus R \oplus R \oplus R \arrow[r, "\mu_1 \circ (\mu_1 \oplus \mu_1)"] & R \arrow[d, equal] \\ R \oplus R \arrow[u, "R \oplus 0 \oplus 0 \oplus R"] \arrow[r, "\mu_1"] & R \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "\mu_0"] & R \arrow[d, equal] \\ 0 \arrow[u, "0"] \arrow[r, "\mu_0"] & R \end{tikzcd}$$

The version of Vertebra (2.48) for $\delta = 1$ together with its version for $\delta = 0$ is framed by the version of Vertebra (2.48) for $\delta = 1$. The cooperadic transition is then given by the following diagram.

$$\begin{tikzcd} R \oplus R \arrow[r, "\mu_1"] \arrow[d, equal] & R \arrow[d, equal] \\ R \oplus R \arrow[r, "\mu_1"] & R \end{tikzcd}$$

Vertebra (2.48) also satisfies Lemma 2.64 when considering the following left factorisation of the stem $\beta_n(\delta) : S_n \to D_n(\delta)$ since the succeeding right cube is a pushout.





Finally, the cases where $\delta = 0$ or 1 makes us realise that the right structure for vertebra (2.48) is that of a node of vertebrae. More specifically, the following vertebra is always reflexive and always frames two copy of itself.

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & D_n \arrow[d] \arrow[dl, phantom, "\ulcorner"] & D_n(0) \\ D_n \arrow[r] & S_n \arrow[ur, "\beta_n(0)"] \arrow[r, "\beta_n(1)"'] & D_n(1) \end{tikzcd}$$

2.4.4. Differential and algebraic geometry.

2.4.4.1. *Synthetic differential geometry.* In synthetic differential geometry, the *Cahiers topos* is a sheaf topos $\mathbf{Sh}(C^\infty\mathbf{Rng}^{\mathrm{op}})$ defined over the category of C^∞-rings, in which the category of smooth manifolds is embedded by mapping a smooth manifold to its set of smooth functions from M to the set of real numbers. By using the internal logic, it is possible to define the following objects for every integer $n \geq 0$.

$$D_n = \{d \mid d^{n+1} = 0\} \qquad D_n(2) = \{(d_1, d_2) \mid d_1^k d_2^{n+1-k} = 0 \text{ for any } 0 \leq k \leq n+1\}$$

These objects stands for infinitesimals. They are used to redefine many concepts of differential geometry in a very categorical way. There are obvious maps $\gamma_n : D_0 \to D_n$ (with mapping $0 \mapsto 0$), $\beta_n : D_n(2) \to D_n$ (with mapping $(d_1, d_2) \mapsto d_1 + d_2$) and $\delta_i^n : D_n \to D_n(2)$ (sending d to $(0, d)$ if $i = 1$ and $(d, 0)$ if $i = 0$) that make the following diagram commute for every integer $n \geq 1$

(2.49)
$$\begin{tikzcd} D_0 \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & D_n \arrow[d, "\delta_1"] & \\ D_n \arrow[r, "\delta_2"'] & D_n(2) \arrow[r, "\beta_n"] & D_n. \end{tikzcd}$$

Interestingly enough, there exist – in the Cahiers topos – objects called *microlinear spaces* for which the previous diagram appears as a vertebra (i.e. the commutative square behaves like a pushout square). For these spaces, the object D_0 also appears as an initial object. The *tangent space* of a microlinear space M at a point $x : D_0 \to M$ is defined as the 'set' of maps $v : D_1 \to M$ making the following diagram commute.

$$\begin{tikzcd} D_0 \arrow[d, "\gamma_1"'] \arrow[dr, "x"] & \\ D_1 \arrow[r, "v"'] & M \end{tikzcd}$$

This set is usually denoted as $T_x(M)$. Now, a morphism of microlinear spaces $f : M \to N$ is a surtraction (resp. intraction) for diagram (2.49) when $n = 1$ if and only if the induced map $T_x(f) : T_x(M) \to T_{f\circ x}(N)$ is a surjection (resp. injection) for every point $x : D_0 \to M$.

Remark 2.75 (Properties)**.** Vertebra (2.49) is reflexive via the trivial reflexive transition $\mathrm{id}_{D_1} : D_1 \to D_1$ and frames two copies of itself via the trivial cooperadic transition $\mathrm{id}_{D_1} : D_1 \to D_1$. In addition, it is not hard to see that it satisfies Lemma 2.64 by taking ι to be β_n since the arrow β_n is seen as an epimorphism from the point of view of microlinear spaces.

Remark 2.76. In differential geometry, a morphism of smooth manifolds $f : M \to N$ for which the map $T_x(f) : T_x(M) \to T_{f(x)}(N)$ is a bijection for every point $x : \mathbf{1} \to M$ corresponds to an *étale map* between smooth manifolds. Specifically, such a morphism satisfies the property that there exists an open neighborhood U of x such that f maps U diffeomorphically onto its image (i.e. it is a local diffeomorphism). This implication corresponds to a generalisation of the usual Inverse Function Theorem.

2.4.4.2. *Algebraic geometry.* In Algebraic Geometry, there is a functor from the category of affine varieties over a field k to the category of schemes $\mathbf{Sch}$ as follows.

$$\begin{array}{ccc} \mathbf{Aff}(k) & \to & \mathbf{Sch} \\ V & \mapsto & \mathrm{Spec}(k[V]) \end{array}$$

We are going to use this functor to describe vertebrae in the category $\mathbf{Sch}$. It is well-known that the affine simplices define affine schemes in $\mathbf{Sch}$. For every integer $n \geq 0$, the affine

n-simplex is defined as follows.

$$\Delta_n = V((\sum_{i=0}^{n} x_i) - 1) \subseteq \mathbb{A}^{n+1},$$

Affine varieties are equipped with a structure of closed sets, which allows us the define the following unions in $\mathbf{Aff}(k)$.

$$\Lambda_n^k = \cup_{j\neq k} V((\sum_{i=0,\neq j}^{n+1} x_i) - 1) \subseteq \mathbb{A}^{n+2},$$

$$\partial\Delta_{n+1} = \cup_j V((\sum_{i=0,\neq j}^{n+1} x_i) - 1) \subseteq \mathbb{A}^{n+2},$$

There are also obvious inclusions $\gamma_n : \partial\Delta_n \to \Delta_n$ and $\gamma_n' : \partial\Delta_n \to \Lambda_n^k$ making the following vertebra commute.

(2.50)
$$\begin{tikzcd}
\partial\Delta_n \arrow[r, "\gamma_n'"] \arrow[d, "\gamma_n"'] & \Lambda_n^k \arrow[d] \\
\Delta_n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \partial\Delta_{n+1} \arrow[r, "\gamma_{n+1}"] & \Delta_{n+1}.
\end{tikzcd}$$

It follows from usual results on pushouts of schemes (for instance, see [**18**]) that the preceding pushout is sent to a pushout of schemes. In other words, diagram (2.50) is sent to a vertebra in **Sch**. Of course, as in section 2.4.2.4, it is possible to extend these vertebrae to more general simplicial constructions in order to obtain more structure. The point of the previous vertebrae is that they are strongly related to the Voedvosky's $\mathbb{A}^1$-homotopy theory. Future work will aim at discussing this point (see next remark).

Remark 2.77. Other type of vertebrae could be defined. In particular, considering 'cubes' instead of 'triangles' could be another option. The advantage of cubes is that they are closer to the notions of topological disc and sphere. Such vertebrae could therefore lead to a better analogy between topological spaces and schemes. In particular, a version of the *Homotopy Hypothesis* relative to these kinds of vertebrae could lead to ∞-groupoidal constructions of Eilenberg-MacLane spaces in **Sch** as well as a notion of 'non-abelian motivic cohomology' – if it made any sense. The construct of Eilenberg-MacLane spaces for schemes is the central part of various results that wait for improvement in motivic cohomology (see [**41**])

Chapter 3

Spines

3.1. Introduction

The aim of the present chapter is to define and explain the theoretical formalism necessary to finish the proof of the two-out-of-six property initiated in Chapter 2. The required properties take the form of cancellation properties. Their proof requires one to distinguish between the case of intractions, which are defined with respect to alliances of nodes of vertebrae, and the case of surtractions, which are defined with respect to extended nodes of vertebrae. More specifically, the theorems towards which the present chapter is heading are of the following form.

Theorem (Cancellation of intractions)**.** Let $f : Y \to Z$ and $g : X \to Y$ be two morphisms in some category $\mathcal{C}$. There exists a notion of sequence of alliances of nodes of vertebrae $\mathfrak{a}_0, \dots, \mathfrak{a}_\ell$ equipped with a sequence of pairs of extended vertebrae $(\mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet), \dots, (\mathfrak{v}^{\ell-1}_\diamond, \mathfrak{v}^{\ell-1}_\bullet)$ such that if g is a surtraction for $\mathfrak{v}^k_\diamond$ and $\mathfrak{v}^k_\bullet$ for every $0 \leq k \leq \ell - 1$ and $f \circ g$ is an intraction of $\mathfrak{a}_\ell$, then f is an intraction for $\mathfrak{a}_0$.

Theorem (Cancellation of surtractions)**.** Let $f : Y \to Z$ and $g : X \to Y$ be two morphisms in some category $\mathcal{C}$. There exists a notion of sequence of extended nodes of vertebrae $\mathfrak{n}_0, \dots, \mathfrak{n}_\ell$ equipped with a sequence of pairs of extended vertebrae $(\mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet), \dots, (\mathfrak{v}^{\ell-1}_\diamond, \mathfrak{v}^{\ell-1}_\bullet)$ such that if g is a surtraction for $\mathfrak{v}^k_\diamond$ and $\mathfrak{v}^k_\bullet$ for every $0 \leq k \leq \ell - 1$ and $f \circ g$ is a surtraction of $\mathfrak{n}_\ell$, then f is a surtraction for $\mathfrak{n}_0$.

The precise statements may be found in Theorem 3.111 and Theorem 3.106. These are expressed in the language of 'spines', which means that the involved structures come along with a notion of 'dimension' (see Chapter 1). The rigorous forms of the previous two theorems also assume some conditions on the components $\mathfrak{a}_0$ and $\mathfrak{n}_0$, which may be seen as 'projectivity' conditions in the homological sense.

Because the preceding results are deeper than those discussed in Chapter 2, it was felt appropriate to give a good intuition of what the following text intends to pursue. We therefore devote an introductory section to discussing the case of topological spaces. This will give me the opportunity to provide some pictural explanation of what is about to follow.

In the case of topological spaces, the vertebrae used to achieve the proof of Theorem 3.111 and Theorem 3.106 at some non-negative dimension n are given by diagram (3.1) for every integer $0 \leq k \leq n - 1$, where the object $\mathbb{S}^{k-1}$ denotes the topological sphere of dimension

$k-1$; the object $\mathbb{D}^k$ denotes the topological disc of dimension k and the arrow $\gamma_k : \mathbb{S}^{k-1} \to \mathbb{D}^k$ denotes the canonical inclusion.

(3.1)
$$V_k := \begin{tikzcd} \mathbb{S}^{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}^k \arrow[d, "\delta_1^k"] & \\ \mathbb{D}^k \arrow[r, "\delta_2^k"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}^k \arrow[r, "\gamma_{k+1}"'] & \mathbb{D}^{k+1} \end{tikzcd}$$

Note that the sequence of vertebrae $V_0, V_1, \dots V_n$ has the particular property that it may be arranged into a sequence of arrows (with respect to the notations of Chapter 2) as follows.

(3.2)
$$\mathbb{S}_{-1} \xrightarrow{V_0} \mathbb{S}_0 \xrightarrow{V_1} \mathbb{S}_1 \xrightarrow{V_2} \mathbb{S}_2 \xrightarrow{V_3} \dots \xrightarrow{V_n} \mathbb{S}_n$$

Throughout this chapter, the previous sequence will be called a *spine of degree n*. Such a structure is defined in section 3.3.1. In this introductory part, this spine will be denoted by S_n. Exactly as in the case of vertebrae, spines may be associated with a zoo, which is defined in section 3.3.6. A *surtraction* for the spine S_n is a surtraction for its *head*, that is to say for its vertebra V_n. Similarly, an *intraction* for the spine S_n is an intraction for the vertebra V_n. With this termonology, a faithful translation of Theorem 3.111 and Theorem 3.106 in the case topological spaces would be the following.

Theorem 3.1 (Cancellation of intractions)**.** Let $f : Y \to Z$ and $g : X \to Y$ be two morphisms in **Top**. If g is a surtraction for the vertebra V_k, for every $0 \le k \le n$, and $f \circ g$ is a intraction for S_n, then f is a intraction for S_n.

Theorem 3.2 (Cancellation of surtractions)**.** Let $f : Y \to Z$ and $g : X \to Y$ be two morphisms in **Top**. If g is a surtraction for the vertebra V_k, for every $0 \le k \le n$, and $f \circ g$ is a surtraction for S_n, then f is a surtraction for S_n.

Note that the so-called sequences of alliances of nodes of vertebrae and extended nodes of vertebrae mentioned in the theorems given at the beginning of the section are not used in the preceding versions. This is because the spine S_n is implicitely equipped with all the properties necessary for the requirement of these sequences.

To help us in the proof of Theorem 3.1 and Theorem 3.2, we will need to introduce some set theoretical terminology. In this respect, if X denotes an object in **Top**, then two paths $x : \mathbb{D}^n \to X$ and $y : \mathbb{D}^n \to X$ picked out in X will be said to be *parallel above* V_n if they make the following left diagram commute.

(3.3)
$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}^n \arrow[d, "y"] \\ \mathbb{D}^n \arrow[r, "x"'] & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}^n \arrow[d] \arrow[rrd, bend left, "y"] & & \\ \mathbb{D}^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] \arrow[rrr, bend right, "x"'] & \mathbb{S}^n \arrow[r, "\gamma_{n+1}"] & \mathbb{D}^{n+1} \arrow[r, dashed] & X \end{tikzcd}$$

Such a relation will be denoted by $x \,||_n\, y$. Note that not all paths are parallel above V_n. Two parallel paths $x : \mathbb{D}^n \to X$ and $y : \mathbb{D}^n \to X$ in X above V_n will then be said to be *equivalent above* V_n, which will be denoted by $x \sim_n y$, if the universal map $x + y : \mathbb{S}^n \to X$ induced by the left diagram of (3.3) over the pushout of diagram (3.1), factorises through the inclusion γ_{n+1} as shown in the right diagram of (3.3).

Remark 3.3. The set of vertebrae consisting of all the V_k's for k running over $\mathbb{N}$ is obviously a canonical choice of vertebrae. There are however other vertebrae of interest. In particular, the following vertebra defined for every $k \geq 0$.

$$V(\gamma_k) := \qquad \begin{tikzcd} \mathbb{S}^{k-1} \arrow[r, equal] \arrow[d, "\gamma_k"'] & \mathbb{S}^{k-1} \arrow[d, "\gamma_k"] \\ \mathbb{D}^k \arrow[r, equal] \arrow[ur, phantom, "\ulcorner" very near end] & \mathbb{D}^k \arrow[r, equal] & \mathbb{D}^k \end{tikzcd}$$

In this case, two paths $x : \mathbb{D}^k \to X$ and $y : \mathbb{S}^{k-1} \to X$ picked out in X will be said to be *parallel above* $V(\gamma_k)$ if they make the following diagram commute.

$$\begin{tikzcd} \mathbb{S}^{k-1} \arrow[r, equal] \arrow[d, "\gamma_k"'] & \mathbb{S}^{k-1} \arrow[d, "y"] \\ \mathbb{D}^k \arrow[r, "x"'] & X \end{tikzcd}$$

Because of the particular form of $V(\gamma_k)$, to be parallel above $V(\gamma_k)$ also means to be equivalent above it. This is why the fact that two paths such as x and y are parallel above $V(\gamma_k)$ will be denoted as the relation $x \sim_{\gamma_k} y$.

We may now use the preceding language to retranslate the notion of intraction and surtraction at a more set theoretical level. Thus, a morphism $f : X \to Y$ is an intraction for S_n (or equivalently for V_n) if and if only the following statement holds.

For every pair $x \,||_n\, y$ in X, if $f(x) \sim_n f(y)$ in Y, then $x \sim_n y$ holds in X.

Such a reformulation is proven in full generality in Proposition 3.20. Similarly, a morphism $f : X \to Y$ is a surtraction for S_n (or equivalently for V_n) if and if only it satisfies the following property.

For every pair $x \sim_{\gamma_n} f(z)$ in Y, there exists
a pair $y \sim_{\gamma_n} z$ in X such that $x \sim_n f(y)$ holds in Y.

Such a reformulation is proven in full generality in Proposition 3.22. The foregoing reformulations are to make the proof of Theorem 3.1 and Theorem 3.1 less cumbersome. Below is the proof of the statement of Theorem 3.1, when n is zero.

Proof of Theorem 3.2, case $n = 0$. According to the preceding reformulation of the definition of surtractions, the goal is to prove that for every pair of paths $x : \mathbb{D}^0 \to Z$ and $z : \mathbb{S}^{-1} \to Y$ for which the relation $x \sim_{\gamma_0} f(z)$ holds, there exists a path $y : \mathbb{D}^0 \to Y$ such that the relations $y \sim_{\gamma_0} z$ and $x \sim_0 f(y)$ hold. In this respect, suppose to be given $x : \mathbb{D}^0 \to Z$ and $z : \mathbb{S}^{-1} \to Y$ such that $x \sim_{\gamma_0} f(z)$ holds. Because $\mathbb{S}^{-1}$ is initial, the following diagram admits a dashed lift making the whole triangle commute.

$$\begin{tikzcd} & X \arrow[d, "g"] \\ \mathbb{S}^{-1} \arrow[r, "z"'] \arrow[ur, dashed, "z'"] & Y \end{tikzcd}$$

Remark 3.4. Such a lifting property is what will be defined as the *projectivity* of the spine S_0, which I referred to at the very beginning of the section. In full generality, the existence of such a lift will be ensured by the fact that the spine S_0 is *projective with respect to the surtraction $g : X \to Y$*. The definition of the notion of projectivity may be found in section 3.2.1 and section 3.3.1.

The preceding lift now provides an equality $z = g(z')$, which turns the equivalence $x \sim_{\gamma_0} f(z)$ into the equivalence $x \sim_{\gamma_0} f \circ g(z')$. Because the composite $f \circ g$ is a surtraction for the spine S_0, the set theoretical reformulation of surtractions implies that there must exist a path $y' : \mathbb{D}^0 \to Y$ such that both relations $y' \sim_{\gamma_0} z'$ and $x \sim_0 f \circ g(y')$ hold. Finally, choosing $y = g(y')$ proves the case for $n = 0$ since

- the relation $x \sim_0 f \circ g(y')$ implies the relation $x \sim_0 f(y)$;
- the relation $y' \sim_{\gamma_0} z'$ implies the relation $g(y') \sim_{\gamma_0} g(z')$, which is exactly the relation $y \sim_{\gamma_0} z$.

The general proof of the case $n = 0$ will be found in the same section in which the notion of projectivity is defined (see Proposition 3.24). □

Because the proof of Theorem 3.1 in the case where n is zero uses the same methods as those used in the proof of Theorem 3.2 when n greater than zero, we will continue this introductory part with the proof Theorem 3.2 for higher cases.

Proof of Theorem 3.2, case $n > 0$. This part of the proof is going to require the fact that g is a surtraction for the vertebra V_k for every $0 \leq k \leq n-1$. The goal is this time to prove that for every pair of paths $x : \mathbb{D}^n \to Z$ and $z : \mathbb{S}^{n-1} \to Y$ for which the relation $x \sim_{\gamma_n} f(z)$ holds, there exists $y : \mathbb{D}^n \to Y$ such that the relation $x \sim_n f(y)$ holds. Note that if we again had a lifting as below, then the text following Remark 3.4 could be used word for word (after replacing 0 with n) to prove the case $n > 0$.

$$\begin{tikzcd} & X \arrow[d, "g"] \\ \mathbb{S}^{n-1} \arrow[ur, dashed, "?"] \arrow[r, "z"'] & Y \end{tikzcd}$$

Unfortunately, the sphere $\mathbb{S}^{n-1}$ has no reason to be projective with respect the surtraction $g : X \to Y$ when n is greater then zero. The idea of the proof is to show that there exists another element $\mathbf{fr}(z) : \mathbb{S}^{n-1} \to Y$, related to z via algebraic operations, for which the following lift exists.

(3.4)
$$\begin{tikzcd} & X \arrow[d, "g"] \\ \mathbb{S}^{n-1} \arrow[ur, dashed, "z'"] \arrow[r, "\mathbf{fr}(z)"'] & Y \end{tikzcd}$$

In the same way as z is mapped to $\mathbf{fr}(z)$, the path x will be associated with a path $\mathbf{fr}(x) : \mathbb{D}^n \to Z$. The construction of $\mathbf{fr}(z)$ and $\mathbf{fr}(x)$ will also associate the lefthand relation, below, with a relation of the same form as given on the right.

$$x \sim_{\gamma_n} f(z) \qquad \Rightarrow \qquad \mathbf{fr}(x) \sim_{\gamma_n} f(\mathbf{fr}(z))$$

Then, the same argument as those given after Remark 3.4 shall prove that there exists $y : \mathbb{D}^n \to Y$ such that the relations $x \sim_{\gamma_n} y$ and $\mathbf{fr}(x) \sim_n f(y)$ hold. Finally, by a process that could be identified as a sort of *transport of structure*, this relation will be transferred to the case of z and x so that the element y will provide a new element $\mathbf{fr}^{-1}(y) : \mathbb{D}^n \to Y$ for which the relation $x \sim_n f(\mathbf{fr}^{-1}(z))$ is true.

The best way of giving an intuition of how to define the 'operation' $\mathbf{fr}(\_)$ is probably to discuss the case where $n = 1$. In this case, the first step consists in constructing a path $\mathbf{fr}(z) : \mathbb{S}^0 \to Y$ out of $z : \mathbb{S}^0 \to Y$. To start with, notice that the domain of $z : \mathbb{S}^0 \to Y$ is the pushout of the following prevertebra.

$$\begin{tikzcd}
\mathbb{S}^{-1} \arrow[r, "\gamma_0"] \arrow[d, "\gamma_0"'] & \mathbb{D}^0 \arrow[d, "\delta_1^0"] \arrow[dr, dashed, bend left, "a"] & \\
\mathbb{D}^0 \arrow[r, "\delta_2^0"] \arrow[rr, dashed, bend right, "b"'] & \mathbb{S}^0 \arrow[r, dashed, "z"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

The precomposition of $z : \mathbb{S}^0 \to Y$ with δ_1^0 and δ_2^0 gives rise to two parallel paths $a : \mathbb{D}^0 \to X$ and $b : \mathbb{D}^0 \to X$, which are nothing but the images of borders of the sphere $\mathbb{S}^0$ in Y via the morphism $z : \mathbb{S}^0 \to Y$. Because the sphere $\mathbb{S}^{-1}$ is projective (or initial), the following diagrams admit a dashed filler making the top parts commute.

$$\begin{tikzcd}
\mathbb{S}^{-1} \arrow[r, dashed, "a'"] \arrow[d, "\gamma_0"'] \arrow[dr] & X \arrow[d, "g"] \\
\mathbb{D}^0 \arrow[r, "a"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{S}^{-1} \arrow[r, dashed, "b'"] \arrow[d, "\gamma_0"'] \arrow[dr] & X \arrow[d, "g"] \\
\mathbb{D}^0 \arrow[r, "b"'] & Y
\end{tikzcd}$$

By definition, the preceding two commutative diagrams imply the following two relations.

$$a \sim_{\gamma_0} g(a') \qquad\qquad b \sim_{\gamma_0} g(b')$$

Note that these relations correspond to the conditional requirements appearing in the set theoretical reformation of the definition of a surtraction for the vertebra V_0. Because g is a surtraction of V_0, we deduce that

1) there exists $y_a : \mathbb{D}^0 \to X$ such that $y_a \sim_{\gamma_0} a'$ and $a \sim_0 g(y_a)$ hold;
2) there exists $y_b : \mathbb{D}^0 \to X$ such that $y_b \sim_{\gamma_0} b'$ and $b \sim_0 g(y_b)$ hold.

In the end, we obtain the following three paths where the relation $x : a \sim_{\mathbb{S}^0} b$ means that x is a sphere with borders a and b.

$$a \sim_0 g(y_a) \qquad x : a \sim_{\mathbb{S}^0} b \qquad b \sim_0 g(y_b) \tag{3.5}$$

'Composing' these three paths then gives a new sphere $\mathbf{fr}(z) : g(y_a) \sim_{\mathbb{S}^0} g(y_b)$ *framing* the whole structure as shown by the next pictural represention.

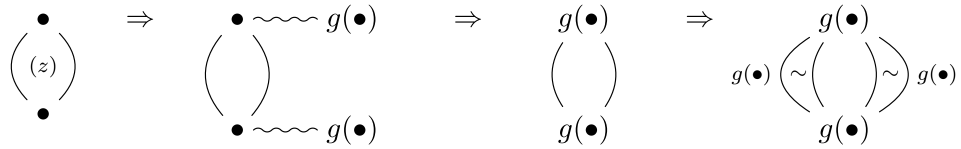
$$\begin{array}{ccccc}
\bullet_a & \Rightarrow & \bullet_a \rightsquigarrow g(\bullet_{y_a}) & \Rightarrow & g(\bullet_{y_a}) \\
 & & & & \\
\bullet_b & & \bullet_b \rightsquigarrow g(\bullet_{y_b}) & & g(\bullet_{y_b})
\end{array}$$

This time, notice that the borders of the sphere $\mathbf{fr}(z)$ are *under* g. The universal property of the pushout $\mathbb{S}^0$ then says that the paths y_a and y_b induce a canonical map $\langle y_a, y_b\rangle : \mathbb{S}^0 \to X$ making the following rightmost triangle commutes.

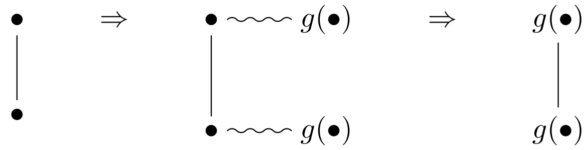
$$\begin{tikzcd}
\mathbb{S}^{-1} \arrow[r, "\gamma_0"] \arrow[d, "\gamma_0"'] & \mathbb{D}^0 \arrow[d, "\delta_1^0"] \arrow[dr, bend left, "y_a"] & \\
\mathbb{D}^0 \arrow[r, "\delta_2^0"] \arrow[rr, bend right, "y_b"'] & \mathbb{S}^0 \arrow[r, dashed, "\langle y_a, y_b\rangle"] \arrow[ul, phantom, "\ulcorner", very near start] & X
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\mathbb{S}^{-1} \arrow[r, "\gamma_0"] \arrow[d, "\gamma_0"'] & \mathbb{D}^0 \arrow[d, "\delta_1^0"] \arrow[dr, bend left, "g(y_a)"] & \\
\mathbb{D}^0 \arrow[r, "\delta_2^0"] \arrow[rr, bend right, "g(y_b)"'] & \mathbb{S}^0 \arrow[r, "\mathbf{fr}(z)"] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & X \arrow[d, "g"] \\
\mathbb{S}^0 \arrow[ur, dashed, "\langle y_a, y_b\rangle"] \arrow[r, "\mathbf{fr}(z)"'] & Y
\end{tikzcd}$$

This last diagram finally produces the wanted lift $z' : \mathbb{S}^0 \to Y$ of diagram (3.4).

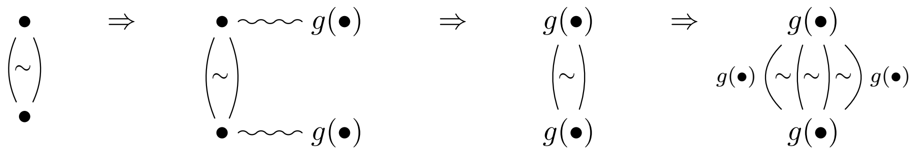
In the cases where $n > 1$, the operation $\mathbf{fr}(\_)$ is similarly made of compositions along paths provided by the surtractivity of g. These operations will later be called *framings* when defined at the level of vertebrae. Below is given a picture of the construction of $\mathbf{fr}(\_)$ when $n = 2$. A first step consists in getting the borders of dimension 1 of z in the image of g by composing the whole sphere $z : \mathbb{S}^1 \to X$ with paths stemming from the surtractivity of g while a second step consists in getting the borders of dimension 2 of the resulting sphere in the image of g.



This process may be generalised at any dimension to obtain a lift $z' : \mathbb{S}^{n-1} \to Y$ for diagram (3.4). Similarly, when $n = 0$, the proof of Theorem 3.1 will use a process resembling the following.



For its part, the case $n = 1$ of Theorem 3.1 will be of the following form.



The proposition that will later provide us with the above sequence of framings is Proposition 3.56.

Notice that the case $n = 1$ as well as the higher cases depend on the notion of dimension. This dimension will be encoded by the spinal structure S_n associated with the vertebra V_n. In particular, the higher cases will require the framing operations to be *functorial*, in the sense that these operations will have to be compatible with all the borders of lower dimension that they frame. Broadly speaking, this condition will be equivalent to requiring *the borders of a framing to be equal to the framings of the borders.*

Two notions of framing will arise at the level of spines from the different needs for the proof of Theorem 3.111 and Theorem 3.106:

1) the notion of *simple framing* will manage the functoriality necessary to prove the cancellation of intractions;
2) the notion of *extensive framings* will manage the functoriality necessary to prove the cancellation of surtraction.

The preceding discussion showed how to associate the sphere $z : \mathbb{S}^{n-1} \to Y$ with another sphere $\mathbf{fr}(z) : \mathbb{S}^{n-1} \to Y$ such that the following lifting exists.

$$\begin{tikzcd} & X \arrow[d, "g"] \\ \mathbb{S}^{n-1} \arrow[ur, dashed, "z'"] \arrow[r, "\mathbf{fr}(z)"'] & Y \end{tikzcd}$$

More specifically, the construction of $\mathbf{fr}(z)$ may be seen as an iteration of elementary framings along paths h_i and h'_i stemming from the surtractivity property of the morphism $g : X \to Y$ as follows.

$$h_n \cdot (h_{n-1} \cdot (\cdots (h_1 \cdot z \cdot h'_1) \ldots) \cdot h'_{n-1}) \cdot h'_n \tag{3.6}$$

The number of framings only depends on the number of times we need to repeat the algorithm so that the resulting sphere $\mathbf{fr}(z)$ has its source and target in the image of g. Recall that we were also provided with a relation of the form $x \sim_{\gamma_n} f(z)$. The two paths $f(z)$ and x were supposed to be parallel above $V(\gamma_n)$ in Z, which means that the equation $f(z) = x \circ \gamma_n$ holds. This may be represented by the next picture.

$$\begin{tikzcd} f(a) \arrow[rr, bend left, no head, "x"] & & f(b) \end{tikzcd}$$

Because the notion of framing will not depend on the object X in which they are operated, we may also use the paths $(h_i)_i$ and $(h'_i)_i$ – or in fact their images via $f : Y \to Z$ – to frame the paths $f(z)$ and x in Y along the paths $(f(h_i))_i$ and $(f(h'_i))_i$ to generate two new elements $\mathbf{fr}(f(z))$ and $\mathbf{fr}(x)$.

$$\begin{tikzcd} f(z) \arrow[d, squiggly, "(f(h_i))_i"', "(f(h'_i))_i"] \\ f(h_n) \cdot (f(h_{n-1}) \cdot (\cdots (f(h_1) \cdot f(z) \cdot f(h'_1) \cdot \ldots) \cdot f(h'_{n-1})) \cdot f(h'_n) \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[d, squiggly, "(f(h_i))_i"', "(f(h'_i))_i"] \\ f(h_n) \cdot (f(h_{n-1}) \cdot (\cdots (f(h_1) \cdot x \cdot f(h'_1) \cdot \ldots) \cdot f(h'_{n-1})) \cdot f(h'_n) \end{tikzcd}$$

For formal reasons, any morphism will be compatible with the structure of framing, which will provide the equality $f(\mathbf{fr}(z)) = \mathbf{fr}(f(z))$. Because the above framings will be 'extensive', they will preserve any relation of the form $\sim_{\gamma_n}$, which means that the following relation will hold.

$$\mathbf{fr}(x) \sim_{\gamma_n} f(\mathbf{fr}(z))$$

This last relation will finally allow us to find the so-called element $y : \mathbb{D}^n \to Y$ as explained a few paragraphs above.

$$y \sim_{\gamma_n} z \qquad\qquad \mathbf{fr}(x) \sim_n f(y)$$

At this stage, it only remains to fetch the paths y to the paths z and x in order to obtain relations of the form $\mathbf{fr}^{-1}(y) \sim_{\gamma_n} z$ and $x \sim_n f(\mathbf{fr}^{-1}(y))$.

Any astute reader would probably guess the process to use to go backwards. The point is that the paths $(h_i)_i$ and $(h'_i)_i$ are all *weakly invertible*. More specifically, there exists weak inverses $(e_i)_i$ and $(e'_i)_i$ to the respective paths $(h_i)_i$ and $(h'_i)_i$ so that successive framings of the form $(e_i \cdot \_ \cdot e'_i)$ will allow us to reverse all the process previously discussed. Such an operation will be called *conjugation* and discussed in section 3.3.5 at the level of vertebrae and in section 3.3.8 at the level of spines. The difficult point will be that the reversing process will be as weak as possible so that giving framings of the form $(e_i \cdot \_ \cdot e'_i)$ will surely not be the last step of the proof of Theorem 3.2.

The property of being weakly invertible will rely on two subnotions called *correspondence* and *mating*, which will both be defined in section 3.3.4. Correspondences are structures allowing the handling of a very particular type of parallelism. This type of parallelism arises from framing a path $h_i \cdot p \cdot h'_i$ along the paths e_i and e'_i, where the symbol p denotes some path of dimension $n \geq i$, so that the resulting path

$$e_i \cdot (h_i \cdot p \cdot h'_i) \cdot e'_i$$

is parallel to p at dimension i. Figure 1 gives a graphical representation of this phenomenon where

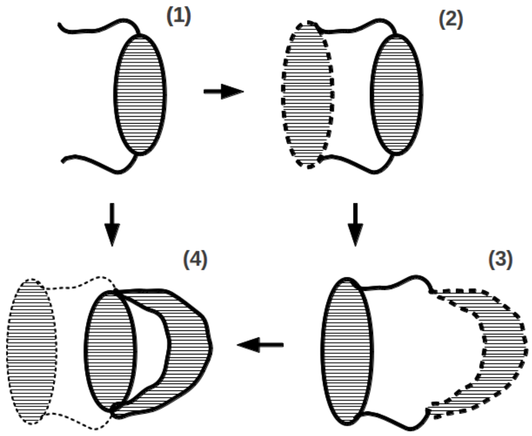


Figure 1. Correspondence stemming from a conjugation of paths

- the path p is represented by the middle cell at step (1);
- the path $h_i \cdot p \cdot h'_i$ is represented by the dashed cell at step (2);
- the path $e_i \cdot (h_i \cdot p \cdot h'_i) \cdot e'_i$ is represented by the dashed cell at step (3)
- and the correspondence is exposed at step (4).

Note that the forms of both parallel paths p and $e_i \cdot (h_i \cdot p \cdot h'_i) \cdot e'_i$ are canonically close to each other. This structural proximity will be expressed in terms of *correspondence* at the level of vertebrae and *memory* at the level of spines. Here, the terminology diverges as a memory will only remember the useful correspondences while other correspondences will have to be ignored in the process of going back. Somehow, the notions of correspondence and memory define a general formalism to talk about parallel pairs of cells between which one would like to see some coherence. The next step then consists in providing the previous pair of parallel cells with a notion of coherency, that is to say linking the two paths via homotopies.

$$e_i \cdot (h_i \cdot p \cdot h'_i) \cdot e'_i \qquad \text{and} \qquad p$$

These homotopies will allow us to fetch all the homotopical information known by $e_i \cdot (h_i \cdot p \cdot h'_i) \cdot e'_i$ towards p. The existence of these homotopies is substantial to the process that will enable us to gradually go back to x from $\mathbf{fr}(x)$. The process of associating a correspondence with a homotopy is named differently depending on whether it is used in the middle of a process or at the end. It will be named *mating* in the middle of a process. In this case, if the path p is not of dimension i, but is at least equipped with a notion of source and target of dimension i (see Figure 1 where p is of dimension 2 and i is equal to 1). Therefore, the correspondence[1]

$$e_i \cdot (h_i \cdot p \cdot h'_i) \cdot e'_i \qquad \asymp \qquad p \tag{3.7}$$

may be broken into two other correspondences relative to the source and target of p at dimension i, say a and b, as follows.

$$e_i \cdot (h_i \cdot a \cdot h'_i) \cdot e'_i \qquad \asymp \qquad a$$
$$e_i \cdot (h_i \cdot b \cdot h'_i) \cdot e'_i \qquad \asymp \qquad b$$

The existence of two respective homotopies, say m_a and m_b, between these two correspondences then suggests a framing of $e_i \cdot h_i \cdot v \cdot h'_i \cdot$ along these two paths. The important point is that this framing results from coherency and should thus be thought of as endowed with nice properties. The pair of paths m_a and m_b is called *pair of mates* of correspondence (3.7).

[1]A correspondence will be denoted by the symbol $\asymp$ when seen as a relation.

Such a terminology stems from the fact that these paths help to recover the lost relationship between the two conjugates of correspondence (3.7) as is explained below. The framing induced by the triple

$$\Big(\quad m_a \quad ; \quad e_i \cdot (h_i \cdot p \cdot h_i') \cdot e_i' \cdot \quad ; \quad m_b \quad \Big)$$

then provides a new paths p' which, this time, does not correspond with p at dimension i, but corresponds with p at dimension $i+1$. We will thus get closer to the paths p by repeating the above process, which is exactly what we want to fetch the so-called element $y : \mathbb{D}_n \to Y$ to the pair x and z. When arriving at the last step of such a process, the notion of correspondence is replaced with the notion of *recollection*, with which is associated a single mate. The particularity of this notion is discussed in section 3.3.7. After such an operation, all the homotopical information is fetched to the paths p. Thus, the successive framings induced by the mates allow us to return to the path p from the information remembered by the path $e_i \cdot (h_i \cdot p \cdot h_i') \cdot e_i'$. Such a construction will be called *chaining (of memories)* and used to define what could be seen as a very general notion of coherence. This very important concept will be discussed in section 3.3.7. □

The underneath comparative table sums up the different sections devoted to define a formalism at the level of vertebrae and generalise it at the level of spines.

Vertebrae	$\Rightarrow$	**Spines**
section 3.2.1 : *Hom-language*	$\Rightarrow$	section 3.3.1 and section 3.3.6 : *Hom-language*
section 3.3.2 : *Framings*	$\Rightarrow$	section 3.3.3 : *Simple & extensive framings*
section 3.3.4 : *Correspondences*	$\Rightarrow$	section 3.3.7 : *Simple & extensive memories*
section 3.3.5 : *Conjugations*	$\Rightarrow$	section 3.3.8 : *(Extended) conjugations*

3.2. Preparation

3.2.1. Theory of vertebrae and hom-language.

3.2.1.1. *Hom-language.* The goal of this section is to recall the underlying language of the hom-sets associated with a category. Some of the proposed notations are unusual but will turn out to be useful later on. Let $\mathcal{C}$ be a category. For every pair of objects A and B in $\mathcal{C}$, the set of morphisms from A to B will be denoted by $\mathcal{C}(A,B)$ and called a *hom-set*. For every object Z and pair of arrows $f : X \to Y$ and $g : A \to B$ in $\mathcal{C}$, the functions defined by

$$\left[\begin{array}{c} \mathcal{C}(B,Z) \to \mathcal{C}(A,Z) \\ u \mapsto u \circ g \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} \mathcal{C}(Z,X) \to \mathcal{C}(Z,Y) \\ u \mapsto f \circ u \end{array}\right]$$

will be denoted by $g \cdot \_$ and $f(\_)$. Thus, a composition $f \circ x \circ g$ may be written as either $f(g \cdot x)$ or $g \cdot f(x)$. Since the composition of $\mathcal{C}$ is associative, a composition $f \circ f' \circ x$ may be written as either $f \circ f'(x)$ or $f(f'(x))$. On the other hand, a composition of the form $x \circ g \circ g'$ may be written as $g' \cdot (g \cdot x)$. This latest formula will be shortened to the expression $(g'g) \cdot x$. Those functions obviously imply a commutative diagram

$$\begin{tikzcd} \mathcal{C}(B,X) \arrow[r, "g\cdot\_"] \arrow[d, "f(\_)"'] & \mathcal{C}(A,X) \arrow[d, "f(\_)"] \\ \mathcal{C}(B,Y) \arrow[r, "g\cdot\_"] & \mathcal{C}(A,Y) \end{tikzcd}$$

in **Set** and thus give rise to a bifunctor $\mathcal{C}(\_,\_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ called *hom-bifunctor of* $\mathcal{C}$. Recall that, by definition of limits and colimits in $\mathcal{C}$, this bifunctor preserves and reflects limits in both variables (see [**34**]).

3.2.1.2. *Hom-language for prevertebrae.* Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. Consider a prevertebra $p := \|\gamma, \gamma'\|$ whose general form is recalled below.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] & \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, dotted] \arrow[ul, phantom, "\ulcorner", very near start] & X \end{tikzcd} \tag{3.8}$$

A pair of two elements $x \in \mathcal{C}(\mathbb{D}_2, X)$ and $y \in \mathcal{C}(\mathbb{D}_1, X)$ will be said to be *parallel in X above p* if the equality $\gamma \cdot x = \gamma' \cdot y$ holds. By universality of the pushout $\mathbb{S}'$, the previous equation induces a unique arrow $h : \mathbb{S}' \to X$ for which the equalities $\delta_2 \cdot h = x$ and $\delta_1 \cdot h = y$ hold. By uniqueness, the arrow h shall be denoted by the symbol $\langle x, y\rangle$, turning the previous relations into $\delta_2 \cdot \langle x, y\rangle = x$ and $\delta_1 \cdot \langle x, y\rangle = y$.

Remark 3.5. For every morphism $f : X \to Y$ in $\mathcal{C}$, the universality of $\mathbb{S}'$ implies that the arrow $f(\langle x, y\rangle)$ equals the arrow $\langle f(x), f(y)\rangle$ in $\mathcal{C}(\mathbb{S}', X)$.

Remark 3.6. It is not hard to see that $\mathrm{id}_{\mathbb{S}'} = \langle \delta_2, \delta_1\rangle$ so that Remark 3.5 implies the equalities $h = h(\mathrm{id}_{\mathbb{S}'}) = \langle h(\delta_2), h(\delta_1)\rangle = \langle \delta_2 \cdot h, \delta_1 \cdot h\rangle$ for every arrow $h : \mathbb{S}' \to X$ in $\mathcal{C}$.

3.2.1.3. *Hom-language for vertebrae.* Let $\mathcal{C}$ be a category and X be some object in $\mathcal{C}$. Consider a vertebra $v := p \cdot \beta$ in $\mathcal{C}$ whose general form is recalled below.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] & & \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"'] \arrow[ul, phantom, "\ulcorner", very near start] & \mathbb{D}' \arrow[r, dotted] & X \end{tikzcd}$$

For every element $h \in \mathcal{C}(\mathbb{D}', X)$, we will write the formal relation $h : x \sim_v y$ if and only if $(\delta_2\beta) \cdot h = x$ and $(\delta_1\beta) \cdot h = y$. The previous relation will be said to *hold in X*. The element h will be said to be a *v-path from x to y in X* to refer to the existence of such a relation. The elements x and y will be called the *source* and *target* of the v-path h. For any element $x \in \mathcal{C}(\mathbb{D}_2, X)$ and $y \in \mathcal{C}(\mathbb{D}_1, X)$, the notation $x \sim_v y$ will be used to mean that there exists some v-path $h : x \sim_v y$ in X. In this case, the element x will be said to be *v-homotopic* to y.

Remark 3.7. Any v-path $h : x \sim_v y$ in X gives rise to a v^{rv}-path $h : y \sim_{v^{\mathrm{rv}}} x$ in X.

Remark 3.8. For every morphism $f : X \to Y$ in $\mathcal{C}$, a relation $h : x \sim_v y$ in X implies a relation $f(h) : f(x) \sim_v f(y)$ in Y.

Any prevertebra $p : \mathbb{S} \multimap \mathbb{S}'$ induces an obvious vertebra $p \cdot \mathrm{id}_{\mathbb{S}'}$ in $\mathcal{C}$. The homotopy relation $\sim_v$ defined for any vertebra v will be denoted by $\sim_p$ in the case where the vertebra v is of the form $p \cdot \mathrm{id}_{\mathbb{S}'}$.

Remark 3.9. Any relation of the form $x \sim_p y$ in X is equivalent to saying that x and y are parallel in X above p.

Proposition 3.10. *Let x and y be two parallel elements in X above p. The relation $h : x \sim_p y$ holds in X if and only if the equality $h = \langle x, y\rangle$ is satisfied in $\mathcal{C}$.*

Proof. Follows from the above definitions and universality of $\mathbb{S}'$. □

Now, consider a morphism $\gamma : \mathbb{S} \to \mathbb{D}_*$ in $\mathcal{C}$. Such an arrow defines two canonical prevertebrae $\|\gamma, \mathrm{id}_{\mathbb{S}}\|$ and $\|\mathrm{id}_{\mathbb{S}}, \gamma\|$. We will associate the first and second one with the homotopy relations $\sim_\gamma$ and $\sim^\gamma$, respectively.

Remark 3.11. Any pair of relations of the form $x \sim_\gamma z$ and $z' \sim^\gamma y$ is entirely determined by the elements x and y, respectively. These relations are equivalent to giving the respective factorisations $z = \gamma \cdot x$ and $z' = \gamma \cdot y$.

Remark 3.12. Any relation of the form $x \sim_\gamma z$ is equivalent to the relation $z \sim^\gamma x$.

Proposition 3.13. *Let $p = \|\gamma, \gamma'\| : \mathbb{S} \multimap \mathbb{S}'$ be a prevertebra in $\mathcal{C}$. The relation $x \sim_p y$ holds in X if and only if there exists a unique $z \in \mathcal{C}(\mathbb{S}, X)$ such that both relations $x \sim_\gamma z$ and $z \sim^{\gamma'} y$ hold in X.*

Proof. Straightforward since $z = \gamma \cdot x = \gamma' \cdot y$. □

Proposition 3.14. *Let $v := p \cdot \beta : \mathbb{S} \multimap (\delta_1, \delta_2)$ be a vertebra in $\mathcal{C}$. The relation $x \sim_\beta z$ holds in X if and only if the relation $x : \delta_2 \cdot z \sim_v \delta_1 \cdot z$ holds in X.*

Proof. The pair of relations $\delta_1 \cdot z = (\delta_1\beta) \cdot x$ and $\delta_2 \cdot z = (\delta_2\beta) \cdot x$ is equivalent to the relation $z = \langle (\delta_2\beta) \cdot x, (\delta_1\beta) \cdot x \rangle = \beta \cdot x$ (see Remark 3.6). □

3.2.1.4. *Hom-language for alliances of vertebrae.* Let $\mathcal{C}$ be a category and X be some object in $\mathcal{C}$. Consider an alliance of vertebrae $(\varkappa, \varrho, \varrho', \varkappa', u) : v \rightsquigarrow \overline{v}$ in $\mathcal{C}$ whose general form is recalled below.

$$
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[rr, "\overline{\gamma}'"] \arrow[dr, "\varkappa"'] \arrow[dd, "\overline{\gamma}"'] & & \overline{\mathbb{D}}_1 \arrow[dr, "\varrho'"] \arrow[dd, "\overline{\delta}_1" near end] & & & & \\
& \mathbb{S} \arrow[rr, "\gamma'" near start] \arrow[dd, "\gamma" near start] & & \mathbb{D}_1 \arrow[dd, "\delta_1" near start] & & & \\
\overline{\mathbb{D}}_2 \arrow[rr, "\overline{\delta}_2" near end] \arrow[dr, "\varrho"'] & & \overline{\mathbb{S}}' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rr, "\overline{\beta}"' near end] \arrow[dr, "\varkappa'"] & & \overline{\mathbb{D}}' \arrow[dr, "u"] & & \\
& \mathbb{D}_2 \arrow[rr, "\delta_2"'] & & \mathbb{S}' \arrow[rr, "\beta"'] & & \mathbb{D}' \arrow[r, dotted] & X
\end{tikzcd}
$$

The following proposition shows how an alliance of vertebrae as above transfers the hom-language of ν to that of $\overline{\nu}$.

Proposition 3.15. *Let X be an object in $\mathcal{C}$ and consider an alliance of vertebrae as above. If the relation $h : x \sim_v y$ holds in X, then the relation $u \cdot h : \varrho \cdot x \sim_{\overline{v}} \varrho' \cdot y$ holds in X. The converse is true when $\varkappa$, ϱ, ϱ' and $\varkappa'$ are identities.*

Proof. Straightforward by looking at the above diagram. □

Remark 3.16. Every alliance of prevertebrae $(\varkappa.\varrho, \varrho', \varkappa') : p \rightsquigarrow \overline{p}$ in $\mathcal{C}$ induces an alliance of vertebrae $(\varkappa.\varrho, \varrho', \varkappa', \varkappa') : p \cdot \mathrm{id}_{\mathbb{S}'} \rightsquigarrow \overline{p} \cdot \mathrm{id}_{\overline{\mathbb{S}}'}$. Proposition 3.15 then implies that any relation $\langle x, y \rangle : x \sim_p y$ in X gives a relation $\varkappa' \cdot \langle x, y \rangle : \varrho \cdot x \sim_{\overline{p}} \varrho' \cdot y$. By Proposition 3.10, the equality $\varkappa' \cdot \langle x, y \rangle = \langle \varrho \cdot x, \varrho' \cdot y \rangle$ follows .

Remark 3.17. Any vertebra $p \cdot \beta$ in $\mathcal{C}$ defines an alliance of vertebrae $(\mathrm{id}_p, \beta) : p \cdot \beta \rightsquigarrow p \cdot \mathrm{id}_{\mathbb{S}'}$ where id_p denotes the identity morphism on the model p in $\mathbf{Mod}_{\mathcal{C}}(\mathtt{Prev})^{\mathrm{op}}$.

Proposition 3.18. *Let $v = p \cdot \beta$ be a vertebra in $\mathcal{C}$. For every pair $x \in \mathcal{C}(\mathbb{D}_2, X)$ and $y \in \mathcal{C}(\mathbb{D}_1, X)$, any relation of the form $x \sim_v y$ implies a relation $x \sim_p y$ in X.*

Proof. Follows from Remark 3.17 and Proposition 3.15. □

Proposition 3.19. *Let $v = p \cdot \beta$ be a vertebra in $\mathcal{C}$ and x and y be two parallel elements in X above p. The relation $h : x \sim_v y$ holds in X if and only if the arrow $\langle x, y\rangle : \mathbb{S}' \to X$ is equal to the composite $h \circ \beta : \mathbb{S}' \to X$.*

Proof. Follows from Remark 3.17, Proposition 3.15 and Proposition 3.10. This may also be shown by using Remark 3.11 and Proposition 3.14. □

3.2.1.5. *Hom-language for nodes of vertebrae.* Let $\mathcal{C}$ be a category and X be some object in $\mathcal{C}$. For every node of vertebrae $\nu := p \cdot \Omega$ in $\mathcal{C}$, we will write the relation $h : x \sim_\nu y$, which will be said to *hold in* X, everytime there exists a vertebra $v \in \nu$ such that the relation $h : x \sim_v y$ holds in X. The element h will then be called a *ν-path* and said to be *from x to y*. The class of ν-paths in X from x to y will be denoted by $\mathcal{C}(\nu, X)(x, y)$. As in the case of vertebrae, we will use a relation of the form $x \sim_\nu y$ to mean that there exists a ν-path $h \in \mathcal{C}(\nu, X)(x, y)$. In this case, the element x will then be said to be *ν-homotopic* to y. The following propositions use the conventional notations introduced in Chapter 2.

Proposition 3.20. *A morphism $f : X \to Y$ in $\mathcal{C}$ is an intraction for an alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ if and only if it satisfies the following property:*

For every pair $x \sim_p y$ in X, if $f(x) \sim_\nu f(y)$ in Y, then $\varrho \cdot x \sim_{\overline{\nu}} \varrho' \cdot y$ in X.

Proof. Suppose that $f : X \to Y$ is an intraction for $\mathfrak{a}$ and consider $x \sim_p y$ in X such that $f(x) \sim_\nu f(y)$ in Y. First, Remark 3.9 implies that x and y are parallel above p and so are $f(x)$ and $f(y)$ by Remark 3.8. By Proposition 3.19 and Remark 3.5, this means that the arrow $f(\langle x, y\rangle) : \mathbb{S}' \to Y$ may be factorised by some $\beta : \mathbb{S}' \to \mathbb{D}' \in \Omega$. Thus, there exists an arrow $h : \mathbb{D}' \to Y$ making the following diagram commute.

$$\begin{array}{ccc} \mathbb{S}' & \xrightarrow{\langle x,y\rangle} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D}' & \xrightarrow[h]{} & Y \end{array}$$

Because f is an intraction for $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$, there exists a stem $\overline{\beta} : \overline{\mathbb{S}}' \to \overline{\mathbb{D}}' \in \overline{\Omega}$ factorising the arrow $\langle x, y\rangle \circ \varkappa' : \overline{\mathbb{S}}' \to X$. By Proposition 3.19 and Remark 3.16, this exactly means that the relation $\varrho \cdot x \sim_{\overline{\nu}} \varrho' \cdot y$ holds in X. Conversely, suppose that the property of the statement holds and consider some stem $\beta : \mathbb{S}' \to \mathbb{D}' \in \Omega$ for which the lefthand diagram, below, commutes. By universality of $\mathbb{S}'$, notice that this commutative square may be rewritten into the following right one as the equality $z = \langle \delta_2 \cdot z, \delta_1 \cdot z\rangle$ holds (see Remark 3.6).

$$\begin{array}{ccc} \mathbb{S}' & \xrightarrow{z} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D}' & \xrightarrow[h]{} & Y \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathbb{S}' & \xrightarrow{\langle \delta_2\cdot z, \delta_1\cdot z\rangle} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D}' & \xrightarrow[h]{} & Y \end{array}$$

By Remark 3.5 and Proposition 3.19, this last diagram exactly means that the relation $f(\delta_2 \cdot z) \sim_\nu f(\delta_1 \cdot z)$ holds in Y. Now, using the property of the statement, we deduce that the relation $(\varrho\delta_2) \cdot z \sim_{\overline{\nu}} (\varrho'\delta_1) \cdot z$ holds in X. By Proposition 3.19 and Remark 3.16, this last relation implies a factorisation of the composite $z \circ \varkappa' : \overline{\mathbb{S}}' \to X$ by a stem $\overline{\beta} \in \overline{\Omega}$. This finally proves that f is simple with respect to the scale $(\Omega, \varkappa', \overline{\Omega})$ and completes the proof of the statement. □

Remark 3.21. It follows from Proposition 3.15 and Remark 3.8 that any alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ and morphism $f : X \to Y$ in $\mathcal{C}$ induce a commutative square of metafunctions as follows for every pair $x \in \mathcal{C}(\mathbb{D}_2, X)$ and $y \in \mathcal{C}(\mathbb{D}_1, X)$.

(3.9)
$$\begin{tikzcd}
\mathcal{C}(\nu, X)(x, y) \arrow[r, "\mathcal{C}(\mathfrak{a},X)"] \arrow[d, "f"'] & \mathcal{C}(\overline{\nu}, X)(\varrho' \cdot x, \varrho' \cdot y) \arrow[d, "f"] \\
\mathcal{C}(\nu, Y)(f(x), f(y)) \arrow[r, "\mathcal{C}(\mathfrak{a},X)"'] & \mathcal{C}(\overline{\nu}, Y)(\varrho' \cdot f(x), \varrho' \cdot f(y))
\end{tikzcd}$$

Proposition 3.20 then says that the morphism f is an intraction for $\mathfrak{a}$ if and only if for every parallel pair $x \in \mathcal{C}(\mathbb{D}_2, X)$ and $y \in \mathcal{C}(\mathbb{D}_1, X)$ above p, if the bottom left class of diagram (3.9) is non-empty, then so is the right top class.

Proposition 3.22. *A morphism $f : X \to Y$ is a surtraction for an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ if and only if it satisfies the following property:*

For every pair $x \sim_\gamma f(z)$ in Y, there exists
a pair $\varkappa \cdot z \sim^{\overline{\gamma}'} y$ in X such that $\varrho \cdot x \sim_{\overline{\nu}} f(y)$ holds in Y.

Proof. Suppose $f : X \to Y$ is a surtraction for ν. We are going to prove that the property holds. By definition of the relation $x \sim_\gamma f(z)$ in Y, the following leftmost diagram must commute. Because f is an surtraction for $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$, it is divisible by the underlying besom of $\mathfrak{n}$. This means that there exist a stem $\overline{\beta} \in \overline{\Omega}$ and two arrows $y : \overline{\mathbb{D}}_1 \to X$ and $h : \overline{\mathbb{D}}' \to Y$ factorising the left diagram into the right one. In particular, this second diagram is equivalent to stating that the relations $\varkappa \cdot z \sim^{\overline{\gamma}'} y$ and $\varrho \cdot x \sim_{\overline{\nu}} f(y)$ hold in X and Y, respectively.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "z"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
\mathbb{D}_2 \arrow[r, "x"] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] \arrow[rr, bend left, "z \circ \varkappa"] & \overline{\mathbb{D}}_1 \arrow[r, "y"] \arrow[d, "\overline{\beta} \circ \overline{\delta}_1"] & X \arrow[d, "f"] \\
\overline{\mathbb{D}}_2 \arrow[r, "\overline{\beta} \circ \overline{\delta}_2"] \arrow[rr, bend right, "x \circ \varrho"'] & \overline{\mathbb{D}}' \arrow[r, "h"] & Y
\end{tikzcd}$$

Now, let us prove that when the property holds, the morphism $f : X \to Y$ is a surtraction for $\mathfrak{n}$. Consider the preceding lefthand commutative square. This diagram exactly says that the relation $x \sim_\gamma f(z)$ holds in Y. By assumption, it follows that there exists a relation $\varkappa \cdot z \sim^{\overline{\gamma}'} y$ in X such that $\varrho \cdot x \sim_{\overline{\nu}} f(y)$ holds in Y. But this exactly means that there exists some $\overline{\beta} : \overline{\mathbb{S}} \to \overline{\mathbb{D}}' \in \overline{\Omega}$ and a path $h : \overline{\mathbb{D}}' \to Y$ making the preceding right diagram commute. This proves that f is divisible by the underlying besom of $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$. □

3.2.1.6. *Projective structures.* Let $\mathcal{C}$ be a category. An object I in $\mathcal{C}$ will be said to be *projective* with respect to a morphism $g : X \to Y$ if for every arrow $u : I \to Y$ in $\mathcal{C}$, there exists a lift (not necessarily unique) as follows.

(3.10)
$$\begin{tikzcd}
& X \arrow[d, "g"] \\
I \arrow[r, "u"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Note that any initial object in $\mathcal{C}$ is projective with respect to every morphism of $\mathcal{C}$. Later on, a prevertebra (resp. vertebra; node of vertebrae) will be said to be *projective* with respect to a morphism $g : X \to Y$ in $\mathcal{C}$ if its domain is projective with respect to g in $\mathcal{C}$. For simplicity, the previous structure will also be said to be *g-projective*. The next proposition shows that projectivity may be induced by surtractions.

Proposition 3.23. *Suppose that $\mathcal{C}$ admits an initial object 0. An object I is projective with respect to an arrow $g : X \to Y$ if and only if g is a surtraction for the following extended (node of) vertebra(s).*

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d, "preseed"'] & 0 \arrow[r] \arrow[d, "seed"] & I \arrow[d, equal] \arrow[dl, phantom, "\ulcorner"] & \\
I \arrow[r, equal] & I \arrow[r, equal] & I \arrow[r, equal, "stem"] & I
\end{tikzcd}$$

Proof. Suppose that $g : X \to Y$ is a surtraction for the extended node of vertebrae given in the statement, say $\mathfrak{n}$. Start with a corner of arrows as the one given below on the left, which may also be seen as a commutative diagram from the initial object 0. Since g is divisible by the besom of $\mathfrak{n}$, there exists an arrow $h : I \to X$ making the middle diagram commute. This diagram is then equivalent to the one given below on the right, which shows that I is projective with respect to g.

$$\begin{tikzcd}
0 \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "g"] \\
I \arrow[r, "y"'] & Y
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
0 \arrow[r] \arrow[d, dashed] \arrow[rr, dashed, bend left] & I \arrow[r, "h"'] \arrow[d, equal] & X \arrow[d, "g"] \\
I \arrow[r, equal] \arrow[rr, bend right, "y"'] & I \arrow[r, "y"] & Y
\end{tikzcd}
\Leftrightarrow
\begin{tikzcd}
0 \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "g"] \\
I \arrow[r, "y"'] \arrow[ur, "h"] & Y
\end{tikzcd}$$

Now, suppose that I is projective with respect to g and let us prove the converse. Start with a commutative square as given on the above left, which aternatively may be seen as the (non-dashed) bottom right corner of the square. By projectivity, there exists a lift $h : I \to X$ making the above rightmost diagram commute, which is equivalent to giving the middle commutative diagram. This finally shows that g is divisible by the besom of $\mathfrak{n}$. $\square$

In the sequel, an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ will be said to be *g-projective* if the domain of its preseed γ is projective.

Proposition 3.24. *Let $g : X \to Y$ and $f : Y \to Z$ be two morphisms in $\mathcal{C}$ and $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ be a g-projective extended node of vertebrae. If $f \circ g$ is a surtraction for $\mathfrak{n}$, then f is a surtraction for $\mathfrak{n}$.*

Proof. We are going to use the characterisation of Proposition 3.22. Consider a pair $x \sim_{\gamma} f(z)$ in Z and let us show that there exists a pair $\varkappa \cdot z \sim^{\overline{\gamma}'} y$ for which the relation $\varrho \cdot x \sim_{\overline{\nu}} f(y)$ holds in Y. By definition, the element z is an arrow $\mathbb{S} \to Y$ in $\mathcal{C}$ where $\mathbb{S}$ is the domain of the preseed γ. By projectivity, there must exist an arrow (lift) $z' : \mathbb{S} \to X$ such that the equation $z = g \circ z' = g(z')$ holds. In other words, our very first relation may now be rewritten as $x \sim_{\gamma} f \circ g(z)$ in Z. Because $f \circ g$ is a surtraction for $\mathfrak{n}$, it follows from this relation and Proposition 3.22 that there exists a pair $\varkappa \cdot z' \sim^{\overline{\gamma}'} y$ in X such that the relation $\varrho \cdot x \sim_{\overline{\nu}} f \circ g(y')$ holds in Y. Remark 3.8 allows us to turn the former relation into the relation $\varkappa \cdot g(z') \sim^{\overline{\gamma}'} g(y')$ in Y, which also gives the relation $\varkappa \cdot z \sim^{\overline{\gamma}'} g(y')$ by the equality $z = g(z')$. This last relation together with the previously obtained relation $\varrho \cdot x \sim_{\overline{\nu}} f(g(y'))$ allows us to conclude by taking $y := g(y')$ and using the equivalence of Proposition 3.22. $\square$

3.2.1.7. *Over-parallelism and under-parallelism.* Let $\mathcal{C}$ be a category and $g : X \to Y$ be a morphism in $\mathcal{C}$. Consider a vertebra $v = \|\gamma, \gamma'\| \cdot \beta$ and a v-path of the form $h : x \sim_v y$ in the object Y. By Remark 3.9 and Proposition 3.18, the elements x and y are known to be parallel above $\|\gamma, \gamma'\|$. The elements x and y will be said to be *parallel over* $g : X \to Y$ if the dashed arrow displayed in the following diagram (denoted by z) factorises through $g : X \to Y$.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] \arrow[dr, dashed, "z"] & \mathbb{D}_1 \arrow[d, "y"] \\ \mathbb{D}_2 \arrow[r, "x"'] & Y \end{tikzcd}$$

Remark 3.25. In hom-language, this exactly means that the unique element z for which both relations $x \sim_\gamma z$ and $z \sim^{\gamma'} y$ hold (see Proposition 3.13) may be written as $z = g(z')$ where $z' \in \mathcal{C}(\mathbb{S}, X)$.

On the other hand, the elements x and y will be said to be *parallel under* $g : X \to Y$ if there exist two arrows $x' : \mathbb{D}_2 \to X$ and $y' : \mathbb{D}_1 \to X$ making the following diagram commute.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "y'"] \arrow[dr, bend left, "y"] & \\ \mathbb{D}_2 \arrow[r, "x'"] \arrow[rr, bend right, "x"'] & X \arrow[r, "g"] & Y \end{tikzcd}$$

Remark 3.26. In hom-language, this exactly means that there exists a parallel pair $x' \in \mathcal{C}(\mathbb{D}_2, X)$ and $y' \in \mathcal{C}(\mathbb{D}_1, X)$ above $\|\gamma, \gamma'\|$ such that $x = g(x')$ and $y = g(y')$. By Remark 3.5, this is equivalent to saying that the relation $\langle x, y\rangle = g(\langle x', y'\rangle)$ is well-defined and holds in $\mathcal{C}$.

Note that being over- and under-parallel above a vertebra $p \cdot \beta$ does not depend on the stem β. It was however felt useful to define such a notion for a general vertebra as the notion of over- and under-parallelism will mostly be used when dealing with general paths. The following two propositions are straightforward.

Proposition 3.27. *If x and y are parallel under g, then they are parallel over g.*

Proposition 3.28. *If v is g-projective, then x and y are parallel over g.*

3.3. Theory of spines

The theory of spines is a natural generalisation of the theory of vertebrae. We shall retrieve the same pattern of presention as in section 3.2.1 and Chapter 2.

3.3.1. Spines.

3.3.1.1. *Prespines.* Let $\mathcal{C}$ be a category and n be a non-negative integer. A *prespine of degree* n in $\mathcal{C}$ consists of a collection of $n+1$ prevertebrae $(p_k : \mathbb{S}_k \multimap \mathbb{S}'_k)_{0 \le k \le n}$ such that for every $0 \le k \le n-1$, the equality $\mathbb{S}'_k = \mathbb{S}_{k+1}$ holds in $\mathcal{C}$. We may thus think of a prespine as a finite sequence of prevertebrae as follows.

$$\mathbb{S}_0 \xrightarrow{p_0} \!\!\circ\, (\delta^0_1, \delta^0_2) \xrightarrow{p_1} \!\!\circ\, (\delta^1_1, \delta^1_2) \xrightarrow{p_2} \!\!\circ \dots \xrightarrow{p_n} \!\!\circ\, (\delta^n_1, \delta^n_2) \tag{3.11}$$

Later on, the prevertebrae p_0 and p_n will be called the *tail* and the *head* of the prespine, respectively. The role played by these last prevertebrae will turn out to be quite specific. For convenience, a prespine of the form $P = (p_k)_{0 \le k \le n}$ will later be defined by a declaration of the form $P = (p_k)$ and said to be *of degree* $n \ge 0$, implying that the indexing by k starts at 0 and stops at n. Besides, any other indexing notation on P will be transferred to the prevertebrae p_k and its attached structure; e.g. $P_* = (p^*_k : \mathbb{S}^*_k \multimap \mathbb{S}^{*\prime}_k)$.

Example 3.29. Prevertebrae in $\mathcal{C}$ are prespines in $\mathcal{C}$ of degrees 0 whose tails are equal to the heads. Later on, we will sometimes see a prevertebra as such.

Remark 3.30. A prespine as defined above implies two collections of n vertebrae of the form $p_k \cdot \gamma_{k+1}$ and $p_k \cdot \gamma'_{k+1}$ in the case where $p_{k+1} = \|\gamma_k, \gamma'_k\|$ for every $0 \leq k \leq n-1$. Note that the only structure of vertebra that may be put on the head is the trivial one $p_n \cdot \mathrm{id}_{\mathbb{S}'_n}$.

3.3.1.2. *Derived prespines.* Let $\mathcal{C}$ be a category and n be a positive integer. Consider a prespine $P = (p_k)$ of degree n in $\mathcal{C}$. For every $0 \leq d \leq n$, the prespine P induces a prespine $\partial^d P := (p_k)$ of degree $n-d$ as shown below.

$$\mathbb{S}_0 \xrightarrow{p_0} \!\circ\, (\delta_1^0, \delta_2^0) \xrightarrow{p_1} \!\circ\, (\delta_1^1, \delta_2^1) \xrightarrow{p_2} \!\circ \dots \xrightarrow{p_{n-d}} \!\circ\, (\delta_1^{n-d}, \delta_2^{n-d})$$

This prespine will later be called the *d-th derived prespine* of P.

3.3.1.3. *Elementary and central cords.* Let $\mathcal{C}$ be a category and $P = (p_k)$ be a prespine of degree $n \geq 0$ in $\mathcal{C}$. We shall consider the following usual notations.

$$p_k = \|\gamma_k, \gamma'_k : \mathbb{S}'_k\| : \mathbb{S}_k \multimap (\delta_1^k, \delta_2^k) \tag{3.12}$$

We will call *k-th elementary cord of P*, for every $0 \leq k \leq n$, the morphism defined by the composite $\mathrm{cd}_k(P) = \delta_2^k \circ \gamma_k = \delta_1^k \circ \gamma'_k$, thus producing an arrow $\mathbb{S}_k \to \mathbb{S}'_k$ in $\mathcal{C}$. Then, the $(k-1)$*-th central cord* will denominate the composite of all the cords of P from the k-th cord to the n-th cord for every non-negative integer k as follows.

$$\Gamma_{k-1}(P) := \begin{cases} \mathrm{id}_{\mathbb{S}'_n} & \text{if } k > n; \\ \Gamma_k(P) \circ \mathrm{cd}_k(P) & \text{if } 0 \leq k \leq n. \end{cases}$$

The composite $\Gamma_k(P)$ is hence an arrow $\mathbb{S}'_k \to \mathbb{S}'_n$ in $\mathcal{C}$ for every $0 \leq k \leq n$. It is not difficult to see that the relation $\Gamma_k(P) = \Gamma_r(P) \circ \Gamma_k(\partial^{n-r} P)$ holds for every $-1 \leq k \leq r \leq n$.

3.3.1.4. *Hom-language for prespines.* Let $\mathcal{C}$ be a category and X be some object in $\mathcal{C}$. Consider a prespine $P = (p_k)$ of degree $n \geq 0$ with the notations of diagram (3.12). A pair of two elements $x \in \mathcal{C}(\mathbb{D}_2^n, X)$ and $y \in \mathcal{C}(\mathbb{D}_1^n, X)$ will be said to be *parallel in X above P* if they are parallel above the head of P. In other words, the relation $x \sim_{p_n} y$ holds in X. The rest of the section introduces a way of presenting such a parallel pair, which will turn out to be much needed in the sequel. First, notice that, by Proposition 3.13, there exists a unique $z \in \mathcal{C}(\mathbb{S}_n, X)$ such that the two relations $x \sim_{\gamma_n} z$ and $z \sim^{\gamma'_n} y$ hold in X. If the integer n is zero, then not much is actually to be said, so suppose that the inequality $n > 0$ holds. Since the equality $\mathbb{S}_n = \mathbb{S}'_{n-1}$ holds in this case, Remark 3.6 enables one to see the element z as a pushout $\langle x_{n-1}, y_{n-1} \rangle$, for which Proposition 3.10 implies a relation $z : x_{n-1} \sim_{p_{n-1}} y_{n-1}$ and allows one to repeat the previous process. One thus inductively defines two collections of elements $x_n, \dots, x_0$ and $y_n, \dots, y_0$ as follows:

$$\begin{cases} x_n := x \text{ and } y_n := y; \\ z_k : x_k \sim_{p_k} y_k \text{ where } x_{k+1} \sim_{\gamma_{k+1}} z_k \text{ and } z_k \sim^{\gamma'_{k+1}} y_{k+1}. \end{cases}$$

In particular, every element $z_k : \mathbb{S}'_k \to X$ may be identifed with the pushout $\langle x_k, y_k \rangle$ for every $0 \leq k \leq n$. It is not hard to see that both elements $x \in \mathcal{C}(\mathbb{D}_2^n, X)$ and $y \in \mathcal{C}(\mathbb{D}_1^n, X)$ entirely determine the previous two sequences via the relations

$$(\delta_2^k \Gamma_k(\partial P) \gamma_n) \cdot x = x_k \qquad \text{and} \qquad (\delta_1^k \Gamma_k(\partial P) \gamma'_n) \cdot y = y_k$$

where $0 \leq k \leq n-1$. It will thus make sense to identify the elements x and y with the vectors $[x_n, \dots, x_0]$ and $[y_n, \dots, y_0]$, respectively. For simplicity, we will also adopt the notations $x := [x_k]_n$ and $y := [y_k]_n$. In addition, for every morphism $f : X \to Y$ in $\mathcal{C}$ and element $x = [x_k]_n \in \mathcal{C}(\mathbb{D}_2^n, X)$, the element $f(x) = [f(x_k)]_n \in \mathcal{C}(\mathbb{D}_2^n, Y)$ will be denoted by $f[x_k]_n$.

Proposition 3.31. *For every $k \geq 0$, the element x_k is parallel to y_k above p_k.*

Proof. Since x and y are parallel above p, the following series of equalities must hold: $\gamma_k \cdot x_k = (\Gamma_{k-1}(\partial P)\gamma_n) \cdot x = (\Gamma_{k-1}(\partial P)\gamma'_n) \cdot y = \gamma'_k \cdot y$. □

3.3.1.5. *Spines.* Let $\mathcal{C}$ be a category and n be a non-negative integer. A *spine* of degree n in $\mathcal{C}$ consists of a prespine $P = (p_k : \mathbb{S}_k \multimap \mathbb{S}'_k)$ of degree n and a further morphism $\beta : \mathbb{S}'_n \to \mathbb{D}'_n$ as follows.

$$\mathbb{S}_0 \overset{p_0}{\multimap} (\delta_1^0, \delta_2^0) \overset{p_1}{\multimap} (\delta_1^1, \delta_2^1) \overset{p_2}{\multimap} \dots \overset{p_n}{\multimap} \mathbb{S}'_n \xrightarrow{\beta} \mathbb{D}' \tag{3.13}$$

The arrow β will be called the *stem* of the spine. Note that this further morphism turns the head p_n of P into a vertebra $p_n \cdot \beta$. Later on, the above spine will be denoted by the symbols $P \cdot \beta$. The prespine P will sometimes be referred to as the *base* of $P \cdot \beta$.

Example 3.32. Every prespine P of degree $n \geq 0$ of the form (3.11) induces a spine of degree n when equipped with the identity morphism $\mathbb{S}'_n \to \mathbb{S}'_n$.

Example 3.33. Every prespine $P = (\|\gamma_k, \gamma'_k\|)_{0 \leq k \leq n}$ of degree $n \geq 1$ gives rise to two canonical spines of degree $n-1$ whose prespines are the derived prespine ∂P and the further morphisms are either the seed or the coseed of the head of P.

$$\begin{tikzcd}
 & & & & & \mathbb{D}_1^n \\
\mathbb{S}_0 \arrow[r, "p_0", -o] & (\delta_1^0, \delta_2^0) \arrow[r, "p_1", -o] & (\delta_1^1, \delta_2^1) \arrow[r, "p_2", -o] \arrow[rr, bend left, "\Gamma_1(\partial P)"] & \dots \arrow[r, "p_{n-1}", -o] & \mathbb{S}'_{n-1} \arrow[ur, "\gamma'_n"] \arrow[dr, "\gamma_n"'] & \\
 & & & & & \mathbb{D}_2^n
\end{tikzcd}$$

More generally, one may keep doing this to produce all sorts of spines of degree $n-d$, where $1 \leq d \leq n$, whose prespines are the derived prespine $\partial^d P$ and whose stems may be chosen among the composite morphisms $\gamma_k \circ \Gamma_{n-d}(\partial^{n-k+1} P)$ and $\gamma'_k \circ \Gamma_{n-d}(\partial^{n-k+1} P)$ for every $k \geq n-d+1$.

3.3.1.6. *Hom-language for spines.* Let $\mathcal{C}$ be a category and X be some object in $\mathcal{C}$. Consider a spine $s = P \cdot \beta$ of degree $n \geq 0$ in $\mathcal{C}$ as follows.

$$\begin{tikzcd}
\mathbb{S}_0 \arrow[r, "p_0", -o] & (\delta_1^0, \delta_2^0) \arrow[r, "p_1", -o] & (\delta_1^1, \delta_2^1) \arrow[r, "p_2", -o] \arrow[rr, bend left, "\Gamma_1(P)"] & \dots \arrow[r, "p_n", -o] & \mathbb{S}'_n \arrow[r, "\beta"'] & \mathbb{D}'
\end{tikzcd}$$

For every element $h \in \mathcal{C}(\mathbb{D}', X)$, we will write $h : x \sim_s y$ if and only if the relation $h : x \sim_{p_n \cdot \beta} y$ holds. In other words, the equations $(\delta_2^n \beta) \cdot h = x$ and $(\delta_1^n \beta) \cdot h = y$ must be satisfied in $\mathcal{C}$. The element h will then be said to be an *s-path from x to y* in X where the elements x and y will be called *source* and *target* as in the case of vertebrae. By Proposition 3.18, the relation $h : x \sim_s y$ implies the relation $x \sim_{p_n} y$, which shows – by definition – that both elements x and y are parallel above the prespine P. If the notations of section 3.3.1.4 are used, the relation $h : x \sim_s y$ may be rewritten as $h : [x_k]_n \sim_s [y_k]_n$, which is equivalent to requiring the following identities to hold in $\mathcal{C}$ for every $0 \leq k \leq n$.

$$(\delta_2^k \Gamma_k(P) \beta) \cdot h = x_k \qquad \text{and} \qquad (\delta_1^k \Gamma_k(P) \beta) \cdot h = y_k \tag{3.14}$$

Note that every morphism $f : X \to Y$ in $\mathcal{C}$ sends a relation $h : [x_k]_n \sim_s [y_k]_n$ in X to a relation $f(h) : f[x_k]_n \sim_s f[y_k]_n$ in Y. For every pair of elements $x \in \mathcal{C}(\mathbb{D}_2^n, X)$ and $y \in \mathcal{C}(\mathbb{D}_1^n, X)$, we will write $x \sim_s y$ whenever there exists an s-path $h : x \sim_s y$ in X. The relation defined at the beginning of the section will be denoted by $\sim_P$ in the case of the spine $P \cdot \mathrm{id}_{\mathbb{S}'_n}$.

Proposition 3.34. *Let $x \in \mathcal{C}(\mathbb{D}_2^n, X)$ and $y \in \mathcal{C}(\mathbb{D}_1^n, X)$. Any relation $h : x \sim_s y$ in X implies a relation $x \sim_P y$ in X. It follows that x and y are parallel in X above P and the equality $\langle x, y \rangle = \beta \cdot h$ holds where $\langle x, y \rangle$ is defined above the head of P.*

Proof. Follows from Proposition 3.18, Remark 3.9 and Proposition 3.19. □

Although a different proof is given, the following statement may be seen as an application of Proposition 3.15 to the obvious alliance of vertebrae $p_n \cdot \beta \rightsquigarrow p_k \cdot (\Gamma_k(P)\beta)$ generated by the arrows defining the prespine P.

Proposition 3.35. *Consider an s-path $h : [x_k]_n \sim_s [y_k]_n$. For every $k \geq 0$, the element x_k is homotopic to y_k above the vertebra $v_k := p_k \cdot (\Gamma_k(P)\beta)$ via the path $h \in \mathcal{C}(\mathbb{D}', X)$, that is to say $h : x_k \sim_{v_k} y_k$.*

Proof. Follows from the equations of (3.14). □

Remark 3.36. Proposition 3.35 may be regarded as a forgetful result in the sense that, for some chosen $0 \leq k \leq n$, a relation of the form $h : a \sim_{v_k} b$ (as considered at the end of the proposition) induces a relation of the form $h : x' \sim_s y'$ where x' and y' can *a priori* only be described by equations of the following form.

$$x' = [?, \dots, ?, a_k, a_{k-1}, \dots, a_0] \qquad y' = [?, \dots, ?, b_k, b_{k-1}, \dots, b_0]$$

In other words, only the last $k+1$ components of x' and y' can be known from the information made available by the source and target $a = [a_i]_k$ and $b = [b_i]_k$.

3.3.1.7. *Nodes of spines.* Let $\mathcal{C}$ be a category and n be a non-negative integer. A *node of spines* of degree n in $\mathcal{C}$ is equivalently

1) a class of spines of dregee n whose prespines are equal;
2) a prespine $P = (p_k)$ of degree n endowed with a class of morphisms Ω such that the domain of every element in Ω is equal to the codomain of the prevertebra p_n.

Note that the class Ω turns the head of P into a node of vertebrae $p_n \cdot \Omega$. Later on, the above node of spines will be denoted by $P \cdot \Omega$. The prespine P will sometimes be referred to as the *base* of $P \cdot \Omega$.

Example 3.37. Every spine defines an obvious node of spines containing itself only.

In the case where n is positive and p_n is of the form $\|\gamma_n, \gamma'_n\|$, the node of spines $P \cdot \Omega$ generates two spines $\partial P \cdot \gamma_n$ and $\partial P \cdot \gamma'_n$ of degrees $n-1$, which will later be called the *spinal seed* and *spinal coseed* of the node of spines $P \cdot \Omega$.

3.3.1.8. *Hom-language for nodes of spines.* Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. For every node of spines σ of degree $n \geq 0$, we will write the relation $h : x \sim_\sigma y$, which will be said to *hold in* X, if there exists a spine s in σ such that the relation $h : x \sim_s y$ holds in X. In that case, the element h will be called a *σ-path* and said to be *from x to y*. The class of σ-paths in X from x to y will be denoted by $\mathcal{C}(\sigma, X)(x, y)$. We will use the notation $x \sim_\sigma y$ to mean that there exists some σ-paths $h \in \mathcal{C}(\sigma, X)(x, y)$. In this case, the element x will be said to be *σ-homotopic* to the element y.

Proposition 3.38. *Suppose that $n > 0$ and denote the spinal seed and coseed of σ by s and s', respectively. Any path $h : x \sim_\sigma y$ in X for which the relation $y : r \sim_{s'} t$ holds in X implies the equalities $x_{n-1} = r$ and $y_{n-1} = t$. There then follows a relation $x : r \sim_s t$ in X.*

Proof. Consider the notation $P := (p_k)$ and suppose that the head of P is of the form $\|\gamma_n, \gamma'_n\|$. By Proposition 3.34, a relation $h : x \sim_\sigma y$ in Y implies a relation $x \sim_{p_n} y$ in Y. Since the inequality $n > 0$ holds, section 3.3.1.4 provides the relations $x \sim_{\gamma_n} \langle x_{n-1}, y_{n-1} \rangle$ and $\langle x_{n-1}, y_{n-1} \rangle \sim^{\gamma'_n} y$ in Y. Using Remark 3.12 and Proposition 3.14 on the latter relation leads to the relation $y : x_{n-1} \sim_{p_{n-1} \cdot \gamma'_n} y_{n-1}$. On the other hand, the path $y : r \sim_{\partial P \cdot \gamma'_n} t$ forces both equalities $r = x_{n-1}$ and $t = y_{n-1}$. Finally, using Proposition 3.14 on the former relation $x \sim_{\gamma_n} \langle x_{n-1}, y_{n-1} \rangle$ provides a path $x : r \sim_{p_{n-1} \cdot \gamma_n} t$ in Y, which proves the statement. □

Let $f : X \to Y$ be a morphism in $\mathcal{C}$, σ be a node of spines of positive degree n and s and s' denote the spinal seed and coseed of σ. The class of pairs (h, y) consisting of a s'-path $y : r \sim_{s'} t$ in X and a σ-path $h : \varrho \cdot x \sim_\sigma f(y)$ in Y will be denoted by $\mathcal{C}(\sigma, f)(r, t)$. It follows from Proposition 3.38 that there exists a metafunction of the form

$$R_\sigma : \mathcal{C}(\sigma, f)(r, t) \longrightarrow \mathcal{C}(s, Y)(f(r), f(t))$$

mapping a pair (h, y) as above to the s-path $x : f(r) \sim_s f(t)$.

3.3.1.9. *Projective structures.* Let $\mathcal{C}$ be a category. A prespine (resp. spine; node of spines) will be said to be *projective* with respect to a morphism $g : X \to Y$ in $\mathcal{C}$ if its tail is projective with respect to the g in $\mathcal{C}$. For simplicity, the previous structure will also be said to be *g-projective.*

3.3.1.10. *Over-parallelism and under-parallelism.* Let $\mathcal{C}$ be a category and $g : X \to Y$ be a morphism in $\mathcal{C}$. Consider a spine $s = P \cdot \beta$ of degree $n \geq 0$ and a s-path of the form $h : [x_k]_n \sim_s [y_k]_n$ in the object Y.

$$\mathbb{S}_0 \xrightarrow{p_0} \circ (\delta_1^0, \delta_2^0) \xrightarrow{p_1} \circ (\delta_1^1, \delta_2^1) \xrightarrow{p_2} \circ \dots \xrightarrow{p_n} \circ \mathbb{S}'_n \xrightarrow{\beta} \mathbb{D}'$$

By Proposition 3.31, the elements x_k and y_k are known to be parallel above $p_k = \|\gamma_k, \gamma'_k\|$ for every $k \geq 0$. The elements x and y will be said to be *k-parallel over* $g : X \to Y$ if the elements x_k and y_k are parallel over $g : X \to Y$.

Remark 3.39. By Remark 3.25, this exactly means that there exists some $z'_k \in \mathcal{C}(\mathbb{S}_k, X)$ such that $\langle x_{k-1}, y_{k-1} \rangle = g(z'_k)$ when $k > 0$.

On the other hand, the elements x and y will be said to be *k-parallel under* $g : X \to Y$ if the elements x_k and y_k are parallel under $g : X \to Y$.

Remark 3.40. By Remark 3.26, this exactly means that there exists a parellel pair $x'_k \in \mathcal{C}(\mathbb{D}_2, X)$ and $y'_k \in \mathcal{C}(\mathbb{D}_1, X)$ above p_k such that the relation $\langle x_k, y_k \rangle = g(\langle x'_k, y'_k \rangle)$ is well-defined and holds in $\mathcal{C}$.

Proposition 3.41. *Suppose that $k > 0$. The elements x and y are $(k-1)$-parallel under $g : X \to Y$ if and only if they are k-parallel over $g : X \to Y$.*

Proof. Follows from Remark 3.39, Remark 3.40 and Remark 3.6. □

Proposition 3.42. *If s is g-projective, then x and y are 0-parallel over g.*

Proof. Follows from Proposition 3.28 □

3.3.2. Framing of vertebrae and nodes of vertebrae.

3.3.2.1. *Categories for vertebral structures.* Let $\mathcal{C}$ be a category. A *morphism of vertebrae* (resp. *nodes of vertebrae*) in $\mathcal{C}$ is defined as an alliance of vertebrae (resp. *nodes of vertebrae*) in $\mathcal{C}$ whose spherical, discal, codiscal and hence cospherical transitions are identities in $\mathcal{C}$. The respective induced categories will be denoted by $\mathbf{Vert}(\mathcal{C})$ and $\mathbf{Nov}(\mathcal{C})$ and their arrows will be written with the symbol $\curvearrowright$. The notation associated with the structure of the above morphisms will follow that of their associated alliances, but for which the trivial data will be removed.

Remark 3.43. Any morphism $u : p \cdot \beta \curvearrowright p \cdot \beta_*$ in $\mathbf{Vert}(\mathcal{C})$ is equivalent to giving a factorisation of the form $\beta = u \circ \beta_*$. Any morphism $(\varphi, u) : p \cdot \Omega \curvearrowright p \cdot \Omega_*$ in $\mathbf{Nov}(\mathcal{C})$ is equivalent to giving a factorisation of every stem $\beta \in \Omega$ of the form $u \circ \varphi(\beta)$.

Remark 3.44. By Remark 3.17, any vertebra $p \cdot \beta$ in $\mathcal{C}$ gives rise to a morphism of vertebrae $\beta : p \cdot \beta \curvearrowright p \cdot \mathrm{id}$ in $\mathbf{Vert}(\mathcal{C})$.

Proposition 3.45. *Let X be an object in $\mathcal{C}$ and $u : v \curvearrowright \overline{v}$ be an arrow in* $\mathbf{Vert}(\mathcal{C})$*. The relation $h : x \sim_v y$ holds in X if and only if so does the relation $u \cdot h : x \sim_{\overline{v}} y$.*

Proof. Follows from Proposition 3.15. □

3.3.2.2. *Semi-extended structures.* In the sequel, the term *sep* (resp. *sev*; *senov*) will stand for *semi-extended prevertebra* (resp. *vertebra*; *node of vertebrae*). This term will refer to any extended prevertebra (resp. vertebra; node of vertebrae) whose spherical transition is an identity. Here, the prefix *semi-* refers to the fact that some part of the structure is trivial. This abbreviation is meant to avoid a too cumbersome phrasing that the whole name would imply. The usual notation $(\varkappa, \varrho)$ will be replaced with $(\mathbb{S}, \varrho)$ when the spherical transition $\varkappa$ is the identity on $\mathbb{S}$. The object $\mathbb{S}$ will be called the domain of the sev $(\mathbb{S}, \varrho)$. Contrary to the usual indexing on extended structures, the indexing notation of a semi-extended structure will usually follow the indexing notation of its associated prevertebra, vertebra or node of vertebrae; e.g. $(\mathbb{S}, \varrho_*) : \gamma \overset{\mathrm{ex}}{\leadsto} p_*$; $(\mathbb{S}, \varrho_*) \cdot \Omega_* : \gamma \overset{\mathrm{ex}}{\leadsto} p_* \cdot \Omega_*$.

3.3.2.3. *Framing of prevertebrae.* Let $\mathcal{C}$ be category, $p = \|\gamma, \gamma'\| : \mathbb{S} \multimap \mathbb{S}'$ be a prevertebra and $(\mathbb{S}, \varrho_\diamond) : \gamma \overset{\mathrm{ex}}{\leadsto} \|\gamma_\diamond, \gamma'_\diamond\|$ and $(\mathbb{S}, \varrho_\bullet) : \gamma' \overset{\mathrm{ex}}{\leadsto} \|\gamma_\bullet, \gamma'_\bullet\|$ be two seps in $\mathcal{C}$. Note that the arrows $\gamma'_\diamond$ and $\gamma'_\bullet$ must have same domain $\mathbb{S}$ in $\mathcal{C}$. A prevertebra will be said to *frame* the prevertebra p *along* the seps $(\mathbb{S}, \varrho_\diamond)$ and $(\mathbb{S}, \varrho_\bullet)$ if it is of the form $\|\gamma'_\diamond, \gamma'_\bullet\|$.

3.3.2.4. *Framing of vertebrae.* Let $\mathcal{C}$ be category and $p = \|\gamma, \gamma'\| : \mathbb{S} \multimap \mathbb{S}'$ be a prevertebra equipped with a framing $p_* = \|\gamma'_\diamond, \gamma'_\bullet\|$ along two seps $(\mathbb{S}, \varrho_\diamond) : \gamma \overset{\mathrm{ex}}{\leadsto} \|\gamma_\diamond, \gamma'_\diamond\|$ and $(\mathbb{S}, \varrho_\bullet) : \gamma' \overset{\mathrm{ex}}{\leadsto} \|\gamma_\bullet, \gamma'_\bullet\|$ in $\mathcal{C}$. A vertebra of the form $p_* \cdot \beta_*$ will be said to *frame* a vertebra of the form $p \cdot \beta$ along two sevs $(\mathbb{S}, \varrho_\diamond) \cdot \beta_\diamond$ and $(\mathbb{S}, \varrho_\bullet) \cdot \beta_\bullet$ if it is equipped with a double pushout[2]

(3.15)
$$\begin{tikzcd}
\mathbb{S}' \arrow[dr, "\beta"] & \mathbb{D}_2^\diamond \arrow[l, "\delta_2 \circ \varrho_\diamond"'] \arrow[r, "\beta_\diamond \circ \delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_2^\bullet \arrow[u, "\delta_1 \circ \varrho_\bullet"] \arrow[d, "\beta_\bullet \circ \delta_2^\bullet"'] & \mathbb{D}' \arrow[dr, "\iota"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"'] & & \mathbb{G}
\end{tikzcd}$$

and a morphism $\pi : \mathbb{D}'_* \to \mathbb{G}$, called *cylinder transition*, making the diagram

(3.16)
$$\begin{tikzcd}
\mathbb{S}'_* \arrow[dr, "\beta_*"] & \mathbb{D}_1^\diamond \arrow[l, "\delta_2^*"'] \arrow[r, "\beta_\diamond \circ \delta_1^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_1^\bullet \arrow[u, "\delta_1^*"] \arrow[d, "\beta_\bullet \circ \delta_1^\bullet"'] & \mathbb{D}'_* \arrow[dr, "\pi"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"'] & & \mathbb{G}
\end{tikzcd}$$

commute. Notice the analogy with framings of extended nodes of vertebrae defined in section 2.3.6.3 (page 70), which also required a pushout and a further morphism making a certain diagram commute. The above structure will later be denoted by $(p \cdot \beta, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright p_* \cdot \beta_*$ when the sevs induced by $(\mathbb{S}, \varrho_\diamond) \cdot \beta_\diamond$ and $(\mathbb{S}, \varrho_\bullet) \cdot \beta_\bullet$ are denoted by $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$, respectively.

[2]The universal cocone is given by the arrows $\iota^\diamond$, ι and $\iota^\bullet$ while the diagram of the colimit is given by the rest of the diagram. Topologically, such a pushout construction should be though of as a sort double mapping cylinder, which turns out to be an actual topological cylinder for the vertebrae of Example 2.4.2.1. This explains the term 'cylinder transition' used afterwards.

3.3.2.5. *Hom-language for framings of vertebrae.* The goal of this section is to translate the notion of framing of vertebrae introduced in section 3.3.2.4 into the hom-language. We shall refer to the diagrams of this section. Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $\mathfrak{f} := (v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_*$ be a framing of vertebrae in $\mathcal{C}$ where $\mathfrak{v}_\diamond : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet$. Suppose given the following set of paths

$$g_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a \qquad h : x \sim_v y \quad \text{and} \quad g_\bullet : \varrho_\bullet \cdot y \sim_{v_\bullet} b$$

in X, where $\varrho_\diamond$ and $\varrho_\bullet$ denote the respective discal transitions of $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$. Note that the previous data are equivalent to giving a commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{S}' \arrow[dr, "\beta"] & \mathbb{D}_2^\diamond \arrow[l, "\delta_2\circ\varrho_\diamond"'] \arrow[r, "\beta_\diamond\circ\delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "g_\diamond"] \\
\mathbb{D}_2^\bullet \arrow[u, "\delta_1\circ\varrho_\bullet"] \arrow[d, "\beta_\bullet\circ\delta_2^\bullet"'] & \mathbb{D}' \arrow[dr, "h"] & \\
\mathbb{D}'_\bullet \arrow[rr, "g_\bullet"'] & & X
\end{tikzcd}$$

By universality of the pushout of diagram (3.15), it follows that there exists a canonical morphism $g_\diamond \star h \star g_\bullet : \mathbb{G} \to X$ making the following diagrams commute.

(3.17)
$$\begin{tikzcd}
\mathbb{D}'_\diamond \arrow[r, "g_\diamond"] \arrow[dr, "\iota_\diamond"'] & X \\
 & \mathbb{G} \arrow[u, dashed, "g_\diamond\star h\star g_\bullet"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}' \arrow[r, "h"] \arrow[dr, "\iota"'] & X \\
 & \mathbb{G} \arrow[u, dashed, "g_\diamond\star h\star g_\bullet"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'_\bullet \arrow[r, "g_\bullet"] \arrow[dr, "\iota_\bullet"'] & X \\
 & \mathbb{G} \arrow[u, dashed, "g_\diamond\star h\star g_\bullet"']
\end{tikzcd}$$

Now, if we denote by $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}}$ the composite morphism $(g_\diamond \star h \star g_\bullet) \circ \pi : \mathbb{D}'_* \to X$, then diagram (3.16) exactly provides a relation of the form $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}} : a \sim_{v_*} b$. Later on, if the framing structure is obvious or makes notations too cumbersome, the notation $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}}$ will be shortened to $\lceil g_\diamond h g_\bullet \rceil$.

Remark 3.46. For every morphism $f : X \to Y$ in $\mathcal{C}$, the path $f(\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}})$ may be identified with the path $\lceil f(g_\diamond) f(h) f(g_\bullet) \rceil_{\mathfrak{f}}$, which is of the form $f(a) \sim_{v_*} f(b)$.

3.3.2.6. *Framings of nodes of vertebrae.* Let $\mathcal{C}$ be category and p be a prevertebra framed by another prevertebra p_* along two sevs $\mathfrak{p}_\diamond : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_\diamond$ and $\mathfrak{p}_\bullet : \gamma' \overset{\mathrm{ex}}{\rightsquigarrow} p_\bullet$ in $\mathcal{C}$. A node of vertebrae of the form $p_* \cdot \Omega_*$ will be said to *frame* a node of vertebrae of the form $p \cdot \Omega$ along two sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ if it is equipped with

1) a metafunction $\psi : \Omega \to \Omega_*$, called its *framing gear*;
2) a framing of vertebrae $(p \cdot \beta, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright p_* \cdot \psi(\beta)$ for every stem $\beta \in \Omega$.

The above structure will later be denoted as $(p \cdot \Omega, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright p_* \cdot \Omega_*$. The framing gear here suggests the following functional notation: if the previous framing of nodes of vertebrae is given a name, say $\mathfrak{f}$, the notation $\mathfrak{f}(\beta)$ will denote the associated framing of vertebrae $(p \cdot \beta, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright p_* \cdot \psi(\beta)$ for the stem $\beta \in \Omega$.

3.3.2.7. *Hom-language for framings of nodes of vertebrae.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $\mathfrak{f} := (\nu, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \nu_*$ be a framing of nodes of vertebrae in $\mathcal{C}$ where $\mathfrak{v}_\diamond : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma' \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet$. The definition of a framing of nodes of vertebrae ensures that any set of paths of the form

$$g_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a \qquad h : x \sim_\nu y \qquad g_\bullet : \varrho_\bullet \cdot y \sim_{v_\bullet} b$$

in an object X of $\mathcal{C}$ gives rise to a (ν_*)-path[3] of the form $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}(\beta)} : a \sim_{\nu_*} b$ in X for some $\beta \in \Omega$. For convenience, a path of the form $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}(\beta)}$ will later be denoted by $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}}$. Note that every pair of paths $e_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a$ and $e_\bullet : \varrho_\bullet \cdot y \sim_{v_\bullet} b$ gives rise to a metafunction

$$\mathcal{C}(\nu, X)(x, y) \to \mathcal{C}(\nu_*, X)(a, b)$$

mapping a ν-path $h : x \sim_v y$ to the (ν_*)-path $\lceil e_\diamond h e_\bullet \rceil_{\mathfrak{f}}$. Such a metafunction will later be denoted by $T_{e_\diamond}^{e_\bullet}$ and called the *tubular operator* of the framing $\mathfrak{f}$. Formally, the structure of a tubular operator such as the previous one will always comprise the associated pair of paths $e_\diamond$ and $e_\bullet$ in its data. The next proposition views the two sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ as senovs, thus allowing the use of the notion of surtraction.

Proposition 3.47. *Let $g : X \to Y$ be a surtraction for $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ and x and y be a parallel pair over g defined above the base of ν in Y. There exists a tubular operator $T_{e_\diamond}^{e_\bullet} : \mathcal{C}(\nu, Y)(x, y) \to \mathcal{C}(\nu_*, Y)(a, b)$ such that a and b are parallel under g.*

Proof. Let $h : x \sim_v y$ be a v-path in Y such that $v \in \nu$ and denote $v = p \cdot \beta$ where p is of the form $\|\gamma, \gamma'\|$. Since x and y are parallel over the morphism $g : X \to Y$ above $p : \mathbb{S} \multimap \mathbb{S}'$ in Y, Remark 3.25 and Remark 3.12 imply that there exists $z' \in \mathcal{C}(\mathbb{S}, X)$ such that the relations $x \sim_\gamma g(z')$ and $y \sim_{\gamma'} g(z')$ hold in Y. Since g is a surtraction for the senovs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$, Proposition 3.22 implies that there exist two paths $z' \sim^{\gamma'_\diamond} x'$ and $z' \sim^{\gamma'_\bullet} y'$ in X such that $\varrho_\diamond \cdot x \sim_{v_\diamond} g(x')$ and $\varrho_\bullet \cdot y \sim_{v_\bullet} g(y')$ hold in Y. First, the last two relations involves the existence of a tubular operator

$$T_{e_\diamond}^{e_\bullet} : \mathcal{C}(\nu, Y)(x, y) \to \mathcal{C}(\nu_*, Y)(g(x'), g(y'))$$

where $e_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} g(x')$ and $e_\bullet : \varrho_\bullet \cdot y \sim_{v_\bullet} g(y')$. Second, the relations given by $z' \sim^{\gamma'_\diamond} x'$ and $z' \sim^{\gamma'_\bullet} y'$ imply, by Proposition 3.13 and Remark 3.12, a relation $x' \sim_{p_*} y'$, where $p_* = \|\gamma'_\diamond, \gamma'_\bullet\|$ is the prevertebra of ν_*. This means, by Remark 3.9, that x' and y' are parallel above p_* and proves that the elements $a := g(x')$ and $b := g(y')$ are parallel under g by using Remark 3.26. □

3.3.2.8. *Morphisms of framings of vertebrae.* Let $\mathcal{C}$ be a category and consider two framings of vertebrae $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_*$ and $(v_\flat, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_\dagger$ in $\mathcal{C}$. Their respective cylinder transitions will be denoted by $\pi : \mathbb{D}'_* \to \mathbb{G}$ and $\pi_\flat : \mathbb{D}'_\dagger \to \mathbb{G}_\flat$. This section requires some additional conditions before introducing a notion of morphism between the two previous framings of vertebrae. First, because both framings have the same pair of semi-extended vertebrae, the vertebrae v and $v_\flat$ must have same base up to canonical isomorphism. Their base will in fact be considered the same, implying that they must have same pushout, say $\mathbb{S}'$. Similarly, the bases of the vertebrae v_* and $v_\dagger$ will be considered equal. Now, by definition, the two previous framings are equipped with two pushouts as follows.

$$\begin{tikzcd}
\mathbb{S}' \arrow[dr, "\beta"] & \mathbb{D}_2^\diamond \arrow[l, "\delta_2 \circ \varrho_\diamond"'] \arrow[r, "\beta_\diamond \circ \delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_2^\bullet \arrow[u, "\delta_1 \circ \varrho_\bullet"] \arrow[d, "\beta_\bullet \circ \delta_2^\bullet"'] & \mathbb{D}' \arrow[dr, "\iota"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"'] & & \mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}' \arrow[dr, "\beta_\flat"] & \mathbb{D}_2^\diamond \arrow[l, "\delta_2 \circ \varrho_\diamond"'] \arrow[r, "\beta_\diamond \circ \delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota_\flat^\diamond"] \\
\mathbb{D}_2^\bullet \arrow[u, "\delta_1 \circ \varrho_\bullet"] \arrow[d, "\beta_\bullet \circ \delta_2^\bullet"'] & \mathbb{D}'_\flat \arrow[dr, "\iota_\flat"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota_\flat^\bullet"'] & & \mathbb{G}_\flat
\end{tikzcd}$$

[3] Here, the brackets are to ease the reading of the prefixed node of vertebrae.

Interestingly, any morphism of vertebrae $u : p \cdot \beta \curvearrowright p \cdot \beta_\flat$ in $\mathcal{C}$ generates, by universality, a canonical arrow $\kappa(u) : \mathbb{G}_\flat \to \mathbb{G}$ making the following diagrams commute (see Remark 3.43).

(3.18)
$$
\begin{tikzcd}
\mathbb{D}'_\diamond \arrow[r, "\iota^\diamond"] \arrow[dr, "\iota_\flat^\diamond"'] & \mathbb{G} \\
& \mathbb{G}_\flat \arrow[u, dashed, "\kappa(u)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}' \arrow[r, "\iota \circ u"] \arrow[dr, "\iota_\flat"'] & \mathbb{G} \\
& \mathbb{G}_\flat \arrow[u, dashed, "\kappa(u)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'_\bullet \arrow[r, "\iota^\bullet"] \arrow[dr, "\iota_\flat^\bullet"'] & \mathbb{G}, \\
& \mathbb{G}_\flat \arrow[u, dashed, "\kappa(u)"']
\end{tikzcd}
$$

A *morphism of framings* from $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_*$ to $(v_\flat, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_\dagger$ consists of two morphisms $u : v \curvearrowright v_\flat$ and $u_* : v_* \curvearrowright v_\dagger$ in $\mathbf{Vert}(\mathcal{C})$ such that the universal arrow $\kappa(u) : \mathbb{G}_\flat \to \mathbb{G}$ makes the following diagram commute.

(3.19)
$$
\begin{tikzcd}
\mathbb{D}'_* \arrow[r, "\pi"] & \mathbb{G} \\
\mathbb{D}'_\dagger \arrow[u, "u_*"] \arrow[r, "\pi_\flat"'] & \mathbb{G}_\flat \arrow[u, "\kappa(u)"']
\end{tikzcd}
$$

The previous notion of morphism, which will be denoted as a pair (u, u_*) and use the symbol $\curvearrowright$ for its arrows, defines a category $\mathbf{Fov}(\mathcal{C})$ whose objects are the framings of vertebrae in $\mathcal{C}$.

Remark 3.48. Let $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_*$ be a framing in $\mathcal{C}$ whose cylinder transition is denoted by π. Then, any pair of isomorphisms $u : v \curvearrowright v_\flat$ and $u_* : v_* \curvearrowright v_\dagger$ (i.e. u and u_* are invertible in $\mathcal{C}$) gives rise to a framing $(v_\flat, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright s_\dagger$, whose cylinder transition is $\pi \circ u$, and an isomorphism $(u, u_*) : (v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_* \curvearrowright (v_\flat, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_\dagger$.

3.3.2.9. *Hom-language for morphisms of framings.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $(u, u_*) : \mathfrak{f} \curvearrowright \mathfrak{f}_\flat$ be a morphism in $\mathbf{Fov}(\mathcal{C})$ whose source and target will be encoded by framings of the form $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_*$ and $(v_\flat, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright v_\dagger$, respectively. Suppose given the set of paths

$$
(3.20) \qquad g_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a \qquad h : x \sim_v y \quad \text{and} \quad g_\bullet : \varrho_\bullet \cdot y \sim_{v_\bullet} b
$$

in X and its associated framing path $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}} : a \sim_{v_*} b$. According to Proposition 3.45, the morphism of vertebrae $u : v_\flat \curvearrowright v$ turns the v-path $h : x \sim_v y$ into a $v_\flat$-path $u \cdot h : x \sim_{s_\flat} y$. Thus, we obtain another set of paths

$$
(3.21) \qquad g_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a \qquad u \cdot h : x \sim_{v_\flat} y \quad \text{and} \quad g_\bullet : \varrho_\bullet \cdot y \sim_{v_\bullet} b
$$

that admits a framing path of the form $\lceil g_\diamond (u \cdot h) g_\bullet \rceil_{\mathfrak{f}_\flat} : a \sim_{v_\dagger} b$. The definition of a morphism of framings of vertebrae then implies the following equality.

$$
\lceil g_\diamond (u \cdot h) g_\bullet \rceil_{\mathfrak{f}_\flat} = u_* \cdot \lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}}
$$

Remark 3.49. Precisely, the above equation follows from the equality

$$
g_\diamond \star (u \cdot h) \star g_\bullet = \kappa(u) \cdot (g_\diamond \star h \star g_\bullet),
$$

which is induced by universality, on comparing the two canonical morphisms $g_\diamond \star (u \cdot h) \star g_\bullet : \mathbb{D}'_\flat \to X$ and $g_\diamond \star h \star g_\bullet : \mathbb{D}' \to X$ via $\kappa(u)$ (see section 3.3.2.5).

3.3.3. Framing for nodes of spines and extended nodes of spines.

3.3.3.1. *From spines to vertebrae.* Let $\mathcal{C}$ be a cateogry and $s = P \cdot \beta$ be a spine of degree $n \geq 0$ in $\mathcal{C}$ with $P = (p_k)$. For every $0 \leq k \leq n$, the structure of s gives rise to the following leftmost diagram in $\mathbf{Vert}(\mathcal{C})$. When $n > 0$, it also induces the corresponding righthand commutative diagram for every $0 \leq k \leq n$.

$$p_k \cdot (\Gamma_k(P)\beta) \xrightarrow{\beta} p_k \cdot \Gamma_k(P)$$

$$\begin{tikzcd} p_k \cdot (\Gamma_k(\partial P)\gamma'_n) \arrow[r, "\gamma'_n"] & p_k \cdot \Gamma_k(\partial P) \\ p_k \cdot (\Gamma_k(P)\beta) \arrow[u, "\beta\circ\delta_1"] \arrow[r, "\beta\circ\delta_2"] & p_k \cdot (\Gamma_k(\partial P)\gamma_n) \arrow[u, "\gamma_n"'] \end{tikzcd}$$

The diagram on the left may be described as a functor $V_s^k : \mathtt{I} \to \mathbf{Vert}(\mathcal{C})$ where $\mathtt{I}$ is the small category consisting of a unique arrow and two objects[4]. The diagram on the right may be seen as a functor $E_s^k : \mathtt{J} \to \mathbf{Vert}(\mathcal{C})$ where $\mathtt{J}$ is a small category consisting of a commutative square of four arrows and four objects. Note that $\mathtt{J}$ is isomorphic to the product category $\mathtt{I} \times \mathtt{I}$, thus providing four inclusions $\mathtt{I} \hookrightarrow \mathtt{J}$. In the sequel, we will denote by $i_{\mathtt{I},\mathtt{J}}$ the inclusion $\mathtt{I} \hookrightarrow \mathtt{J}$ restricting the functor E_s^k to $V_{P\cdot\gamma_n}^k$ (see above diagrams). Throughout this paper, the category $\mathtt{I}$ will be encoded as a category of the form $\{0 \to 1\}$.

3.3.3.2. *From nodes of spines to nodes of vertebrae.* Let $\mathcal{C}$ be a cateogry and σ be a spine. In the sequel, for every $0 \leq k \leq n$, we will denote by $V_\sigma^k(0)$ and $V_\sigma^k(1)$ the nodes of vertebrae encoded by the collections $\{V_s^k(0)\}_{s\in\sigma}$ and $\{V_s^k(1)\}_{s\in\sigma}$, respectively.

3.3.3.3. *Compatibility of prespines.* Let $\mathcal{C}$ be a category and $P = (p_k)$ and $P_* = (p_k^*)$ be two prespines in $\mathcal{C}$ of degrees $n \geq 0$ and $m \geq 0$, respectively. For every non-negative integer q, the two prespines P and P_* will be said to be *q-compatible* if the equality $p_k = p_k^*$ holds for every $0 \leq k \leq q-1$ provided that the prevertebrae p_k and p_k^* are well-defined. In particular, when the two inequalities $q \leq n+1$ and $q \leq m+1$ hold, this is equivalent to requiring the equation $\partial^{n-q+1}P = \partial^{m-q+1}P_*$.

3.3.3.4. *Framing of prespines.* Let $\mathcal{C}$ be a category and $P = (p_k)$ and $P_* = (p_k^*)$ be two prespines in $\mathcal{C}$ of non-negative degrees. Consider a non-negative integer q less than or equal to the degrees of P and P_* and suppose that the prevertebra p_q is of the form $\|\gamma_q, \gamma'_q\|$. The prespine P will be said to *frame* the prespine P_* *at rank q along* a pair of seps $\mathfrak{p}_\diamond : \gamma_q \overset{\mathrm{ex}}{\rightsquigarrow} p_\diamond$ and $\mathfrak{p}_\bullet : \gamma'_q \overset{\mathrm{ex}}{\rightsquigarrow} p_\bullet$ if

1) the prespine P is q-compatible with P_*;
2) the prevertebra p_q frames the prevertebra p_q^* along $\mathfrak{p}_\diamond$ and $\mathfrak{p}_\bullet$.

The above structure will be denoted as $(P, \mathfrak{p}_\diamond, \mathfrak{p}_\bullet) \triangleright_q P_*$ and called a *q-framing of prespines.*

3.3.3.5. *Framing of functors.* Let $\mathcal{C}$ be a category, L be a connected small category and $A : L \to \mathbf{Vert}(\mathcal{C})$ and $A_* : L \to \mathbf{Vert}(\mathcal{C})$ be two functors. The functor A will be said to *frame* the functor A_* along two sevs of the form $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ if it is equipped with a functor $L \to \mathbf{Fov}(\mathcal{C})$ with the following mapping rules on objects and arrows.

$$d \mapsto (A(d), \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright A_*(d) \qquad (t : d \to d') \mapsto (A(t), A_*(t))$$

Such a structure will be denoted by $(A, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright A_*$ and said to be *defined over L.*

Remark 3.50. The use of the same pair of sevs for the framings of vertebrae is ensured by the fact that L is connected, thereby implying that the base of all vertebrae in the image of A are equal by definition of a morphism in $\mathbf{Fov}(\mathcal{C})$.

Definition 3.51 (Simple framing)**.** Let $q \geq 0$ and $(P, \mathfrak{p}_\diamond, \mathfrak{p}_\bullet) \triangleright_q P_*$ be a q-framing of prespines. A spine $s = P\cdot\beta$ will be said to *simply frame* a spine $s_* = P_*\cdot\beta_*$ *at rank q along* two sevs of the form $\mathfrak{v}_\diamond = \mathfrak{p}_\diamond \cdot \beta_\diamond$ and $\mathfrak{v}_\bullet = \mathfrak{p}_\bullet \cdot \beta_\bullet$ if it is equipped with a framing of functors $(V_s^q, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright V_{s_*}^q$ over $\mathtt{I}$. Such a structure will later be denoted by $(s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q^V s_*$ and called a *simple q-framing of spines* in $\mathcal{C}$.

[4] The symbol $\mathtt{I}$ is deliberately chosen in place of $\mathbf{2}$ to enhance the fact $\mathtt{I}$ specifically refers to the domain of V_s^k.

Definition 3.52 (Extensive framing)**.** Let $q \geq 0$ and $(P, \mathfrak{p}_\diamond, \mathfrak{p}_\bullet) \triangleright_q P_*$ be a q-framing of prespines. A spine $s = P \cdot \beta$ will be said to *extensively frame* a spine $s_* = P_* \cdot \beta_*$ *at rank* q *along* two sevs of the form $\mathfrak{v}_\diamond = \mathfrak{p}_\diamond \cdot \beta_\diamond$ and $\mathfrak{v}_\bullet = \mathfrak{p}_\bullet \cdot \beta_\bullet$ if it is equipped with a framing of functors $(E^q_s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright E^q_{s_*}$ over $\mathtt{J}$. Such a structure will be denoted by $(s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^E_q s_*$ and called an *extensive q-framing of spines* in $\mathcal{C}$. Note that restricting the previous framing along $i_{\mathtt{I},\mathtt{J}}$ provides a simple q-framing of spines $(P \cdot \gamma_n, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^V_q P_* \cdot \gamma^*_n$ (see section 3.3.3.1).

3.3.3.6. *Hom-language for simple framings.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $(s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^V_q s_*$ be a simple q-framing of spines as defined in Definition 3.51 where $\mathfrak{v}_\diamond : \gamma_q \overset{\text{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma'_q \overset{\text{ex}}{\rightsquigarrow} v_\bullet$. Suppose given the set of paths

$$g_\diamond : \varrho_\diamond \cdot x_q \sim_{v_\diamond} a \qquad h : [x_k]_n \sim_s [y_k]_n \quad \text{and} \quad g_\bullet : \varrho_\bullet \cdot y_q \sim_{v_\bullet} b$$

in X, where $\varrho_\diamond$ and $\varrho_\bullet$ denote the discal transitions of the sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$. By Proposition 3.35 and definition of a q-framing, the previous set of paths gives rise to a $V^q_{s_*}(0)$-path $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ from a to b if we denote $\mathfrak{f}_q := (V^q_s(0), \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q V^q_{s_*}(0)$. By Remark 3.36, this element may also be seen as an (s_*)-path $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ whose source and target are of the respective form $[\dots, [a_i]_q]$ and $[\dots, [b_i]_q]$, where the first $m - q$ components of both source and target are unknown if s_* is of degree m. However, the definition of a q-framing of spines allows one to obtain more information about the source and target of the (s_*)-path $\lceil g_\diamond h g_\bullet \rceil$. By definition of a morphism of framings of spines in section 3.3.2.9, the following equality of $V^q_{\sigma_*}(1)$-paths must hold if we denote $\mathfrak{f}'_q := (V^q_s(1), \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q V^q_{s_*}(1)$.

$$\lceil g_\diamond (\beta \cdot h) g_\bullet \rceil_{\mathfrak{f}'_q} = \beta_* \cdot \lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q} \tag{3.22}$$

Because the left and right hand sides of the foregoing equation are morphisms in $\mathcal{C}(\mathbb{S}^{*\prime}_n, X)$, this equation is also an equality of $(P_* \cdot \mathrm{id})$-paths, which, from this point of view, involves other sources and targets. Note that because the relation $\langle x, y \rangle : [x_k]_n \sim_s [y_k]_n$ holds, the framing $\lceil g_\diamond \langle x, y \rangle g_\bullet \rceil_{\mathfrak{f}'_q}$ is well-defined along the paths $g_\diamond$ and $g_\bullet$. Equation (3.22) allows one to prove the following result.

Proposition 3.53. *The source and target of the (s_*)-path $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ are equal to the source and target of the $(P_* \cdot \mathrm{id})$-path $\lceil g_\diamond \langle x, y \rangle g_\bullet \rceil_{\mathfrak{f}'_q}$, respectively.*

Proof. First, by Proposition 3.45 and Remark 3.44, the source and target of the (s_*)-path $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ must be the same as those of the $(P_* \cdot \mathrm{id})$-path $\beta_* \cdot \lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$. Equation (3.22) also implies that these are exactly the source and target of the $(P_* \cdot \mathrm{id})$-path $\lceil g_\diamond (\beta \cdot h) g_\bullet \rceil_{\mathfrak{f}'_q}$. This finally proves the statement since Proposition 3.34 implies the equality of $(P \cdot \mathrm{id})$-paths $\beta \cdot h = \langle x, y \rangle$. ☐

3.3.3.7. *Framing of nodes of spines.* Let $\mathcal{C}$ be a category, q a non-negative integer and P be a prespine equipped with a q-framing P_* along a pair of seps $\mathfrak{p}_\diamond$ and $\mathfrak{p}_\bullet$. A node of spines $\sigma := P \cdot \Omega$ will be said to *simply* (resp. *extensively*) *frame* a node of spines $\sigma_* := P_* \cdot \Omega_*$ along two sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ if it is equipped with a

1) a metafunction $\psi : \Omega \to \Omega_*$, called its *framing gear*;
2) a simple (resp. extensive) q-framing of spines $(P \cdot \beta, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^V_q P_* \cdot \psi(\beta)$ (resp. $(P \cdot \beta, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^E_q P_* \cdot \psi(\beta)$) for every stem $\beta \in \Omega$.

Such a structure will be denoted by $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^V_q \sigma_*$ (resp. $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^E_q \sigma_*$) and called a *simple* (resp. *extensive*) *q-framing of nodes of spines* in $\mathcal{C}$.

3.3.3.8. *Hom-language for simple framings of nodes of spines.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^V_q \sigma_*$ be a simple q-framing of nodes of spines where $\mathfrak{v}_\diamond : \gamma_q \overset{\text{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma'_q \overset{\text{ex}}{\rightsquigarrow} v_\bullet$. Suppose given the paths

$$g_\diamond : \varrho_\diamond \cdot x_q \sim_{v_\diamond} a \qquad h : [x_k]_n \sim_\sigma [y_k]_n \quad \text{and} \quad g_\bullet : \varrho_\bullet \cdot y_q \sim_{v_\bullet} b$$

in X, where $\varrho_\diamond$ and $\varrho_\bullet$ denote the discal transitions of the sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$. We deduce from section 3.3.3.6 and section 3.3.3.7 that the previous set of paths gives rise to a $V^q_{\sigma_*}(0)$-path $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ from a to b if we denote $\mathfrak{f}_q := (V^q_\sigma(0), \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q V^q_{\sigma_*}(0)$. Proposition 3.22 also shows that the source and target of $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$, when seen as a (σ_*)-path, are equal to the respective source and target of the $(P_* \cdot \mathrm{id})$-path $\lceil g_\diamond \langle x, y \rangle g_\bullet \rceil$. In other words, the source and target of $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ do not depend on the encoding vertebra of $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q}$ as a (σ_*)-path. Thus, if we denote by x' and y' the source and target of $\lceil g_\diamond \langle x, y \rangle g_\bullet \rceil$, the preceding discussion shows that for every pair of paths $g_\diamond : \varrho_\diamond \cdot x_q \rightsquigarrow_{v_\diamond} a$ and $g_\bullet : \varrho_\bullet \cdot y_q \rightsquigarrow_{v_\bullet} b$, there exists a diagram of metafunctions

$$\begin{tikzcd} \mathcal{C}(V^q_\sigma(0), X)(x_q, y_q) \arrow[r, "T^{g_\bullet}_{g_\diamond}"] & \mathcal{C}(V^q_{\sigma_*}(0), X)(a, b) \\ \mathcal{C}(\sigma, X)(x, y) \arrow[u, "\subseteq"] \arrow[r, dashed] & \mathcal{C}(\sigma_*, X)(x', y') \arrow[u, "\subseteq"'] \end{tikzcd} \tag{3.23}$$

whose bottom arrow maps any σ-path $h : x \rightsquigarrow_\sigma y$ to the (σ_*)-paths $\lceil g_\diamond h g_\bullet \rceil_{\mathfrak{f}_q} : x' \rightsquigarrow_{\sigma_*} y'$ where $x'_q = a$ and $y'_q = b$. The dashed arrow of diagram (3.23) will also be denoted by $T^{g_\bullet}_{g_\diamond}$. The structure defined by the previous metafunctions together with the associated pair of paths $g_\diamond$ and $g_\bullet$ will be called a *simple q-tubular operator*.

Proposition 3.54. *Let $g : X \to Y$ be a surtraction for $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ and x and y be a q-parallel pair over g above the base of σ. There exists a simple q-tubular operator $T^{e_\bullet}_{e_\diamond} : \mathcal{C}(\sigma, Y)(x, y) \to \mathcal{C}(\sigma_*, Y)(x', y')$ such that x' and y' are q-parallel under g.*

Proof. Follows from Proposition 3.47 and the equations $x'_q = a$ and $y'_q = b$. □

Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and consider the paths $g_\diamond$ and $g_\bullet$ as defined at the beginning of the section. The application of f on the two paths $g_\diamond$ and $g_\bullet$ in X provides two other paths in Y as follows.

$$f(g_\diamond) : \varrho_\diamond \cdot f(x_q) \rightsquigarrow_{v_\diamond} f(a) \qquad f(g_\bullet) : \varrho_\bullet \cdot f(y_q) \rightsquigarrow_{v_\bullet} f(b)$$

Consider a σ-path $h : f(x) \rightsquigarrow_\sigma f(y)$ in Y. As above, the source and target of the $(P_* \cdot \mathrm{id})$-path $\lceil g_\diamond \langle x, y \rangle g_\bullet \rceil$ will be denoted by x' and y', respectively. First, the framing of vertebrae $\mathfrak{f}_q$ implies a $V^q_{\sigma_*}(0)$-path $\lceil f(g_\diamond) h f(g_\bullet) \rceil_{\mathfrak{f}_q}$. Because $\mathfrak{f}_q$ is induced by the simple q-framing of nodes of spines $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright^V_q \sigma_*$, it follows from Proposition 3.53 that the source and target of $\lceil f(g_\diamond) h f(g_\bullet) \rceil_{\mathfrak{f}_q}$, when seen as a (σ_*)-path, are equal to the source and target of the following $(P_* \cdot \mathrm{id})$-path in Y.

$$\lceil f(g_\diamond)(\beta \cdot h) f(g_\bullet) \rceil = \lceil f(g_\diamond) \langle f(x), f(y) \rangle f(g_\bullet) \rceil$$

By Remark 3.46, this last element is equal to $f(\lceil g_\diamond \langle x, y \rangle g_\bullet \rceil)$, which is, by assumption, of the form $f(x') \rightsquigarrow_{\sigma_*} f(y')$ when seen as a (σ_*)-path. In other words, the above discussion provides a simple q-tubular operator of the following form.

$$T^{f(g_\bullet)}_{f(g_\diamond)} : \mathcal{C}(\sigma, Y)(f(x), f(y)) \to \mathcal{C}(\sigma_*, Y)(f(x'), f(y')) \tag{3.24}$$

More specifically, the following proposition holds.

Proposition 3.55. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Any simple q-tubular operator of the form $T^{g_\bullet}_{g_\diamond} : \mathcal{C}(\sigma, X)(x, y) \to \mathcal{C}(\sigma_*, X)(x', y')$ implies another one of the form (3.24) making the following diagram commute.*

$$\begin{tikzcd} \mathcal{C}(\sigma, X)(x, y) \arrow[r, "T^{g_\bullet}_{g_\diamond}"] \arrow[d, "f"'] & \mathcal{C}(\sigma_*, X)(x', y') \arrow[d, "f"] \\ \mathcal{C}(\sigma, Y)(f(x), f(y)) \arrow[r, "T^{f(g_\bullet)}_{f(g_\diamond)}"] & \mathcal{C}(\sigma_*, Y)(f(x'), f(y')) \end{tikzcd}$$

Proof. The first part of the statement is implied by the discussion preceding the present proposition. The commutative diagram is, for its part, equivalent to the fact that the identity $\lceil f(g_\diamond)f(h)f(g_\bullet)\rceil_{\mathfrak{f}_q} = f(\lceil g_\diamond h g_\bullet\rceil_{\mathfrak{f}_q})$ holds for every path $h \in \mathcal{C}(\sigma, X)(x, y)$, which is a consequence of Remark 3.46. $\square$

3.3.3.9. *Hom-language for extensive framing of nodes of spines.* Let $\mathcal{C}$ be a category, $f : X \to Y$ be an object in $\mathcal{C}$ and $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \rhd_q^E \sigma_*$ be an extensive q-framing of nodes of spines where $\mathfrak{v}_\diamond : \gamma_q \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma'_q \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet$. We will denote the node of spines σ by $P \cdot \Omega$ and its respective spinal seed and coseed by s and s'. Similarly, the spine σ_* will be written as $P_* \cdot \Omega_*$ and its respective spinal seed and coseed will be denoted by s_* and s'_*. Consider a pair (h, y) in $\mathcal{C}(\sigma, f)(r, t)$ with the notations $y : [t_k]_n \sim_{s'} [r_k]_n$ and two paths $g_\diamond : \varrho_\diamond \cdot r_q \sim_{v_\diamond} a$ and $g_\bullet : \varrho_\bullet \cdot t_q \sim_{v_\bullet} b$ in X where $\varrho_\diamond$ and $\varrho_\bullet$ denote the discal transitions of the sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$. We deduce from Definition 3.52 and section 3.3.3.8 that the extensive q-framing applied on the paths of the pair (h, y) gives rise to a (σ_*)-path $\lceil f(g_\diamond)hf(g_\bullet)\rceil$ in Y and an (s_*)-path $\lceil g_\diamond y g_\bullet\rceil$ in X. Note that the former path makes sense regarding the conditions on sources and targets involved in section 3.3.3.8 by application of Proposition 3.38. By definition of a node of spines, we may assume that the σ-path h is encoded by a $(P \cdot \beta)$-path for some $\beta \in \Omega$. It follows from Definition 3.52 and section 3.3.2.9 that the source of the (σ_*)-path $\lceil f(g_\diamond)hf(g_\bullet)\rceil$ is equal to

$$\lceil f(g_\diamond)((\delta_2\beta) \cdot h)f(g_\bullet)\rceil = \lceil f(g_\diamond)xf(g_\bullet)\rceil$$

Similarly, the target of the (σ_*)-path $\lceil f(g_\diamond)hf(g_\bullet)\rceil$ is equal to

$$\lceil f(g_\diamond)((\delta_1\beta) \cdot h)f(g_\bullet)\rceil = \lceil f(g_\diamond)f(y)f(g_\bullet)\rceil = f(\lceil g_\diamond y g_\bullet\rceil).$$

Regarding the induced simple q-framings $(s', \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \rhd_q^V s'_*$ and $(s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \rhd_q^V s_*$, we also know from Proposition 3.53 that the source and target of the (s'_*)-path $\lceil g_\diamond y g_\bullet\rceil$ are the same as those of the $(P_* \cdot \mathrm{id})$-path $\lceil g_\diamond\langle r, t\rangle g_\bullet\rceil$ and the source and target of the (s_*)-path $\lceil f(g_\diamond)xf(g_\bullet)\rceil$ are the same as those of the $(P_* \cdot \mathrm{id})$-path

$$\lceil f(g_\diamond)\langle f(r), f(t)\rangle f(g_\bullet)\rceil = \lceil f(g_\diamond)f(\langle r, t\rangle)f(g_\bullet)\rceil = f(\lceil g_\diamond\langle r, t\rangle g_\bullet\rceil).$$

Thus, if we denote by r' and t' the source and target of $\lceil g_\diamond\langle r, t\rangle g_\bullet\rceil$, the preceding discussion shows that for every pair of paths $g_\diamond : \varrho_\diamond \cdot r_q \sim_{v_\diamond} a$ and $g_\bullet : \varrho_\bullet \cdot t_q \sim_{v_\bullet} b$, there exists a diagram of metafunctions

$$\begin{tikzcd} \mathcal{C}(s, Y)(f(r), f(t)) \arrow[r, "T^{f(g_\bullet)}_{f(g_\diamond)}"] & \mathcal{C}(s_*, Y)(f(r'), f(t')) \\ \mathcal{C}(\sigma, f)(r, t) \arrow[u, "R_\sigma"] \arrow[r, dashed] & \mathcal{C}(\sigma_*, f)(r', t') \arrow[u, "R_{\sigma_*}"'] \end{tikzcd} \tag{3.25}$$

whose bottom arrow maps any pair (h, y) to the pair of paths defined by

$$(\lceil f(g_\diamond)hf(g_\bullet)\rceil, \lceil g_\diamond y g_\bullet\rceil).$$

The dashed arrow of diagram (3.25) will be denoted by $T^{g_\bullet}_{g_\diamond}$. The structure defined by the previous metafunctions together with the associated pair of paths $g_\diamond$ and $g_\bullet$ will be called an *extensive q-tubular operator.*

3.3.3.10. *Sequences of framings of nodes of spines.* In the sequel, the terms *framing of nodes of spines* will be shortened to *fonos.* Let $\mathcal{C}$ be a category and ℓ be a positive integer. A *sequence of simple fonos's of length ℓ* in $\mathcal{C}$ consists of a sequence of ℓ simple fonos's $\mathfrak{F} := \{\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_{\ell-1}\}$ in $\mathcal{C}$ such that, for every $0 \leq i \leq \ell - 1$, the framing $\mathfrak{f}_i$ is a simple i-framing of nodes of spines of the form $(\sigma_i, \mathfrak{v}^i_\diamond, \mathfrak{v}^i_\bullet) \rhd_i^V \sigma_{i+1}$. In other words, we have a sequence of consecutive framings of the following form.

$$(\sigma_0, \mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet) \quad \rhd_0^V \quad (\sigma_1, \mathfrak{v}^1_\diamond, \mathfrak{v}^1_\bullet) \quad \rhd_1^V \dots \rhd_{\ell-2}^V \quad (\sigma_{\ell-1}, \mathfrak{v}^{\ell-1}_\diamond, \mathfrak{v}^{\ell-1}_\bullet) \quad \rhd_{\ell-1}^V \quad \sigma_\ell$$

The set of extended vertebrae consisting of $\mathfrak{v}_\diamond^0, \ldots, \mathfrak{v}_\diamond^{\ell-1}$ and $\mathfrak{v}_\bullet^0, \ldots, \mathfrak{v}_\bullet^{\ell-1}$ will be called the *tubular shell* of the sequence. A morphism that is a surtraction for every extended (node of) vertebrae in the tubular shell of $\mathfrak{F}$ will be called a *tubular surtraction* for $\mathfrak{F}$. Denote by n_i the degree of the nodes of spines σ_i for every $0 \leq i \leq \ell$. The sequence of simple fonos's $\mathfrak{F}$ will be said to be *convergent* if the relation $n_\ell = \ell - 1$ holds.

Lemma 3.56. *Let $\mathfrak{F}$ be a sequence of simple fonos's of length ℓ as above, $g : X \to Y$ be a tubular surtraction for $\mathfrak{F}$ and x and y be a pair of 0-parallel elements over g above the base of σ_0 in Y. There exists a composition of ℓ tubular operators*

$$\mathcal{C}(\sigma_0, Y)(x, y) \to \mathcal{C}(\sigma_1, Y)(x_1, y_1) \to \cdots \to \mathcal{C}(\sigma_{n_\ell}, Y)(x_\ell, y_\ell)$$

such that x_ℓ and y_ℓ are $(\ell-1)$-parallel under g above the base of σ_ℓ in Y.

Proof. Follows from successive uses of Proposition 3.54 and Proposition 3.41. $\square$

Remark 3.57. If sequence $\mathfrak{F}$ of Lemma 3.56 is convergent, then x_ℓ and y_ℓ are (n_ℓ)-parallel under g above the base of σ_ℓ in Y, which means that they are parallel under g above the head of σ_ℓ in Y. In other words, there exists a pair of parallel elements x' and y' above the head of σ_ℓ in X such that $x_\ell = g(x')$ and $y_\ell = g(y')$.

3.3.4. Correspondences of vertebrae.

3.3.4.1. *Correspondences of vertebrae.* The reader might want to refer to section 3.1 regarding the intuition behind correspondences, which, in a few words, is that of a pair of vertebrae that are canonically related in the sense that one of the vertebrae should be viewed as the image of the other via algebraic operations. Let $\mathcal{C}$ be a category and $v = p \cdot \beta$ and $\overline{v} = \overline{p} \cdot \overline{\beta}$ be two vertebrae in $\mathcal{C}$ with respective domains $\mathbb{S}$ and $\overline{\mathbb{S}}$. Denote by $\mathbb{D}'$ and $\overline{\mathbb{D}}'$ the respective codomains of the stems β and $\overline{\beta}$. A *correspondence of vertebrae* between v and $\overline{v}$ consists of an arrow $\varkappa : \overline{\mathbb{S}} \to \mathbb{S}$ in $\mathcal{C}$, an object $\mathbb{M}$ in $\mathcal{C}$, called the *messenger*, and two morphisms $u : \mathbb{D} \to \mathbb{M}$ and $\overline{u} : \overline{\mathbb{D}}' \to \mathbb{M}$ in $\mathcal{C}$ such that the relation

$$u \circ \beta \circ \Gamma_{-1}(p) \circ \varkappa = \overline{u} \circ \overline{\beta} \circ \Gamma_{-1}(\overline{p}) \tag{3.26}$$

holds in $\mathcal{C}$ (see section 3.3.1.3 for Γ_{-1}). Such a correspondence will be denoted by the symbols $(u, \overline{u}) \vdash v \asymp \overline{v}$. The pair $(u, \overline{u})$ will be replaced with a triple $(\varkappa, u, \overline{u})$ when the morphism $\varkappa$ needs to be specified.

3.3.4.2. *Strong correspondences of vertebrae.* Let $\mathcal{C}$ be a category and $v = p \cdot \beta$ and $\overline{v} = p \cdot \overline{\beta}$ be two vertebrae in $\mathcal{C}$. Denote by $\mathbb{D}'$ and $\overline{\mathbb{D}}'$ the respective codomains of the stems β and $\overline{\beta}$. A *strong correspondence of vertebrae* between v and $\overline{v}$ consists of an alliance of prevertebrae $\mathfrak{p} := (\varkappa, \rho, \rho', \varkappa') : p \rightsquigarrow \overline{p}$, an object $\mathbb{M}$ in $\mathcal{C}$, called the *messenger* and two arrows $u : \mathbb{D} \to \mathbb{M}$ and $\overline{u} : \overline{\mathbb{D}}' \to \mathbb{M}$ in $\mathcal{C}$ such that the relation

$$u \circ \beta \circ \varkappa' = \overline{u} \circ \overline{\beta}$$

holds in $\mathcal{C}$. Such a structure will be denoted by the symbols $(u, \overline{u}) \vdash v \mathrel{\tilde{\asymp}} \overline{v}$. The pair $(u, \overline{u})$ will be replaced with a triple $(\mathfrak{p}, u, \overline{u})$ when the alliance $\mathfrak{p}$ needs to be specified. Note that any strong correspondence of the form $(\mathfrak{p}, u, \overline{u}) \vdash v \mathrel{\tilde{\asymp}} \overline{v}$ in $\mathcal{C}$ gives rise to a correspondence $(\varkappa, u, \overline{u}) \vdash v \asymp \overline{v}$ in $\mathcal{C}$ with same messenger.

3.3.4.3. *Morphisms of correspondences.* Let $\mathcal{C}$ be a category and $(\varkappa, u, \overline{u}) \vdash p \cdot \beta \asymp \overline{p} \cdot \overline{\beta}$ and $(\varkappa, u_*, \overline{u}_*) \vdash p \cdot \beta_* \asymp \overline{p} \cdot \overline{\beta}_*$ (resp. $(\mathfrak{p}, u, \overline{u}) \vdash p \cdot \beta \mathrel{\tilde{\asymp}} \overline{p} \cdot \overline{\beta}$ and $(\mathfrak{p}, u_*, \overline{u}_*) \vdash p \cdot \beta_* \mathrel{\tilde{\asymp}} \overline{p} \cdot \overline{\beta}_*$) be two (resp. strong) correspondences of vertebrae in $\mathcal{C}$ whose respective messengers will be denoted by $\mathbb{M}$ and $\mathbb{M}_*$. A *morphism of correspondences* from $(u, \overline{u})$ to $(u_*, \overline{u}_*)$ consists of

1) two arrows $a : p \cdot \beta \curvearrowright p \cdot \beta_*$ and $b : \overline{p} \cdot \overline{\beta} \curvearrowright \overline{p} \cdot \overline{\beta}_*$ in $\mathbf{Vert}(\mathcal{C})$;

2) an arrow $\kappa : \mathbb{M}_* \to \mathbb{M}$ making the following diagram commute.

(3.27)
$$\begin{tikzcd}
\mathbb{D}'_* \arrow[r, "u_*"] \arrow[d, "a"'] & \mathbb{M}_* \arrow[d, "\kappa"'] & \overline{\mathbb{D}}'_* \arrow[l, "\overline{u}_*"'] \arrow[d, "b"] \\
\mathbb{D}' \arrow[r, "u"'] & \mathbb{M} & \overline{\mathbb{D}}' \arrow[l, "\overline{u}"]
\end{tikzcd}$$

This notion of morphisms defines a category of (resp. strong) correspondences of vertebrae in $\mathcal{C}$ that will be denoted by $\mathbf{Corov}(\mathcal{C})$ (resp. $\mathbf{Scov}(\mathcal{C})$). A morphism of correspondences as defined above will be denoted by $[\kappa, a, b] : (u, \overline{u}) \Rightarrow (u_*, \overline{u}_*)$.

Remark 3.58. Let $(u, \overline{u}) \vdash p \cdot \beta \asymp \overline{p} \cdot \overline{\beta}$ be a correspondence in $\mathbf{Corov}(\mathcal{C})$ whose messenger will be denoted by $\mathbb{M}$. It follows that post-composing the morphisms u and $\overline{u}$ with any arrow $\kappa : \mathbb{M} \to \mathbb{M}'$ in $\mathcal{C}$ provides a correspondence $(\kappa \circ u, \kappa \circ \overline{u}) \vdash p \cdot \beta \asymp \overline{p} \cdot \overline{\beta}$ and a morphism $[\kappa, \mathrm{id}, \mathrm{id}] : (\kappa \circ u, \kappa \circ \overline{u}) \Rightarrow (u, \overline{u})$ in $\mathbf{Corov}(\mathcal{C})$

Remark 3.59. Let $(u, \overline{u}) \vdash p \cdot \beta \asymp \overline{p} \cdot \overline{\beta}$ be a correspondence in $\mathbf{Corov}(\mathcal{C})$ whose messenger will be denoted by $\mathbb{M}$ and consider a pair of arrows $a : p \cdot \beta \curvearrowright p \cdot \beta_*$ and $b : \overline{p} \cdot \overline{\beta} \curvearrowright \overline{p} \cdot \overline{\beta}_*$ in $\mathbf{Vert}(\mathcal{C})$. It follows that pre-composing the morphisms u and $\overline{u}$ with a and b provides a correspondence $(u \circ a, \overline{u} \circ b) \vdash p_* \cdot \beta_* \asymp \overline{p}_* \cdot \overline{\beta}_*$ and a morphism $[\mathrm{id}, a, b] : (u, \overline{u}) \Rightarrow (u \circ a, \overline{u} \circ b)$ in $\mathbf{Corov}(\mathcal{C})$.

3.3.4.4. *Mates for correspondences.* Let $\mathcal{C}$ be a category and $(\varkappa, u, \overline{u}) \vdash p \cdot \beta \asymp \overline{p} \cdot \overline{\beta}$ be a correspondence in $\mathbf{Corov}(\mathcal{C})$ equipped with a messenger $\mathbb{M}$. The version of formula (3.26) for the correspondence $(u, \overline{u})$ implies that the next diagrams commute.

(3.28)
$$\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma\circ\varkappa"] \arrow[d, "\overline{\gamma}"'] & \mathbb{D}_1 \arrow[d, bend left, "u\circ\beta\circ\delta_2"] \\
\overline{\mathbb{D}}_2 \arrow[r, bend right, "\overline{u}\circ\overline{\beta}\circ\overline{\delta}_2"'] & \mathbb{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma'\circ\varkappa"] \arrow[d, "\overline{\gamma}'"'] & \mathbb{D}_2 \arrow[d, bend left, "u\circ\beta\circ\delta_1"] \\
\overline{\mathbb{D}}_1 \arrow[r, bend right, "\overline{u}\circ\overline{\beta}\circ\overline{\delta}_1"'] & \mathbb{M}
\end{tikzcd}$$

A *pair of mates* for the correspondence $(u, \overline{u}) \vdash p \cdot \beta \asymp \overline{p} \cdot \overline{\beta}$ consists of a pair of vertebrae $v_\diamond := \|\overline{\gamma}, \gamma \circ \varkappa\| \cdot \beta_\diamond : \mathbb{S} \multimap \mathbb{S}'_\diamond$ and $v_\bullet := \|\overline{\gamma}', \gamma' \circ \varkappa\| \cdot \beta_\bullet : \mathbb{S} \multimap \mathbb{S}'_\bullet$ together with a pair of morphisms $\varpi_\diamond : \mathbb{D}'_\diamond \to \mathbb{D}'$ and $\varpi_\bullet : \mathbb{D}'_\bullet \to \mathbb{D}'$ in $\mathcal{C}$ such that the diagrams given in (3.28) factorise as follows.

(3.29)
$$\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma\circ\varkappa"] \arrow[d, "\overline{\gamma}"'] & \mathbb{D}_1 \arrow[d, "\delta_1^\diamond"] \arrow[rrr, bend left, "u\circ\beta\circ\delta_2"] & & & \\
\overline{\mathbb{D}}_2 \arrow[r, "\delta_2^\diamond"] \arrow[rrr, bend right, "\overline{u}\circ\overline{\beta}\circ\overline{\delta}_2"'] & \mathbb{S}'_\diamond \arrow[r, "\beta_\diamond"] & \mathbb{D}'_\diamond \arrow[r, "\varpi_\diamond"] & \mathbb{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma'\circ\varkappa"] \arrow[d, "\overline{\gamma}'"'] & \mathbb{D}_2 \arrow[d, "\delta_1^\bullet"] \arrow[rrr, bend left, "u\circ\beta\circ\delta_1"] & & & \\
\overline{\mathbb{D}}_1 \arrow[r, "\delta_2^\bullet"] \arrow[rrr, bend right, "\overline{u}\circ\overline{\beta}\circ\overline{\delta_1}"'] & \mathbb{S}'_\bullet \arrow[r, "\beta_\bullet"] & \mathbb{D}'_\bullet \arrow[r, "\varpi_\bullet"] & \mathbb{M}
\end{tikzcd}$$

This being defined, a quick rearrangement of the previous diagrams gives the next one, which reminds of the type of diagrams considered for framings of vertebrae.

(3.30)
$$\begin{tikzcd}
\overline{\mathbb{S}}' \arrow[dr, "\overline{\beta}"] & \mathbb{D}_2 \arrow[l, "\overline{\delta}_2"'] \arrow[r, "\beta_\diamond \circ \delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\varpi_\diamond"] \\
\mathbb{D}_2 \arrow[u, "\overline{\delta}_1"] \arrow[d, "\beta_\bullet \circ \delta_2^\bullet"'] & \overline{\mathbb{D}}' \arrow[dr, "\overline{u}"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\varpi_\bullet"'] & & \mathbb{M}
\end{tikzcd}$$

Indeed, if one sees the vertebrae $v_\diamond$ and $v_\bullet$ as semi-extended vertebrae with trivial discal transitions, section 3.3.2.5 shows that this commutative diagram fulfills all the conditions required to suit a framing of $\overline{p}\cdot\overline{\beta}$ along the pair of (semi-extended) vertebrae $(v_\diamond, v_\bullet)$. The use of the notation $\lceil \varpi_\diamond \overline{u} \varpi_\bullet \rceil$ shall thus make sense. The next proposition shows that morphisms of correspondences are well-behaved with respect to pair of mates.

Proposition 3.60. *Let $[\kappa, a, b] : (u, \overline{u}) \Rightarrow (u_*, \overline{u}_*)$ be an arrow in $\mathbf{Corov}(\mathcal{C})$. If the codomain $(u_*, \overline{u}_*)$ is equipped with a pair of mates, then so is the domain $(u, \overline{u})$. In addition, the associated vertebrae are the same for both correspondences.*

Proof. Denote by $(v_\diamond, \varpi_\diamond)$ and $(v_\bullet, \varpi_\bullet)$ the pair of mates associated with $(u_*, \overline{u}_*)$. Then, post-compose the versions of diagrams (3.29) for the correspondence $(u_*, \overline{u}_*)$ with the arrow $\kappa : \mathbb{M}_* \to \mathbb{M}$ and use the relations available in diagram (3.27) as well as the equations defining the morphisms of vertebrae a and b (see Remark 3.43) to rearrange the resulting diagrams into the following ones.

$$\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma_* \circ \varkappa"] \arrow[d, "\overline{\gamma}_*"'] & \mathbb{D}_1 \arrow[d, "\delta_1^\diamond"] \arrow[rrd, bend left, "u \circ \beta \circ \delta_2"] & & \\
\overline{\mathbb{D}}_2 \arrow[r, "\delta_2^\diamond"] \arrow[rrr, bend right, "\overline{u} \circ \overline{\beta} \circ \overline{\delta}_2"'] & \mathbb{S}'_\diamond \arrow[r, "\beta_\diamond"] & \mathbb{D}'_\diamond \arrow[r, "\kappa \circ \varpi_\diamond"] & \mathbb{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma'_* \circ \varkappa"] \arrow[d, "\overline{\gamma}'_*"'] & \mathbb{D}_2 \arrow[d, "\delta_1^\bullet"] \arrow[rrd, bend left, "u \circ \beta \circ \delta_1"] & & \\
\overline{\mathbb{D}}_1 \arrow[r, "\delta_2^\bullet"] \arrow[rrr, bend right, "\overline{u} \circ \overline{\beta} \circ \overline{\delta}_1"'] & \mathbb{S}'_\bullet \arrow[r, "\beta_\bullet"] & \mathbb{D}'_\bullet \arrow[r, "\kappa \circ \varpi_\bullet"] & \mathbb{M}
\end{tikzcd}$$

This shows that $(v_\diamond, \kappa \circ \varpi_\diamond)$ and $(v_\bullet, \kappa \circ \varpi_\bullet)$ defines a pair of mates for $(u, \overline{u})$ as the equalities of prevertebrae $\|\gamma_*, \gamma'_*\| = \|\gamma, \gamma'\|$ and $\|\overline{\gamma}_*, \overline{\gamma}'_*\| = \|\overline{\gamma}, \overline{\gamma}'\|$ hold by definition of the arrows a and b in $\mathbf{Vert}(\mathcal{C})$. We can see that the pair of vertebrae associated with $(u, \overline{u})$ is indeed the same as that of $(u_*, \overline{u}_*)$. $\square$

The previous proposition shows that it is possible to transmit a pair of mates from the codomain to the domain. On the other hand, transmitting a pair of mates from domain to codomain requires the morphism κ between the two messengers to be an identity (see Remark 3.59), which will rarely be the case. This last point motivates the following definition. We will later denote by $\mathbf{Mcov}(\mathcal{C})$ the category whose objects are correspondences c in $\mathcal{C}$ equipped with a pair of mates μ and whose arrows $(c, \mu) \Rightarrow (c_*, \mu_*)$ are morphisms $c \Rightarrow c_*$ in $\mathbf{Corov}(\mathcal{C})$ such that the pair of mates μ is induced by μ_* as shown in Proposition 3.60.

3.3.4.5. *Spans of correspondences.* Let $\mathcal{C}$ be a category and (c, μ) and $c_\dagger$ be two objects in $\mathbf{Mcov}(\mathcal{C})$ and $\mathbf{Scov}(\mathcal{C})$, respectively. The pair $((c, \mu), c_\dagger)$ will be said to form a *span* if both correspondences c and $c_\dagger$ are of respective form $(\varkappa, u, \overline{u}) \vdash v \asymp \overline{v}$ and $(\mathfrak{p}, u, u_\dagger) \vdash v \stackrel{\sim}{\asymp} \overline{v}_\dagger$ where $\varkappa$ is the spherical transition of $\mathfrak{p}$. Note that their messengers must be represented by the same object $\mathbb{M}$ since the morphism u, going to the messenger $\mathbb{M}$, is part of both correspondences. A *morphism of spans of correspondences* from $((c, \mu), c_\dagger)$ to $((c_*, \mu_*), c_\sharp)$ consists of a morphism

$[\kappa, a, b] : (c, \mu) \Rightarrow (c_*, \mu_*)$ in $\mathbf{Mcov}(\mathcal{C})$ and a morphism $[\kappa, a, b_\dagger] : c_\dagger \Rightarrow c_\sharp$ in $\mathbf{Scov}(\mathcal{C})$. Notice that the arrow $\kappa : \mathbb{M}_* \to \mathbb{M}$ and the morphism of vertebrae $a : v \curvearrowright v_*$ – when supposing that c and c_* are of the form $v \asymp \overline{v}$ and $v_* \asymp \overline{v}_*$ with messengers $\mathbb{M}$ and $\mathbb{M}_*$, respectively – are common to both morphisms.

3.3.4.6. *Framings of correspondences of vertebrae.* Let $\mathcal{C}$ be a category and (c, μ) be an object in $\mathbf{Mcov}(\mathcal{C})$ where $c := (\varkappa, u, \overline{u}) \vdash v \asymp \overline{v}$ and $\mu := (v_\diamond, \varpi_\diamond, v_\bullet, \varpi_\bullet)$. A vertebra $\overline{v}_\dagger$ will be said to *frame* the pair (c, μ) if it is equipped with a framing of vertebrae $\mathfrak{f} := (\overline{v}, v_\diamond, v_\bullet) \triangleright \overline{v}_\dagger$ (see section 3.3.4.4 for the use of vertebrae instead of sevs in framings). By definition of the pair of mates μ and the framing $\mathfrak{f}$, if the prevertebra $v = p \cdot \beta$ is of the form $\|\gamma, \gamma' : \mathbb{S}'\|$, the prevertebra of $\overline{v}_\dagger = \overline{p}_\dagger \cdot \overline{\beta}_\dagger$ must be of the form $\|\gamma \circ \varkappa, \gamma' \circ \varkappa\|$ (see section 3.3.2.3). The universality of the codomain of $\overline{p}_\dagger$ then implies the existence of a canonical arrow θ (see next diagram) inducing an alliance of prevertebrae $\mathfrak{p} := (\varkappa, \mathrm{id}, \mathrm{id}, \theta) : p \rightsquigarrow \overline{p}_\dagger$.

(3.31)
$$\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\gamma' \circ \varkappa"] \arrow[d, "\gamma \circ \varkappa"'] & \mathbb{D}_1 \arrow[r, equal] \arrow[d, "\overline{\delta}_1^\dagger"] & \mathbb{D}_1 \arrow[dd, "\delta_1"] \\
\mathbb{D}_2 \arrow[r, "\overline{\delta}_2^\dagger"'] \arrow[d, equal] & \overline{\mathbb{S}}'_\dagger \arrow[dr, dashed, "\theta"] & \\
\mathbb{D}_2 \arrow[rr, "\delta_2"'] & & \mathbb{S}'
\end{tikzcd}$$

Also, recall that the discussion of section 3.3.4.4 showed that the framing $\mathfrak{f}$ implied the existence of an $(\overline{v}_\dagger)$-path $\overline{u}_\dagger := \lceil \varpi_\diamond \overline{u} \varpi_\bullet \rceil_{\mathfrak{f}} : \overline{\mathbb{D}}_\dagger \to \mathbb{M}$ (see diagram (3.30)).

Proposition 3.61. *The previously defined data* $(\mathfrak{p}, u, \overline{u}_\dagger)$ *defines a strong correspondence with messenger* $\mathbb{M}$*. If follows that the pair* (c, μ) *and the strong correspondence* $(\mathfrak{p}, u, \overline{u}_\dagger)$ *define a span of correspondences.*

Proof. This may be shown after some straightforward calculations by using the diagrammatic relations given in *(i)* the top right parts of both diagrams of (3.29) for the pair of mates μ and *(ii)* the versions of diagram (3.16) and diagram (3.15) for the framing of vertebrae $(\overline{v}, v_\diamond, v_\bullet) \triangleright \overline{v}_\dagger$, to expose both arrows $u \circ \beta \circ \theta$ and $\lceil \varpi_\diamond \overline{u} \varpi_\bullet \rceil_{\mathfrak{f}} \circ \overline{\beta}_\dagger$ as solutions of the following commutative problem (see diagram (3.31) for the diagrammatic relations associated with the arrow θ).

$$\begin{tikzcd}
\overline{\mathbb{S}}'_\dagger \arrow[ddrr, dashed] & \mathbb{D}_1 \arrow[l, "\overline{\delta}_2^\dagger"'] \arrow[r, "\beta_\diamond \circ \delta_1^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\varpi_\diamond"] \\
\mathbb{D}_1 \arrow[u, "\overline{\delta}_1^\dagger"] \arrow[d, "\beta_\bullet \circ \delta_1^\bullet"'] & & \\
\mathbb{D}'_\bullet \arrow[rr, "\varpi_\bullet"'] & & \mathbb{M}
\end{tikzcd}$$

The universality of the pushout $\mathbb{S}'$ then leads to the uniqueness of such a solution and provides the wanted correspondence. □

Denote the strong correspondence $(\mathfrak{p}, u, \overline{u}_\dagger)$ by $c_\dagger$. In the sequel, the framing of the pair (c, μ) by the vertebra $\overline{v}_\dagger$ will be denoted by the symbols $(c, \mu) \triangleright c_\dagger$, thereby exposing the structure of span produced by Propoposition 3.61. The correspondence $c_\dagger$ will then be said to *frame* the pair (c, μ). Such a structure will be referred to as a *framing of correspondences of vertebrae.*

3.3.4.7. *Convention of notations.* In the sequel, we shall associate any correspondence of the form $(\varkappa_{\text{-}}, u_{\text{-}}, \overline{u}_{\text{-}}) \vdash v_{\text{-}} \asymp \overline{v}_{\text{-}}$ with the symbol $c_{\text{-}}$ and any pair of mates $(v_\diamond^{\text{-}}, \varpi_\diamond^{\text{-}}, v_\bullet^{\text{-}}, \varpi_\bullet^{\text{-}})$ with the symbol $\mu_{\text{-}}$.

3.3.4.8. *Hom-language for framing of correspondences of vertebrae.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $c := (u, \overline{u}) \vdash v \asymp \overline{v}$ be a correspondence of vertebrae in $\mathcal{C}$ with messenger $\mathbb{M}$. Consider an element h in $\mathcal{C}(\mathbb{M}, X)$ and two paths of the form $u \cdot h : x \sim_v y$ and $\overline{u} \cdot h : \overline{x} \sim_{\overline{v}} \overline{y}$. When c is equipped with a pair of mates μ, both diagrams (3.29) induce a $v_\diamond$-path $\varpi_\diamond : (\overline{\delta_2 \beta}) \cdot \overline{u} \sim_{v_\diamond} (\delta_2\beta) \cdot u$ and a $v_\bullet$-path $\varpi_\bullet : (\overline{\delta_1\beta}) \cdot \overline{u} \sim_{v_\bullet} (\delta_1\beta) \cdot u$. By applying the arrow $h : \mathbb{M} \to X$ on them, we obtain the following two paths.

$$\varpi_\diamond \cdot h : \overline{x} \sim_{v_\diamond} x \quad \text{and} \quad \varpi_\bullet \cdot h : \overline{y} \sim_{v_\bullet} y$$

Finally, equipping the pair (c, μ) with a framing of correspondences $(c, \mu) \triangleright c_\dagger$ gives the following series of equalities (see Remark 3.46).

$$\lceil (\varpi_\diamond \cdot h)(\overline{u} \cdot h)(\varpi_\bullet \cdot h) \rceil = \lceil \varpi_\diamond \overline{u} \varpi_\bullet \rceil \cdot h = \overline{u}_\dagger \cdot h$$

Note that the equality involved on both extremities looks like the type of formula obtained for morphisms of framings of vertebrae when expressed in the hom-language.

3.3.4.9. *Morphisms of framings of correspondences of vertebrae.* The terms *framing of correspondences of vertebrae* will be shortened to *focov*. Let $\mathcal{C}$ be a category and $(c, \mu) \triangleright c_\dagger$ and $(c_*, \mu_*) \triangleright c_\sharp$ be two focovs in $\mathcal{C}$. A *morphism of focovs* from the former to the latter consists of a morphism $[\kappa, a, b] : (c, \mu) \Rightarrow (c_*, \mu_*)$ in $\mathbf{Mcov}(\mathcal{C})$ and a morphism of framings $(b, b_\dagger) : (\overline{v}, v_\diamond, v_\bullet) \triangleright \overline{v}_\dagger \curvearrowright (\overline{v}_*, v_\diamond, v_\bullet) \triangleright \overline{v}_\sharp$ in $\mathbf{Fov}(\mathcal{C})$.

Proposition 3.62. *The pair $([\kappa, a, b], [\kappa, a, b_\dagger])$ defines a morphism of spans of correspondences from $((c, \mu), c_\dagger)$ to $((c_*, \mu_*), c_\sharp)$.*

Proof. The main difficulty is to prove that the diagram below commutes, where $\varpi_\diamond \star \overline{u}_* \star \varpi_\bullet$ denotes the canonical arrow induced by the version of diagram (3.30) for the framing $(c_*, \mu_*) \triangleright c_\sharp$. It will then be easy to deduce the structure of morphism of spans of correspondences from its outer commutative square.

$$\begin{tikzcd}
\mathbb{M}_* \arrow[d, "\kappa"'] & \mathbb{G}'_* \arrow[l, "\varpi_\diamond \star \overline{u}_* \star \varpi_\bullet"'] \arrow[d, dashed, "\kappa(b)"] \arrow[dll, "(\kappa\circ\varpi_\diamond)\star(b\cdot\overline{u})\star(\kappa\circ\varpi_\bullet)"'] & \mathbb{D}'_\sharp \arrow[l, "\pi_*"'] \arrow[d, "b_\dagger"] \\
\mathbb{M} & \mathbb{G}' \arrow[l, "(\kappa\circ\varpi_\diamond)\star\overline{u}\star(\kappa\circ\varpi_\bullet)"] & \mathbb{D}'_\dagger \arrow[l, "\pi"]
\end{tikzcd}$$

First, the top left triangle follows from Remark 3.46 and the equation $\kappa \circ \overline{u}_* = \overline{u} \circ b$ given by the definition of morphism $[\kappa, a, b]$ in $\mathbf{Mcov}(\mathcal{C})$. The bottom left triangle follows from Remark 3.49 applied on the morphism $(b, b_\dagger)$ in $\mathbf{Fov}(\mathcal{C})$. The rightmost square is, for its part, given by definition of the morphism $(b, b_\dagger)$ in $\mathbf{Fov}(\mathcal{C})$. The above commutative diagram then implies the one, below, where the right-hand square is given by the outer commutative square of the above diagram and the left-hand square is given by the definition of the morphism $[\kappa, a, b]$ in $\mathbf{Mcov}(\mathcal{C})$.

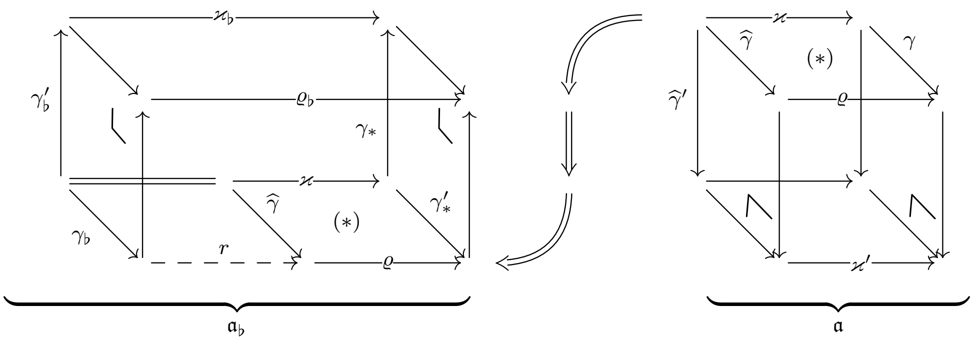
$$\begin{tikzcd}
\mathbb{D}' \arrow[r, "u_*"] \arrow[d, "a"'] & \mathbb{M}_* \arrow[d, "\kappa"'] & \mathbb{D}'_\sharp \arrow[l, "\lceil \varpi_\diamond \overline{u}_* \varpi_\bullet \rceil"'] \arrow[d, "b_\dagger"] \\
\mathbb{D}' \arrow[r, "u"'] & \mathbb{M} & \mathbb{D}'_\dagger \arrow[l, "\lceil (\kappa\circ\varpi_\diamond)\overline{u}(\kappa\circ\varpi_\bullet) \rceil"]
\end{tikzcd}$$

According to section 3.3.4.6 and section 3.3.4.5, this last diagram together with the morphism $[\kappa, a, b]$ define a morphism of spans of correspondences $((c, \mu), c_\dagger) \Rightarrow ((c_*, \mu_*), c_\sharp)$. □

This notion of morphism induces a category $\mathbf{Focov}(\mathcal{C})$ whose objects are the focovs in $\mathcal{C}$.

3.3.5. Conjugation of vertebrae.

3.3.5.1. *Conjugable pairs of alliances of prevertebrae.* Let $\mathcal{C}$ be category and consider two alliances of prevertebrae $\mathfrak{a} : p \rightsquigarrow \widehat{p}$ and $\mathfrak{a}_\flat : p_* \rightsquigarrow p_\flat^{\mathrm{rv}}$ in $\mathcal{C}$ such that the seed of p is equal to the coseed of p_*. Suppose that $\mathfrak{a}$ and $\mathfrak{a}_\flat$ are of the form $(\varkappa, \varrho, \varrho', \varkappa')$ and $(\varkappa_\flat, \varrho_\flat, \varrho'_\flat, \varkappa'_\flat)$, respectively.



The alliance $\mathfrak{a}_\flat$ will be said to be *conjugable* with the alliance $\mathfrak{a}$ along a morphism $r : \mathbb{D}_2^\flat \to \widehat{\mathbb{D}}_2$ if the arrow $\mathbf{seed}(\mathfrak{a}_\flat) : \gamma_\flat \Rightarrow \gamma$ factorises through $\mathbf{seed}(\mathfrak{a}) : \widehat{\gamma} \Rightarrow \gamma$ along a commutative square $\gamma_\flat \Rightarrow \widehat{\gamma}$ whose top arrow is an identity and whose bottom arrow is given by r (see preceding diagram). In particular, this implies the factorisation $\varrho'_\flat = \varrho \circ r$. Similarly, an alliance of vertebrae $\mathfrak{a}_\flat : v_* \rightsquigarrow v_\flat^{\mathrm{rv}}$ will be said to be *conjugable* with another one $\mathfrak{a} : v \rightsquigarrow \widehat{v}$ if so are their underlying alliances of prevertebrae.

Remark 3.63. By definition, the arrow $r : \mathbb{D}_2^\flat \to \widehat{\mathbb{D}}_2$ induces an obvious communication $t : \widehat{\gamma} \rightsquigarrow \gamma_\flat$, which induces an obvious semi-extended vertebra $(\mathbb{S}_\flat, r) : \widehat{\gamma} \overset{\mathrm{ex}}{\rightsquigarrow} v_\flat$.

3.3.5.2. *Conjugation of vertebrae.* Let $\mathcal{C}$ be category and consider four vertebrae v, $\overline{v}$, $\ddot{v}$, $\widehat{v}$ and an alliance of vertebrae $\mathfrak{a} : \ddot{v} \rightsquigarrow \widehat{v}$ in $\mathcal{C}$. The triple $(v, \mathfrak{a}, \overline{v})$ will be said to *form a conjugation of vertebrae in* $\mathcal{C}$ if it is equipped with

1) two semi-extended vertebrae $\mathfrak{v}_\diamond : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma' \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet$;
2) two reflections of vertebrae $\mathfrak{a}_\flat : v_\diamond \rightsquigarrow v_\flat^{\mathrm{rv}}$ and $\mathfrak{a}_\dagger : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$ conjugable with the alliances of vertebrae $\mathfrak{a}$ and $\mathfrak{a}^{\mathrm{rv}}$ along morphisms
$$r_\flat : \mathbb{D}_2^\flat \to \widehat{\mathbb{D}}_1 \quad \text{and} \quad r_\dagger : \mathbb{D}_2^\dagger \to \widehat{\mathbb{D}}_2,$$
respectively;
3) two framings of vertebrae $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \ddot{v}$ and $(\widehat{v}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright \overline{v}$ where $\mathfrak{v}_\flat$ and $\mathfrak{v}_\dagger$ are the underlying sevs $(\mathbb{S}_\flat, r_\flat) : \widehat{\gamma} \overset{\mathrm{ex}}{\rightsquigarrow} v_\flat$ and $(\mathbb{S}_\dagger, r_\dagger) : \widehat{\gamma}' \overset{\mathrm{ex}}{\rightsquigarrow} v_\dagger$ (see Remark 3.63).

The above structure will be denoted by $(v, \mathfrak{a}, \overline{v})$ and said to *be defined along the pairs* $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ *and* $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$. By definition of a framing, the domains of the vertebrae v, $\ddot{v}$ and sevs $\mathfrak{v}_\diamond$, $\mathfrak{v}_\bullet$ are the same, say $\mathbb{S}$, and, similarly, the domains of the vertebrae $\widehat{v}$, $\overline{v}$ and sevs $\mathfrak{v}_\flat$, $\mathfrak{v}_\dagger$ must also be equal to the same object, say $\overline{\mathbb{S}}$. The spherical transition of $\mathfrak{a}$ is then an arrow $\overline{\mathbb{S}} \to \mathbb{S}$, which will be denoted by $\varkappa$.

Proposition 3.64. *A conjugation of vertebrae* $(p \cdot \beta, \mathfrak{a}, \overline{p} \cdot \overline{\beta})$ *as above induces an alliance of prevertebrae* $p \rightsquigarrow \overline{p}$ *whose spherical transition is* $\varkappa : \overline{\mathbb{S}} \to \mathbb{S}$ *and whose discal and codiscal transitions are the discal transitions of semi-extended vertebrae* $\mathfrak{a}_\flat \circledcirc \mathfrak{v}_\diamond$ *and* $\mathfrak{a}_\dagger \circledcirc \mathfrak{v}_\bullet$*, respectively.*

Proof. First, the semi-extended vertebrae $\mathfrak{v}_\diamond : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma' \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet$ give the factorisations $\gamma = \varrho_\diamond \circ \gamma_\diamond$ and $\gamma' = \varrho_\bullet \circ \gamma_\bullet$ where $\varrho_\diamond$ and $\varrho_\bullet$ denote the respective discal transitions of $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$. Similarly, the alliances of vertebrae $\mathfrak{a}_\flat : v_\diamond \rightsquigarrow v_\flat^{\mathrm{rv}}$ and $\mathfrak{a}_\dagger : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$ give the factorisations $\gamma_\diamond \circ \varkappa = \varrho_\flat \circ \gamma'_\flat$ and $\gamma_\bullet \circ \varkappa = \varrho_\dagger \circ \gamma'_\dagger$ where $\varrho_\flat$ and $\varrho_\dagger$ denote the respective discal transitions of $\mathfrak{a}_\flat$ and $\mathfrak{a}_\dagger$. This implies the following commutative diagram where, by definition, the prevertebra $\|\gamma'_\flat, \gamma'_\dagger\|$ corresponds to the prevertebra $\overline{p}$ and the dashed arrow θ is the canonical morphism generated by the underlying universal problem over the codomain of $\overline{p}$.

(3.32)
$$\begin{tikzcd}
\mathbb{S} \arrow[drrr, bend left, "\gamma"] \arrow[dddr, bend right, "\gamma'"'] & & & \\
& \mathbb{S} \arrow[ul, "\varkappa"'] \arrow[r, "\gamma'_\dagger"] \arrow[d, "\gamma'_\flat"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbb{D}_1^\dagger \arrow[r, "\varrho_\bullet \circ \varrho_\dagger"'] \arrow[d, "\delta_1^\dagger"] & \mathbb{D}_1 \arrow[dd, "\delta_1"] \\
& \mathbb{D}_2^\flat \arrow[r, "\delta_2^\flat"'] \arrow[d, "\varrho_\diamond \circ \varrho_\flat"] & \overline{\mathbb{S}}' \arrow[dr, dashed, "\theta"] & \\
& \mathbb{D}_2 \arrow[rr, "\delta_2"] & & \mathbb{S}'
\end{tikzcd}$$

Finally, the above diagram provides an alliance $(\varkappa, \varrho_\diamond \circ \varrho_\flat, \varrho_\bullet \circ \varrho_\dagger, \theta) : p \rightsquigarrow \overline{p}$ □

3.3.5.3. *Morphisms of alliances of vertebrae.* Let $\mathcal{C}$ be category and $\mathfrak{a} : v \rightsquigarrow \overline{v}$ and $\mathfrak{a}_* : v_* \rightsquigarrow \overline{v}_*$ be two alliances of vertebrae in $\mathcal{C}$ such that $\mathfrak{a} = (\mathfrak{p}, u)$ and $\mathfrak{a}_* = (\mathfrak{p}, u_*)$. A *morphism of alliances of vertebrae* $\mathfrak{a} \Rightarrow \mathfrak{a}_*$ consists of two morphisms of vertebrae $w : v \curvearrowright v_*$ and $\overline{w} : \overline{v} \curvearrowright \overline{v}_*$ for which the next diagram commute in $\mathcal{C}$.

(3.33)
$$\begin{tikzcd}
\overline{\mathbb{D}}'_* \arrow[r, "u_*"] \arrow[d, "\overline{w}"'] & \mathbb{D}'_* \arrow[d, "w"] \\
\overline{\mathbb{D}}' \arrow[r, "u"'] & \mathbb{D}'
\end{tikzcd}$$

This is also equivalent to saying that the equation $\mathfrak{a} \circledcirc w = \overline{w} \circledcirc \mathfrak{a}_*$ holds in $\mathbf{Ally}(\mathcal{C})$ when the arrows w and $\overline{w}$ in $\mathbf{Vert}(\mathcal{C})$ are seen as alliances of vertebrae. Thus, a morphism of alliances is a particular morphism in the arrow category of $\mathbf{Ally}(\mathcal{C})$. The category whose objects are alliances of vertebrae and whose arrows are morphisms of vertebrae will be denoted by $\mathbf{Alov}(\mathcal{C})$.

3.3.5.4. *Morphisms of conjugations of vertebrae.* Let $\mathcal{C}$ be category and consider two conjugations of spines $(v, \mathfrak{a}, \overline{v})$ and $(v_*, \mathfrak{a}_*, \overline{v}_*)$ defined along the same pairs $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ and $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$ and where $\mathfrak{a} : \ddot{v} \rightsquigarrow \widehat{v}$ and $\mathfrak{a}_* : \ddot{v}_* \rightsquigarrow \widehat{v}_*$. Suppose that $\mathfrak{a}$ and $\mathfrak{a}_*$ are of the form $(\mathfrak{p}, u)$ and $(\mathfrak{p}, u_*)$, respectively. A *morphism of conjugations* from the former to the latter consists of two morphisms of framings of vertebrae $(w, \ddot{w}) : (v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \ddot{v} \curvearrowright (v_*, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \ddot{v}_*$ and $(\widehat{w}, \overline{w}) : (\widehat{v}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright \overline{v} \curvearrowright (\widehat{v}_*, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright \overline{v}_*$ in $\mathbf{Fov}(\mathcal{C})$ such that the pair $(\ddot{w}, \widehat{w})$ defines a morphism of alliances of vertebrae $\mathfrak{a} \Rightarrow \mathfrak{a}_*$ in $\mathbf{Alov}(\mathcal{C})$. This notion of morphism induces a category $\mathbf{Conj}(\mathcal{C})$ whose objects are the conjugations of vertebrae in $\mathcal{C}$.

3.3.5.5. *From conjugations to strong correspondences.* Let $\mathcal{C}$ be a category. The aim of this section is to define a functor $\mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$. Let us first define the mapping on objects. Consider a conjugation of vertebrae $(v, \mathfrak{a}, \overline{v})$ defined along some pairs $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ and $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$, whose components will be denoted by $(\mathbb{S}_{\_}, \varrho_{\_})$, $(\varkappa_{\_}, \varrho_{\_}, \varrho'_{\_}, \varkappa'_{\_}, \tau_{\_})$ and $(\overline{\mathbb{S}}_{\_}, r_{\_})$, respectively (see section 3.3.5.2 for the notations of the respective sources and targets). We will suppose $\mathfrak{a} : \ddot{v} \rightsquigarrow \widehat{v}$ where $\mathfrak{a} = (\varkappa, \varrho, \varrho', \varkappa', u)$ and denote by π and $\widehat{\pi}$ the cylinder transitions of the first and second framings of along $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$ and $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$. First is recalled the version of diagram (3.16) for the framing of vertebrae $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \ddot{v}$ on left-hand side of the following implication. On the right-hand side is the resulting diagram after a pre-composition with the discal transitions

$\varrho_\flat$ and $\varrho_\dagger$ on the top and left faces, respectively.

(3.34)
$$\begin{tikzcd}
\ddot{\mathbb{S}}' \arrow[dr, "\ddot{\beta}"] & \mathbb{D}_1^\diamond \arrow[l, "\ddot{\delta}_2"'] \arrow[r, "\beta_\diamond\circ\delta_1^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_1^\bullet \arrow[u, "\ddot{\delta}_1"] \arrow[d, "\beta_\bullet\circ\delta_1^\bullet"'] & \ddot{\mathbb{D}}' \arrow[dr, "\pi"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"] & & \mathbb{G}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\ddot{\mathbb{S}}' \arrow[dr, "\ddot{\beta}"] & \mathbb{D}_2^\flat \arrow[l, "\ddot{\delta}_2\circ\varrho_\flat"'] \arrow[r, "\beta_\flat\circ\delta_2^\flat"] & \mathbb{D}'_\flat \arrow[dd, "\iota^\diamond\circ\tau_\flat"] \\
\mathbb{D}_2^\dagger \arrow[u, "\ddot{\delta}_1\circ\varrho_\dagger"] \arrow[d, "\beta_\dagger\circ\delta_2^\dagger"'] & \ddot{\mathbb{D}}' \arrow[dr, "\pi"] & \\
\mathbb{D}'_\dagger \arrow[rr, "\iota^\bullet\circ\tau_\dagger"] & & \mathbb{G}
\end{tikzcd}$$

Note that the occurence of the arrows $\tau_\flat$ and $\tau_\dagger$ in the right-hand diagram comes from the use of the commutative square $\mathbf{triv}(\mathfrak{a}_\flat)$ and $\mathbf{triv}(\mathfrak{a}_\dagger)$. It turns out that the right-hand side diagram of (3.34) defines a universal problem for the version of diagram (3.15) associated with the framing $(\widehat{v}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright \overline{v}$, which is given below in diagram (3.35). This follows from the factorisations $\varrho_\flat = \varrho \circ r_\flat$ and $\varrho_\dagger = \varrho' \circ r_\dagger$ given by conjugability (see section 3.3.5.1) and the identities $u \circ \widehat{\beta} \circ \widehat{\delta}_2 = \ddot{\beta} \circ \ddot{\delta}_2 \circ \varrho$ and $u \circ \widehat{\beta} \circ \widehat{\delta}_1 = \ddot{\beta} \circ \ddot{\delta}_1 \circ \varrho'$ given by the alliance $\mathfrak{a} : \ddot{v} \rightsquigarrow \widehat{v}$.

(3.35)
$$\begin{tikzcd}
\widehat{\mathbb{S}}' \arrow[dr, "\widehat{\beta}"] & \mathbb{D}_2^\flat \arrow[l, "\widehat{\delta}_2\circ r_\flat"'] \arrow[r, "\beta_\flat\circ\delta_2^\flat"] & \mathbb{D}'_\flat \arrow[dd, "\widehat{\iota}^\flat"] \\
\mathbb{D}_2^\dagger \arrow[u, "\widehat{\delta}_1\circ r_\dagger"] \arrow[d, "\beta_\dagger\circ\delta_2^\dagger"'] & \widehat{\mathbb{D}}' \arrow[dr, "\widehat{\iota}"] & \\
\mathbb{D}'_\dagger \arrow[rr, "\widehat{\iota}^\dagger"] & & \widehat{\mathbb{G}}
\end{tikzcd}$$

It then follows the existence of a canonical arrow $\zeta : \widehat{\mathbb{G}} \to \mathbb{G}$ making the following three diagrams commute.

(3.36)
$$\begin{tikzcd}
\mathbb{D}'_\flat \arrow[r, "\iota^\diamond\circ\tau_\flat"] \arrow[dr, "\widehat{\iota}^\flat"'] & \mathbb{G} \\
& \widehat{\mathbb{G}} \arrow[u, dashed, "\zeta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathbb{D}}' \arrow[r, "\pi\circ u"] \arrow[dr, "\widehat{\iota}"'] & \mathbb{G} \\
& \widehat{\mathbb{G}} \arrow[u, dashed, "\zeta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'_\dagger \arrow[r, "\iota^\bullet\circ\tau_\dagger"] \arrow[dr, "\widehat{\iota}^\dagger"'] & \mathbb{G} \\
& \widehat{\mathbb{G}} \arrow[u, dashed, "\zeta"']
\end{tikzcd}$$

Now, consider the version of diagram (3.16) for the framing $(\widehat{v}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright \overline{v}$ (see the left-hand side diagram of (3.37)) and apply the leftmost and rightmost relations of (3.36) on it after post-composing with ζ. This provides the right-hand side commutative diagram of (3.37).

(3.37)
$$\begin{tikzcd}
\overline{\mathbb{S}}' \arrow[dr, "\overline{\beta}"] & \mathbb{D}_1^\flat \arrow[l, "\overline{\delta}_2"'] \arrow[r, "\beta_\flat\circ\delta_1^\flat"] & \mathbb{D}'_\flat \arrow[dd, "\widehat{\iota}^\flat"] \\
\mathbb{D}_1^\dagger \arrow[u, "\overline{\delta}_1"] \arrow[d, "\beta_\dagger\circ\delta_1^\dagger"'] & \overline{\mathbb{D}}' \arrow[dr, "\widehat{\pi}"] & \\
\mathbb{D}'_\dagger \arrow[rr, "\widehat{\iota}^\dagger"'] & & \widehat{\mathbb{G}}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\overline{\mathbb{S}}' \arrow[dr, "\overline{\beta}"] & \mathbb{D}_1^\flat \arrow[l, "\overline{\delta}_2"'] \arrow[r, "\beta_\flat\circ\delta_1^\flat"] & \mathbb{D}'_\flat \arrow[dd, "\iota^\diamond\circ\tau_\flat"] \\
\mathbb{D}_1^\dagger \arrow[u, "\overline{\delta}_1"] \arrow[d, "\beta_\dagger\circ\delta_1^\dagger"'] & \overline{\mathbb{D}}' \arrow[dr, "\zeta\circ\widehat{\pi}"] & \\
\mathbb{D}'_\dagger \arrow[rr, "\iota^\bullet\circ\tau_\dagger"'] & & \mathbb{G}
\end{tikzcd}$$

Lastly, if we consider the version of diagram (3.15) for the framing $(v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \widehat{v}$, we obtain the left-hand side of (3.38). Pre-composing this diagram with the codiscal transitions $\varrho'_\flat$ and

$\varrho'_\dagger$ on the top and left faces then gives the diagram on the right.

(3.38)
$$\begin{tikzcd}
\mathbb{S}' & \mathbb{D}_2^\diamond \arrow[l, "\delta_2\circ\varrho_\diamond"'] \arrow[r, "\beta_\diamond\circ\delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_2^\bullet \arrow[u, "\delta_2\circ\varrho_\bullet"] \arrow[d, "\beta_\bullet\circ\delta_2^\bullet"'] & \mathbb{D}' \arrow[dr, "\iota"] \arrow[from=ul, "\beta"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"'] & & \mathbb{G}
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\mathbb{S}' & \mathbb{D}_1^\flat \arrow[l, "\delta_2\circ\varrho_\diamond\circ\varrho'_\flat"'] \arrow[r, "\beta_\flat\circ\delta_1^\flat"] & \mathbb{D}'_\flat \arrow[dd, "\iota^\diamond\circ\tau_\flat"] \\
\mathbb{D}_1^\dagger \arrow[u, "\delta_1\circ\varrho_\bullet\circ\varrho'_\dagger"] \arrow[d, "\beta_\dagger\circ\delta_1^\dagger"'] & \mathbb{D}' \arrow[dr, "\iota"] \arrow[from=ul, "\beta"] & \\
\mathbb{D}'_\dagger \arrow[rr, "\iota^\bullet\circ\tau_\dagger"'] & & \mathbb{G}
\end{tikzcd}$$

Again, the occurence of the arrows $\tau_\flat$ and $\tau_\dagger$ in the right-hand diagram comes from the use of the commutative square $\mathbf{triv}(\mathfrak{a}_\flat)$ and $\mathbf{triv}(\mathfrak{a}_\dagger)$. Interestingly, the respective right diagrams of (3.37) and (3.38) provide two solutions for the following universal problem over the pushout $\mathbb{S}'$, namely $\zeta\circ\widehat{\pi}\circ\overline{\beta}$ and $\iota\circ\beta\circ\theta$, where θ is the codiscal transition of the alliance of prevertebrae, say $\mathfrak{p}_\ddagger$, defined by Proposition 3.64 (see diagram (3.32) for the same notations).

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\overline{\gamma}"'] & \mathbb{D}_1^\flat \arrow[d, "\overline{\delta}_1"] \arrow[dr, bend left, "\iota^\diamond\circ\tau_\diamond\circ\beta_\flat\circ\delta_1^\flat"] & \\
\mathbb{D}_2^\dagger \arrow[r, "\overline{\delta}_2"] \arrow[rr, bend right, "\iota^\bullet\circ\tau_\bullet\circ\beta_\dagger\circ\delta_1^\dagger"'] & \overline{\mathbb{S}}' \arrow[r, dashed] & \mathbb{G}
\end{tikzcd}$$

By universality, we deduce the identity $\iota\circ\beta\circ\theta = \zeta\circ\widehat{\pi}\circ\overline{\beta}$. This equality defines a strong correspondence of vertebrae in $\mathbf{Scov}(\mathcal{C})$ with messenger $\mathbb{G}$ as follows.

(3.39)
$$\mathcal{S}_{\mathrm{cor}}(v,\mathfrak{a},\overline{v}) := \quad (\mathfrak{p}_\ddagger, \iota, \zeta\circ\widehat{\pi}) \vdash p\cdot\beta \mathrel{\ddot{\simeq}} \overline{p}\cdot\overline{\beta}$$

We are now going to send every morphism $(w,\ddot{w},\widehat{w},\overline{w}) : (v,\mathfrak{a},\overline{v}) \curvearrowright (v_*,\mathfrak{a}_*,\overline{v}_*)$ defined along the same pairs $\mathfrak{v}_\diamond,\mathfrak{v}_\bullet$, $\mathfrak{a}_\flat,\mathfrak{a}_\dagger$ and $\mathfrak{v}_\flat,\mathfrak{v}_\dagger$ in $\mathbf{Conj}(\mathcal{C})$ to a morphism of correspondences $\mathcal{S}_{\mathrm{cor}}(v,\mathfrak{a},\overline{v}) \Rightarrow \mathcal{S}_{\mathrm{cor}}(v_*,\mathfrak{a}_*,\overline{v}_*)$ in $\mathbf{Scov}(\mathcal{C})$. This morphism will arise in the following form.

(3.40)
$$\begin{tikzcd}
\mathbb{D}'_* \arrow[r, "\iota_*"] \arrow[d, "w"'] & \mathbb{G}_* \arrow[d, "\kappa(w)"] & \overline{\mathbb{D}}'_* \arrow[l, "\zeta_*\circ\widehat{\pi}_*"'] \arrow[d, "\overline{w}"] \\
\mathbb{D}' \arrow[r, "\iota"] & \mathbb{G} & \overline{\mathbb{D}}' \arrow[l, "\zeta\circ\widehat{\pi}"']
\end{tikzcd}$$

First, considering the versions of diagram (3.18) and diagram (3.19) for the morphism of framings $(w,\ddot{w}) : (v,\mathfrak{v}_\diamond,\mathfrak{v}_\bullet)\rhd\ddot{v} \curvearrowright (v_*,\mathfrak{v}_\diamond,\mathfrak{v}_\bullet)\rhd\ddot{v}_*$ and the version of diagram (3.36) for the image $\mathcal{S}_{\mathrm{cor}}(v_*,\mathfrak{a}_*,\overline{v}_*)$ provides the following diagrams.

$$\begin{tikzcd}
\mathbb{D}'_\diamond \arrow[r, "\iota^\diamond"] \arrow[dr, "\iota_*^\diamond"'] & \mathbb{G} \\
& \mathbb{G}_* \arrow[u, "\kappa(w)"']
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{D}'_\bullet \arrow[r, "\iota^\bullet"] \arrow[dr, "\iota_*^\bullet"'] & \mathbb{G} \\
& \mathbb{G}_* \arrow[u, "\kappa(w)"']
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{D}' \arrow[r, "\iota"] & \mathbb{G} \\
\mathbb{D}'_* \arrow[u, "w"] \arrow[r, "\iota_*"'] & \mathbb{G}_* \arrow[u, "\kappa(w)"']
\end{tikzcd}
\quad
\begin{tikzcd}
\ddot{\mathbb{D}}' \arrow[r, "\pi"] & \mathbb{G} \\
\ddot{\mathbb{D}}'_* \arrow[u, "\ddot{w}"] \arrow[r, "\pi_*"'] & \mathbb{G}_* \arrow[u, "\kappa(w)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}'_\flat \arrow[r, "\iota_*^\diamond\circ\tau_\flat"] \arrow[dr, "\widehat{\iota}_*^\flat"'] & \mathbb{G}_* \\
& \widehat{\mathbb{G}}_* \arrow[u, dashed, "\zeta_*"']
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{D}'_\dagger \arrow[r, "\iota_*^\bullet\circ\tau_\dagger"] \arrow[dr, "\widehat{\iota}_*^\dagger"'] & \mathbb{G}_* \\
& \widehat{\mathbb{G}}_* \arrow[u, dashed, "\zeta_*"']
\end{tikzcd}
\quad
\text{(blank)}
\quad
\begin{tikzcd}
\widehat{\mathbb{D}}'_* \arrow[r, "\pi_*\circ u_*"] \arrow[dr, "\widehat{\iota}_*"'] & \mathbb{G}_* \\
& \widehat{\mathbb{G}}_* \arrow[u, dashed, "\zeta_*"']
\end{tikzcd}$$

Then, considering the version of diagram (3.36) for the image $\mathcal{S}_{\mathrm{cor}}(v,\mathfrak{a},\overline{v})$ and the versions of diagrams (3.18, 3.19) for the morphism of framings $(\widehat{w},\overline{w}) : (\widehat{v},\mathfrak{v}_\flat,\mathfrak{v}_\dagger)\rhd\overline{v} \curvearrowright (\widehat{v}_*,\mathfrak{v}_\flat,\mathfrak{v}_\dagger)\rhd\overline{v}_*$

provides the following diagrams.

$$
\begin{tikzcd}
\mathbb{D}'_\flat \arrow[r, "\iota^\diamond\circ\tau_\flat"] \arrow[dr, "\widehat{\iota}^\flat"'] & \mathbb{G} \\
 & \widehat{\mathbb{G}}_* \arrow[u, dashed, "\zeta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'_\dagger \arrow[r, "\iota^\bullet\circ\tau_\dagger"] \arrow[dr, "\widehat{\iota}^\dagger"'] & \mathbb{G} \\
 & \widehat{\mathbb{G}} \arrow[u, dashed, "\zeta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathbb{D}}' \arrow[r, "\pi\circ u"] \arrow[dr, "\widehat{\iota}"'] & \mathbb{G} \\
 & \widehat{\mathbb{G}} \arrow[u, dashed, "\zeta"']
\end{tikzcd}
\qquad
\text{(blank)}
$$

$$
\begin{tikzcd}
\mathbb{D}'_\flat \arrow[r, "\widehat{\iota}^\flat"] \arrow[dr, "\widehat{\iota}^\flat_*"'] & \widehat{\mathbb{G}} \\
 & \widehat{\mathbb{G}}_* \arrow[u, "\kappa(\widehat{w})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'_\dagger \arrow[r, "\widehat{\iota}^\dagger"] \arrow[dr, "\widehat{\iota}^\dagger_*"'] & \widehat{\mathbb{G}} \\
 & \widehat{\mathbb{G}}_* \arrow[u, "\kappa(\widehat{w})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathbb{D}}' \arrow[r, "\widehat{\iota}"] & \widehat{\mathbb{G}} \\
\widehat{\mathbb{D}}'_* \arrow[u, "\widehat{w}"] \arrow[r, "\widehat{\iota}_*"'] & \widehat{\mathbb{G}}_* \arrow[u, "\kappa(\widehat{w})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{D}}' \arrow[r, "\widehat{\pi}"] & \widehat{\mathbb{G}} \\
\overline{\mathbb{D}}'_* \arrow[u, "\overline{w}"] \arrow[r, "\widehat{\pi}_*"'] & \widehat{\mathbb{G}}_* \arrow[u, "\kappa(\widehat{w})"']
\end{tikzcd}
$$

After pasting the previous two sets of diagrams as suggested by their arrangements and using the version of diagram (3.33) for the pair $(\ddot{w}, \widehat{w})$, it follows that $\kappa(w)\circ\zeta_*$ and $\zeta\circ\kappa(\widehat{w})$ are solutions of the following universal problem over diagram (3.35).

$$
\begin{tikzcd}
\mathbb{D}'_\flat \arrow[r, "\iota^\diamond\circ\tau_\flat"] \arrow[dr, "\widehat{\iota}^\flat_*"'] & \widehat{\mathbb{G}} \\
 & \widehat{\mathbb{G}}_* \arrow[u, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathbb{D}}'_* \arrow[r, "\pi\circ\widehat{w}\circ u"] \arrow[dr, "\widehat{\iota}_*"'] & \widehat{\mathbb{G}} \\
 & \widehat{\mathbb{G}}_* \arrow[u, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}'_\dagger \arrow[r, "\iota^\bullet\circ\tau_\dagger"] \arrow[dr, "\widehat{\iota}^\dagger_*"'] & \widehat{\mathbb{G}} \\
 & \widehat{\mathbb{G}}_* \arrow[u, dashed]
\end{tikzcd}
$$

By uniqueness, this implies an equality $\kappa(w)\circ\zeta_* = \zeta\circ\kappa(\widehat{w})$. Using this last equation with the two squares that were not subject to a pasting in the previous process (pasting with a blank diagram), we finally deduce the commutative diagram (3.40). The functoriality of $\mathcal{S}_{\mathrm{cor}}$ finally follows from that of the construction $\kappa(\_)$ (see section 3.3.2.8). This therefore defines a functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$.

3.3.5.6. *Hom-language for conjugation of vertebrae.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $(v, \mathfrak{a}, \overline{v})$ be a conjugation of vertebrae along pairs $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ and $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$ as defined in section 3.3.5.2. We will describe the strong correspondence $\mathcal{S}_{\mathrm{orr}}(v, \mathfrak{a}, \overline{v})$ by the data $(\mathfrak{p}_\ddagger, u, \overline{u}) \vdash v \mathrel{\widetilde{\asymp}} \overline{v}$ where the alliance of prevertebrae $\mathfrak{p}_\ddagger$ will be supposed to be of the form $(\varkappa, \rho, \rho', \theta) : p \rightsquigarrow \overline{p}$. According to Proposition 3.47, the framing of (nodes of) vertebrae $\mathfrak{f} := (v, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \ddot{v}$ provides a tubular operator

$$T^{e_\bullet}_{e_\diamond} :\ \mathcal{C}(v, X)(x, y) \longrightarrow \mathcal{C}(\ddot{v}, X)(a, b) \tag{3.41}$$

for every pair of paths $e_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a$ and $e_\bullet : \varrho_\bullet \cdot x \sim_{v_\bullet} b$. Then, Proposition 3.15 implies that the alliance $\mathfrak{a} : \ddot{v} \rightsquigarrow \widehat{v}$ induces a metafunction of the following form.

$$\mathcal{C}(\mathfrak{a}, X) :\ \mathcal{C}(\ddot{v}, X)(a, b) \longrightarrow \mathcal{C}(\widehat{v}, X)(\varrho\cdot a, \varrho'\cdot b) \tag{3.42}$$

By Proposition 3.15 and Remark 3.7, the two reflections $\mathfrak{a}_\flat : v_\diamond \rightsquigarrow v^{\mathrm{rv}}_\flat$ and $\mathfrak{a}_\dagger : v_\bullet \rightsquigarrow v^{\mathrm{rv}}_\dagger$ turn the paths $e_\diamond$ and $e_\bullet$ into two paths as follows.

$$\tau_\flat\cdot e_\diamond : \varrho'_\flat\cdot a \sim_{v_\flat} (\varrho_\flat\varrho_\flat)\cdot x \qquad\qquad \tau_\dagger\cdot e_\bullet : \varrho'_\dagger\cdot b \sim_{v_\diamond} (\varrho_\dagger\varrho_\dagger)\cdot y$$

By conjugability, we know that the factorisations $\varrho'_\flat = \varrho\circ r_\flat$ and $\varrho'_\dagger = \varrho'\circ r_\dagger$ hold in $\mathcal{C}$. Besides, Proposition 3.64 implies the identities $\rho = \varrho_\diamond\circ\varrho_\flat$ and $\rho' = \varrho_\bullet\circ\varrho_\dagger$. This gives the two following expressions.

$$\tau_\flat\cdot e_\diamond : r_\flat\cdot(\varrho\cdot a) \sim_{v_\flat} \rho\cdot x \qquad\qquad \tau_\dagger\cdot e_\bullet : r_\dagger\cdot(\varrho'\cdot b) \sim_{v_\dagger} \rho'\cdot y$$

Because we are provided with a framing $\widehat{\mathfrak{f}} := (\widehat{v}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright \overline{v}$ where $\mathfrak{v}_\flat = (\mathbb{S}_\flat, r_\flat)$ and $\mathfrak{v}_\dagger = (\mathbb{S}_\dagger, r_\dagger)$, the previous two paths induce a tubular operator as follows.

$$T^{\tau_\dagger\cdot e_\bullet}_{\tau_\flat\cdot e_\diamond} :\ \mathcal{C}(\widehat{v}, X)(\varrho\cdot a, \varrho'\cdot b) \longrightarrow \mathcal{C}(\overline{v}, X)(\rho\cdot x, \rho'\cdot y) \tag{3.43}$$

We can see that the previous three arrows given in (3.41), (3.42) and (3.43) form a composable set of metafunctions. We will denote their composite as follows.

$$U^{e_\bullet}_{e_\diamond} : \mathcal{C}(v,X)(x,y) \to \mathcal{C}(\overline{v},X)(\rho\cdot x,\rho'\cdot y)$$

Consider a v-path $h : x \sim_v y$ in X. By construction, the metafunction $U^{e_\bullet}_{e_\diamond}$ maps the path $h \in \mathcal{C}(v,X)(x,y)$ to the $\overline{v}$-path given on the left-hand side of equation (3.44). The series of equations that then follows shows that this path may be expressed in the form of a $\overline{v}$-path $(\widehat{\pi}\zeta)\cdot h'$ where h' is the canonical arrow $e_\diamond \star h \star e_\bullet$ defined in section 3.3.2.5 for the framing $\mathfrak{f} := (v,\mathfrak{v}_\diamond,\mathfrak{v}_\bullet) \triangleright \ddot{v}$.

$$\Big\lceil(\tau_\flat\cdot e_\diamond)\Big(u\cdot(e_\diamond h e_\bullet)_{\mathfrak{f}}\Big)(\tau_\dagger\cdot e_\bullet)\Big\rceil_{\widehat{\mathfrak{f}}} = \Big\lceil(\tau_\flat\cdot e_\diamond)\Big((u\pi)\cdot h'\Big)(\tau_\dagger\cdot e_\bullet)\Big\rceil_{\widehat{\mathfrak{f}}} \tag{3.44}$$

$$= \Big\lceil((\tau_\flat\iota^\diamond)\cdot h')\Big((u\pi)\cdot h'\Big)((\tau_\dagger\iota^\bullet)\cdot h')\Big\rceil_{\widehat{\mathfrak{f}}} \tag{3.45}$$

$$= \lceil(\tau_\flat\iota^\diamond)(u\pi)(\tau_\dagger\iota^\bullet)\rceil_{\widehat{\mathfrak{f}}}\cdot h' \tag{3.46}$$

$$= (\widehat{\pi}\zeta)\cdot h' \tag{3.47}$$

Equation (3.44) and equation (3.45) are given by the notations set up in section 3.3.2.5. Equation (3.46) follows from Remark 3.46 and the conventions set up by section 3.2.1.1. Finally, equation (3.47) is given by the universal property involved in the definition of the diagrams of (3.36) . Note that the construction made in section 3.3.2.5 also implies the identity $\iota\cdot h' = h$ (see middle diagram of (3.17)). The preceding discussion then shows the next proposition.

Proposition 3.65. *Any v-path h in X and its image $U^{e_\bullet}_{e_\diamond}(h)$ may be expressed in terms of a v-path $\iota\cdot h'$ and a $\overline{v}$-path $(\widehat{\pi}\zeta)\cdot h'$ where the pair $(\iota,\widehat{\pi}\zeta)$ forms a strong correspondence according to equation (3.39) and $h' = e_\diamond \star h \star e_\bullet$.*

3.3.6. Allied and extended nodes of spines.

3.3.6.1. *Alliance of prespines.* Let $\mathcal{C}$ be category and n a non-negative integer. An *alliance of prespines* of degree n consists of two prespines $P = (p_k)$ and $\overline{P} = (\overline{p}_k)$ of degrees n and an alliance of prevertebrae of the form

$$\mathfrak{p}_k := (\varkappa_k,\varrho_k,\varrho'_k,\varkappa_{k+1}) : p_k \rightsquigarrow \overline{p}_k \tag{3.48}$$

for every $0 \leq k \leq n$. Note that the cospherical transition of an alliance $\mathfrak{p}_k$ is the spherical transition of the alliance $\mathfrak{p}_{k+1}$. Such a structure will be denoted as an arrow $(\mathfrak{p}_\_) : P \rightsquigarrow \overline{P}$ and said to *be of degree n*. The alliance of prevertebrae $\mathfrak{p}_n$ will later be called the *head of the alliance* $(\mathfrak{p}_\_) : P \rightsquigarrow \overline{P}$. Finally, for any alliance of prespines $\mathfrak{p}_\_$ of positive degree n, it will later come in handy to denote by $\partial\mathfrak{p}_\_$ the alliance of prespines of degree $n-1$ consisting of the alliances $\mathfrak{p}_0$, $\mathfrak{p}_1$, … and $\mathfrak{p}_{n-1}$.

3.3.6.2. *Allied structures.* Let $\mathcal{C}$ be category and n a non-negative integer. An *alliance of spines* (resp. *nodes of spines*) of degree n consists of two spines $s = P\cdot\beta$ and $\overline{s} = \overline{P}\cdot\overline{\beta}$ (resp. nodes of spines $\sigma = P\cdot\Omega$ and $\overline{\sigma} = \overline{P}\cdot\overline{\Omega}$) of degrees n together with an alliance of prespines $(\mathfrak{p}_\_) : P \rightsquigarrow \overline{P}$ of degree n such that its head is equipped with a structure of alliance of vertebrae $(\mathfrak{p}_n,u) : p_n\cdot\beta \rightsquigarrow \overline{p}_n\cdot\overline{\beta}$ (resp. of nodes of vertebrae $(\mathfrak{p}_n,\phi,u) : p_n\cdot\Omega \rightsquigarrow \overline{p}_n\cdot\overline{\Omega}$). The above structure will be denoted as an arrow $(\mathfrak{p}_\_,u) : s \rightsquigarrow \overline{s}$ (resp. $(\mathfrak{p}_\_,\phi,u) : \sigma \rightsquigarrow \overline{\sigma}$) and said to *be of degree n*.

Remark 3.66. An alliance of nodes of spines $(\mathfrak{p}_\_,\phi,u) : \sigma \rightsquigarrow \overline{\sigma}$ is completely encoded by a collection of alliance of spines $(\mathfrak{p}_\_,u_\beta) : P\cdot\beta \rightsquigarrow \overline{P}\cdot\phi(\beta)$ where β runs over Ω.

Remark 3.67. Consider some non-negative integer $q \leq n$ as well as the notation $\mathtt{I} = \{0 \to 1\}$. Any alliance of spines $\mathfrak{a} := (\mathfrak{p}_\_,u) : s \rightsquigarrow \overline{s}$ of degree n, where $\mathfrak{p}_\_$ is as given in formula (3.48), induces two alliances of vertebrae $\mathfrak{a}_q(0) := (\mathfrak{p}_q,u) : V^q_s(0) \rightsquigarrow V^q_{\overline{s}}(0)$ and

$\mathfrak{a}_q(1) := (\mathfrak{p}_q, \varkappa_{n+1}) : V_s^q(1) \rightsquigarrow V_{\overline{s}}^q(1)$. The alliance of vertebrae associated with $\mathfrak{a}$ also provides a morphism $\mathfrak{a}_q(0) \Rightarrow \mathfrak{a}_q(1)$ in $\mathbf{Alov}(\mathcal{C})$ given by the following commutative square.

$$\begin{tikzcd} \overline{\mathbb{S}}'_n \arrow[r, "\varkappa_{n+1}"] \arrow[d, "\overline{\beta}"'] & \mathbb{S}'_n \arrow[d, "\beta"] \\ \overline{\mathbb{D}}' \arrow[r, "u"'] & \mathbb{D}' \end{tikzcd}$$

The previous data then defines a functor $a_q(\_) : \mathtt{I} \to \mathbf{Alov}(\mathcal{C})$ mapping the objects 0 and 1 to $\mathfrak{a}_q(0)$ and $\mathfrak{a}_q(1)$, respectively. Similarly, an alliance of nodes of spines $\mathfrak{a} := (\mathfrak{p}_\_, \phi, u) : \sigma \rightsquigarrow \overline{\sigma}$ induces two alliances of nodes of vertebrae $\mathfrak{a}_q(0) : V_\sigma^q(0) \rightsquigarrow V_{\overline{\sigma}}^q(0)$ and $\mathfrak{a}_q(1) : V_\sigma^q(1) \rightsquigarrow V_{\overline{\sigma}}^q(1)$ whose components are induced by the construction of Remark 3.66 on $\mathfrak{a} := (\mathfrak{p}_\_, \phi, u) : \sigma \rightsquigarrow \overline{\sigma}$ and the first construction of the present remark. Because of its particular form, the alliance $\mathfrak{a}_q(1)$ turns out to be an alliance of vertebrae $(\mathfrak{p}_q, \varkappa_{n+1}) : V_s^q(1) \rightsquigarrow V_{\overline{s}}^q(1)$ for any $s \in \sigma$ and $\overline{s} \in \overline{\sigma}$.

3.3.6.3. *Zoo of an alliance of nodes of spines.* The zoo associated with an alliance of nodes of spines $\mathfrak{a} := (\mathfrak{p}_\_, \phi, u) : \sigma \rightsquigarrow \overline{\sigma}$ of some degree $n \geq 0$ corresponds to the zoo associated with the alliance of nodes of vertebrae $\mathfrak{a}_n(0)$.

3.3.6.4. *Hom-language for alliance of nodes of spines.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and n be a non-negative integer. Consider an alliance of nodes of spines $\mathfrak{a} := (\mathfrak{p}_\_, \phi, u) : \sigma \rightsquigarrow \overline{\sigma}$ of degree n with $\mathfrak{p}_\_ = (\varkappa_k, \varrho_k, \varrho'_k, \varkappa_{k+1}) : p_k \rightsquigarrow \overline{p}_k$.

Proposition 3.68. *Every relation $h : [x_k]_n \sim_\sigma [y_k]_n$ in X implies a relation $u \cdot h : [\varrho_k \cdot x_k] \sim_{\overline{\sigma}} [\varrho'_k \cdot y_k]_n$ in X where $[\varrho_k \cdot x_k] = \varrho_n \cdot x_n$ and $[\varrho'_k \cdot y_k] = \varrho'_n \cdot y_n$.*

Proof. It suffices to apply the operation $\varrho_k \cdot \_$ and $\varrho'_k \cdot \_$ on the equations of (3.14), for every $0 \leq k \leq n$, and use the relations provided by the alliance $\mathfrak{a}$. ☐

By construction, the alliance of nodes of vertebrae $\mathfrak{a}_q(0)$ is an arrow $V_\sigma^q(0) \rightsquigarrow V_{\overline{\sigma}}^q(0)$. The previous proposition then shows that, for every parallel pair x and y above the base of σ in X, the following diagram commutes.

(3.49)
$$\begin{tikzcd} \mathcal{C}(V_\sigma^q(0), X)(x_q, y_q) \arrow[r, "\mathcal{C}(\mathfrak{a}_q(0),X)"] & \mathcal{C}(V_{\overline{\sigma}}^q(0), X)(\varrho_q \cdot x_q, \varrho_q \cdot y_q) \\ \mathcal{C}(\sigma, X)(x, y) \arrow[r, "\mathcal{C}(\mathfrak{a},X)"'] \arrow[u, "\subseteq"] & \mathcal{C}(\overline{\sigma}, X)(\varrho_n \cdot x, \varrho_n \cdot y) \arrow[u, "\subseteq"'] \end{tikzcd}$$

Note that, by definition, the vertical arrows of the previous diagram become identities when $q = n$. This last point together with Remark 3.21 imply that a morphism $f : X \to Y$ in $\mathcal{C}$ is an intraction for the alliance of nodes of spines $\mathfrak{a} : \sigma \rightsquigarrow \overline{\sigma}$ if and only if for every parallel pair $x \in \mathcal{C}(\mathbb{D}_2, X)$ and $y \in \mathcal{C}(\mathbb{D}_1, X)$ above p, if the bottom left class of diagram (3.50) is non-empty, then so is the right top class.

(3.50)
$$\begin{tikzcd} \mathcal{C}(\sigma, X)(x, y) \arrow[r, "\mathcal{C}(\mathfrak{a},X)"] \arrow[d, "f"'] & \mathcal{C}(\overline{\sigma}, X)(\varrho'_n \cdot x, \varrho'_n \cdot y) \arrow[d, "f"] \\ \mathcal{C}(\sigma, Y)(f(x), f(y)) \arrow[r, "\mathcal{C}(\mathfrak{a},X)"'] & \mathcal{C}(\overline{\sigma}, Y)(\varrho'_n \cdot f(x), \varrho'_n \cdot f(y)) \end{tikzcd}$$

3.3.6.5. *Extended structures.* Let $\mathcal{C}$ be a category and n be a positive integer. An *extended spine (resp. node of spines) of degree* n in $\mathcal{C}$ consists of a spine s of degree $n-1$ and a spine $\overline{s}$ (resp. node of spines $\overline{\sigma}$) of degree n equipped with an alliance of spines $(\mathfrak{p}_{\_}, \varrho) : s \rightsquigarrow s_*$ of degree $n-1$ where s_* denotes the spinal seed of $\overline{s}$ (resp. spinal seed of $\overline{\sigma}$) . This structure will later be denoted as an arrow $(\mathfrak{p}_{\_}, \varrho) : s \overset{\mathrm{EX}}{\rightsquigarrow} \overline{s}$ (resp. $(\mathfrak{p}_{\_}, \varrho) : s \overset{\mathrm{EX}}{\rightsquigarrow} \overline{\sigma}$) and said to *be of degree* n.

Remark 3.69. Considering the notations $s := P \cdot \gamma$, $\overline{\sigma} := \overline{P} \cdot \Omega$ and $\overline{P} = (\overline{p}_k)$, the extended node of spines $(\mathfrak{p}_{\_}, \varrho)$ induces an extended node of vertebrae $\gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{p}_n \cdot \Omega$ encoded by the pair $(\varkappa_n, \varrho)$, where $\varkappa_n$ is the cospherical transition of $\mathfrak{p}_{n-1}$.

3.3.6.6. *Zoo of an extended node of spines.* The zoo associated with an extended node of spines $(\mathfrak{p}_{\_}, \varrho) : P \cdot \gamma \overset{\mathrm{EX}}{\rightsquigarrow} \overline{P} \cdot \overline{\Omega}$ corresponds to the zoo associated with its underlying extended nodes of vertebrae $(\varkappa_n, \varrho) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{p}_n \cdot \Omega$ (see Remark 3.69).

3.3.6.7. *Hom-language for extended nodes of spines.* Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and n be a positive integer. Consider an extended node of spines $\varsigma := (\mathfrak{p}_{\_}, \varrho) : s \overset{\mathrm{EX}}{\rightsquigarrow} \overline{\sigma}$ of degree n with $\mathfrak{p}_{\_} = (\varkappa_k, \varrho_k, \varrho'_k, \varkappa_{k+1}) : p_k \rightsquigarrow \overline{p}_k$. The spinal seed of the node of spines $\overline{\sigma}$ will be denoted by s_* and the induced alliance of spines $(\mathfrak{p}_{\_}, \varrho) : s \rightsquigarrow s_*$ will be denoted by $\mathfrak{a}$. It follows from Proposition 3.22 and the construction of section 3.3.1.8 that a morphism $f : X \to Y$ is a surtraction for $\varsigma : s \overset{\mathrm{EX}}{\rightsquigarrow} \overline{\sigma}$ if and only if for every choice of element $x : \mathbf{1} \to \mathcal{C}(s, X)(f(r), f(t))$ where r and t are parallel above the base of s in X, the following dashed arrow exists and makes the following diagram commute.

$$\begin{tikzcd}
 & \mathbf{1} \arrow[r, dashed] \arrow[d, "x"'] & \mathcal{C}(\overline{\sigma}, f)(\varrho_{n-1} \cdot r, \varrho'_{n-1} \cdot t) \arrow[d, "R_{\overline{\sigma}}"] & (h, f(y)) \arrow[d, mapsto] \\
\mathcal{C}(s, Y)(f(r), f(t)) \arrow[rr, "\mathcal{C}(\mathfrak{a}, X)"'] & & \mathcal{C}(s_*, Y)(\varrho_{n-1} \cdot f(r), \varrho'_{n-1} \cdot f(t)) & \varrho \cdot x
\end{tikzcd}$$

To show that the above lifting exists from Proposition 3.22, set $z := \langle r, t\rangle$ and use Remark 3.16 to show the equation $\varkappa'_n \cdot z = \langle \varrho_{n-1} \cdot r, \varrho'_{n-1} \cdot t\rangle$. Then, Proposition 3.14 and section 3.2.1.2 allow one to conclude quite easily. To prove the other direction, take z as in Proposition 3.22 and set $r := \delta_2 \cdot z$ and $t := \delta_1 \cdot z$. Note that the pair formed by r and t is parallel above s in X. Remark 3.16 then provides the following equations.

$$\varkappa'_n \cdot z = \langle (\varrho_{n-1}\delta_2) \cdot z, (\varrho'_{n-1}\delta_1) \cdot z\rangle = \langle \varrho_{n-1} \cdot r, \varrho'_{n-1} \cdot t\rangle$$

Finally, the use of Proposition 3.14 and section 3.2.1.2 allows one to deduce Proposition 3.22 from the above lifting property.

3.3.7. Memories of spines and extended spines. The idea behind the concept of memory is that of a 'pair of sets of vertebrae that partially correspond with each other' in the sense that the correspondences will only appear for the vertebrae that one would like to remember through the various framing that one would like to use and the additional vertebrae are only there to be carried along via the framings, meaning that only their existence suffices. For more intuition, see section 3.1.

3.3.7.1. *Memories of functors.* Let $\mathcal{C}$ be a category, L and L' be two small categories and K be a connected subcategory of L and L'. Consider two functors $A : L \to \mathbf{Vert}(\mathcal{C})$ and $B : L' \to \mathbf{Vert}(\mathcal{C})$. The functor B will be said to *(resp. strongly) remember the functor* A *over* K if it is equipped with a functor $\vartheta : K \to \mathbf{Corov}(\mathcal{C})$ (resp. $\vartheta : K \to \mathbf{Scov}(\mathcal{C})$) with the following mapping rules on objects (left-hand side) and arrows (right-hand side).

$$d \mapsto (\varkappa, u_d, \overline{u}_d) \vdash A(d) \asymp B(d) \qquad (t : d \to d') \mapsto [\kappa_t, A(t), B(t)]$$
$$(\text{resp. } (\mathfrak{p}, u_d, \overline{u}_d) \vdash A(d) \overset{\sim}{\asymp} B(d))$$

Such a structure will be called a (resp. *strong*) *memory* and denoted by $(\varkappa, u, \overline{u}) \vdash A \asymp B$ (resp. $(\mathfrak{p}, u, \overline{u}) \vdash A \mathrel{\tilde{\asymp}} B$) where the triple $(\varkappa, u, \overline{u})$ (resp. the triple $(\mathfrak{p}, u, \overline{u})$) will sometimes be replaced with the symbol ϑ. The connectedness of K ensures that the components $\varkappa$ and $\mathfrak{p}$ used in the definition of ϑ do not vary over K by definition of the morphisms in $\mathbf{Corov}(\mathcal{C})$ and $\mathbf{Scov}(\mathcal{C})$. Let q be a non-negative integer and s and $\overline{s}$ be two spines in $\mathcal{C}$ whose respective prespines are given by P and $\overline{P}$. Both prespines will be supposed to be of non-negative degrees n and m, respectively. The next two definitions assume that $L = L' = \mathtt{I}$ and K is the one-object category containing the terminal object of $\mathtt{I}$ (see section 3.3.3.1 for the definitions of $\mathtt{I}$ and $\mathtt{J}$).

Definition 3.70 (Simple memories of spines)**.** A *simple q-memory of spines* between s and $\overline{s}$ consists of an alliance of prespines $(\mathfrak{p}_{\_}) : \partial^{n-q+1}P \rightsquigarrow \partial^{m-q+1}\overline{P}$ of degree $q-1$ together with a memory of functors of the form $(\varkappa_q, u, \overline{u}) \vdash V_s^q \asymp V_{\overline{s}}^q$ over K, where $\varkappa_q$ is the cospherical transition of $\mathfrak{p}_{q-1}$. Such a simple q-memory of spines will later be denoted as $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \asymp_q \overline{s}$.

Definition 3.71 (Simple strong memories of spines)**.** A *simple strong q-memory of spines* between s and $\overline{s}$ consists of an alliance of prespines $(\mathfrak{p}_{\_}) : \partial^{n-q}P \rightsquigarrow \partial^{m-q}\overline{P}$ of degree q together with a memory of functors of the form $(\mathfrak{p}_q, u, \overline{u}) \vdash V_s^q \mathrel{\tilde{\asymp}} V_{\overline{s}}^q$ over K, where $\mathfrak{p}_q$ is the head of the previous alliance of prespines. Such a simple strong q-memory of spines will later be denoted as a correspondence of vertebrae $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \mathrel{\tilde{\asymp}}_q \overline{s}$.

Remark 3.72. Considering the notation $\mathtt{I} := \{0 \to 1\}$, a simple q-memory of spines is equivalent to requiring a correspondence of vertebrae $(\varkappa_q, u, \overline{u}) \vdash V_s^q(1) \asymp V_{\overline{s}}^q(1)$. Because the vertebrae $V_s^q(1)$ and $V_{\overline{s}}^q(1)$ do not depend on the stems of s and $\overline{s}$ (see section 3.3.3.1), Definition 3.70 may even be viewed as a correspondence of the form $(\varkappa_q, u, \overline{u}) \vdash V_{P\cdot\mathrm{id}}^q(1) \asymp V_{\overline{P}\cdot\mathrm{id}}^q(1)$. The only subtlety is that the functors V_s^q and $V_{\overline{s}}^q$ carries the information of the stems, which will play a substantial role in the notion of framing. This remark also holds for strong q-memories of spines when replacing the symbol $\asymp$ with the symbol $\tilde{\asymp}$.

The integer m is now going to be assumed to be positive. The next two definitions suppose that $L = K = \mathtt{I}$ and $L' = \mathtt{J}$ so that the inclusion $K \subseteq L'$ is given by $i_{\mathtt{I},\mathtt{J}}$ (see the end of section 3.3.3.1).

Definition 3.73 (Extensive memories of spines)**.** An *extensive q-memory of spines* between s and $\overline{s}$ consists of an alliance of prespines $(\mathfrak{p}_{\_}) : \partial^{n-q+1}P \rightsquigarrow \partial^{m-q+1}\overline{P}$ of degree $q-1$ together with a memory of functors of the form $(\varkappa_q, u, \overline{u}) \vdash V_s^q \asymp E_{\overline{s}}^q$ over K, where $\varkappa_q$ is the cospherical transition of $\mathfrak{p}_{q-1}$. Such an extensive q-memory of spines will later be denoted as $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \frown_q \overline{s}$.

Definition 3.74 (Extensive memories of spines)**.** An *extensive strong q-memory of spines* between s and $\overline{s}$ consists of an alliance of prespines $(\mathfrak{p}_{\_}) : \partial^{n-q}P \rightsquigarrow \partial^{m-q}\overline{P}$ of degree q together with a memory of functors of the form $(\mathfrak{p}_q, u, \overline{u}) \vdash V_s^q \mathrel{\tilde{\asymp}} E_{\overline{s}}^q$ over K, where $\mathfrak{p}_q$ is the head of the previous alliance of prespines. Such an extensive strong q-memory of spines will later be denoted as $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \mathrel{\tilde{\frown}}_q \overline{s}$.

Remark 3.75. Because $\overline{s}$ is of positive degree, the spinal seed of $\overline{s}$ exists and will be denoted by s_*. Considering the notation $\mathtt{I} := \{0 \to 1\}$, an extensive q-memory of spines is equivalent to requiring two correspondences of vertebrae $(\varkappa_q, u_0, \overline{u}_0) \vdash V_s^q(0) \asymp V_{s_*}^q(0)$ and $(\varkappa_q, u_1, \overline{u}_1) \vdash V_s^q(1) \asymp V_{s_*}^q(1)$ as well as a morphism of correspondences from the former to the latter given by the commutative diagram, below, for which the notations $s = P \cdot \gamma$ and $s_* = \partial\overline{P} \cdot \gamma_*$ have been used.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "u_1"] \arrow[d, "\gamma"'] & \mathbb{M}_1 \arrow[d, "\kappa"] & \mathbb{S}_* \arrow[l, "\overline{u}_1"'] \arrow[d, "\gamma_*"] \\
\mathbb{D}_2 \arrow[r, "u_0"'] & \mathbb{M}_0 & \mathbb{D}_2^* \arrow[l, "\overline{u}_0"]
\end{tikzcd}
$$

The only subtlety is that the functor $E_{\overline{s}}^q$ carries the information of the antiseed, stem, trivial stem and costem of the spine $\overline{s}$, which will play a substantial role in the next notion of framing. This remark also holds for strong extensive q-memories of spines when replacing the symbol with $\asymp$ the symbol $\stackrel{\sim}{\asymp}$.

Proposition 3.76. *Let s and $\overline{s}$ be two spines in $\mathcal{C}$. Any strong memory of functors $(\mathfrak{p}, u, \overline{u}) \vdash V_s^q \stackrel{\sim}{\asymp} V_{\overline{s}}^q$ over $\mathtt{I}$ gives rise to a memory of functors $(\varkappa', u, \overline{u}) \vdash V_s^{q+1} \asymp V_{\overline{s}}^{q+1}$ over $\mathtt{I}$ where $\varkappa'$ is the cospherical transition of $\mathfrak{p}$. The same result holds for memories defined over a subcategory of $\mathtt{I}$ (instead of $\mathtt{I}$).*

Proof. Represent the category $\mathtt{I}$ by the arrow $\{0 \xrightarrow{t} 1\}$ and consider the notations $s = P \cdot \beta$ and $\overline{s} = \overline{P} \cdot \overline{\beta}$ where $P = (p_k)$ and $\overline{P} = (\overline{p}_k)$. By assumption, the alliance $\mathfrak{p}$ must be of the form $(\varkappa, \varrho, \varrho', \varkappa') : p_q \rightsquigarrow \overline{p}_q$. By definition, a strong memory $(\mathfrak{p}, u, \overline{u}) \vdash V_s^q \stackrel{\sim}{\asymp} V_{\overline{s}}^q$ amounts to considering the next equalities.

$$u_0 \circ \beta \circ \Gamma_q(P) \circ \varkappa' = \overline{u}_0 \circ \overline{\beta} \circ \Gamma_q(\overline{P}) \qquad u_1 \circ \Gamma_q(P) \circ \varkappa' = \overline{u}_1 \circ \Gamma_q(\overline{P})$$

A quick rearrangement of these equations then gives the following ones (see section 3.3.1.3 for the various relations satisfied by Γ_q).

$$\begin{cases} u_0 \circ (\beta \circ \Gamma_{q+1}(P)) \circ \Gamma_{-1}(p_{q+1}) \circ \varkappa' = \overline{u}_0 \circ (\overline{\beta} \circ \Gamma_{q+1}(\overline{P})) \circ \Gamma_{-1}(\overline{p}_{q+1}) \\ u_1 \circ \Gamma_{q+1}(P) \circ \Gamma_{-1}(p_{q+1}) \circ \varkappa' = \overline{u}_1 \circ \Gamma_{q+1}(\overline{P}) \circ \Gamma_{-1}(\overline{p}_{q+1}) \end{cases}$$

This exactly means that the correspondence $(\varkappa', u_d, \overline{u}_d) \vdash V_s^{q+1}(d) \stackrel{\sim}{\asymp} V_{\overline{s}}^{q+1}(d)$ holds for any object d in $\mathtt{I}$. Now, the morphism $[\kappa_t, V_s^q(t), V_{\overline{s}}^q(t)]$ is equivalent to giving a diagram as follows.

$$
\begin{tikzcd}
\mathbb{S}' \arrow[r, "u_1"] \arrow[d, "\beta"'] & \mathbb{M}_1 \arrow[d, "\kappa_t"'] & \overline{\mathbb{S}}' \arrow[l, "\overline{u}_1"'] \arrow[d, "\overline{\beta}"] \\
\mathbb{D}' \arrow[r, "u_0"'] & \mathbb{M}_0 & \overline{\mathbb{D}}' \arrow[l, "\overline{u}_0"]
\end{tikzcd}
$$

Because the previous data do not depend on both q and the structure of P and $\overline{P}$, this is also equivalent to giving a morphism of correspondences of the form $[\kappa_t, V_s^{q+1}(t), V_{\overline{s}}^{q+1}(t)]$. The strong memory $(\mathfrak{p}, u, \overline{u}) \vdash V_s^q \stackrel{\sim}{\asymp} V_{\overline{s}}^q$ thus induces a functor $\mathtt{I} \to \mathbf{Corov}(\mathcal{C})$ encoding a memory $(\varkappa', u, \overline{u}) \vdash V_s^{q+1} \asymp V_{\overline{s}}^{q+1}$. The second part of the statement is straightforward from what precedes. $\square$

The last statement of Proposition 3.76 implies the next two propositions.

Proposition 3.77. *Any simple strong q-memory $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \stackrel{\sim}{\asymp}_q \overline{s}$ gives rise to a simple $(q+1)$-memory $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \asymp_{q+1} \overline{s}$.*

Proposition 3.78. *Any extensive strong q-memory $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \stackrel{\sim}{\frown}_q \overline{s}$ gives rise to an extensive $(q+1)$-memory $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \stackrel{\sim}{\frown}_{q+1} \overline{s}$.*

3.3.7.2. *Mates for memories of functors.* Let $\mathcal{C}$ be a category, L and L' be two small categories and K be a non-empty subcategory of L and L'. Suppose that L' and K are connected categories and consider two functors $A : L \to \mathbf{Vert}(\mathcal{C})$ and $B : L' \to \mathbf{Vert}(\mathcal{C})$. A memory $\vartheta \vdash A \asymp B$ over K in $\mathcal{C}$ will be said to *admit a pair of mates* if its functor $\vartheta : K \to \mathbf{Corov}(\mathcal{C})$ is equipped with a lift to $\mathbf{Mcov}(\mathcal{C})$ as follows.

$$\begin{tikzcd} & \mathbf{Mcov}(\mathcal{C}) \arrow[d, two heads] & (c,\mu) \arrow[d, mapsto] \\ K \arrow[ur, dashed, "M_\vartheta"] \arrow[r, "\vartheta"'] & \mathbf{Corov}(\mathcal{C}) & c \end{tikzcd}$$

Such a memory will later be denoted as $(\vartheta, M_\vartheta) \vdash A \asymp B$.

Remark 3.79. When K has a terminal object, say $\mathbf{1}$, Proposition 3.60 implies that it suffices to equip $\vartheta(\mathbf{1})$ with a pair of mates to be able to lift the whole functor $\vartheta : K \to \mathbf{Corov}(\mathcal{C})$ to $\mathbf{Mcov}(\mathcal{C})$.

Remark 3.80. Because K is non-empty, there exists at least an object in the image of ϑ, say $c \vdash v \asymp \overline{v}$, that admits a pair of mates, say μ. Because K is connected, the definition of a morphism in $\mathbf{Mcov}(\mathcal{C})$ implies that the pairs of vertebrae provided by the pairs of mates in the image of M_ϑ are all equal and *a fortiori* equal to that of μ. Similarly, because L' is connected, the definition of a morphism in $\mathbf{Vert}(\mathcal{C})$ implies that the bases of the vertebrae in the image of B are all equal and *a fortiori* equal to the base of $\overline{v}$, say $\overline{p}$. Now, recall that one of the conditions in order for the pair (c, μ) to admit a framing $(c, \mu) \triangleright c_\dagger$ is that the prevertebra $\overline{p}$ is framed by another one, say p_*, along the vertebrae associated with μ.

$$(c,\mu) \triangleright c_\dagger \quad \Rightarrow \quad (\overline{p}, v_\diamond, v_\bullet) \triangleright p_* \quad \Rightarrow \quad (B(d), v_\diamond, v_\bullet) \triangleright B_*(d)$$

In particular, this implies that the bases of all vertebrae in the image of B, which are equal to $\overline{p}$, are framed by the prevertebra p_* along the vertebrae associated with μ. In other words, because the notion of framing of vertebrae relies on the notion of framing of prevertebrae (see section 3.3.2.4), any framing of a pair (c, μ) in the image of ϑ prepares a potential framing of all the vertebrae in the image of B along the vertebrae of μ. This shows that even if K consists of a small portion of L', a framing over K entails a potential framing over L' by connectedness.

Definition 3.81. A *pair of mates* for either an extensive or a simple (resp. strong) memory of spines is a pair of mates for its underlying memory of functors.

3.3.7.3. *Spans of memories.* Let $\mathcal{C}$ be a category, (ϑ, M_ϑ) be a memory in $\mathcal{C}$ equipped with a pair of mates over some category K and $\vartheta_\dagger$ be a strong memory over K in $\mathcal{C}$. The pair $((\vartheta, M_\vartheta), \vartheta_\dagger)$ will be said to form a *span* over K if both memories ϑ and $\vartheta_\dagger$ are of respective forms $(\varkappa, u, \overline{u}) \vdash A \asymp B$ and $(\mathfrak{p}, u, \overline{u}_\dagger) \vdash A \overset{\sim}{\asymp} B_\dagger$ such that both functors B and $B_\dagger$ are functors $L' \to \mathbf{Vert}(\mathcal{C})$ and $\varkappa$ is equal to the spherical transition of the alliance $\mathfrak{p}$. If the functor A is defined on a category L, the span will be said to be *defined along the triple* (K, L, L') .

Remark 3.82. According to section 3.3.7.2, the use of the pair (ϑ, M_ϑ) implies that the categories K and L' must be non-empty and connected, which is always the case for simple and extensive memories of spine.

Definition 3.83. A *span* of simple (resp. extensive) q-memories of spines consists of an alliance of prespines $\mathfrak{p}$ of degree q, a simple (resp. extensive) q-memory of spines $(\partial\mathfrak{p}_{\text{-}}, u, \overline{u}) \vdash s \asymp_q \overline{s}$ and a simple (resp. extensive) strong q-memory of spines $(\mathfrak{p}_{\text{-}}, u, \overline{u}) \vdash s \overset{\sim}{\asymp}_q s_\dagger$ such that the underlying memories of functors of the two q-memories form a span of memories as defined above.

3.3.7.4. *Framing of memories.* Let $\mathcal{C}$ be a category and $((\vartheta, M_\vartheta), \vartheta_\dagger)$ be a span of memories in $\mathcal{C}$ defined along a triple (K, L, L'). The strong memory $\vartheta_\dagger$ will be said to *frame* the memory ϑ along the lift M_ϑ if it is equipped with a functor $K \to \mathbf{Focov}(\mathcal{C})$ with the mapping rules

$$d \mapsto M_\vartheta(d) \triangleright \vartheta_\dagger(d) \qquad\qquad (t : d \to d') \mapsto [M_\vartheta(t), B_\dagger(t)]$$

and a functor $L \to \mathbf{Fov}(\mathcal{C})$ with the mapping rules

$$d \mapsto (B(d), v_\bullet, v_\diamond) \triangleright B_\dagger(d) \qquad\qquad (t : d \to d') \mapsto (B(t), B_\dagger(t))$$

Recall that the notion of span was the structure chosen to express the data of the category $\mathbf{Focov}(\mathcal{C})$ (see section 3.3.4.9). Also, remark that the last functor does make sense regarding Remark 3.80. The above structure will later be denoted by $(\vartheta, M_\vartheta) \triangleright \vartheta_\dagger$ and referred to as a *framing of memories.*

Definition 3.84. A *framing* of simple (resp. extensive) q-memories of spines consists of a span of simple (resp. extensive) q-memories of spines whose underlying memories of functors form a framing of memories as defined above.

3.3.7.5. *Simple memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer and $\sigma = P \cdot \Omega$ and $\overline{\sigma} = \overline{P} \cdot \overline{\Omega}$ be two nodes of spines in $\mathcal{C}$ of degrees n and m, respectively. A *simple q-memory* between σ and $\overline{\sigma}$ is a simple q-memory of spines between $P \cdot \mathrm{id}$ and $\overline{P} \cdot \mathrm{id}$ (see Remark 3.85). More precisely, such a structure consists of an alliance of prespines $(\mathfrak{p}_\_) : \partial^{n-q+1} P \rightsquigarrow \partial^{m-q+1} \overline{P}$ of degree $q-1$ together with a correspondence of vertebrae

$$(\varkappa_q, u, \overline{u}) \vdash p_q \cdot \Gamma_q(P) \asymp \overline{p}_q \cdot \Gamma_q(\overline{P})$$

where $\varkappa_q$ denotes the cospherical transition of the alliance of prevertebrae $\mathfrak{p}_{q-1}$. The previous data will be referred to by the symbols $(\mathfrak{p}_k, u, \overline{u}) \vdash \sigma \asymp_q \overline{\sigma}$.

Remark 3.85. The point of the above definition is that it defines a structure of simple q-memory of the form $\vartheta \vdash P \cdot \beta \asymp_q \overline{P} \cdot \overline{\beta}$ independent of any choice of stem β and $\overline{\beta}$ in $\mathcal{C}$ (see Remark 3.72). Specifying the stems β and $\overline{\beta}$ will however play a substantial role for the notion of framing of simple q-memories. In particular, we will denote by $\vartheta(\overline{\beta})$ the simple q-memory of spines of type $P \cdot \mathrm{id} \asymp_q \overline{P} \cdot \overline{\beta}$ encoded by the triple $(\varkappa_q, u, \overline{u})$ for every $\overline{\beta} \in \overline{\Omega}$.

3.3.7.6. *Simple strong memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer and $\sigma = P \cdot \Omega$ and $\overline{\sigma} = \overline{P} \cdot \overline{\Omega}$ be two nodes of spines in $\mathcal{C}$ of degrees n and m, respectively. A *simple strong q-memory* between σ and $\overline{\sigma}$ is a simple strong q-memory of spines between $P \cdot \mathrm{id}$ and $\overline{P} \cdot \mathrm{id}$ (see Remark 3.86). More precisely, such a structure consists of an alliance of prespines $(\mathfrak{p}_\_) : \partial^{n-q} P \rightsquigarrow \partial^{m-q} \overline{P}$ of degree q together with a strong correspondence of vertebrae

$$(\mathfrak{p}_q, u, \overline{u}) \vdash p_q \cdot \Gamma_q(P) \stackrel{\sim}{\asymp} \overline{p}_q \cdot \Gamma_q(\overline{P})$$

where $\mathfrak{p}_q$ refers to the head of the previous alliance of prespines. The previous data will be referred to by the symbols $(\mathfrak{p}_\_, u, \overline{u}) \vdash \sigma \stackrel{\sim}{\asymp}_q \overline{\sigma}$.

Remark 3.86. The point of the above definition is that it defines a structure of simple strong q-memory $\vartheta \vdash P \cdot \beta \stackrel{\sim}{\asymp}_q \overline{P} \cdot \overline{\beta}$ independent of any choice of stem β and $\overline{\beta}$ in $\mathcal{C}$ (see Remark 3.72). Specifying the stems β and $\overline{\beta}$ will however play a substantial role for the notion of framing of simple strong q-memories. In particular, we will denote by $\vartheta(\overline{\beta})$ the simple strong q-memory of spines of type $P \cdot \mathrm{id} \asymp_q \overline{P} \cdot \overline{\beta}$ encoded by the triple $(\mathfrak{p}_q, u, \overline{u})$ for every stem $\overline{\beta} \in \overline{\Omega}$.

3.3.7.7. *Mates for simple memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer and $\vartheta := (\mathfrak{p}_{\_}, u, \overline{u}) \vdash \sigma \asymp_q \overline{\sigma}$ denote a simple q-memory of nodes of spines in $\mathcal{C}$ where $\sigma = P \cdot \Omega$ and $\overline{\sigma} = \overline{P} \cdot \overline{\Omega}$. A *pair of mates* for the q-memory ϑ consists of a pair of mates for the underlying correspondence of vertebrae $(\varkappa_q, u, \overline{u})$, where $\varkappa_q$ denotes the cospherical transition of the alliance of prevertebrae $\mathfrak{p}_{q-1}$.

Remark 3.87. This is equivalent to equipping all simple q-memories of spines $\vartheta(\overline{\beta})$ where $\overline{\beta} \in \overline{\Omega}$ with a common pair of mates (see Remark 3.85).

3.3.7.8. *Framing of simple memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer, $\vartheta \vdash \sigma \asymp_q \overline{\sigma}$ be a simple q-memory of nodes of spines in $\mathcal{C}$ equipped with a pair of mates μ and $\vartheta_\dagger \vdash \sigma \stackrel{\sim}{\asymp}_q \overline{\sigma}_\dagger$ be a simple strong q-memory of nodes of spines. The memory $\vartheta_\dagger$ will be said to *frame* the memory ϑ along the pair of mates μ if it is equipped with

1) a metafunction $\psi : \overline{\Omega} \to \overline{\Omega}_\dagger$, called its *framing gear*;
2) a framing of q-memories of spines $(\vartheta(\overline{\beta}), \mu) \triangleright \vartheta_\dagger(\psi(\overline{\beta}))$ for every $\overline{\beta} \in \overline{\Omega}$.

The above structure will later be denoted by the symbols $(\vartheta, \mu) \triangleright_q \vartheta_\dagger$.

Remark 3.88. According to section 3.3.7.4, such a framing involves a simple q-framing of the node of spines σ by the node of spines $\overline{\sigma}$ along the vertebrae of μ (see Definition 3.51 and section 3.3.3.7).

3.3.7.9. *Simple chaining of nodes of spines.* Let $\mathcal{C}$ be a category, σ be a nodes of spines in $\mathcal{C}$ and ℓ and q be non-negative integers. A *simple q-chaining of nodes of spines of length ℓ above σ* consists of a sequence of $\ell + 1$ nodes of spines $\Sigma := \{\sigma_0, \sigma_1, \dots, \sigma_\ell\}$ in $\mathcal{C}$ together with $\ell + 1$ simple strong memories of nodes of spines of the form

$$\vartheta_i := (\mathfrak{p}^i_{\_}, u, u_i) \vdash \sigma \stackrel{\sim}{\asymp}_{q+i} \sigma_i$$

for every $0 \leq i \leq \ell$ and a sequence of ℓ framings of memories as follows.

$$(\vartheta_0, \mu_0) \triangleright_{q+1} (\vartheta_1, \mu_1) \triangleright_{q+2} (\vartheta_2, \mu_2) \triangleright_{q+3} \cdots \triangleright_{q+\ell-1} (\vartheta_{\ell-1}, \mu_{\ell-1}) \triangleright_{q+\ell} \vartheta_\ell$$

Note that the previous sequence of framings of simple memories makes sense by Proposition 3.77 as a simple strong $(q+i)$-memory ϑ_i may be seen as a simple $(q+i+1)$-memory, thus allowing the next framing. Then, section 3.3.4.6 forces all memory ϑ_i to have a common component u as assumed above. In particular, this implies that the messengers $\mathbb{M}_i$ of the correspondences associated with the memories ϑ_i must be equal to one and the same object, say $\mathbb{M}$, for every $0 \leq i \leq \ell$. Such a chaining will later be denoted by $(\sigma, \Sigma, \vartheta_{\_}, \mu_{\_})$, where $\vartheta_{\_}$ stands for the collection of correspondences $\vartheta_0, \dots, \vartheta_\ell$ and $\mu_{\_}$ is the symbol used to denote the pairs of mates along which the framings are done. A chaining such as $(\sigma, \Sigma, \vartheta_{\_}, \mu_{\_})$ will be said to be *convergent* if the degrees of the spines σ_ℓ and σ equal $q + \ell$.

Proposition 3.89. *Suppose that $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ is convergent for $\ell > 0$. Let $h' \in \mathcal{C}(\mathbb{M}, X)$ for some object X in $\mathcal{C}$ and consider (x, y) and (x', y') two parallel pairs above the bases of σ and σ_0, respectively. If $\langle x, y\rangle = u \cdot h'$ and $\langle x', y'\rangle = u_0 \cdot h'$, then there exists a sequence of composable $(q+i)$-tubular operators defined along the pairs of paths $\varpi^i_\diamond \cdot h'$ and $\varpi^i_\bullet \cdot h'$, for every $0 \leq i \leq \ell - 1$, whose composite is of the form $\mathcal{C}(\sigma_0, X)(x', y') \to \mathcal{C}(\sigma_\ell, X)(x, y)$.*

Proof. Suppose that the prespines of the nodes of spines $\sigma := P \cdot \Omega$ and $\sigma_i := P_i \cdot \Omega_i$ are of degrees n and n_i, for every $0 \leq i \leq \ell$, respectively. First, the equalities $\langle x, y\rangle = u \cdot h'$ and $\langle x', y'\rangle = u_0 \cdot h'$ involve two paths of the following forms.

$$u \cdot h' : [x_k]_n \sim_P [y_k]_n \qquad u_0 \cdot h' : [x'_k]_{n_0} \sim_{P_0} [y'_k]_{n_0}$$

For reasons of definition regarding the framing $(\vartheta_0, \mu_0) \triangleright_{q+1} \vartheta_1$, the degree of P_0 must satisfy the inequality $n_0 \geq q+1$. Since the chaining is convergent and $\ell > 0$, the inequality $n > q$ holds as well. It follows from Proposition 3.35 that the previous two paths give the next ones.

$$(3.51) \qquad u \cdot h' : x_{q+1} \sim_{V^{q+1}_{P\cdot\mathrm{id}}(0)} y_{q+1} \qquad\qquad u_0 \cdot h' : x'_{q+1} \sim_{V^{q+1}_{P_0\cdot\mathrm{id}}(0)} y'_{q+1}$$

By assumption, the simple strong q-memory ϑ_0 also provides the following correspondence of vertebrae (see Proposition 3.77 and Remark 3.72).

$$(3.52) \qquad (u, u_0) \vdash V^{q+1}_{P\cdot\mathrm{id}}(1) \asymp V^{q+1}_{P_0\cdot\mathrm{id}}(1)$$

Applying the first part of the discussion of section 3.3.4.8 on the pair of paths given in (3.51) and the correspondence (3.52) shows the existence of two paths $\varpi^0_\diamond \cdot h' : x_{q+1} \sim_{v^0_\diamond} x'_{q+1}$ and $\varpi^0_\bullet \cdot h' : y_{q+1} \sim_{v^0_\bullet} y'_{q+1}$. Because the framing of simple q-memories of nodes of spines $(\vartheta_0, \mu_0) \triangleright_{q+1} \vartheta_1$ involves a simple q-framing of nodes of spines $(\sigma_0, v^0_\diamond, v^0_\bullet) \triangleright_{q+1} \sigma_1$ (see Remark 3.88), the previous pair of paths defines a tubular operator

$$T : \mathcal{C}(\sigma_0, X)(x', y') \to \mathcal{C}(\sigma_1, X)(x^{(2)}, y^{(2)})$$

where $x^{(2)}_{q+1} = x_{q+1}$ and $y^{(2)}_{q+1} = y_{q+1}$ (see section 3.3.3.8). Section 3.3.3.8 also shows that, since the element $\langle x', y' \rangle$ is well-defined, the elements $x^{(2)}$ and $y^{(2)}$ are parallel above the base of σ_1. The definition of $x^{(2)}$ and $y^{(2)}$ discussed in section 3.3.3.8 and the end of section 3.3.4.8 then implies the following equations (see the following discussion for more details).

$$\langle x^{(2)}, y^{(2)} \rangle = \lceil (\varpi^0_\diamond \cdot h') \langle x', y' \rangle (\varpi^0_\bullet \cdot h') \rceil = u_1 \cdot h'$$

The first identity is a consequence of Proposition 3.34 when seeing the middle element as a $(P_1 \cdot \mathrm{id})$-path. The second identity is a consequence of section 3.3.4.8 when seeing the middle element as a $(V^{q+1}_{P_1\cdot\mathrm{id}}(0))$-path. We thus come back to our initial situation, but with the two equalities $\langle x, y \rangle = u \cdot h'$ and $\langle x^{(2)}, y^{(2)} \rangle = u_1 \cdot h'$. Repeating the above arguments $\ell - 1$ times in regard to the memories $\vartheta_1, \dots, \vartheta_{\ell-1}$ then leads to the existence of a chain of ℓ tubular operators along the pairs of paths $\varpi^k_\diamond \cdot h'$ and $\varpi^k_\bullet \cdot h'$, for every $0 \leq k \leq \ell$, whose composite is of the form

$$\mathcal{C}(\sigma_0, X)(x', y') \to \mathcal{C}(\sigma_\ell, X)(x^{(\ell+1)}, y^{(\ell+1)})$$

where $x^{(\ell+1)}_{q+\ell} = x_{q+\ell}$ and $y^{(\ell+1)}_{q+\ell} = y_{q+\ell}$. Since the chaining is convergent, these last two equations may be rewritten as $x^{(\ell+1)} = x$ and $y^{(\ell+1)} = y$. This finally proves the statement. □

3.3.7.10. *Extensive memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer, $s = P \cdot \gamma$ be a spine and $\overline{\sigma} := \overline{P} \cdot \overline{\Omega}$ be a node of spines in $\mathcal{C}$ of positive degree. An *extensive q-memory* between s and $\overline{\sigma}$ is an extensive q-memory between s and $\overline{P} \cdot \mathrm{id}$ (see Remark 3.90). More precisely, if we denote the head of $\overline{P}$ by $\|\gamma_*, \gamma'_*\|$, such a structure consists of an alliance of prespine $(\mathfrak{p}_{\_}) : P \rightsquigarrow \partial\overline{P}$ of degree $q-1$, two correspondences of vertebrae

$$\begin{cases} (\varkappa_q, u, \overline{u}) & \vdash \quad p_q \cdot (\Gamma_q(P)\gamma) \asymp \overline{p}_q \cdot (\Gamma_q(\partial\overline{P})\gamma_*) \\ (\varkappa_q, u_!, \overline{u}_!) & \vdash \quad p_q \cdot \Gamma_q(P) \asymp \overline{p}_q \cdot \Gamma_q(\partial\overline{P}) \end{cases}$$

where $\varkappa_q$ refers to the cospherical transition of the alliance $\mathfrak{p}_{q-1}$ and a morphism of correspondences from the former to the latter encoded by a diagram as follows.

$$\begin{array}{ccccc} \mathbb{S} & \xrightarrow{u_!} & \mathbb{M}_! & \xleftarrow{\overline{u}_!} & \mathbb{S}_* \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\gamma_*} \\ \mathbb{D}_2 & \xrightarrow[u]{} & \mathbb{M} & \xleftarrow[\overline{u}]{} & \mathbb{D}^*_2 \end{array}$$

The previous data will be referred to by the symbols $\binom{u_!, \overline{u}_!}{u, \overline{u}}[\mathfrak{p}_{\_}, \kappa] \vdash s \frown_q \overline{\sigma}$.

Remark 3.90. The point of the above definition is that it defines a structure of extensive q-memory $\vartheta \vdash P \cdot \gamma \asymp_q \overline{P} \cdot \overline{\beta}$ independent of any choice of stem $\overline{\beta}$ in $\mathcal{C}$ (see Remark 3.75). Specifying the stem $\overline{\beta}$ will however play a substantial role for the notion of framing of simple q-memories. We will later denote by $\vartheta(\overline{\beta})$ the extensive q-memory of spines of type $P \cdot \gamma \asymp_q \overline{P} \cdot \overline{\beta}$ encoded by the triple $(\varkappa_q, u, \overline{u})$ for every $\overline{\beta} \in \overline{\Omega}$.

3.3.7.11. *Extensive strong memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer, $s = P \cdot \gamma$ be a spine and $\overline{\sigma} := \overline{P} \cdot \overline{\Omega}$ be a node of spines in $\mathcal{C}$ of positive degree. An *extensive strong q-memory* between s and $\overline{\sigma}$ is an extensive strong q-memory between s and $\overline{P} \cdot \mathrm{id}$ (see Remark 3.91). More precisely, if we denote the head of $\overline{P}$ by $\|\gamma_*, \gamma_*'\|$, such a structure consists of an alliance of prespine $(\mathfrak{p}_{\_}) : P \rightsquigarrow \partial\overline{P}$ of degree q, two strong correspondences of vertebrae

$$\begin{cases} (\mathfrak{p}_q, u, \overline{u}) & \vdash \quad p_q \cdot (\Gamma_q(P)\gamma) \mathrel{\tilde{\asymp}} \overline{p}_q \cdot (\Gamma_q(\partial\overline{P})\gamma_*) \\ (\mathfrak{p}_q, u_!, \overline{u}_!) & \vdash \quad p_q \cdot \Gamma_q(P) \mathrel{\tilde{\asymp}} \overline{p}_q \cdot \Gamma_q(\partial\overline{P}) \end{cases}$$

and a morphism of correspondences from the former to the latter encoded by a diagram as follows.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "u_!"] \arrow[d, "\gamma"'] & \mathbb{M}_! \arrow[d, "\kappa"] & \mathbb{S}_* \arrow[l, "\overline{u}_!"'] \arrow[d, "\gamma_*"] \\ \mathbb{D}_2 \arrow[r, "u"'] & \mathbb{M} & \mathbb{D}_2^* \arrow[l, "\overline{u}"] \end{tikzcd}$$

The previous data will be referred to by the symbols $\binom{u_!, \overline{u}_!}{u, \overline{u}}[\mathfrak{p}_{\_}, \kappa] \vdash s \mathrel{\tilde{\frown}}_q \overline{\sigma}$.

Remark 3.91. The point of the above definition is that it defines a structure of extensive-strong q-memory $\vartheta \vdash P \cdot \gamma \mathrel{\tilde{\asymp}}_q \overline{P} \cdot \overline{\beta}$ independent of any choice of stem $\overline{\beta}$ in $\mathcal{C}$ (see Remark 3.75). Specifying the stem $\overline{\beta}$ will however play a substantial role for the notion of framing of simple q-memories. We will later denote by $\vartheta(\overline{\beta})$ the extensive q-memory of spines of type $P \cdot \gamma \mathrel{\tilde{\asymp}}_q \overline{P} \cdot \overline{\beta}$ encoded by the triple $(\varkappa_q, u, \overline{u})$ for every $\overline{\beta} \in \overline{\Omega}$.

3.3.7.12. *Mates for extensive memories of nodes of spines.* Let $\mathcal{C}$ be a category and q be a non-negative integer. Consider an extensive q-memory of nodes of spines as follows.

$$\vartheta := \begin{pmatrix} u_!, \overline{u}_! \\ u, \overline{u} \end{pmatrix} [\mathfrak{p}_{\_}, \kappa] \vdash s \frown_q \overline{\sigma}$$

A *pair of mates* for the q-memory ϑ consists of a pair of mates for the underlying correspondence of vertebrae encoded by the triple $(\varkappa_q, u_!, \overline{u}_!)$, where $\varkappa_q$ denotes the cospherical transition of $\mathfrak{p}_{q-1}$.

Remark 3.92. By Remark 3.79, this is equivalent to equipping all extensive q-memories of spines $\vartheta(\overline{\beta})$ where $\overline{\beta} \in \overline{\Omega}$ with a common pair of mates.

3.3.7.13. *Framing of extensive memories of nodes of spines.* Let $\mathcal{C}$ be a category, q be a non-negative integer, $\vartheta \vdash s \frown_q \sigma_*$ be a extensive q-memory of nodes of spines in $\mathcal{C}$ equipped with a pair of mates μ and $\vartheta_\dagger \vdash s \mathrel{\tilde{\frown}}_q \sigma_\dagger$ be an extensive strong q-memory of nodes of spines. The classes of stems associated with σ_* and $\sigma_\dagger$ will be denoted by $\overline{\Omega}$ and $\overline{\Omega}_\dagger$, respectively. The memory $\vartheta_\dagger$ will be said to *frame* the memory ϑ along the pair of mates μ if it is equipped with

1) a metafunction $\psi : \overline{\Omega} \to \overline{\Omega}_\dagger$, called its *framing gear*;
2) following the notations of Remark 3.90 and Remark 3.91, a framing of extensive q-memories of spines $(\vartheta(\overline{\beta}), \mu) \triangleright \vartheta_\dagger(\psi(\overline{\beta}))$ for every $\overline{\beta} \in \overline{\Omega}$.

The above structure will later be denoted by the symbols $(\vartheta, \mu) \triangleright_q \vartheta_\dagger$.

Remark 3.93. According to section 3.3.7.4, such a framing involves an extensive q-framing of the node of spines σ_* by the node of spines $\sigma_\dagger$ along the vertebrae associated with the pair of mates μ. (see Definition 3.52 and section 3.3.3.7).

3.3.7.14. *Recollections.* Let $\mathcal{C}$ be a category, n be a positive integer, s be a spine of degree $n-1$ and $\overline{\sigma}$ be a node of spines of degree n in $\mathcal{C}$. The spinal seed of $\overline{\sigma}$ will be denoted by s_*. A *recollection* between s and $\overline{\sigma}$ is a simple strong $(n-1)$-memory of spines between the spines s and s_*. More explicitly, such a recollection consists of an alliance of prespines $(\mathfrak{p}_\_) : P \rightsquigarrow \partial\overline{P}$ of degree $n-1$ whose head is equipped with a strong correspondence of vertebrae $(\mathfrak{p}_{n-1}, u, \overline{u}) \vdash V_s^{n-1}(0) \overset{\sim}{\frown}_{n-1} V_{s_*}^{n-1}(0)$ in $\mathcal{C}$. Such a recollection will later be denoted by $(\mathfrak{p}_\_, u, \overline{u}) \vdash s \overset{\sim}{\smile} \overline{\sigma}$ and said to *be of height n*.

Remark 3.94. For notations $s := P \cdot \gamma$ and $s_* := \partial\overline{P} \cdot \gamma_*$, the previous recollection involves an object $\mathbb{M}$ in $\mathcal{C}$, standing for the messenger of the correspondence, and two morphisms $u : \mathbb{D}_2 \to \mathbb{M}$ and $\overline{u} : \mathbb{D}_2^* \to \mathbb{M}$ in $\mathcal{C}$ for which the following identity holds in $\mathcal{C}$, where $\varkappa_n$ denotes the cospherical transition of $\mathfrak{p}_{n-1}$.

$$u \circ \gamma \circ \varkappa_n = \overline{u} \circ \gamma_*$$

3.3.7.15. *Mates for recollections.* Let $\mathcal{C}$ be a category and $(\mathfrak{p}_\_, u, \overline{u}) \vdash s \overset{\sim}{\smile} \overline{\sigma}$ be a recollection of height n in $\mathcal{C}$ with notations $s = P\cdot\gamma$ and $\overline{\sigma} = \overline{P}\cdot\overline{\Omega}$ where the head of $\overline{P}$ will be supposed to be of the form $p_* := \|\gamma_*, \gamma'_*\|$. The messenger of the recollection and the cospherical transition of $\mathfrak{p}_{n-1}$ will be denoted by $\mathbb{M}$ and $\varkappa_n$, respectively. By Remark 3.94, the following diagram must commute.

(3.53)
$$\begin{tikzcd}
\overline{\mathbb{S}}_n \arrow[r, "\gamma\circ\varkappa_n"] \arrow[d, "\gamma_*"'] & \mathbb{D}_2 \arrow[dr, bend left, "u"] & \\
\overline{\mathbb{D}}_2^n \arrow[rr, bend right, "\overline{u}"'] & & \mathbb{M}
\end{tikzcd}$$

A *mate* for the recollection $(\mathfrak{p}_\_, u, \overline{u}) \vdash s \overset{\sim}{\smile} \overline{\sigma}$ consists of a vertebra $v_\circ$ of the form $\|\gamma\circ\varkappa_n, \gamma_*\| \cdot \beta_\circ : \mathbb{S} \multimap \mathbb{S}'_\circ$ together with a morphism $\varpi_\circ : \mathbb{D}'_\circ \to \mathbb{D}'$ in $\mathcal{C}$ such that the diagram given in (3.53) factorises as follows.

(3.54)
$$\begin{tikzcd}
\overline{\mathbb{S}}_n \arrow[r, "\gamma\circ\varkappa_n"] \arrow[d, "\gamma_*"'] & \mathbb{D}_2 \arrow[d, "\delta_2^\circ"] \arrow[rrd, bend left, "u"] & & \\
\overline{\mathbb{D}}_2^n \arrow[r, "\delta_1^\circ"] \arrow[rrr, bend right, "\overline{u}"'] & \mathbb{S}'_\circ \arrow[r, "\beta_\circ"] & \mathbb{D}'_\circ \arrow[r, "\varpi_\circ"] & \mathbb{M}
\end{tikzcd}$$

In order to prepare the next section on framings of recollections, note that the previous data produce a pair of communicating extended nodes of vertebrae $\mathfrak{n}_\circ : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\circ$ and $\mathfrak{n}_* : \gamma_* \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \overline{\Omega}$, where the spherical and discal transitions of the former are given by $\varkappa_n$ and an identity and those of the latter are given by identities, respectively.

3.3.7.16. *Framing of recollections.* Let $\mathcal{C}$ be a category and $(\mathfrak{p}_\_, u, \overline{u}) \vdash s \overset{\sim}{\smile} \overline{\sigma}$ be a recollection of height n in $\mathcal{C}$ where $\overline{\sigma} := \overline{P}\cdot\overline{\Omega}$ and $s := P\cdot\gamma$. Suppose that the recollection $(\mathfrak{p}_\_, u, \overline{u})$ admits a mate $m_\circ$. We will keep the same notations as in section 3.3.7.15. Let $P_\sharp$ be a prespine of degree n that is n-compatible with $\overline{P}$ and whose head is denoted by $p_\sharp$. An extended node of spines of the form $(\mathfrak{p}_\_, \varrho) : P\cdot\gamma \overset{\mathrm{EX}}{\rightsquigarrow} P_\sharp \cdot \Omega_\sharp$ will be said to *frame* the recollection $(\mathfrak{p}_\_, u, \overline{u})$ along $m_\circ$ if its underlying extended node of vertebrae $(\varkappa_n, \varrho) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_\sharp \cdot \Omega_\sharp$ (see Remark 3.69) frames the communicating pair of extended nodes of vertebrae $\mathfrak{n}_\circ : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\circ$ and $\mathfrak{n}_* : \gamma_* \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \overline{\Omega}$ in the sense of Chapter 2 (see section 2.3.6.3 and section 2.3.6.4).

Remark 3.95. By definition of a framing, the discal and spherical transitions of $(\varkappa_n, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} p_\sharp \cdot \Omega_\sharp$ must be the same as those of $\mathfrak{n}_\circ : \gamma \overset{\text{ex}}{\rightsquigarrow} v_\circ$ – which forces ϱ to be an identity – and the prevertebra $p_\sharp$ must be of the form $\|\gamma \circ \varkappa_n, \gamma'_*\|$.

3.3.7.17. *Hom-language for framing of recollections.* This section aims at translating the content of section 3.3.7.16 into the hom-language. Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $(\mathfrak{p}_\lrcorner, u, \overline{u}) \vdash s \overset{\smile}{\sim} \overline{\sigma}$ be a recollection of height n in $\mathcal{C}$ as defined in section 3.3.7.14. Suppose that $(\mathfrak{p}_\lrcorner, u, \overline{u})$ admits a mate $m_\circ$ as given in section 3.3.7.15 and is framed by an extended node of spines $\varsigma_\sharp := (\mathfrak{p}_\lrcorner, \mathrm{id}) : P \cdot \gamma \overset{\text{EX}}{\rightsquigarrow} P_\sharp \cdot \Omega_\sharp$ along $m_\circ$ as defined in section 3.3.7.16. We will keep the same notations as in these sections and denote the node of vertebrae given by $p_\sharp \cdot \Omega_\sharp$ as $\nu_\sharp$. Consider a path of the form $h : \overline{u} \cdot x \sim_{\overline{\sigma}} y$ in X. In that case, the element x must be an arrow $\mathbb{M} \to X$ in $\mathcal{C}$. Note that diagram (3.54) also provides a path $\varpi_\circ : u \sim_{v_\circ} \overline{u}$ in $\mathbb{M}$, which produces, after applying x on it, a path $\varpi_\circ \cdot x : u \cdot x \sim_{v_\circ} \overline{u} \cdot x$ in X. The source and target of the paths h and $\varpi_\circ \cdot x$ imply that there exists a stem $\overline{\beta} \in \overline{\Omega}$ for which the following diagram commutes.

$$\begin{tikzcd} \mathbb{D}^*_2 \arrow[r, "\beta_\circ \circ \delta^\circ_1"] \arrow[d, "\overline{\beta} \circ \overline{\delta}_2"'] & \mathbb{D}'_\circ \arrow[d, "x \circ \varpi_\circ"] \\ \overline{\mathbb{D}}' \arrow[r, "h"'] & X \end{tikzcd}$$

Because the extended node of spines $\varsigma_\sharp$ frames the recollection $(\mathfrak{p}_\lrcorner, u, \overline{u})$ along $m_\circ$, there exists a stem $\beta_\sharp : \mathbb{S}_\sharp \to \mathbb{D}'_\sharp$ in $\Omega_\sharp$ such that the vertebra $p_\sharp \cdot \beta_\sharp$ is equipped with a pushout

(3.55)
$$\begin{tikzcd} \mathbb{D}^*_2 \arrow[r, "\beta_\circ \circ \delta^\circ_1"] \arrow[d, "\overline{\beta} \circ \overline{\delta}_2"'] & \mathbb{D}'_\circ \arrow[d, "\epsilon_1"] \\ \overline{\mathbb{D}}' \arrow[r, "\epsilon_2"'] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbb{E} \end{tikzcd}$$

and a cooperadic transition $\eta : \mathbb{D}'_\sharp \to \mathbb{E}$ satisfying the following equalities (see section 2.3.6.3).

(3.56)
$$\epsilon_2 \circ \beta \circ \delta_1 = \eta \circ \beta_\sharp \circ \delta^\sharp_1 \qquad \epsilon_1 \circ \beta \circ \delta_2 = \eta \circ \beta_\sharp \circ \delta^\sharp_2$$

By universality of the pushout of diagram (3.55), it follows that there exists a canonical morphism $(\varpi_\circ \cdot x) \star h : \mathbb{E} \to X$ making the following diagram commute.

$$\begin{tikzcd} \mathbb{D}'_\circ \arrow[r, "x \circ \varpi_\circ"] \arrow[dr, "\epsilon_1"'] & X \\ & \mathbb{E} \arrow[u, "(\varpi_\circ \cdot x) \star h"'] \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{D}}' \arrow[r, "h"] \arrow[dr, "\epsilon_2"'] & X \\ & \mathbb{E} \arrow[u, "(\varpi_\circ \cdot x) \star h"'] \end{tikzcd}$$

Now, if we denote by $\lceil(\varpi_\circ \cdot x)h\rceil$ the composite arrow $((\varpi_\circ \cdot x) \star h) \circ \eta : \mathbb{D}'_\sharp \to X$, the equations of (3.56) show that the element $\lceil(\varpi_\circ \cdot x)h\rceil$ is a $(\nu_\sharp)$-path of the form $u \cdot x \sim_{\nu_\sharp} y$ in X. We are now going to translate the above discussion in terms of hom-language for extended nodes of spines. For convenience, denote by s_* and $s_\natural$ the spinal seeds of $\overline{\sigma}$ and $\sigma_\sharp$, respectively.

Remark 3.96. By Remark 3.95, the spines s_* and $s_\natural$ must be equal to the spines $\partial\overline{P} \cdot \gamma_*$ and $\partial\overline{P} \cdot (\varkappa_n \gamma)$, respectively. The remark also implies that the spinal coseed of $\sigma_\sharp$ must be equal to that of $\overline{\sigma}$, which is of the form $\partial\overline{P} \cdot \gamma'_*$.

For every morphism $f : X \to Y$ in $\mathcal{C}$ and parallel pair of elements (r, t) in X above $\partial\overline{P}$, denote by $\mathcal{C}(s_*, Y, \overline{u})(f(r), f(t))$ the subclass of $\mathcal{C}(s_*, Y)(f(r), f(t))$ whose elements may be written as paths $\overline{u} \cdot x$ for some $x : \mathbb{M} \to Y$. Similarly, denote by $\mathcal{C}(\overline{\sigma}, f, \overline{u})(r, t)$ the subclass of $\mathcal{C}(\overline{\sigma}, f)(r, t)$ whose elements are sent via

$$R_{\overline{\sigma}} : \mathcal{C}(\overline{\sigma}, f)(r, t) \to \mathcal{C}(s_*, Y)(f(r), f(t))$$

to the class $\mathcal{C}(s_*, Y, \overline{u})(f(r), f(t))$. It follows from Remark 3.96 and diagram (3.53) holding for the recollection $(\mathfrak{p}_{\_}, u, \overline{u}) \vdash s \stackrel{\sim}{\smile} \overline{\sigma}$ that there exists a inclusion functor

$$L_{m_\circ} : \mathcal{C}(s_*, Y, \overline{u})(f(r), f(t)) \hookrightarrow \mathcal{C}(s_\natural, Y)(f(r), f(t))$$

mapping an element $\overline{u} \cdot x : f(r) \sim_{s_*} f(t)$ to the element $u \cdot x : f(r) \sim_{s_\natural} f(t)$. Now, the point of previous discussion was to show that there also exists a metafunction

$$M_{m_\circ} : \mathcal{C}(\overline{\sigma}, f, \overline{u})(r, t) \to \mathcal{C}(\sigma_\sharp, f)(r, t)$$

mapping a pair (h, y) to the element $(\lceil(\varpi_\circ \cdot x)h\rceil, y)$, where $y : r \sim_{s'_*} t$. The construction of this last metafunction induces the following commutative diagram.

$$\begin{tikzcd} \mathcal{C}(s_*, Y, \overline{u})(f(r), f(t)) \arrow[r, "L_{m_\circ}"] & \mathcal{C}(s_\natural, Y)(f(r), f(t)) \\ \mathcal{C}(\overline{\sigma}, f, \overline{u})(r, t) \arrow[r, "M_{m_\circ}"] \arrow[u, hook, "R_{\overline{\sigma}}"] & \mathcal{C}(\sigma_\sharp, f)(r, t) \arrow[u, hook, "R_{\sigma_\sharp}"'] \end{tikzcd} \tag{3.57}$$

The above commutative diagram together with the mate $m_\circ$ will be called a *reminiscent operator.*

3.3.7.18. *Extensive chainings of nodes of spines.* Let $\mathcal{C}$ be a category, s be a spine in $\mathcal{C}$ and ℓ and q be non-negative integers. An *extensive q-chaining of nodes of spines of length ℓ above s* consists of a sequence of $\ell+1$ nodes of spines $\Sigma := \{\sigma_0, \sigma_1, \dots, \sigma_\ell\}$ in $\mathcal{C}$ together with $\ell+1$ extensive strong memories of nodes of spines of the form

$$\vartheta_i := \begin{pmatrix} \tilde{u}, \tilde{u}_i \\ u, u_i \end{pmatrix} [\mathfrak{p}^i_{\_}, \kappa] \vdash s \stackrel{\sim}{\frown}_{q+i} \sigma_i \quad \text{with} \quad \sigma_i := P_i \cdot \Omega_i \quad \text{and} \quad s := P \cdot \gamma$$

for every $0 \leq i \leq \ell$ and a sequence of ℓ framings of memories as follows.

$$(\vartheta_0, \mu_0) \rhd_{q+1} (\vartheta_1, \mu_1) \rhd_{q+2} (\vartheta_2, \mu_2) \rhd_{q+3} \cdots \rhd_{q+\ell-1} (\vartheta_{\ell-1}, \mu_{\ell-1}) \rhd_{q+\ell} \vartheta_\ell$$

Note that the previous sequence of framings of memories makes sense by Proposition 3.78 as an extensive strong $(q+i)$-memory ϑ_i may be seen as an extensive $(q+i+1)$-memory, thus allowing the next framing. Note that the arrow κ is the same for every extensive memory, which is forced by the notion of morphism in $\mathbf{Focov}(\mathcal{C})$ (section 3.3.4.9). We will later denote the morphism κ as an arrow $\tilde{\mathbb{M}} \to \mathbb{M}$ where $\tilde{\mathbb{M}}$ and $\mathbb{M}$ denotes the respective messengers of the correspondences encoded by $(\mathfrak{p}^i_{\_}, \tilde{u}, \tilde{u}_i)$ and $(\mathfrak{p}^i_{\_}, u, u_i)$, for every $0 \leq i \leq \ell$. Such a chaining will later be denoted by $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$, where $\vartheta_{\_}$ stands for the collection of correspondences $\vartheta_0, \dots, \vartheta_\ell$ and $\mu_{\_}$ is the symbol used to denote the pairs of mates along which the framings are done. An extensive chaining such as $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ will be said to be *semi-convergent* if the degrees of s and s_ℓ equal $q+\ell$.

Remark 3.97. For every $0 \leq i \leq \ell$, denote the spinal seed of σ_i by s_i. By construction, the extensive $(q+i)$-memory of nodes of spines ϑ_i comprises a simple $(q+1)$-memory of (nodes of) spines given by

$$\tilde{\vartheta}_i := (\mathfrak{p}^i_{\_}, \tilde{u}, \tilde{u}_i) \vdash s \stackrel{\sim}{\asymp}_{q+i} s_i$$

(see, for instance, Remark 3.72 and Remark 3.75). The pair of mates μ_i then induces a pair of mates $\tilde{\mu}_i$ for the $(q+i)$-memory $\tilde{\vartheta}_i$. It follows that the extensive chaining $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ induces a simple chaining of (nodes of) spines as follows.

$$(\tilde{\vartheta}_0, \tilde{\mu}_0) \rhd_{q+1} (\tilde{\vartheta}_1, \tilde{\mu}_1) \rhd_{q+2} (\tilde{\vartheta}_2, \tilde{\mu}_2) \rhd_{q+3} \cdots \rhd_{q+\ell-1} (\tilde{\vartheta}_{\ell-1}, \tilde{\mu}_{\ell-1}) \rhd_{q+\ell} \tilde{\vartheta}_\ell$$

It also turns out that when the extensive chaining $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ is semi-convergent, the simple chaining of spines $(s, S, \tilde{\vartheta}_{\_}, \tilde{\mu}_{\_})$, where $S := \{s_0, \dots, s_\ell\}$, is convergent.

Remark 3.98. If the extensive chaining $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ is semi-convergent, then the strong $(q+\ell)$-memory ϑ_ℓ provides a recollection of height $q+\ell+1$ encoded by its associated strong $(q+\ell)$-correspondence $(\mathfrak{p}^\ell_{\_}, u, u_\ell) \vdash s \stackrel{\sim}{\asymp}_{q+\ell} s_\ell$ (see section 3.3.7.14).

The extensive chaining $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ will be said to be *convergent* if it is semi-convergent and the recollection mentioned in Remark 3.98 is equipped with a mate as well as a framing along that mate. By definition, a framing for the recollection ϑ_ℓ consists of an extended node of spines of the form $\varsigma_\sharp : s \overset{\mathrm{EX}}{\rightsquigarrow} \sigma_\sharp$. The extended node of spines $\varsigma_\sharp$ will later be called the *closure* of the extensive chaining.

Let $f : X \to Y$ be a morphism in $\mathcal{C}$, z be an element of $\mathcal{C}(\tilde{\mathbb{M}}, X)$ and (r_i, t_i) be any parallel pair in X above the base of s_i, for some $0 \leq i \leq \ell$. In the next proposition, we will denote by $\mathcal{C}(s_i, Y, \vartheta_i, z)(f(r_i), f(t_i))$ the subclass of $\mathcal{C}(s_i, Y, u_i)(f(r_i), f(t_i))$ whose elements may be written as paths $u_i \cdot x$ for some elements $x \in \mathcal{C}(\mathbb{M}, Y)$ and for which both identities $f(z) = \kappa \cdot x$ and $\langle r_i, t_i \rangle = \tilde{u}_i \cdot z$ hold. We will also denote by $\mathcal{C}(\sigma_i, f, \vartheta_i, z)(r_i, t_i)$ the subclass of $\mathcal{C}(\sigma_i, f)(r_i, t_i)$ whose elements are sent via $R_{\sigma_i} : \mathcal{C}(\sigma_i, f)(r_i, t_i) \to \mathcal{C}(s_i, Y)(f(r_i), f(t_i))$ to the class $\mathcal{C}(s_i, Y, \vartheta_i, z)(f(r_i), f(t_i))$. Because the next proposition considers a convergent chaining, the closure will be denoted as above and the spinal seed of the node of spines $\sigma_\sharp$ will be denoted by $s_\natural$. By construction, the extended node of spines $\varsigma_\sharp$ must be of the form $(\mathfrak{p}^\ell_{\_}, \mathrm{id})$ (see Remark 3.95). The head of the alliance of prespines $(\mathfrak{p}^\ell_{\_}) : P \rightsquigarrow \partial P_\ell$ will be encoded by the data $(\varkappa, \rho, \rho', \varkappa')$.

Remark 3.99. If $\ell > 0$, then the discal and codiscal transitions ρ and ρ' must be identities in $\mathcal{C}$. This is forced by the form of the alliance of prevertebrae constructed in section 3.3.4.6 for a framing of correspondences of vertebrae.

Proposition 3.100. *Suppose that $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ is convergent for $\ell \geq 0$ with closure $\varsigma_\sharp$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and $z \in \mathcal{C}(\tilde{\mathbb{M}}, X)$. Consider (r, t) and (r', t') two parallel pairs in X above the bases of s and s_0, respectively. If $\langle r, t \rangle = \tilde{u} \cdot z$ and $\langle r', t' \rangle = \tilde{u}_0 \cdot z$, then there exists a finite sequence of composable tubular operators and a reminiscent operator whose composite is of the following form.*

$$\begin{tikzcd}
\mathcal{C}(s_0, Y, \vartheta_0, z)(f(r'), f(t')) \arrow[r] & \mathcal{C}(s_\natural, Y)(\rho \cdot f(r), \rho' \cdot f(t)) \\
\mathcal{C}(\sigma_0, f, \vartheta_0, z)(r', t') \arrow[r] \arrow[u, "R_{\sigma_0}"] & \mathcal{C}(\sigma_\sharp, f)(\rho \cdot r, \rho' \cdot t) \arrow[u, "R_{\sigma_\sharp}"']
\end{tikzcd}$$

The top arrow then sends any element of the form $u_0 \cdot x$, where $x \in \mathcal{C}(\mathbb{M}, Y)$, to the element $u \cdot x$.

Proof. Suppose that $\ell = 0$. By semi-convergence, the spines s and s_0 must be of degree q. Proposition 3.19 applied on the relation $\langle r, t \rangle = \tilde{u} \cdot z$ implies a $(P \cdot \mathrm{id})$-path $\tilde{u} \cdot z : r \sim_P t$ in X. Note that the alliance of prespines $(\mathfrak{p}^0_{\_}) : P \rightsquigarrow \partial P_0$ induces an alliance of spines $(\mathfrak{p}^0_{\_}, \varkappa') : P \cdot \mathrm{id} \rightsquigarrow \partial P_0 \cdot \mathrm{id}$. By Proposition 3.3.6.4, this last alliance turns the $(P \cdot \mathrm{id})$-path $\tilde{u} \cdot z$ into a $(\partial P_0 \cdot \mathrm{id})$-path in X as follows.

$$(\varkappa' \tilde{u}) \cdot z : \rho \cdot r \sim_{\partial P_0} \rho' \cdot t$$

On the one hand, the strong q-correspondence of spines $(\mathfrak{p}^0_{\_}, \tilde{u}, \tilde{u}_0) \vdash s \overset{\sim}{\simeq}_q s_0$ provides an identity $\tilde{u} \circ \varkappa' = \tilde{u}_0$, thus turning the previous path into a path $\tilde{u}_0 \cdot z : \rho \cdot r \sim_{\partial P_0} \rho' \cdot t$ in X. On the other hand, Proposition 3.19 applied on the equality $\langle r', t' \rangle = \tilde{u}_0 \cdot z$ also shows that the relation $\tilde{u}_0 \cdot z : r' \sim_{\partial P_0} t'$ must hold in X. These two expressions of the path $\tilde{u}_0 \cdot z$ imply that the equalities $r' = \rho \cdot r$ and $t' = \rho' \cdot t$ must hold. In the end, applying the construction of section 3.3.7.17 on the recollection provided by ϑ_0 as well as the morphism $f : X \to Y$ and the parallel pair (r', t') above ∂P_0 provides the statement for $\ell = 0$. Now, suppose the inequality $\ell > 0$ holds. In that case, it is possible to apply Proposition 3.89 on the convergent simple chaining of (nodes of) spines $(s, S, \tilde{\vartheta}_{\_}, \tilde{\mu}_{\_})$ defined in Remark 3.97 with respect to the equations $\langle r, t \rangle = \tilde{u} \cdot z$ and $\langle r', t' \rangle = \tilde{u}_0 \cdot z$. There then follows a sequence of simple tubular operators

$$T_i : \mathcal{C}(s_i, X)(r_i, t_i) \longrightarrow \mathcal{C}(s_{i+1}, X)(r_{i+1}, t_{i+1})$$

for every $0 \leq i \leq \ell - 1$ where $r_0 = r'$ and $t_0 = t'$ by convention and $r_\ell = r$ and $t_\ell = t$ by convergence. The metafunction T_i maps a (s_i)-path x in X to the (s_{i+1})-path $\lceil(\varpi^i_\diamond \cdot z)x(\varpi^i_\bullet \cdot z)\rceil$ in X. By Proposition 3.55, the previous sequence of simple tubular operators induces another one consisting of metafunctions of the form

$$T_i^f : \mathcal{C}(s_i, Y)(f(r_i), f(t_i)) \longrightarrow \mathcal{C}(s_{i+1}, Y)(f(r_{i+1}), f(t_{i+1}))$$

mapping an (s_i)-path x in Y to the (s_{i+1})-path $\lceil f(\varpi^i_\diamond \cdot z)xf(\varpi^i_\bullet \cdot z)\rceil$ in Y. As mentioned in Remark 3.93, the involved framings are also extensive framings of nodes of spines. Because the paths along which the tubular operators T_i^f are defined are in the image of $f(\_)$, the previous metafunctions may be lifted to extensive tubular operators of nodes of spines as follows (see section 3.3.3.9).

$$\begin{tikzcd}
\mathcal{C}(s_i, Y)(f(r_i), f(t_i)) \arrow[r, "T_i^f"] & \mathcal{C}(s_{i+1}, Y)(f(r_{i+1}), f(t_{i+1})) \\
\mathcal{C}(\sigma_i, f)(r_i, t_i) \arrow[u, "R_{\sigma_i}"] \arrow[r, dashed] & \mathcal{C}(\sigma_{i+1}, f)(r_{i+1}, t_{i+1}) \arrow[u, "R_{\sigma_{i+1}}"']
\end{tikzcd}$$

Composing the above sequence of tubular operators then provides the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C}(s_0, Y)(f(r'), f(t')) \arrow[r] & \mathcal{C}(s_\ell, Y)(f(r), f(t)) \\
\mathcal{C}(\sigma_0, f)(r', t') \arrow[u, "R_{\sigma_0}"] \arrow[r] & \mathcal{C}(\sigma_\ell, f)(r, t) \arrow[u, "R_{\sigma_\ell}"']
\end{tikzcd} \tag{3.58}$$

Note that, by section 3.3.4.8, restricting the tubular operator T_i^f to the set[5]

$$\mathcal{C}(s_i, Y, \vartheta_i, z)(f(r_i), f(t_i))$$

provides a metafunction

$$\mathcal{C}(s_i, Y, \vartheta_i, z)(f(r_i), f(t_i)) \to \mathcal{C}(s_{i+1}, Y, \vartheta_{i+1}, z)(f(r_{i+1}), f(t_{i+1}))$$

mapping a path $u_i \cdot x$ to the path $\lceil(\varpi^i_\diamond \cdot f(z))(u_i \cdot x)(\varpi^i_\bullet \cdot f(z))\rceil$, which is equal to

$$\lceil(\varpi^i_\diamond \cdot (\kappa \cdot x))(u_i \cdot x)(\varpi^i_\bullet \cdot (\kappa \cdot x))\rceil = u_{i+1} \cdot x$$

since the equality $f(z) = \kappa \cdot x$ holds and $\kappa \circ \varpi^i_\diamond$ and $\kappa \circ \varpi^i_\bullet$ are the morphisms associated with the pair of mates of the correspondence of vertebrae $(\varkappa^i_{q+i+1}, u, u_i)$, where $\varkappa^i_{q+i+1}$ denotes the cospherical transition of $\mathfrak{p}^i_{q+i}$ (see Proposition 3.60 and the definition of $\mathbf{Mcov}(\mathcal{C})$ that follows it). Along that restriction, diagram (3.58) leads to the following commutative diagram, where the top arrow maps any (s_0)-path $u_0 \cdot x$ to the (s_ℓ)-path $u_\ell \cdot x$ (see the definitions of section 3.3.7.17).

$$\begin{tikzcd}
\mathcal{C}(s_0, Y, \vartheta_0, z)(f(r'), f(t')) \arrow[r] & \mathcal{C}(s_\ell, Y, u_\ell)(f(r), f(t)) \\
\mathcal{C}(\varsigma_0, f, \vartheta_0, z)(r', t') \arrow[u, "R_{\varsigma_0}"] \arrow[r] & \mathcal{C}(\varsigma_\ell, f, u_\ell)(r, t) \arrow[u, "R_{\varsigma_\ell}"']
\end{tikzcd}$$

Finally, the statement follows by applying the construction of section 3.3.7.17 on the recollection induced by ϑ_ℓ (see Remark 3.98) and composing the version of diagram (3.57) thus

[5] The class of stems of a spine is always a singleton and a fortiori a set. It follows that the class of paths associated with a spine is a set.

obtained with the above commutative square. The source and target involved are coherent with the statement by Remark 3.99. □

3.3.8. Conjugations of nodes of spines.

3.3.8.1. *Conjugation of nodes of spines.* Let $\mathcal{C}$ be category and q be a non-negative integer. Consider four nodes of spines $\sigma = P \cdot \Omega$, $\overline{\sigma} = \overline{P} \cdot \overline{\Omega}$, $\ddot{\sigma} = \ddot{P} \cdot \ddot{\Omega}$ and $\widehat{\sigma} = \widehat{P} \cdot \widehat{\Omega}$ of same degree $n \geq 0$ and an alliance of nodes of spines $\mathfrak{a} : \ddot{\sigma} \rightsquigarrow \widehat{\sigma}$ in $\mathcal{C}$ of degree n. The triple $(\sigma, \mathfrak{a}, \overline{\sigma})$ will be said to *form a q-conjugation of nodes of spines in $\mathcal{C}$* if it is equipped with

1) two semi-extended vertebrae $\mathfrak{v}_\diamond : \gamma \overset{\text{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma' \overset{\text{ex}}{\rightsquigarrow} v_\bullet$;
2) two reflections of vertebrae $\mathfrak{a}_\flat : v_\diamond \rightsquigarrow v_\flat^{\text{rv}}$ and $\mathfrak{a}_\dagger : v_\bullet \rightsquigarrow v_\dagger^{\text{rv}}$ conjugable with the alliances of vertebrae $\mathfrak{a}_q(1)$ and $\mathfrak{a}_q(1)^{\text{rv}}$ along morphisms
$$r_\flat : \mathbb{D}_2^\flat \to \widehat{\mathbb{D}}_1 \quad \text{and} \quad r_\dagger : \mathbb{D}_2^\dagger \to \widehat{\mathbb{D}}_2,$$
respectively;
3) two simple q-framings of nodes of spines $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q \ddot{\sigma}$ and $(\widehat{\sigma}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright_q \overline{\sigma}$ where $\mathfrak{v}_\flat$ and $\mathfrak{v}_\dagger$ are the sevs $(\mathbb{S}_\flat, r_\flat) : \widehat{\gamma}_q \overset{\text{ex}}{\rightsquigarrow} v_\flat$ and $(\mathbb{S}_\dagger, r_\dagger) : \widehat{\gamma}'_q \overset{\text{ex}}{\rightsquigarrow} v_\dagger$ (see Remark 3.63).

The preceding structure will be denoted by $(\sigma, \mathfrak{a}, \overline{\sigma})$ and said to be *defined along the pairs* $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$*,* $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ *and* $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$. In the sequel, the discal transitions of $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ will be denoted by $\varrho_\diamond$ and $\varrho_\bullet$, respectively.

Proposition 3.101. *The q-conjugation of nodes of spines $(\sigma, \mathfrak{a}, \overline{\sigma})$ provides a conjugation of vertebrae $(p_q \cdot \Gamma_q(P), \mathfrak{a}_q(1), \overline{p}_q \cdot \Gamma_q(\overline{P}))$ defined along the pairs $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ and $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$, where $\mathfrak{a}_q(1) : \ddot{p}_q \cdot \Gamma_q(\ddot{P}) \rightsquigarrow \widehat{p}_q \cdot \Gamma_q(\widehat{P})$.*

Proof. Directly follows from the definitions of the q-framings $(\sigma, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q \ddot{\sigma}$ and $(\widehat{\sigma}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright_q \overline{\sigma}$ and the definition of a conjugation of vertebrae (see section 3.3.5.2). □

We are going to show that the conjugation of Proposition 3.101 induces a simple strong q-memory of nodes of spines between σ and $\overline{\sigma}$. First, applying the functor $\mathcal{S}_{\text{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$ of section 3.3.5.5 on the conjugation of Proposition 3.101 provides the strong correspondence of vertebrae of (3.59) where $\mathfrak{p}'_q$ is an alliance of prevertebrae that has the same spherical transition as the alliance $\mathfrak{a}_q(1)$.

$$(\mathfrak{p}'_q, u, u_0) \vdash p_q \cdot \Gamma_q(P) \overset{\sim}{\asymp} \overline{p}_q \cdot \Gamma_q(\overline{P}) \tag{3.59}$$

By Definition 3.51, the two simple q-framings associated with the conjugation of nodes of spines $(\sigma, \mathfrak{a}, \overline{\sigma})$ force the pairs of prespines $(P, \ddot{P})$ and $(\widehat{P}, \overline{P})$ to be q-compatible. This compatibility implies that the first q alliances of prevertebrae of $\mathfrak{a} : \ddot{\sigma} \rightsquigarrow \widehat{\sigma}$ induce an alliance of prespines of degree $q-1$ as follows.

$$(\mathfrak{p}_{\text{-}}) : \partial^{n+1-q} P \rightsquigarrow \partial^{n+1-q} \overline{P} \tag{3.60}$$

By construction, the spherical transition of the alliance of prevertebrae $\mathfrak{p}'_q$ is equal to the spherical transition of $\mathfrak{p}_q$, which is also equal to the cospherical transition of $\mathfrak{p}_{q-1}$. If, for the sake of fitting the usual notations, we denote the alliance $\mathfrak{p}_k$ by the symbol $\mathfrak{p}'_k$ for every $0 \leq k \leq q-1$, the correspondence $(\mathfrak{p}'_q, u, u_0)$ obtained in (3.59) together with the alliance of prespines obtained in (3.60) define a simple strong q-memory of nodes of spines (see section 3.3.7.6) as follows.

$$\vartheta_0 := (\mathfrak{p}'_{\text{-}}, u, u_0) \vdash \sigma \overset{\sim}{\asymp}_q \overline{\sigma}$$

For convenience, denote the non-negative integer $n - q$ by the symbol ℓ. The conjugation $(\sigma, \mathfrak{a}, \overline{\sigma})$ will be said to be *convergent* if it is equipped with a convergent simple q-chaining of nodes of spines of length ℓ whose first memory is ϑ_0. The conjugation is therefore equipped with a sequence of framings as follows.

$$(\vartheta_0, \mu_0) \triangleright_{q+1} (\vartheta_1, \mu_1) \triangleright_{q+2} (\vartheta_2, \mu_2) \triangleright_{q+3} \cdots \triangleright_{q+\ell-1} (\vartheta_{\ell-1}, \mu_{\ell-1}) \triangleright_{q+\ell} \vartheta_\ell$$

From now on, the discussion assumes that the conjugation $(\sigma, \mathfrak{a}, \overline{\sigma})$ is convergent and considers the notations $\vartheta_i := (\mathfrak{p}^i_-, u, u_i) \vdash \sigma \mathrel{\widetilde{\gtrsim}}_{q+i} \sigma_i$ for every $0 \leq i \leq \ell$. Because the equality $n = q + \ell$ holds, the alliance of prespines associated with the strong n-memory ϑ_ℓ does not involve any derivation; i.e. it is of the form $(\mathfrak{p}^\ell_-) : P \rightsquigarrow P_\ell$. Here, P_ℓ denotes the prespine of the node of spines σ_ℓ, which the convergence assumption forces to be of degree n (see section 3.3.7.9).

Remark 3.102. By definition of a framing of simple memories of nodes of spines, and more specifically, its compatibility condition, the equality $\mathfrak{p}^i_k = \mathfrak{p}^{i+1}_k$ holds for any $0 \leq k \leq q + i$ and $0 \leq i \leq \ell$. In particular, this forces the alliance of prevertebra $\mathfrak{p}^i_k$ to be equal to the previously defined alliance $\mathfrak{p}'_k$ for any $0 \leq k \leq q$ and $0 \leq i \leq \ell$.

Remark 3.103. Although this remark holds for a general context, the notations used herein will serve our next discussion as they correspond to the notations that have previously been introduced. Let $\sigma = P \cdot \Omega$ and $\sigma_\ell = P_\ell \cdot \Omega_\ell$ be two nodes of spines of positive degrees n where ℓ stands for some given symbol. Equipping the two previous nodes of spines with an alliance of prespines $(\mathfrak{p}^\ell_-) : P \rightsquigarrow P_\ell$ of degree n gives rise to an alliance of nodes of spines $(\mathfrak{p}^\ell_-, \phi, \mathrm{id}) : \sigma \rightsquigarrow \sigma_\sharp$ of degree n where

- the node of spines $\sigma_\sharp$ is of the form $P_\ell \cdot \Omega_\sharp$;
- the class $\Omega_\sharp$ consists of the elements of Ω augmented by the class of elements of the form $\beta \circ \varkappa_{n+1}$ for every $\beta \in \Omega_\ell$, where $\varkappa_{n+1}$ denotes the cospherical transition of the alliance of prevertebrae $\mathfrak{p}_n$;
- the metafunction $\phi : \Omega \to \Omega_\sharp$ maps a stem β to the stem $\beta \circ \varkappa_{n+1}$.

The assumptions of Remark 3.103 exactly correspond to the situation at the end of the above discussion when the conjugation $(\sigma, \mathfrak{a}, \overline{\sigma})$ is convergent. In that case, the alliance of nodes of spines defined by Remark 3.103 will be denoted as $\mathfrak{all}_0(\sigma, \mathfrak{a}, \overline{\sigma}) : \sigma \rightsquigarrow \sigma_\sharp$. Note that the data of Remark 3.103 also define an alliance of nodes of spines $(\mathrm{id}_{P_\ell}, \subseteq, \mathrm{id}) : \sigma_\ell \rightsquigarrow \sigma_\sharp$ encoded by the inclusion $\Omega_\ell \subseteq \Omega_\sharp$ and the identity alliance $\mathrm{id}_{P_\ell} : P_\ell \rightsquigarrow P_\ell$. This second alliance will be denoted by $\mathfrak{all}_1(\sigma, \mathfrak{a}, \overline{\sigma}) : \sigma_\ell \rightsquigarrow \sigma_\sharp$ in the case of a convergent conjugation $(\sigma, \mathfrak{a}, \overline{\sigma})$.

3.3.8.2. *Hom-language for conjugation of nodes of spines.* The aim of the present section is to translate the constructions of section 3.3.8.1 into the hom-language. Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and q be an non-negative integer. Consider a q-conjugation of nodes of spines $(\sigma, \mathfrak{a}, \overline{\sigma})$ as defined in section 3.3.8.1. The conjugation will be supposed to be convergent. We shall use the same notations as in section 3.3.8.1. Let x and y be a parallel pair in X above the base of σ. By definition, the q-conjugation $(\sigma, \mathfrak{a}, \overline{\sigma})$ provides two framings of nodes of vertebrae

$$\mathfrak{f} := (V^q_\sigma(0), \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q V^q_{\ddot{\sigma}}(0) \qquad \text{and} \qquad \widehat{\mathfrak{f}} := (V^q_{\widehat{\sigma}}(0), \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright_q V^q_{\overline{\sigma}}(0)$$

and an alliance of nodes of vertebrae $\mathfrak{a}_q(0) : V^q_{\ddot{\sigma}}(0) \rightsquigarrow V^q_{\widehat{\sigma}}(0)$. We are now going to mimick the first part of section 3.3.5.6 for this set of data. To do so, denote by

- ϱ_n and ϱ'_n the respective discal and codiscal transitions of $\mathfrak{a}_n(0)$;
- ϱ_q and ϱ'_q the respective discal and codiscal transitions of $\mathfrak{a}_q(0)$;
- $\varrho_\diamond$ and $\varrho_\bullet$ the discal transitions of the sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$
- ρ_0 and ρ'_0 the discal and codiscal transitions of the prevertebra $\mathfrak{p}'_q$

and consider two paths as follows.

$$e_\diamond : \varrho_\diamond \cdot x \sim_{v_\diamond} a \qquad\qquad e_\bullet : \varrho_\bullet \cdot x \sim_{v_\bullet} b$$

As explained in section 3.3.5.6, it follows from Proposition 3.15 and Remark 3.7 that the two reflections $\mathfrak{a}_\flat : v_\diamond \rightsquigarrow v_\flat^{\mathrm{rv}}$ and $\mathfrak{a}_\dagger : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$ turn the paths $e_\diamond$ and $e_\bullet$ into two paths as follows.

$$\tau_\flat \cdot e_\diamond : \varrho_\flat' \cdot a \sim_{v_\flat} (\varrho_\flat \varrho_\flat) \cdot x \qquad\qquad \tau_\dagger \cdot e_\bullet : \varrho_\dagger' \cdot b \sim_{v_\dagger} (\varrho_\dagger \varrho_\dagger) \cdot y$$

By conjugability, the factorisations $\varrho_\flat' = \varrho_q \circ r_\flat$ and $\varrho_\dagger' = \varrho_q' \circ r_\dagger$ must hold in $\mathcal{C}$. Besides, Proposition 3.64 implies the identities $\rho_0 = \varrho_\diamond \circ \varrho_\flat$ and $\rho_0' = \varrho_\bullet \circ \varrho_\dagger$. The two expressions

$$\tau_\flat \cdot e_\diamond : r_\flat \cdot (\varrho_q \cdot a) \sim_{v_\flat} \rho_0 \cdot x \qquad \text{and} \qquad \tau_\dagger \cdot e_\bullet : r_\dagger \cdot (\varrho_q' \cdot b) \sim_{v_\dagger} \rho_0' \cdot y$$

then follow. Because $\mathfrak{v}_\flat$ and $\mathfrak{v}_\dagger$ are encoded by the pairs $(\mathbb{S}_\flat, r_\flat)$ and $(\mathbb{S}_\dagger, r_\dagger)$, Proposition 3.47, Proposition 3.15 and the previous two paths provide the following metafunctions.

$$\begin{cases} T_{e_\diamond}^{e_\bullet} & : \ \mathcal{C}(V_\sigma^q(0), X)(x_q, y_q) \to \mathcal{C}(V_{\ddot{\sigma}}^q(0), X)(a, b) \\ \mathcal{C}(\mathfrak{a}_q(0), X) & : \ \mathcal{C}(V_{\ddot{\sigma}}^q(0), X)(a, b) \to \mathcal{C}(V_{\widehat{\sigma}}^q(0), X)(\varrho_q \cdot a, \varrho_q' \cdot b) \\ T_{\tau_\flat \cdot e_\diamond}^{\tau_\dagger \cdot e_\bullet} & : \ \mathcal{C}(V_{\widehat{\sigma}}^q(0), X)(\varrho_q \cdot a, \varrho_q' \cdot b) \to \mathcal{C}(V_{\overline{\sigma}}^q(0), X)(\rho_0 \cdot x_q, \rho_0' \cdot y_q) \end{cases}$$

Notice that the previous three metafunctions are composable. Their composition will be denoted by $V_{e_\diamond}^{e_\bullet}$. According to section 3.3.3.8 and section 3.3.6.4, the three metafunctions defining $V_{e_\diamond}^{e_\bullet}$ may be lifted to the hom-language of nodes of spines, so that the metafunction $V_{e_\diamond}^{e_\bullet}$ itself may also be lifted to this language as follows.

$$\begin{tikzcd} \mathcal{C}(V_\sigma^q(0), X)(x_q, y_q) \arrow[rr, "V_{e_\diamond}^{e_\bullet}"] & & \mathcal{C}(V_{\overline{\sigma}}^q(0), X)(\rho_0 \cdot x_q, \rho_0' \cdot y_q) \\ \mathcal{C}(\sigma, X)(x, y) \arrow[u, "\subseteq"] \arrow[rr, dashed, "W_{e_\diamond}^{e_\bullet}"'] & & \mathcal{C}(\overline{\sigma}, X)(x', y') \arrow[u, "\subseteq"'] \end{tikzcd} \tag{3.61}$$

Since $\langle x, y\rangle$ is well-defined, section 3.3.3.8 ensures that the $\ddot{P}$-path $\lceil e_\diamond \langle x, y\rangle e_\bullet \rceil$ is well-defined, which will be supposed of the form $x'' \sim_{\ddot{P}} y''$. In particular, this implies that the component $\mathcal{C}(\mathfrak{a}, X)$ of the metafunction $W_{e_\diamond}^{e_\bullet}$ is an arrow of the following form (see diagram (3.49)).

$$\mathcal{C}(\mathfrak{a}, X) : \mathcal{C}(\ddot{\sigma}, X)(x'', y'') \to \mathcal{C}(\widehat{\sigma}, X)(\varrho_n \cdot x'', \varrho_n' \cdot y'') \tag{3.62}$$

In this case, the discussion of section 3.3.3.8 applied to the tubular operator $T_{\tau_\flat \cdot e_\diamond}^{\tau_\dagger \cdot e_\bullet}$ forces the following relation.

$$\lceil (\tau_\flat \cdot e_\diamond)\langle \varrho_n \cdot x'', \varrho_n' \cdot y''\rangle(\tau_\dagger \cdot e_\bullet)\rceil : x' \sim_{\overline{P}} y' \tag{3.63}$$

It follows from Proposition 3.34 applied on the $\ddot{P}$-path $\lceil e_\diamond \langle x, y\rangle e_\bullet \rceil$ and the path (3.63) as well as Remark 3.16 applied on the head of the alliance $\mathfrak{a}$ that the following identity holds, where $\varkappa_{n+1}$ denotes the cospherical transition of $\mathfrak{a}_n(0)$.

$$\langle x', y'\rangle = \lceil (\tau_\flat \cdot e_\diamond)\big(\varkappa_{n+1} \cdot \lceil e_\diamond \langle x, y\rangle e_\bullet \rceil\big)(\tau_\dagger \cdot e_\bullet)\rceil \tag{3.64}$$

Similarly, it is possible to apply the construction of section 3.3.5.6 to the conjugation of vertebrae provided by Proposition 3.101 (see conjugation χ below) for the paths $e_\diamond : \varrho_\diamond \cdot x_q \sim_{v_\diamond} a$ and $e_\diamond : \varrho_\diamond \cdot y_q \sim_{v_\diamond} b$.

$$\chi := (p_q \cdot \Gamma_q(P), \mathfrak{a}_q(1), \overline{p}_q \cdot \Gamma_q(\overline{P}))$$

By doing so, we obtain another metafunction as follows.

$$\mathcal{C}(p_q \cdot \Gamma_q(P), X)(x_q, y_q) \xrightarrow{U_{e_\diamond}^{e_\bullet}} \mathcal{C}(\overline{p}_q \cdot \Gamma_q(\overline{P}), X)(\rho_0 \cdot x_q, \rho_0' \cdot y_q)$$

By Proposition 3.35, the P-path $\langle x,y\rangle$ may be seen as an element in the domain of the metafunction $U^{e_\bullet}_{e_\diamond}$. Since the alliance $\mathfrak{a}_q(1)$ is of the form $(\mathfrak{p}_q,\varkappa_{n+1}) : V^q_{\widehat{\sigma}}(1) \rightsquigarrow V^q_{\overline{\sigma}}(1)$ (see Remark 3.67), the definition of the metafunction $U^{e_\bullet}_{e_\diamond}$ implies that the following identity holds.

$$U^{e_\bullet}_{e_\diamond}(\langle x,y\rangle) = \lceil(\tau_\flat\cdot e_\diamond)\big(\varkappa_{n+1}\cdot\lceil e_\diamond\langle x,y\rangle e_\bullet\rceil\big)(\tau_\dagger\cdot e_\bullet)\rceil$$

Equation (3.64) then turns the above identity into the following one.

$$U^{e_\bullet}_{e_\diamond}(\langle x,y\rangle) = \langle x',y'\rangle \tag{3.65}$$

In order to conclude this section, we will have to distinguish between the cases where $\ell = 0$ and $\ell > 0$. First, suppose that $\ell = 0$. In this case, the identity $q = n$ holds, which implies the equalities $x' = \rho_0\cdot x_n$ and $y' = \rho'_0\cdot y_n$ (see diagram (3.61)). The condition $\ell = 0$ also implies the equalities $\overline{\sigma} = \sigma_0 = \sigma_\ell$. Section 3.3.8.1 hence provides an alliance of nodes of spines $\mathfrak{all}_1(\sigma,\mathfrak{a},\overline{\sigma}) : \overline{\sigma} \rightsquigarrow \sigma_\sharp$, which induces the following dashed metafunction.

$$\mathcal{C}(\sigma,X)(x,y) \xrightarrow{W^{e_\bullet}_{e_\diamond}} \mathcal{C}(\overline{\sigma},X)(x',y') \dashrightarrow \mathcal{C}(\sigma_\sharp,X)(\rho_0\cdot x_n,\rho'_0\cdot y_n) \tag{3.66}$$

Now, suppose that $\ell > 0$. Recall that the strong correspondence associated with the q-memory $\vartheta_0 = (\mathfrak{p}'_q,u,u_0)$ is defined as the image $\mathcal{S}_{\mathrm{cor}}(\chi)$. It follows from Proposition 3.65 and equation (3.65) that the relations $\langle x,y\rangle = u\cdot h'$ and $\langle x',y'\rangle = u_0\cdot h'$ holds for $h' = e_\diamond\star\langle x,y\rangle\star e_\bullet$. This last identities enable us to apply Proposition 3.89 on the spines σ and $\sigma_0 = \overline{\sigma}$, which provides a composite of tubular operators of the following form.

$$T:\ \mathcal{C}(\overline{\sigma},X)(x',y') \longrightarrow \mathcal{C}(\sigma_\ell,X)(x,y)$$

Furthermore, the condition $\ell > 0$ implies that the alliance $\mathfrak{p}^\ell_n$ is built up from section 3.3.4.6. This means that the discal and codiscal transitions of $\mathfrak{p}^\ell_n$, say ρ_ℓ and ρ'_ℓ, are identities (see diagram (3.31)). Hence, the equalities $x = \rho_\ell\cdot x$ and $y = \rho'_\ell\cdot y$ hold. The alliance $\mathfrak{all}_1(\sigma,\mathfrak{a},\overline{\sigma}) : \sigma_\ell \rightsquigarrow \sigma_\sharp$ then induces the following dashed metafunction.

$$\mathcal{C}(\sigma,X)(x,y) \xrightarrow{T\circ W^{e_\bullet}_{e_\diamond}} \mathcal{C}(\sigma_\ell,X)(x,y) \dashrightarrow \mathcal{C}(\sigma_\sharp,X)(\rho_\ell\cdot x,\rho'_\ell\cdot y) \tag{3.67}$$

Finally, diagram (3.66) and diagram (3.67) show that for any $\ell \geq 0$, there exists a metafunction of the form

$$\mathcal{C}(\sigma,X)(x,y) \xrightarrow{S^{e_\bullet}_{e_\diamond}} \mathcal{C}(\sigma_\sharp,X)(\rho_\ell\cdot x,\rho'_\ell\cdot y)$$

that may be factorised by $W^{e_\bullet}_{e_\diamond}$, where ρ_ℓ and ρ'_ℓ denotes the discal and codiscal transitions of $\mathfrak{p}^\ell_n$, respectively.

Remark 3.104. Interestingly, the metafunction induced by the alliance $\mathfrak{all}_0(\sigma,\mathfrak{a},\overline{\sigma}) : \sigma \rightsquigarrow \sigma_\sharp$ for the pair (x,y), which is parallel above the base of σ, has the same domain and codomain as the metafunction $S^{e_\bullet}_{e_\diamond}$ since its head is equal to $\mathfrak{p}^\ell_n$.

$$\mathcal{C}(\mathfrak{all}_0(\sigma,\mathfrak{a},\overline{\sigma}),X) : \mathcal{C}(\sigma,X)(x,y) \to \mathcal{C}(\sigma_\sharp,X)(\rho_\ell\cdot x,\rho'_\ell\cdot y) \tag{3.68}$$

Proposition 3.105. *If the codomain of the metafunction $\mathcal{C}(\mathfrak{a},X)$ evaluated at the parallel pair (x'',y'') (see diagram (3.62)) is non-empty , then so is the codomain of the metafunction $\mathcal{C}(\mathfrak{all}_0(\sigma,\mathfrak{a},\overline{\sigma}),X)$ evaluated at the parallel pair (x,y) (see diagram (3.68)).*

Proof. By construction, the metafunction $W^{e_\bullet}_{e_\diamond}$ may be written as a composite of metafunctions comprising $\mathcal{C}(\mathfrak{a},X)$. It follows that if the codomain of $\mathcal{C}(\mathfrak{a},X)$ is non-empty, then so is the codomain of $W^{e_\bullet}_{e_\diamond}$. Because $W^{e_\bullet}_{e_\diamond}$ factorises $S^{e_\bullet}_{e_\diamond}$, the codomain of $S^{e_\bullet}_{e_\diamond}$ is non-empty. Since the codomain of $S^{e_\bullet}_{e_\diamond}$ is the same as that of the metafunction $\mathcal{C}(\mathfrak{all}_0(\sigma,\mathfrak{a},\overline{\sigma}),X)$ evaluated at the pair (x,y) (see Remark 3.104), the statement follows. $\square$

3.3.8.3. *Sequences of conjugations of nodes of spines.* Let $\mathcal{C}$ be a category and ℓ be a positive integer. A *sequence of conjugations of nodes of spines of length* ℓ in $\mathcal{C}$ consists, for every $0 \leq i \leq \ell - 1$, of a convergent i-conjugation of nodes of spines

$$\chi_i := (\sigma_i, \mathfrak{a}^{i+1}, \overline{\sigma}_i)$$

such that the alliance of nodes of spines $\mathfrak{a}^{i+1}$ is of the form $\sigma_{i+1} \rightsquigarrow \widehat{\sigma}_{i+1}$, for every $0 \leq i \leq \ell-1$, and the relation

$$\mathfrak{a}^i = \mathfrak{all}_0(\chi_i)$$

holds for every $1 \leq i \leq \ell-1$. We will later denote by $\mathfrak{a}^0$ the alliance $\mathfrak{all}_0(\chi_0)$. By construction, the alliance $\mathfrak{a}^0$ is an arrow of the form $\sigma_0 \rightsquigarrow \sigma_\sharp$ where $\sigma_\sharp$ is defined as in section 3.3.8.1. The conjugation χ_i will be supposed to be defined along pairs $\mathfrak{v}^i_\diamond, \mathfrak{v}^i_\bullet$, $\mathfrak{a}^i_\flat, \mathfrak{a}^i_\dagger$ and $\mathfrak{v}^i_\flat, \mathfrak{v}^i_\dagger$. By definition of a conjugation of nodes of spines, if σ_0 is of degree n, then all nodes of spines $\sigma_0, \ldots, \sigma_\ell$, $\widehat{\sigma}_0, \ldots, \widehat{\sigma}_\ell$ and $\sigma_\sharp$ are of degree n. The sequence will be said to be *closed* if the identity $n = \ell - 1$ holds. In that case, the underlying sequence of simple framings of nodes of spines

$$(\sigma_0, \mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet) \quad \triangleright^V_0 \quad (\sigma_1, \mathfrak{v}^1_\diamond, \mathfrak{v}^1_\bullet) \quad \triangleright^V_1 \ \ldots \ \triangleright^V_{\ell-2} \quad (\sigma_{\ell-1}, \mathfrak{v}^{\ell-1}_\diamond, \mathfrak{v}^{\ell-1}_\bullet) \quad \triangleright^V_{\ell-1} \quad \sigma_\ell$$

is convergent. The above sequence of framings is, below, denoted by $\mathfrak{F}$.

Theorem 3.106. *Let $\chi_0, \ldots, \chi_{\ell-1}$ be a closed sequence of conjugations of nodes of spines of length ℓ as above and $f : Y \to Z$ and $g : X \to Y$ be two morphisms in $\mathcal{C}$. Suppose that g is a tubular surtraction for $\mathfrak{F}$ and the node of spines σ_0 is g-projective. If $f \circ g$ is an intraction for $\mathfrak{a}^\ell$, then f is an intraction for $\mathfrak{a}^0$.*

Proof. According to section 3.3.6.4, the morphism $f : Y \to Z$ is an intraction for $\mathfrak{a}^0 : \sigma_0 \rightsquigarrow \sigma_\sharp$ if for every pair of parallel elements x and y above the base of σ_0 in Y, it satisfies the property that if the class $\mathcal{C}(\sigma_0, Z)(f(x), f(y))$ is non-empty, then so is the codomain of the metafunction $\mathcal{C}(\mathfrak{a}^0, Y)$ evaluated at the pair (x, y), namely the class $\mathcal{C}(\sigma_\sharp, Y)(\rho_0 \cdot x, \rho'_0 \cdot y)$ where ρ_0 and ρ'_0 are the respective discal and codiscal transitions of $\mathfrak{a}^0_n(0)$. To prove such a statement, consider a pair of parallel elements x and y above the base of σ_0 in Y and suppose that $\mathcal{C}(\sigma_0, Z)(f(x), f(y))$ is non-empty. If P_0 denotes the prespine of σ_0, then Remark 3.9 provides the relation $x \sim_{P_0} y$. Because the prespine P_0 is g-projective, Proposition 3.42 implies that the pair x and y is 0-parallel over g in Y. By Lemma 3.56 and Remark 3.57, there exists a pair of elements x' and y' parallel above σ_ℓ in Y and a sequence of tubular operators of the form

$$T^{e^i_\bullet}_{e^i_\diamond} : \ \mathcal{C}(\sigma_i, Y)(x_i, y_i) \longrightarrow \mathcal{C}(\sigma_{i+1}, Y)(x_{i+1}, y_{i+1}),$$

for every $0 \leq i \leq \ell-1$, such that $x_0 = x$ and $y_0 = y$ and $x_\ell = g(x')$ and $y_\ell = g(y')$. As showed in section 3.3.8.2 for any given pair of paths $e^i_\diamond$ and $e^i_\bullet$ in Y, the above tubular operators come along with metafunctions

$$\mathcal{C}(\mathfrak{a}^{i+1}, Y) : \mathcal{C}(\sigma_{i+1}, Y)(x_{i+1}, y_{i+1}) \to \mathcal{C}(\widehat{\sigma}_{i+1}, Y)(\rho_{i+1} \cdot x_{i+1}, \rho'_{i+1} \cdot y_{i+1})$$

satisfying the property that if the codomain of $\mathcal{C}(\mathfrak{a}^{i+1}, Y)$ is non-empty, then that of $\mathcal{C}(\mathfrak{a}^i, Y)$ is non-empty (see Proposition 3.105). This implies, by induction, that if the codomain of $\mathcal{C}(\mathfrak{a}^\ell, Y)$ is non-empty, then that of $\mathcal{C}(\mathfrak{a}^0, Y)$ is non-empty. Because $x_0 = x$ and $y_0 = y$, this means that if the codomain of $\mathcal{C}(\mathfrak{a}^\ell, Y)$ is non-empty, then f is an intraction for $\mathfrak{a}^0$ (see above discussion). To conclude the proof, it thus suffices to prove that the codomain of $\mathcal{C}(\mathfrak{a}^\ell, Y)$ is non-empty. Note that applying the morphism $f : Y \to Z$ on the pairs of paths $e^i_\diamond$ and $e^i_\bullet$ in Y, for every $0 \leq i \leq \ell - 1$, induces another sequence of tubular operators as follows.

$$T^{f(e^i_\bullet)}_{f(e^i_\diamond)} : \ \mathcal{C}(\sigma_i, Z)(f(x_i), f(y_i)) \longrightarrow \mathcal{C}(\sigma_{i+1}, Z)(f(x_{i+1}), f(y_{i+1}))$$

By assumption, the class $\mathcal{C}(\sigma_0, Z)(f(x), f(y))$ is non-empty, so the last class of the sequence $\mathcal{C}(\sigma_\ell, Z)(fg(x'), fg(y'))$ is also non-empty. Because the composite $f \circ g$ is an intraction for

$\mathfrak{a}_\ell : \sigma_\ell \rightsquigarrow \widehat{\sigma}_\ell$, this implies that the class $\mathcal{C}(\widehat{\sigma}_\ell, X)(\rho_\ell \cdot x', \rho'_\ell \cdot y')$ is non-empty. Applying $g : X \to Y$ on the elements of this class shows that the class

$$\mathcal{C}(\widehat{\sigma}_\ell, Y)(\rho_\ell \cdot g(x'), \rho'_\ell \cdot g(y')) = \mathcal{C}(\widehat{\sigma}_\ell, Y)(\rho_\ell \cdot x_\ell, \rho'_\ell \cdot y_\ell)$$

is non-empty. Because this last class is the codomain of the metafunction $\mathcal{C}(\mathfrak{a}^\ell, Y)$ defined on $\mathcal{C}(\widehat{\sigma}_\ell, Y)(x_\ell, y_\ell)$, what precedes shows that f is an intraction for $\mathfrak{a}^0$. □

3.3.8.4. *Extended conjugation of nodes of spines.* Let $\mathcal{C}$ be category, n be a positive integer and q be a non-negative integer. Consider a spine $s = P \cdot \gamma$ of degree $n-1$, a node of spines $\overline{\sigma} = \overline{P} \cdot \overline{\Omega}$ of degree n and an extended node of spines $\varsigma = \ddot{s} \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}$ of degree n in $\mathcal{C}$. The underlying alliance of spines of ς of degree $n-1$ will be denoted by $\mathfrak{a} : \ddot{s} \rightsquigarrow \hat{s}$ where $\hat{s}$ denotes the spinal seed of $\widehat{\sigma}$. The triple $(s, \varsigma, \overline{\sigma})$ will be said to form an *extended q-conjugation of node of spines* in $\mathcal{C}$ if it is equipped with

1) two semi-extended vertebrae $\mathfrak{v}_\diamond : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond$ and $\mathfrak{v}_\bullet : \gamma' \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet$;
2) two reflections of vertebrae $\mathfrak{a}_\flat : v_\diamond \rightsquigarrow v_\flat^{\mathrm{rv}}$ and $\mathfrak{a}_\dagger : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$ conjugable with the alliances of vertebrae $\mathfrak{a}_q(1)$ and $\mathfrak{a}_q(1)^{\mathrm{rv}}$ along morphisms

$$r_\flat : \mathbb{D}_2^\flat \to \ddot{\mathbb{D}}_1 \quad \text{and} \quad r_\dagger : \mathbb{D}_2^\dagger \to \ddot{\mathbb{D}}_2,$$

respectively.
3) a simple q-framing of spines $(s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q \ddot{s}$ and an extensive q-framing of nodes of spines $(\widehat{\sigma}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright_q \overline{\sigma}$ where $\mathfrak{v}_\flat$ and $\mathfrak{v}_\dagger$ are the sevs $(\mathbb{S}_\flat, r_\flat) : \widehat{\gamma}_q \overset{\mathrm{ex}}{\rightsquigarrow} v_\flat$ and $(\mathbb{S}_\dagger, r_\dagger) : \widehat{\gamma}'_q \overset{\mathrm{ex}}{\rightsquigarrow} v_\dagger$.

The above structure will be denoted by $(s, \varsigma, \overline{\sigma})$ and said to be *defined along the pairs* $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ *and* $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$. In the sequel, the discal transitions of $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ will be denoted by $\varrho_\diamond$ and $\varrho_\bullet$, respectively.

Remark 3.107. Denote by s_* the spinal seed of $\overline{\sigma}$. It follows from the definition of the simple q-framing of spines $(s, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright_q \ddot{s}$ and the extensive q-framing of nodes of spines $(\widehat{\sigma}, \mathfrak{v}_\flat, \mathfrak{v}_\dagger) \triangleright_q \overline{\sigma}$ that the extended conjugation $(s, \varsigma, \overline{\sigma})$ comes along with a functor $\mathtt{I} \to \mathbf{Conj}(\mathcal{C})$ whose mapping rules on objects and arrows are as follows.

$$d \mapsto (V_s^q(d), \mathfrak{a}_q(d), V_{s_*}^q(d)) \qquad (t : d \to d') \mapsto (V_s^q(t), V_{\ddot{s}}^q(t), V_{\widehat{s}}^q(t), V_{s_*}^q(t))$$

The underlying conjugations are defined along the pairs $\mathfrak{v}_\diamond, \mathfrak{v}_\bullet$, $\mathfrak{a}_\flat, \mathfrak{a}_\dagger$ and $\mathfrak{v}_\flat, \mathfrak{v}_\dagger$.

We are going to show that the functor of conjugations defined in Remark 3.107 induces an extensive strong q-memory of nodes of spines between the spine s and the node of spines $\overline{\sigma}$. First, composing the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Cong} \to \mathbf{Scov}(\mathcal{C})$ of section 3.3.5.5 with the functor $\mathtt{I} \to \mathbf{Conj}(\mathcal{C})$ of Remark 3.107 provides two strong correspondences of vertebrae

$$(\mathfrak{p}'_q, u, u_0) \vdash V_s^q(0) \overset{\approx}{\approx} V_{s_*}^q(0) \quad \text{and} \quad (\mathfrak{p}'_q, \tilde{u}, \tilde{u}_0) \vdash V_s^q(1) \overset{\approx}{\approx} V_{s_*}^q(1)$$

where $\mathfrak{p}'_q$ is an alliance of prevertebrae that has the same spherical transition as the alliances $\mathfrak{a}_q(1)$ and $\mathfrak{a}_q(1)$ and a morphism of strong correspondences of vertebrae from the former to the latter encoded by the following diagram (see diagram (3.40)).

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\tilde{u}"] \arrow[d, "V_s^q(t)=\gamma"'] & \mathbb{G}_1 \arrow[d, "\kappa(\gamma)"'] & \mathbb{S}_* \arrow[l, "\tilde{u}_0"'] \arrow[d, "\gamma_*=V_{s_*}^q(t)"] \\
\mathbb{D}_2 \arrow[r, "u"'] & \mathbb{G}_0 & \mathbb{D}_2^* \arrow[l, "u_0"]
\end{tikzcd}$$

For convenience, the morphism $\kappa(\gamma)$ will later be denoted as κ. Denote by $\ddot{P}$ and $\widehat{P}$ the respective prespines of $\ddot{s}$ and $\widehat{\sigma}$. By Definition 3.51 and Definition 3.52, the two q-framings associated with the extended conjugation of nodes of spines $(s, \varsigma, \overline{\sigma})$ force the pairs of prespines

$(P, \ddot{P})$ and $(\widehat{P}, \overline{P})$ to be q-compatible. This compatibility implies that the first q alliances of prevertebrae of $\mathfrak{a} : \ddot{s} \rightsquigarrow \hat{s}$ induces an alliance of prespines of degree $q-1$ as follows.

$$(\mathfrak{p}_{\text{-}}) : \partial^{n+1-q}P \rightsquigarrow \partial^{n+1-q}\overline{P} \tag{3.69}$$

By construction, the spherical transition of the alliance of prevertebrae $\mathfrak{p}'_q$ is equal to the spherical transition of $\mathfrak{p}_q$, which is also equal to the cospherical transition of $\mathfrak{p}_{q-1}$. If, for the sake of fitting the usual notations, we denote the alliance $\mathfrak{p}_k$ by the symbol $\mathfrak{p}'_k$ for every $0 \leq k \leq q-1$, the correspondences $(\mathfrak{p}'_q, u, u_0)$ and $(\mathfrak{p}'_q, \tilde{u}, \tilde{u}_0)$ together with the alliance of prespines obtained in (3.69) define an extensive strong q-memory of nodes of spines (see section 3.3.7.11) as follows.

$$\vartheta_0 := \begin{pmatrix} \tilde{u}, \tilde{u}_0 \\ u, u_0 \end{pmatrix} [\mathfrak{p}'_{\text{-}}, \kappa] \vdash s \mathrel{\tilde{\frown}_q} \overline{\sigma}$$

For convenience, denote the non-negative integer $n-1-q$ by the symbol ℓ. The extended conjugation $(s, \varsigma, \overline{\sigma})$ will be said to be *convergent* if it is equipped with a convergent extensive q-chaining of nodes of spines of length ℓ whose first memory is ϑ_0. The conjugation is therefore equipped with a sequence of framings as follows.

$$(\vartheta_0, \mu_0) \triangleright_{q+1} (\vartheta_1, \mu_1) \triangleright_{q+2} (\vartheta_2, \mu_2) \triangleright_{q+3} \cdots \triangleright_{q+\ell-1} (\vartheta_{\ell-1}, \mu_{\ell-1}) \triangleright_{q+\ell} \vartheta_\ell$$

From now on, the discussion assumes that the conjugation $(s, \varsigma, \overline{\sigma})$ is convergent and considers the notations

$$\vartheta_i := \begin{pmatrix} \tilde{u}, \tilde{u}_i \\ u, u_i \end{pmatrix} [\mathfrak{p}^i_{\text{-}}, \kappa] \vdash s \mathrel{\tilde{\frown}_{q+i}} \sigma_i$$

for every $0 \leq i \leq \ell$. Because the equality $n-1 = q+\ell$ holds, the extensive strong $(n-1)$-memory ϑ_ℓ induces a recollection of height n (see Remark 3.98), which, by convergence, is framed by an extended nodes of spines – the closure of the chaining, which will be denoted by $\varsigma_\sharp : s \overset{\text{EX}}{\rightsquigarrow} \sigma_\sharp$ – along a given mate.

3.3.8.5. *Hom-language for extended conjugation of nodes of spines.* The aim of the present section is to translate the constructions of section 3.3.8.4 into the hom-language. Let $\mathcal{C}$ be a category, $f : X \to Y$ be a morphism in $\mathcal{C}$ and q be an non-negative integer. Consider an extended q-conjugation of nodes of spines $(s, \varsigma, \overline{\sigma})$ as defined in section 3.3.8.4. The conjugation will be supposed to be convergent. We shall use the same notations as in section 3.3.8.4. Let r and t be a parallel pair in X above the base of s. First, applying the construction of section 3.3.5.6 (see following explanations) on the conjugation of vertebrae $(V^q_s(0), \mathfrak{a}_q(0), V^q_{\overline{s}}(0))$ provided by Remark 3.107 produces a metafunction of the form

$$\mathcal{C}(V^q_s(0), X)(r_q, t_q) \xrightarrow{V^{e_\bullet}_{e_\diamond}} \mathcal{C}(V^q_{\overline{s}}(0), X)(\rho_0 \cdot r_q, \rho'_0 \cdot t_q)$$

for every pair of paths of the form $e_\diamond : \varrho_\diamond \cdot r_q \sim_{v_\diamond} a$ and $e_\bullet : \varrho_\bullet \cdot t_q \sim_{v_\diamond} b$ in X. The morphisms $\varrho_\diamond$ and $\varrho_\bullet$ are given by the discal transitions of the sevs $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ and the morphisms ρ_0 and ρ'_0 define the discal and codiscal transitions of the prevertebra $\mathfrak{p}'_q$. The metafunction $V^{e_\bullet}_{e_\diamond}$ is then the composition of three metafunctions

$$\begin{cases} T^{e_\bullet}_{e_\diamond} & : \quad \mathcal{C}(V^q_s(0), X)(r_q, t_q) \to \mathcal{C}(V^q_{\ddot{s}}(0), X)(a, b) \\ \mathcal{C}(\mathfrak{a}_q(0), X) & : \quad \mathcal{C}(V^q_{\ddot{s}}(0), X)(a, b) \to \mathcal{C}(V^q_{\hat{s}}(0), X)(\varrho_q \cdot a, \varrho'_q \cdot b) \\ T^{\tau_\dagger \cdot e_\bullet}_{\tau_\flat \cdot e_\diamond} & : \quad \mathcal{C}(V^q_{\hat{s}}(0), X)(\varrho_q \cdot a, \varrho'_q \cdot b) \to \mathcal{C}(V^q_{\overline{s}}(0), X)(\rho_0 \cdot r_q, \rho'_0 \cdot t_q) \end{cases}$$

where $T^{e_\bullet}_{e_\diamond}$ and $T^{\tau_\dagger \cdot e_\bullet}_{\tau_\flat \cdot e_\diamond}$ are tubular operators and ϱ_q and ϱ'_q are the discal and codiscal transition of $\mathfrak{a}_q(0)$. Proposition 3.65 of the same section then states that the metafunction $V^{e_\bullet}_{e_\diamond}$ maps any $V^q_s(0)$-path x from r_q to t_q in X to a $V^q_{\overline{s}}(0)$-path of the form $u_0 \cdot x'$ in X where $x = u \cdot x'$

and $x' = e_\diamond \star x \star e_\bullet$. Now, according to section 3.3.3.8 and section 3.3.6.4, such a composite allows one to lift the metafunction $V_{e_\diamond}^{e_\bullet}$ as follows.

$$\begin{tikzcd} \mathcal{C}(V_s^q(0),X)(r_q,t_q) \arrow[r, "V_{e_\diamond}^{e_\bullet}"] & \mathcal{C}(V_{\overline{s}}^q(0),X)(\rho_0\cdot r_q,\rho_0'\cdot t_q) \\ \mathcal{C}(s,X)(r,t) \arrow[u, "\subseteq"] \arrow[r, dashed, "W_{e_\diamond}^{e_\bullet}"'] & \mathcal{C}(\overline{s},X)(r',t') \arrow[u, "\subseteq"'] \end{tikzcd} \tag{3.70}$$

The metafunction $W_{e_\diamond}^{e_\bullet}$ then maps any s-path x from r to t in X to the $\overline{s}$-path $u_0 \cdot x'$ in X where the identity $x = u \cdot x'$ and $x' = e_\diamond \star h \star e_\bullet$ holds by definition of $V_{e_\diamond}^{e_\bullet}$. Since $\langle r,t\rangle$ is well-defined, the elements r' and t' are parallel above the base of $\overline{s}$, so that the following identity holds (see section 3.3.3.8 and Proposition 3.34).

$$\langle r',t'\rangle = \lceil (\tau_\flat\cdot e_\diamond)\big(\varkappa_n \cdot \lceil e_\diamond\langle r,t\rangle e_\bullet\rceil\big)(\tau_\dagger\cdot e_\bullet)\rceil \tag{3.71}$$

The symbol $\varkappa_n$ denotes the cospherical transition of the head of $\mathfrak{a}$ and the operation $\varkappa_n \cdot \_$ appearing in formula (3.71) is deduced from Remark 3.16 in regard to the action of $\mathcal{C}(\mathfrak{a}_q(0),X)$ at the level of the spines (see diagram (3.49)). Note that the above data, holding in X, may also be imitated in Y by applying Proposition 3.55 on the metafunctions defining $W_{e_\diamond}^{e_\bullet}$. We thus obtain a metafunction

$$W_{f(e_\diamond)}^{f(e_\bullet)} : \mathcal{C}(s,Y)(f(r),f(t)) \longrightarrow \mathcal{C}(\overline{s},Y)(f(r'),f(t')) \tag{3.72}$$

that is the composition of three other metafunctions

$$\begin{cases} T_{f(e_\diamond)}^{f(e_\bullet)} & : \ \mathcal{C}(s,Y)(f(r),f(t)) \to \mathcal{C}(\ddot{s},Y)(f(a'),f(b')) \\ \mathcal{C}(\mathfrak{a},Y) & : \ \mathcal{C}(\ddot{s},Y)(f(a'),f(b')) \to \mathcal{C}(\hat{s},Y)(\varrho_{n-1}\cdot f(a'),\varrho_{n-1}'\cdot f(b')) \\ T_{\tau_\flat\cdot f(e_\diamond)}^{\tau_\dagger\cdot f(e_\bullet)} & : \ \mathcal{C}(\hat{s},Y)(\varrho_{n-1}\cdot f(a'),\varrho_{n-1}'\cdot f(b')) \to \mathcal{C}(\overline{s},Y)(f(r'),f(t')) \end{cases}$$

where ϱ_{n-1} and ϱ_{n-1}' denote the respective discal and codiscal transition of $\mathfrak{a}_{n-1}(0)$ and a' and b' denote the source and target of the $(\ddot{P}\cdot\mathrm{id})$-path $\lceil e_\diamond\langle r,t\rangle e_\bullet\rceil$, respectively. Proposition 3.65 then states that the metafunction of (3.72) maps any s-path $x : f(r) \sim_s f(t)$ in Y to the $\overline{s}$-path $u_0\cdot x'$ in Y where $x = u\cdot x'$ and $x' = f(e_\diamond)\star x\star f(e_\bullet)$. Note that such a mapping implies that the image of the metafunction given in diagram (3.72) is included in the following class (this type of class is defined in section 3.3.7.17).

$$\mathcal{C}(\overline{s},Y,u_0)(f(r'),f(t'))$$

Finally, note that it is also possible to apply the construction of section 3.3.5.6 on the other conjugation of vertebrae $\chi := (V_s^q(1),\mathfrak{a}_q(1),V_{\overline{s}}^q(1))$ given by Remark 3.107 along the paths $e_\diamond : \varrho_\diamond\cdot r_q \sim_{v_\diamond} a$ and $e_\diamond : \varrho_\diamond\cdot t_q \sim_{v_\diamond} b$ in X. We thus obtain another metafunction as follows.

$$\mathcal{C}(V_s^q(1),X)(r_q,t_q) \xrightarrow{U_{e_\diamond}^{e_\bullet}} \mathcal{C}(V_{\overline{s}}^q(1),X)(\rho_0\cdot r_q,\rho_0'\cdot t_q)$$

By Proposition 3.35, the P-path $\langle r,t\rangle$ may be seen as an element in the domain of the metafunction $U_{e_\diamond}^{e_\bullet}$. Since the alliance $\mathfrak{a}_q(1)$ is of the form $(\mathfrak{p}_q,\varkappa_n) : V_{\hat{s}}^q(1) \rightsquigarrow V_{\ddot{s}}^q(1)$ (see Remark 3.67), the definition of the metafunction $U_{e_\diamond}^{e_\bullet}$ implies that the following identity holds.

$$U_{e_\diamond}^{e_\bullet}(\langle r,t\rangle) = \lceil (\tau_\flat\cdot e_\diamond)\big(\varkappa_n \cdot \lceil e_\diamond\langle r,t\rangle e_\bullet\rceil\big)(\tau_\dagger\cdot e_\bullet)\rceil$$

Equation (3.71) then turns this last identity into the identity $U_{e_\diamond}^{e_\bullet}(\langle r,t\rangle) = \langle r',t'\rangle$. Proposition 3.65 then states that the relations $\langle r,t\rangle = \tilde{u}\cdot z$ and $\langle r',t'\rangle = \tilde{u}_0\cdot z$ hold for $z = e_\diamond \star \langle r,t\rangle \star e_\bullet$.

Proposition 3.108. *The image of the metafunction (3.72) is contained in the following class.*

$$\mathcal{C}(\overline{s},Y,\vartheta_0,z)(f(r'),f(t'))$$

Recall that this type of class is defined in section 3.3.7.18.

Proof. Because the equation $\langle r', t'\rangle = u_0 \cdot z$ holds and the image of the metafunction (3.72) is already included in $\mathcal{C}(\overline{s}, Y, u_0)(f(r'), f(t'))$, the statement is proven if the identity $f(z) = \kappa \cdot x$ can be shown to hold. To do so, consider the notation $s := P \cdot \gamma$. The identity $f(z) = \kappa \cdot x$ is then deduced from the following series of equalities.

$$\begin{aligned} f(z) &= f(e_\diamond \star \langle r, t\rangle \star e_\bullet) && \text{(by definition of } z\text{)}\\ &= f(e_\diamond) \star \langle f(r), f(t)\rangle \star f(e_\bullet) && \text{(by universality)}\\ &= f(e_\diamond) \star (\gamma \cdot x) \star f(e_\bullet) && \text{(by Proposition 3.34)}\\ &= \kappa(\gamma) \cdot (f(e_\diamond) \star x \star f(e_\bullet)) && \text{(by Remark 3.49)} \end{aligned}$$

Because we renamed the arrow $\kappa(\gamma)$ by κ in section 3.3.8.4, the statement follows. □

Recall that the extensive chaining $(s, \Sigma, \vartheta_{\_}, \mu_{\_})$ of section 3.3.8.4 admits a closure $\varsigma_\sharp : s \overset{\text{EX}}{\rightsquigarrow} \sigma_\sharp$. The spinal seed of the node of spines $\sigma_\sharp$ will be denoted by $s_\natural$ and the underlying alliance of spines $s \rightsquigarrow s_\natural$ will be denoted by $\mathfrak{a}_\sharp$. By construction, the extended node of spines $\varsigma_\sharp$ and the alliance $\mathfrak{a}_\sharp$ must be of the form $(\mathfrak{p}^\ell_{\_}, \mathrm{id})$ (see Remark 3.95). The head of the alliance of prespines $(\mathfrak{p}^\ell_{\_}) : P \rightsquigarrow \partial P_\ell$ will be encoded by the data $(\varkappa, \rho, \rho', \varkappa')$. Because both relations $\langle r, t\rangle = \tilde{u} \cdot z$ and $\langle r', t'\rangle = \tilde{u}_0 \cdot z$ hold, Proposition 3.100 provides a commutative square

$$\text{(3.73)} \qquad \begin{tikzcd} \mathcal{C}(s_0, Y, \vartheta_0, z)(f(r'), f(t')) \arrow[r, "T"] & \mathcal{C}(s_\natural, Y)(\rho \cdot f(r), \rho' \cdot f(t)) \\ \mathcal{C}(\sigma_0, f, \vartheta_0, z)(r', t') \arrow[u, "R_{\sigma_0}"] \arrow[r, "T'"] & \mathcal{C}(\sigma_\sharp, f)(\rho \cdot r, \rho' \cdot t) \arrow[u, "R_{\sigma_\sharp}"'] \end{tikzcd}$$

whose top arrow sends any element of the form $u_0 \cdot x'$ to the element $u \cdot x'$. Since the spine $\overline{s}$ is equal to the spine s_0, the composite

$$\text{(3.74)} \qquad T \circ W^{f(e_\bullet)}_{f(e_\diamond)} : \mathcal{C}(s, Y)(f(r), f(t)) \longrightarrow \mathcal{C}(s_\natural, Y)(\rho \cdot f(r), \rho' \cdot f(t))$$

maps any s-path $x : f(r) \sim_s f(t)$ to the $s_\natural$-path $x : \rho \cdot f(r) \sim_{\overline{s}} \rho' \cdot f(t)$. This mapping rule comes from the identity $x = u \cdot x'$ where $x' = f(e_\diamond) \star x \star f(e_\bullet)$.

Remark 3.109. The fact that the mapping rule of the previous metafunction is trivial (i.e. of the form $x \mapsto x$) does make sense at the level of the spines since, in the case where s is denoted by $P \cdot \gamma$, then $s_\natural = \partial P_\sharp \cdot (\varkappa' \gamma)$ (see Remark 3.95).

Now, because the alliance $\mathfrak{a}_\sharp$ is of the form $(\mathfrak{p}^\ell_{\_}, \mathrm{id})$, the mapping rule of the metafunction, below, must be the same as that of diagram (3.74). In other words, the metafunction of diagram (3.74) is equal to the following one.

$$\mathcal{C}(\mathfrak{a}_\sharp, X) : \mathcal{C}(s, Y)(f(r), f(t)) \longrightarrow \mathcal{C}(s_\natural, Y)(\rho \cdot f(r), \rho' \cdot f(t))$$

The above discussion finally leads to the following proposition.

Proposition 3.110. *If for any choice of element $x : \mathbf{1} \to \mathcal{C}(\ddot{s}, Y)(f(a'), f(b'))$, the dashed arrow of the next diagram exists and makes the commutative square commute*

$$\begin{tikzcd} 1 \arrow[r, dashed] \arrow[d, "x"'] & \mathcal{C}(\widehat{\sigma}, f)(\varrho_{n-1} \cdot a', \varrho'_{n-1} \cdot b') \arrow[d, "R_{\widehat{\sigma}}"] \\ \mathcal{C}(\ddot{s}, Y)(f(a'), f(b')) \arrow[r, "\mathcal{C}(\mathfrak{a}, X)"'] & \mathcal{C}(\hat{s}, Y)(\varrho_{n-1} \cdot f(a'), \varrho'_{n-1} \cdot f(b')) \end{tikzcd}$$

then for any choice of element $x' : \mathbf{1} \to \mathcal{C}(s, Y)(f(r), f(t))$, the dashed arrow of the following diagram exists and makes the commutative square commute.

(3.75)
$$\begin{tikzcd} & 1 \arrow[r, dashed] \arrow[d, "x'"'] & \mathcal{C}(\sigma_\sharp, f)(\rho \cdot r, \rho' \cdot t) \arrow[d, "R_{\sigma_\sharp}"] \\ & \mathcal{C}(s, Y)(f(r), f(t)) \arrow[r, "\mathcal{C}(\mathfrak{a}_\sharp, X)"'] & \mathcal{C}(s_\natural, Y)(\rho \cdot f(r), \rho' \cdot f(t)) \end{tikzcd}$$

Proof. To prove this statement, suppose that the hypothesis holds and consider a choice of element as follows.

$$x' : \mathbf{1} \to \mathcal{C}(s, Y)(f(r), f(t))$$

The goal is to lift the composite $\mathcal{C}(\mathfrak{a}_\sharp, X) \circ x'$ along $R_{\sigma_\sharp}$ as shown in diagram (3.75). By the previous discussion, we know that the metafunction $\mathcal{C}(\mathfrak{a}_\sharp, X)$ may be expressed as a composite of the following form.

(3.76)
$$T \circ W^{f(e_\bullet)}_{f(e_\diamond)} = T \circ T^{\tau_\dagger \cdot f(e_\bullet)}_{\tau_\flat \cdot f(e_\diamond)} \circ \mathcal{C}(\mathfrak{a}, Y) \circ T^{f(e_\bullet)}_{f(e_\diamond)}$$

By assumption, we may lift the composite

(3.77)
$$\mathcal{C}(\mathfrak{a}, Y) \circ T^{f(e_\bullet)}_{f(e_\diamond)} \circ x' : \mathbf{1} \to \mathcal{C}(\hat{s}, Y)(\varrho_{n-1} \cdot f(a'), \varrho'_{n-1} \cdot f(b'))$$

along the metafunction $R_{\widehat{\sigma}}$. By section 3.3.3.9, because the paths $\tau_\flat \cdot f(e_\diamond) = f(\tau_\flat \cdot e_\diamond)$ and $\tau_\dagger \cdot f(e_\bullet) = f(\tau_\dagger \cdot e_\bullet)$ are in the image of f, the following diagram exists and commutes.

$$\begin{tikzcd} \mathcal{C}(\widehat{\sigma}, f)(\varrho_{n-1} \cdot a', \varrho'_{n-1} \cdot b') \arrow[r, "T^{\tau_\dagger \cdot e_\bullet}_{\tau_\flat \cdot e_\diamond}"] \arrow[d, "R_{\widehat{\sigma}}"'] & \mathcal{C}(\sigma_0, f)(\rho \cdot r, \rho' \cdot t) \arrow[d, "R_{\sigma_0}"] \\ \mathcal{C}(\hat{s}, Y)(\varrho_{n-1} \cdot f(a'), \varrho'_{n-1} \cdot f(b')) \arrow[r, "T^{\tau_\dagger \cdot f(e_\bullet)}_{\tau_\flat \cdot f(e_\diamond)}"] & \mathcal{C}(s_0, Y)(f(r'), f(t')) \end{tikzcd}$$

The above commutative diagram then transforms the lift associated with the choice of element (3.77) into a lift associated with the following choice of element along the metafunction R_{σ_0}.

(3.78)
$$T^{\tau_\dagger \cdot f(e_\bullet)}_{\tau_\flat \cdot f(e_\diamond)} \circ \mathcal{C}(\mathfrak{a}, Y) \circ T^{f(e_\bullet)}_{f(e_\diamond)} \circ x' : \mathbf{1} \to \mathcal{C}(s_0, Y)(f(r'), f(t'))$$

By Proposition 3.108, the above choice lies in the class $\mathcal{C}(s_0, Y, \vartheta_0, z)(f(r'), f(t'))$. Finally, by using diagram (3.73) and equation (3.76) expressing the metafunction $\mathcal{C}(\mathfrak{a}_\sharp, X)$, it is not hard to see that the lift associated with the choice (3.78) along R_{σ_0} gives rise to a lift as shown in diagram (3.75). $\square$

3.3.8.6. *Sequences of conjugations of extended nodes of spines.* Let $\mathcal{C}$ be a category and ℓ be a positive integer. A *sequence of conjugations of extended nodes of spines of length ℓ* in $\mathcal{C}$ consists, for every $0 \leq i \leq \ell - 1$, of a convergent extended i-conjugation of nodes of spines

$$\chi_i := (s_i, \varsigma^{i+1}, \overline{\sigma}_i)$$

such that the extended nodes of spines ς^{i+1} is of the form $s_{i+1} \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}_{i+1}$, for every $0 \leq i \leq \ell - 1$, and ς_i defines the closure of the chaining associated with the convergent conjugation χ_i for every $1 \leq i \leq \ell - 1$. The closure of the convergent conjugation χ_0 will be denoted by ς_0 and the underlying alliance of spines of ς^0 will be denoted by $\mathfrak{a}^0$. By construction, the alliance $\mathfrak{a}^0$ is an arrow of the form $s_0 \overset{\mathrm{EX}}{\rightsquigarrow} \sigma_\sharp$ where $\sigma_\sharp$ is defined as in section 3.3.8.4. The conjugation χ_i will be supposed to be defined along pairs $\mathfrak{v}^i_\diamond, \mathfrak{v}^i_\bullet$, $\mathfrak{a}^i_\flat, \mathfrak{a}^i_\dagger$ and $\mathfrak{v}^i_\flat, \mathfrak{v}^i_\dagger$. By definition of a conjugation of nodes of spines, if ς_0 is of degree n, then all spines $s_0, \dots, s_\ell$ are of degree $n - 1$ and all nodes of spines $\widehat{\sigma}_0, \dots, \widehat{\sigma}_\ell$ and $\sigma_\sharp$ are of degree n. The sequence will be said to be *closed* if the identity $n = \ell$ holds. In that case, the underlying sequence of simple framings of nodes of spines

$$(s_0, \mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet) \quad \rhd^V_0 \quad (s_1, \mathfrak{v}^1_\diamond, \mathfrak{v}^1_\bullet) \quad \rhd^V_1 \dots \rhd^V_{\ell-2} \quad (s_{\ell-1}, \mathfrak{v}^{\ell-1}_\diamond, \mathfrak{v}^{\ell-1}_\bullet) \quad \rhd^V_{\ell-1} \quad s_\ell$$

is convergent. The above sequence of framings is, below, denoted by $\mathfrak{F}$.

Theorem 3.111. *Let $\chi_0, \dots, \chi_{\ell-1}$ be a closed sequence of conjugations of extended nodes of spines of length ℓ as above and $f : Y \to Z$ and $g : X \to Y$ be two morphisms in $\mathcal{C}$. Suppose that g is a tubular surtraction for $\mathfrak{F}$ and the spine s_0 is g-projective. If $f \circ g$ is a surtraction for the extended nodes of spines ς^ℓ, then f is a surtraction for the extended nodes of spines ς^0.*

Proof. The spinal seeds of the nodes of spines $\widehat{\sigma}_0, \dots, \widehat{\sigma}_\ell$ and $\sigma_\sharp$ will be denoted by $\widehat{s}_0, \dots, \widehat{s}_\ell$ and $s_\natural$, respectively. According to section 3.3.6.7, the morphism $f : Y \to Z$ is a surtraction for $\varsigma^0 : s_0 \overset{\mathrm{EX}}{\rightsquigarrow} \sigma_\sharp$ if for every pair of parallel elements r and t above the base of s_0 in Y and any choice of element $x : \mathbf{1} \to \mathcal{C}(s_0, Z)(f(r), f(t))$, the dashed arrow, below, exists and makes the following diagram commutes.

$$\begin{tikzcd} & 1 \arrow[r, dashed] \arrow[d, "x"'] & \mathcal{C}(\widehat{\sigma}, f)(\rho_0 \cdot r, \rho'_0 \cdot t) \arrow[d, "R_{\widehat{\sigma}}"] \\ & \mathcal{C}(s_0, Z)(f(r), f(t)) \arrow[r, "\mathcal{C}(\mathfrak{a}^0, X)"'] & \mathcal{C}(s_\natural, Z)(\rho_0 \cdot f(r), \rho'_0 \cdot f(t)) \end{tikzcd}$$

The notation ρ_0 and ρ'_0 refer to the discal and codiscal transitions of the head of the alliance $\mathfrak{a}^0$, respectively. Consider a pair of parallel elements r and t above the base of s_0 in Y. By Remark 3.9, the relation $r \sim_{P_0} t$ holds if P_0 denotes the prespine of s_0. Because the prespine P_0 is g-projective, Proposition 3.42 implies that the pair of elements r and t is 0-parallel over g in Y. By Lemma 3.56 and Remark 3.57, there exists a pair of elements r' and t' parallel above s_ℓ in Y and a sequence of tubular operators of the form

$$T^{e^i_\bullet}_{e^i_\diamond} : \mathcal{C}(s_i, Y)(r_i, t_i) \longrightarrow \mathcal{C}(s_{i+1}, Y)(r_{i+1}, t_{i+1}),$$

for every $0 \leq i \leq \ell - 1$, such that $r_0 = r$ and $t_0 = t$ and $r_\ell = g(r')$ and $t_\ell = g(t')$. Section 3.3.8.2 shows that for any given pair of paths $e^i_\diamond$ and $e^i_\bullet$ in Y, the previous tubular operator come along with metafunctions $\mathcal{C}(\mathfrak{a}^{i+1}, Z)$ of the form

$$\mathcal{C}(s_{i+1}, Z)(f(r_{i+1}), f(t_{i+1})) \to \mathcal{C}(\widehat{s}_{i+1}, Z)(\rho_{i+1} \cdot f(r_{i+1}), \rho'_{i+1} \cdot f(t_{i+1}))$$

satisfying the property that if for any choice of element

$$x_{i+1} : \mathbf{1} \to \mathcal{C}(s_{i+1}, Z)(f(r_{i+1}), f(t_{i+1})),$$

the dashed arrow of the next diagram exists and makes the commutative square commute (3.79)

$$\begin{tikzcd} 1 \arrow[r, dashed] \arrow[d, "x_{i+1}"'] & \mathcal{C}(\widehat{\sigma}_{i+1}, f)(\rho_{i+1} \cdot f(r_{i+1}), \rho'_{i+1} \cdot f(t_{i+1})) \arrow[d, "R_{\widehat{\sigma}_{i+1}}"] \\ \mathcal{C}(s_{i+1}, Z)(f(r_{i+1}), f(t_{i+1})) \arrow[r, "\mathcal{C}(\mathfrak{a}^{i+1}, X)"'] & \mathcal{C}(\widehat{s}_{i+1}, Z)(\rho_{i+1} \cdot f(r_{i+1}), \rho'_{i+1} \cdot f(t_{i+1})) \end{tikzcd}$$

then for any choice of element $x_i : \mathbf{1} \to \mathcal{C}(s_i, Z)(f(r_i), f(t_i))$, the dashed arrow of the following diagram exists and makes the commutative square commute (see Proposition 3.110).

$$\begin{tikzcd}
 & 1 \arrow[r, dashed] \arrow[d, "x_i"'] & \mathcal{C}(\widehat{\sigma}_i, f)(\rho_i \cdot f(r_i), \rho'_i \cdot f(t_i)) \arrow[d, "R_{\widehat{\sigma}_i}"] \\
 & \mathcal{C}(s_i, Z)(f(r_i), f(t_i)) \arrow[r, "\mathcal{C}(\mathfrak{a}^i, X)"'] & \mathcal{C}(\widehat{s}_i, Z)(\rho_i \cdot f(r_i), \rho'_i \cdot f(t_i))
\end{tikzcd}$$

Because the equalities $r_0 = r$ and $t_0 = t$ hold, the inductive process underlying the above implication says that if diagram (3.79) exists for any choice x_ℓ (take $i = \ell - 1$), then f is a surtraction for ς^0. But this is the case, since we know that both relations $r_\ell = g(r')$ and $t_\ell = g(t')$ hold and $f \circ g$ is a surtraction for ς^ℓ, so that section 3.3.6.7 implies that diagram (3.79) evaluated at $i = \ell - 1$ exists for any choice x_ℓ. ☐

3.4. Examples of everyday spines

3.4.1. Higher category theories and topological spaces. The introduction of the present chapter already gave a taste of what the previous structures look like in **Top**. In fact, the construction of spinal framings, conjugations, correspondences and convergent chainings of memories will be done in Chapter 6 in a unified and detailed way for higher category theories and topological spaces.

3.4.2. Simplicial sets. This section describes how to generate the multiple structures seen in the present chapter for the case of simplicial sets. The considered vertebrae are those defined in section 2.4.2.4 of Chapter 2, all contained in the class denoted by $\mathcal{E}$. As noticed in the aforementioned section, all our structures will arise from Γ-factorisations, that is to say factorisations of the form $p \circ i$ where p is in $\mathbf{llp}(\mathbf{rlp}(\Gamma))$ and i is in $\mathbf{rlp}(\Gamma)$ where Γ is the following set of arrows.

$$\{\gamma_n : \partial\Delta_n \to \Delta_n\}_{n \in \omega}$$

To illustrate this, consider a vertebra $\|g_n, g'_n\| \cdot \beta$ in **sSet** whose notations are conform with those of section 2.4.2.4. To define a framing of this vertebra along two other vertebrae $\|g_n, g'_\diamond\| \cdot \beta_\diamond$ and $\|g'_n, g'_\bullet\| \cdot \beta_\bullet$, first form the following left pushout in **sSet**.

$$\begin{tikzcd}
\mathbb{S}_{q+1} \arrow[dr, "\beta"] & \mathbb{D}_2^q \arrow[l, "\delta_2^q"'] \arrow[r, "\beta_\diamond \circ \delta_2^\diamond"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_1^q \arrow[u, "\delta_1^q"] \arrow[d, "\beta_\bullet \circ \delta_2^\bullet"'] & \mathbb{D}' \arrow[dr, "\iota"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"'] & & \mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}_q \arrow[r, "g'_q"] \arrow[d, "g_q"'] & \mathbb{D}_1^q \arrow[d, "\delta_1^q"] \\
\mathbb{D}_2^q \arrow[r, "\delta_2^q"'] & \mathbb{S}_{q+1}
\end{tikzcd}$$

Pasting the above pushout diagram with the prevertebra given on the right along their common part comprising the cospan made of δ_1^q and δ_2^q then provides the following commutative diagram. The dashed arrow is generated from the universality of the pushout $\mathbb{S}_{q+1}$.

$$\begin{tikzcd}
\mathbb{S}_q \arrow[r, "g'_q"] \arrow[d, "g_q"'] & \mathbb{D}_1^q \arrow[r, "\beta_\diamond \circ \delta_1^\diamond"] \arrow[d, "\delta_1^q"] & \mathbb{D}'_\diamond \arrow[dd, "\iota^\diamond"] \\
\mathbb{D}_2^q \arrow[r, "\delta_2^q"'] \arrow[d, "\beta_\bullet \circ \delta_1^\bullet"'] & \mathbb{S}_{q+1} \arrow[dr, dashed, "u"] & \\
\mathbb{D}'_\bullet \arrow[rr, "\iota^\bullet"'] & & \mathbb{G}
\end{tikzcd}$$

Finally, factorising the arrow $u : \mathbb{S}_{q+1} \to \mathbb{G}$ into an arrow $\beta_* : \mathbb{S}_{q+1} \to \mathbb{D}'_*$ in $\mathbf{llp}(\mathbf{rlp}(\Gamma))$ and a morphism $\pi : \mathbb{D}'_* \to \mathbb{G}$ in $\mathbf{rlp}(\Gamma)$ exactly gives a vertebra $\|g_n, g'_n\| \cdot \beta_*$ framing the vertebra $\|g_n, g'_n\| \cdot \beta$ along the pair $\|g_n, g'_\diamond\| \cdot \beta_\diamond$ and $\|g'_n, g'_\bullet\| \cdot \beta_\bullet$ where the cylinder transition is provided by $\pi : \mathbb{D}'_* \to \mathbb{G}$.

Remark 3.112. It may be shown that the above framing is compatible with the morphisms of vertebrae. This follows from the fact that the cylinder transitions are in the class $\mathbf{rlp}(\Gamma)$. Specifically, considering two framings of vertebrae $\mathfrak{f} := (p \cdot \beta, v_\diamond, v_\bullet) \triangleright p_* \cdot \beta_*$ and $\mathfrak{f}_\flat := (p_\flat \cdot \beta_\flat, v_\diamond, v_\bullet) \triangleright p_\dagger \cdot \beta_\dagger$ equipped with respective cylinder transitions $\pi : \mathbb{D}'_* \to \mathbb{G}$ and $\pi : \mathbb{D}'_\dagger \to \mathbb{G}_\flat$ and a morphism of vertebrae $u : p \cdot \beta \curvearrowright p_\flat \cdot \beta_\flat$ always provides a commutative diagram as given below on the left (see also section 3.3.2.8 for the notations). The fact that $\beta_\dagger$ is in $\mathbf{llp}(\mathbf{rlp}(\Gamma))$ and the cylinder transition π is in $\mathbf{rlp}(\Gamma)$ then leads to the existence of a lift as given below on the right.

$$\begin{tikzcd}
\mathbb{S} \arrow[rr, "\beta_*"] \arrow[d, "\beta_\dagger"'] & & \mathbb{D}'_* \arrow[d, "\pi"] \\
\mathbb{D}'_\dagger \arrow[r, "\pi_\flat"'] & \mathbb{G}_\flat \arrow[r, "\kappa(u)"'] & \mathbb{G}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[rr, "\beta_*"] \arrow[d, "\beta_\dagger"'] & & \mathbb{D}'_* \arrow[d, "\pi"] \\
\mathbb{D}'_\dagger \arrow[r, "\pi_\flat"'] \arrow[urr, dashed, "u_*"] & \mathbb{G}_\flat \arrow[r, "\kappa(u)"'] & \mathbb{G}
\end{tikzcd}$$

The lift $u_* : \mathbb{D}'_\dagger \to \mathbb{D}'_*$ exactly defines a morphism of vertebrae $u_* : p_* \cdot \beta_* \curvearrowright p_\dagger \cdot \beta_\dagger$ together with a morphism of framings $(u, u_*) : \mathfrak{f} \curvearrowright \mathfrak{f}_\flat$.

Similarly, it is possible to generate reflections of vertebrae. Let $v := \|g_n, g'_n\| \cdot \beta$ be a vertebra in **sSet** as defined in section 2.4.2.4. There is a trivial alliance of prevertebrae $\|g_n, g'_n\| \rightsquigarrow (\|g'_n, g_n\|^{\mathrm{rv}})^{\mathrm{rv}}$. It may be shown that a straightforward Γ-factorisation provides an arrow β_* in $\mathbf{llp}(\mathbf{rlp}(\Gamma))$ inducing an alliance of vertebrae as follows.

$$\|g_n, g'_n\| \cdot \beta \rightsquigarrow (\|g'_n, g_n\|^{\mathrm{rv}} \cdot \beta_*)^{\mathrm{rv}}$$

All the above constructions may be used to generate conjugations of vertebrae, spines or nodes of spines. The correspondences resulting from this conjugations after apllication of the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Cong} \to \mathbf{Scov}(\mathcal{C})$ comes along with commutative squares as given in diagram (3.28). Again, Γ-factorisations may be used to fill these squares in with structures of mates as shown in diagram (3.29). The ability of generating an unlimited number of mates and framings eventually allows the construction of convergent chainings and convergent conjugations of spines and nodes of spines. All the conditions of functoriality regarding simple and extensive framings, chainings and other similar structures are ensured by right lifting properties as illustrated in Remark 3.112.

In the end, this shows that Γ-factorisations are an efficient way of constructing the elementary structure permitting the application of Theorem 3.111 and Theorem 3.111.

3.4.3. Chain complexes. Let R be a ring. This section only discusses the form of the framings in the case of the category of non-negatively graded chain complexes $\mathbf{Ch}_R$ for the (nodes) of vertebrae defined in section 2.4.3.4. First, recall that the vertebrae used are of the following from, where $R(\delta)$ is $R^{\oplus 2}$ when $\delta = 1$ and 0 when $\delta = 0$.

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & D_n \arrow[d] & \\ D_n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S_n \arrow[r, "\beta_n(\delta)"] & D_n(\delta) \end{tikzcd}$$

These vertebrae are obviously projective. Denote by $v(\delta)$ the above vertebra for every $\delta \in \{0,1\}$. It may be shown that a vertebra $v(\delta_4)$ frames a vertebra $v(\delta_1)$ along two vertebrae $v(\delta_1)$ and $v(\delta_2)$ for any 4-tuple $(\delta_1, \delta_2, \delta_3) \rhd \delta_4$ and cylinder transitions as follows (when truncated at rank n and $n+1$).

$$\begin{tikzcd} R \oplus R \oplus R \arrow[r, "\mu\circ\mu"] & R \arrow[d, equal] \\ R \oplus R \arrow[u, "R\oplus 0\oplus R"] \arrow[r, "\mu"'] & R \end{tikzcd}$$
(1,1,1)▷1

$$\begin{tikzcd} R \oplus R \oplus 0 \arrow[r, "\mu"] \arrow[d, equal] & R \arrow[d, equal] \\ R \oplus R \arrow[r, "\mu"'] & R \end{tikzcd}$$
(1,1,0)▷1

$$\begin{tikzcd} 0 \oplus R \oplus R \arrow[r, "\mu"] \arrow[d, equal] & R \arrow[d, equal] \\ R \oplus R \arrow[r, "\mu"'] & R \end{tikzcd}$$
(1,0,1)▷1

$$\begin{tikzcd} 0 \oplus R \oplus R \arrow[r, "\mu"] \arrow[d, equal] & R \arrow[d, equal] \\ R \oplus R \arrow[r, "\mu"'] & R \end{tikzcd}$$
(1,0,0)▷1

$$\begin{tikzcd} R \oplus 0 \oplus R \arrow[r, "\mu"] \arrow[d, equal] & R \arrow[d, equal] \\ R \oplus R \arrow[r, "\mu"'] & R \end{tikzcd}$$
(1,0,1)▷1

$$\begin{tikzcd} R \oplus 0 \oplus 0 \arrow[r, equal] & R \arrow[d, equal] \\ 0 \arrow[u, "!"] \arrow[r, "!"'] & R \end{tikzcd}$$
(0,1,0)▷0

$$\begin{tikzcd} 0 \oplus 0 \oplus R \arrow[r, equal] & R \arrow[d, equal] \\ 0 \arrow[u, "!"] \arrow[r, "!"'] & R \end{tikzcd}$$
(0,0,1)▷0

$$\begin{tikzcd} 0 \oplus 0 \oplus 0 \arrow[r, equal] & R \arrow[d, equal] \\ 0 \arrow[u, "!"] \arrow[r, "!"'] & R \end{tikzcd}$$
(0,0,0)▷0

The construction of the other structures is really an exercise of universal algebra, which is left to the motivated reader.

Chapter 4

Vertebral and Spinal Categories

4.1. Introduction

The present chapter organises the different structures of Chapters 2 & 3 into homotopy theories called *vertebral* and *spinal categories*. Their properties are close to those of usual homotopy theories (such as model categories and categories of fibrant objects). The work established herein obviously paves the way for possible generalisations to homotopy theories in bicategories and other higher categories (up to adequate generalisation of the concept of vertebra). In the spirit of the theory of Quillen's equivalences (see [**38, 12**]) and Crans' transfers (see [**9**]), vertebral and spinal categories will be associated with a theory of functors. In particular, the proof of the Homotopy Hypothesis will be sketched in general terms. The chapter will finish with a series of examples.

Chapter 2 showed that a notion of homotopy resulted from the notion of vertebra – or, in fact, the notions of extended node of vertebrae and alliance of nodes of vertebrae. It was shown that the zoos with which these were associated had properties reminiscent of the kind underpinning usual homotopy theories. This precisely happened when

1) the different vertebral structures interacted between each other via compositional actions stemming from the composition of diagrams; e.g. right action of alliances of nodes of vertebrae on extended nodes of vertebrae.
2) the vertebral structures were equipped with additional colimits and morphisms; e.g. reflexive vertebrae and framings of extended nodes of vertebrae.

In both cases, the zoo of the solicited structures was used in such a way that some assumptions on items pertaining to certain zoos lead to conclusions on items of other zoos (for instance, see the propositions of section 2.3.4.2, which never concerns the same zoo).

This instability suggests the need of a bigger structure in which the different interactions between vertebral structures would be stable. This would allow the definition of a unique zoo satisfying properties independent of the involved structures – at least, in appearance.

Our first homotopical structure is called *vertebral category* and relies on the results of Chapter 1. For a chosen category $\mathcal{C}$, it consists of *(i)* a structure E containing extended nodes of vertebrae in $\mathcal{C}$; *(ii)* two structures A and A' containing alliances of nodes of vertebrae in $\mathcal{C}$. The extended nodes of vertebrae in E may diagrammatically be composed, thus giving rise to a monoid-like structure

$$\underbrace{E \otimes E \to E}_{\textit{'pasting' of two extended nodes of vertebrae}}$$

the structures A and A' will act on the right of E and be related via a particular type of map called a *prolinear map*.

$$\underbrace{E \otimes A \to E \qquad\qquad E \otimes A' \to E}_{\textit{'pasting' of alliances on extended nodes of vertebrae}} \qquad\qquad \underbrace{A' \nrightarrow A}_{\textit{prolinear map}}$$

Finally, following the example of the structure of category, a vertebral category will be equipped with a main notion of composition, given by framings of extended nodes of vertebrae, and a notion of identity, given by reflexive nodes of vertebrae.

$$\underbrace{\Sigma_0 E \otimes \Sigma_1 E \to \Sigma_\star E}_{\textit{framings of extended nodes of vertebrae}} \qquad\qquad \underbrace{\mathbf{1} \to E}_{\textit{picks out reflexive nodes of vertebrae}}$$

The symbols Σ_0, Σ_1 and $\Sigma_\star$ here stands for operations on E that will be made explicit later on. Such a vertebral category $(\mathcal{C}, E, A, A')$ will be associated with a zoo where

- the (trivial) fibrations will be defined with respect to the alliances of A';
- the intractions will be defined with respect to the alliances of A;
- the surtractions and pseudofibrations will be defined with respect to E.

The algebraic operations previously defined will then enable stable interactions between the different protagonists of the zoo.

The problem that arises regarding vertebral categories is that they are not sufficient to show the entire two-out-of-six property, which requires the notion of spine defined in Chapter 3. Roughly, a spine is a vertebra equipped with a notion of dimension encoded by a prespine. To permit the two-out-of-six property in vertebral categories, we will therefore need to equip the nodes of vertebrae of the vertebral category with structures of spine.

$$\underset{\textit{(node of spines)}}{\sigma} \longmapsto \underset{\textit{(node of vertebrae)}}{\nu}$$

Such attributions require one to choose a dimension for every vertebra in the vertebral category. We will thus have maps associating spines with vertebrae such that this mappings ensures that every vertebra has a lifting to a spine of some dimension.

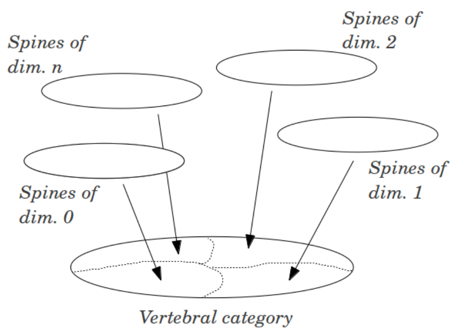


Figure 1. Picture of a spinal category

We will thus need to *cover* vertebral categories with elementary structures characteristic of a particular dimension. Because the notion of zoo associated with the vertebral category will have to be transported at the level of spines, these structures will be given by pairs (A_n, E_n) for every $n \in \omega$, where

- every spine in A_n and E_n is of dimension n;
- the objects of A_n are sent to the objects of A;
- the objects of E_n are sent to the objects of E.

A vertebral category equipped with a covering of its elements by the previously described structures will be called a *spinal category* when equipped with some additional operations. These operations will enable us to achieve the proof of the two-out-of-six property. Precisely, these will be ternary operations encoding the framings of spines along semi-extended nodes of vertebrae as defined in Chapter 3 (see the following pair of operations).

$$T_{\ltimes}(E) \otimes A_n \otimes T_{\rtimes}(E) \to A_n \qquad\qquad T_{\ltimes}(E) \otimes E_n \otimes T_{\rtimes}(E) \to E_n$$

The operations $T_{\ltimes}(\_)$ and $T_{\rtimes}(\_)$ appearing in the preceding maps take the extended node of vertebrae of E and *derive* them into semi extended-nodes of vertebrae.

$$\begin{array}{ccc} \mathfrak{n} & \longmapsto & \mathfrak{b} \circledcirc d\mathfrak{n} \\ \textit{(extended vertebra)} & & \textit{(semi-extended vertebra)} \end{array}$$

Somehow, the structures $T_{\ltimes}(E)$ and $T_{\ltimes}(E)$ will resemble tangent bundles for these derivations. We will finally require the image of the previous framings to be equipped with an inverse process, which will be defined in terms of convergent conjugations. The cancellation theorems of Chapter 3 will then allow the proof of the two-out-of-six property in spinal categories.

Finally, we will discuss various notions of functors that transfer the zoo of vertebral and spinal categories to other categories. The fact that only the zoo is potentially transferred is here important as functors do not transport the spinal and vertebral structures in general. The notion of transfer will take the form of functors $F : \mathcal{C} \to \mathcal{D}$ that send the elements of E, A and A' to similar structure in the category $\mathcal{D}$, which will give rise to classes of elements $F(E)$, $F(A)$ and $F(A')$. This will allow us to make sense of a notion of zoo in the category $\mathcal{D}$. Then, any spinal or vertebral structure on $\mathcal{C}$ will provide the zoo of $\mathcal{D}$ with properties similar to those obtained for spinal or vertebral categories.

4.2. Preparation

4.2.1. Warming-up on vertebrae and spines.

4.2.1.1. *Reminder of the second chapter.* In Chapter 2, the concept of framing concerned the notion of extended nodes of vertebrae. Specifically, we had two extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and $\mathfrak{n}_\flat : \gamma'_* \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ where the preseed γ'_* of the latter was equal to the coseed of the node of vertebrae ν_*. This last condition on the two extended nodes of vertebrae presented the pair $(\mathfrak{n}, \mathfrak{n}_\flat)$ as a *communicating pair*. From such a setting was then defined the notion of framing for such a pair, which consisted of a third extended node of vertebrae $\mathfrak{n}_\bullet : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\bullet$ where the coseed of $\nu_\bullet$ was equal to that of $\nu_\flat$. To resume, if we denote by $\mathcal{E}(\gamma, \gamma'_*)$ the class of extended nodes of vertebrae whose preseed and coseed are equal to γ and γ'_*, respectively, the concept of framing of extended node of vertebrae takes any pair in the left product of classes, below, and associate it with an extended node of vertebrae in the corresponding right class.

$$\mathcal{E}(\gamma, \gamma'_*) \times \mathcal{E}(\gamma'_*, \gamma'_\flat) \qquad \rightsquigarrow \qquad \mathcal{E}(\gamma, \gamma'_\flat)$$

Later on, we will be interested in providing such a process with specific choices so that we obtain a metafunction of the form $\mathcal{E}(\gamma, \gamma'_*) \times \mathcal{E}(\gamma'_*, \gamma'_\flat) \to \mathcal{E}(\gamma, \gamma'_\flat)$.

4.2.1.2. *Reminder of the third chapter.* In Chapter 3, the concept of framing of nodes of spines considered the combination of two semi-extended nodes of vertebrae together with a node of spines and gave another node of spines as output. Specifically, for every fixed non-negative integer q, the operation of q-framing considered a node of spines $\sigma = (p_k) \cdot \Omega$ of degree greater than or equal to q along two semi-extended nodes of vertebrae $\mathfrak{n}_\diamond : \gamma_q \rightsquigarrow \nu_\diamond$ and $\mathfrak{n}_\bullet : \gamma'_q \rightsquigarrow \nu_\bullet$ where the prevertebra p_q was of the form $\|\gamma_q, \gamma'_q\|$. The output was then a node of spines $\sigma_* = (p^*_k) \cdot \Omega_*$ where

- **(compatibility)** the equality $p_q = p^*_k$ held for every $0 \leq k \leq q-1$;
- **(framing)** the prevertebra p^*_q was of the form $\|\gamma'_\bullet, \gamma'_\diamond\|$ where $\gamma'_\bullet$ and $\gamma'_\diamond$ were the coseeds of $\nu_\diamond$ and $\nu_\bullet$, respectively.

In other words, if we denote by $\mathcal{O}_k(\gamma, \gamma')$ the class of nodes of spines $(p_k) \cdot \Omega$ whose prevertebra p_k is of the form $\|\gamma, \gamma'\|$ and by $\mathcal{T}(\gamma, \gamma')$ the class of semi-extended nodes of vertebrae whose preseeds and coseeds are equal to γ and γ', respectively, the concept of framing of spines takes any triple in the left product of classes, below, and associate it with a node of spines in the corresponding right class.

$$\mathcal{T}(\gamma, \gamma'_\diamond) \times \mathcal{O}_k(\gamma, \gamma') \times \mathcal{T}(\gamma', \gamma'_\bullet) \quad \rightsquigarrow \quad \mathcal{O}_k(\gamma'_\diamond, \gamma'_\bullet)$$

Later on, we will be interested in providing such a process with specific choices so that we obtain metafunctions of the form $\mathcal{T}(\gamma, \gamma'_\diamond) \times \mathcal{O}_k(\gamma, \gamma') \times \mathcal{T}(\gamma', \gamma'_\bullet) \to \mathcal{O}_k(\gamma'_\diamond, \gamma'_\bullet)$. The images of these operations will be equipped with structures of convergent conjugation, that is to say some extra structure allowing the 'reverse' of the involved framing. Such assumptions will suffice to achieve the wanted properties for which spinal categories are defined.

4.2.1.3. *Conventions on hom-sets.* For convenience, the hom-sets of the category of nodes of vertebrae $\mathbf{Ally}(\mathcal{C})$ will later be denoted as $\mathbf{Ally}(\nu, \nu_*)$.

4.2.1.4. *Notations.* Let $\mathcal{C}$ be a category and n be a non-negative integer. In the sequel, we shall let $\mathbf{Aos}(\mathcal{C}, n)$ denote the category whose objects are spines of degree n and whose morphisms are alliances between them. Similarly, we shall let $\mathbf{Anos}(\mathcal{C}, n)$ denote the category whose objects are nodes of spines of degree n and whose morphisms are alliances between them.

4.2.2. Factorisation properties.

4.2.2.1. *Purpose of this section.* The goal of this section is to reformulate the property already defined in Chapter 2, but in terms of factorisation games. The reason for these reformulations is to hide the cumbersome parts of future calculations. For illustration, it follows from Yoneda's Lemma that a morphism $f : X \to Y$ in some category $\mathcal{C}$ has the rlp with respect to a morphism $\gamma : A \to B$ if and only if for every commutative diagram of the form given below on the left, there exists a 'lift' $h : \mathbf{1} \to \mathcal{C}(B, X)$ making the diagram on the right commute.

$$\begin{tikzcd} \mathbf{1} \arrow[r, "x"] \arrow[d, "y"'] & \mathcal{C}(A,X) \arrow[d, "\mathcal{C}(A,f)"] \\ \mathcal{C}(B,Y) \arrow[r, "\mathcal{C}(\gamma,Y)"'] & \mathcal{C}(A,Y) \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} \mathbf{1} \arrow[dr, dashed, "h"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\ & \mathcal{C}(B,X) \arrow[r, "\mathcal{C}(\gamma,X)"] \arrow[d, "\mathcal{C}(B,f)"'] & \mathcal{C}(A,X) \arrow[d, "\mathcal{C}(A,f)"] \\ & \mathcal{C}(B,Y) \arrow[r, "\mathcal{C}(\gamma,Y)"'] & \mathcal{C}(A,Y) \end{tikzcd}$$

The previous implication essentially consists in stating that any natural transformation of the form $\mathbf{1} \Rightarrow F$ over the category made of two opposite arrows $\bullet \to \bullet \leftarrow \bullet$ lifts to a natural transformation $\mathbf{1} \Rightarrow G$ over the product category $\mathbf{2} \times \mathbf{2}$. This type of language will be that of factorisation games and will turn out to be extremely useful to prove our future properties.

4.2.2.2. *Factorisation games.* A *factorisation game* consists of an inclusion of small categories $i : \mathtt{P} \hookrightarrow \mathtt{O}$, a class Ω, called *attacking configuration*, a class Ω', called *defending configuration*, a collection of functors

$$\{A_\beta : \mathtt{P} \to \mathbf{Set}\}_{\beta\in\Omega}$$

called *collection of attacking moves* and a collection of functors

$$\{D_{(\beta,\beta')} : \mathtt{O} \to \mathbf{Set}\}_{(\beta,\beta')\in\Omega\times\Omega'}$$

called *collection of defending moves*, such that the restriction of $D_{(\beta,\beta')}$ along $i : \mathtt{P} \hookrightarrow \mathtt{O}$ equals the functor A_β for every $(\beta,\beta') \in \Omega \times \Omega'$.

$$\begin{tikzcd} \mathtt{P} \arrow[r, "A_\beta"] \arrow[d, "i"'] & \mathbf{Set} \\ \mathtt{O} \arrow[ur, "D_{(\beta,\beta')}"'] & \end{tikzcd}$$

Such a structure will later be denoted as a 4-tuple (A, D, Ω, Ω') and said to be *of type* $i : \mathtt{P} \hookrightarrow \mathtt{O}$. This will be denoted as an arrow $(A, D, \Omega, \Omega') \dashv \mathtt{P} \hookrightarrow \mathtt{O}$. A *play* for a factorisation game $(A, D, \Omega, \Omega') \dashv \mathtt{P} \hookrightarrow \mathtt{O}$ consists of an element $\beta \in \Omega$ and a morphism $\Delta_{\mathtt{P}}(\mathbf{1}) \Rightarrow A_\beta$ in $\mathbf{Set}^{\mathtt{P}}$. A factorisation game $(A, D, \Omega, \Omega') \dashv \mathtt{P} \hookrightarrow \mathtt{O}$ will then be said to *have a winning strategy* if for every play $\wp : \Delta_{\mathtt{P}}(\mathbf{1}) \Rightarrow A_\beta$ in $\mathbf{Set}^{\mathtt{P}}$, there exists $\beta' \in \Omega'$ and an arrow $\wp' : \Delta_{\mathtt{O}}(\mathbf{1}) \Rightarrow D_{(\beta,\beta')}$ in $\mathbf{Set}^{\mathtt{O}}$ whose image via the functor $\mathbf{Set}^i : \mathbf{Set}^{\mathtt{O}} \to \mathbf{Set}^{\mathtt{P}}$ induced by the pre-composition with $i : \mathtt{P} \hookrightarrow \mathtt{O}$ is equal to the play $\wp$.

$$\begin{tikzcd} \mathbf{Set}^{\mathtt{O}} \arrow[d, "\_\circ i"'] & \Delta_{\mathtt{O}}(\mathbf{1}) \arrow[r, Rightarrow, "\wp'"] \arrow[d, mapsto, "\_\circ i"'] & D_{(\beta,\beta')} \arrow[d, mapsto, "\_\circ i"] \\ \mathbf{Set}^{\mathtt{P}} & \Delta_{\mathtt{P}}(\mathbf{1}) \arrow[r, Rightarrow, "\wp"] & A_\beta \end{tikzcd}$$

4.2.2.3. *Equivalences of factorisation games.* An *equivalence of factorisation games* from a factorisation game (A, D, Ω, Ω_*) to factorisation game $(A^\flat, D^\flat, \Omega_\flat, \Omega_\dagger)$ of the same type $i : \mathtt{P} \hookrightarrow \mathtt{O}$ consists of

1) two surjective metafunctions $\psi : \Omega \to \Omega_\flat$ and $\psi_* : \Omega_* \to \Omega_\dagger$;
2) a collection of isomorphisms

$$a_\beta : A_\beta \cong A^\flat_{\psi(\beta)}$$

in $\mathbf{Set}^{\mathtt{P}}$ for every $\beta \in \Omega$ as well as a collection of isomorphisms

$$d_{(\beta,\beta_*)} : D_{(\beta,\beta_*)} \cong D^\flat_{(\psi(\beta),\psi_*(\beta_*))}$$

in $\mathbf{Set}^{\mathtt{O}}$ for every $(\beta,\beta_*) \in \Omega \times \Omega_*$ such that the image of $d_{(\beta,\beta_*)}$ via the functor $\mathbf{Set}^i : \mathbf{Set}^{\mathtt{O}} \to \mathbf{Set}^{\mathtt{P}}$ is equal to a_β

The point of equivalent factorisation games is the following proposition.

Proposition 4.1. *Let (A, D, Ω, Ω_*) and $(A^\flat, D^\flat, \Omega_\flat, \Omega_\dagger)$ be two equivalent factorisation games of type $i : \mathtt{P} \hookrightarrow \mathtt{O}$ as defined above. The factorisation game (A, D, Ω, Ω_*) admits a winning strategy if and only if $(A^\flat, D^\flat, \Omega_\flat, \Omega_\dagger)$ admits a winning strategy.*

Proof. Let us prove that if the factorisation game (A, D, Ω, Ω_*) admits a winning strategy, then so does $(A^\flat, D^\flat, \Omega_\flat, \Omega_\dagger)$. Consider the play, below, with $\beta_\flat \in \Omega_\flat$ for the factorisation game $(A^\flat, D^\flat, \Omega_\flat, \Omega_\dagger)$.

$$\wp : \Delta_{\mathtt{P}}(\mathbf{1}) \Rightarrow A^\flat_{\beta_\flat}$$

Because the metafunction $\psi : \Omega \to \Omega_\flat$ is surjective, there must exist $\beta \in \Omega$ such that the equality $\psi(\beta) = \beta_\flat$ holds. Then, using the isomorphism a_β, we may define the following play for (A, D, Ω, Ω_*).

$$\Delta_{\mathtt{P}}(\mathbf{1}) \xRightarrow{\wp} A^\flat_{\beta_\flat} \xRightarrow{a_\beta^{-1}} A_\beta$$

Because (A, D, Ω, Ω_*) admits a winning strategy, there exists some $\beta_* \in \Omega_*$ for which the preceding play may be lifted to $\mathbf{Set}^{\mathtt{P}}$ as follows.

$$\begin{tikzcd}
\mathbf{Set}^{\mathtt{O}} \arrow[d, "_\circ i"'] & \Delta_{\mathtt{O}}(\mathbf{1}) \arrow[rr, Rightarrow, "\wp'"] \arrow[d, mapsto, "_\circ i"'] & & D_{(\beta,\beta_*)} \arrow[d, mapsto, "_\circ i"] \\
\mathbf{Set}^{\mathtt{P}} & \Delta_{\mathtt{P}}(\mathbf{1}) \arrow[r, Rightarrow, "\wp"] & A^\flat_{\beta_\flat} \arrow[r, Rightarrow, "a_\beta^{-1}"] & A_\beta
\end{tikzcd}$$

Using the isomorphism $d_{(\beta,\beta_*)}$, we may finally expose a lift of the initial play $\wp : \Delta_{\mathtt{P}}(\mathbf{1}) \Rightarrow A^\flat_{\beta_\flat}$ to $\mathbf{Set}^{\mathtt{O}}$ as follows.

$$\begin{tikzcd}
\mathbf{Set}^{\mathtt{O}} \arrow[d, "_\circ i"'] & \Delta_{\mathtt{O}}(\mathbf{1}) \arrow[rr, Rightarrow, "\wp'"] \arrow[d, mapsto, "_\circ i"'] & & D_{(\beta,\beta_*)} \arrow[r, Rightarrow, "d_{(\beta,\beta_*)}"] \arrow[d, mapsto, "_\circ i"] & D_{(\beta_\flat,\psi_*(\beta_*))} \arrow[d, mapsto, "_\circ i"] \\
\mathbf{Set}^{\mathtt{P}} & \Delta_{\mathtt{P}}(\mathbf{1}) \arrow[r, Rightarrow, "\wp"] \arrow[rrr, Rightarrow, bend right, "\wp"'] & A^\flat_{\beta_\flat} \arrow[r, Rightarrow, "a_\beta^{-1}"] & A_\beta \arrow[r, Rightarrow, "a_\beta"] & A^\flat_{\beta_\flat}
\end{tikzcd}$$

Conversely, consider a play $\wp : \Delta_{\mathtt{P}}(\mathbf{1}) \Rightarrow A_\beta$ for the factorisation game (A, D, Ω, Ω_*). Because the factorisation game $(A^\flat, D^\flat, \Omega_\flat, \Omega_\dagger)$ admits a winning strategy, there exists some $\beta_\dagger \in \Omega_\dagger$ for which the following lift holds.

$$\begin{tikzcd}
\mathbf{Set}^{\mathtt{O}} \arrow[d, "_\circ i"'] & \Delta_{\mathtt{O}}(\mathbf{1}) \arrow[rr, Rightarrow, "\wp'"] \arrow[d, mapsto, "_\circ i"'] & & D_{(\psi(\beta),\beta_\dagger)} \arrow[d, mapsto, "_\circ i"] \\
\mathbf{Set}^{\mathtt{P}} & \Delta_{\mathtt{P}}(\mathbf{1}) \arrow[r, Rightarrow, "\wp"] & A_\beta \arrow[r, Rightarrow, "a_\beta"] & A^\flat_{\psi(\beta)}
\end{tikzcd}$$

Since $\psi_* : \Omega_* \to \Omega_\dagger$ is surjective, there must exists some $\beta_* \in \Omega_*$ such that the equality $\psi_*(\beta_*) = \beta_\dagger$ holds. Finally, the inverse of d_{β,β_*} allows us to build a lift of the play $\wp : \Delta_{\mathtt{P}}(\mathbf{1}) \Rightarrow F$ in $\mathbf{Set}^{\mathtt{O}}$ as follows.

$$\begin{tikzcd}
\mathbf{Set}^{\mathtt{O}} \arrow[d, "_\circ i"'] & \Delta_{\mathtt{O}}(\mathbf{1}) \arrow[rr, Rightarrow, "\wp'"] \arrow[d, mapsto, "_\circ i"'] & & D_{(\psi(\beta),\beta_\dagger)} \arrow[r, Rightarrow, "d^{-1}_{(\beta,\beta_*)}"] \arrow[d, mapsto, "_\circ i"] & D_{(\beta,\beta_*)} \arrow[d, mapsto, "_\circ i"] \\
\mathbf{Set}^{\mathtt{P}} & \Delta_{\mathtt{P}}(\mathbf{1}) \arrow[r, Rightarrow, "\wp"] \arrow[rrr, Rightarrow, bend right, "\wp"'] & A_\beta \arrow[r, Rightarrow, "a_\beta"] & A^\flat_{\psi(\beta)} \arrow[r, Rightarrow, "a_\beta^{-1}"] & A_\beta
\end{tikzcd}$$

This finishes the proof of the statement. □

4.2.2.4. *Lifting properties as factorisation games.* Let $\mathcal{C}$ be a category. If follows from the Yoneda Lemma and the definition of section 2.2.1.1 that a morphism $f : X \to Y$ has the right lifting property with respect to a commutative square

$$\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[d, "\gamma"] \\
B' \arrow[r, "\theta'"'] & B
\end{tikzcd} \tag{4.1}$$

which will later be denoted as an arrow $\mathbf{s} : \gamma' \Rightarrow \gamma$ in $\mathcal{C}^2$, if and only if for every pair of functions $x : \mathbf{1} \to \mathcal{C}(A, X)$ and $y : \mathbf{1} \to \mathcal{C}(B, Y)$ making the following diagram commute

(4.2)
$$\begin{tikzcd}
& \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(\theta,X)"] \arrow[dd, "\mathcal{C}(A,f)"] & & \mathcal{C}(A',X) \arrow[dd, "\mathcal{C}(A',f)"] \\
\mathbf{1} \arrow[ur, "y"] \arrow[dd, "y"'] & & & \\
& \mathcal{C}(A,Y) \arrow[rr, "\mathcal{C}(\theta,Y)"] & & \mathcal{C}(A',Y) \\
\mathcal{C}(B,Y) \arrow[ur, "\mathcal{C}(\gamma,Y)"] \arrow[rr, "\mathcal{C}(\theta',Y)"] & & \mathcal{C}(B',Y) \arrow[ur, "\mathcal{C}(\gamma',Y)"'] &
\end{tikzcd}$$

there exists a function (lift) $h : \mathbf{1} \to \mathcal{C}(B', X)$ such that the following diagram commutes.

(4.3)
$$\begin{tikzcd}
& \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(\theta,X)"] \arrow[dd] & & \mathcal{C}(A',X) \arrow[dd, "\mathcal{C}(A',f)"] \\
\mathbf{1} \arrow[ur, "x"] \arrow[rr, dashed, "h"] \arrow[dd, "y"'] & & \mathcal{C}(B',X) \arrow[ur, "\mathcal{C}(\gamma',X)"'] \arrow[dd] & \\
& \mathcal{C}(A,Y) \arrow[rr] & & \mathcal{C}(A',Y) \\
\mathcal{C}(B,Y) \arrow[ur, "\mathcal{C}(\gamma,Y)"] \arrow[rr, "\mathcal{C}(\theta',Y)"] & & \mathcal{C}(B',Y) \arrow[ur, "\mathcal{C}(\gamma',Y)"'] &
\end{tikzcd}$$

Let now $\mathcal{C}(\mathbf{s}, f)_0^A$ denote the diagram resulting from the removal of the object $\mathbf{1}$ and the morphisms x and y in (4.2). Similarly, let $\mathcal{C}(\mathbf{s}, f)_0^D$ denote the result of removing $\mathbf{1}$, x, y and h from (4.3).

Proposition 4.2. *The morphism $f : X \to Y$ has the right lifting property with respect to diagram (4.1) if and only if the factorisation game (A, D, Ω, Ω_*), where*

- *Ω and Ω_* are singletons equal to $\{0\}$;*
- *A_0 is given by the diagram of hom-sets $\mathcal{C}(\mathbf{s}, f)_0^A$;*
- *D_0 is given by the diagram of hom-sets $\mathcal{C}(\mathbf{s}, f)_0^D$,*

has a winning strategy.

Proof. The previous definition shows how any natural transformation of the form $(x, y) : \mathbf{1} \Rightarrow \mathcal{C}(\mathbf{s}, f)_0^A$ implies a lift $(x, y, h) : \mathbf{1} \Rightarrow \mathcal{C}(\mathbf{s}, f)_0^D$ that restricts to the previous natural transformation. $\square$

4.2.2.5. *Simplicity as factorisation gaming.* Let $\mathcal{C}$ be a category and $\mathcal{S} := (\Omega, \varkappa, \Omega')$ be a scale in $\mathcal{C}$ where $k : A' \to A$. Recall that the two classes Ω and Ω' contain morphisms in $\mathcal{C}$ whose domains are equal to A and A', respectively. It follows from the Yoneda Lemma that a morphism $f : X \to Y$ in $\mathcal{C}$ is simple with respect to $\mathcal{S}$ if and only if for every morphism $\beta : A \to B$ in Ω and every pair of functions $x : \mathbf{1} \to \mathcal{C}(A, X)$ and $y : \mathbf{1} \to \mathcal{C}(B, Y)$ making the following diagram commute

(4.4)
$$\begin{tikzcd}
 & \mathcal{C}(A,X) \arrow[r, "\mathcal{C}(k,X)"] \arrow[dd, "\mathcal{C}(A,f)"] & \mathcal{C}(A',X) \arrow[dd, "\mathcal{C}(A',f)"] \\
\mathbf{1} \arrow[ur, "x"] \arrow[dd, "y"'] & & \\
 & \mathcal{C}(A,Y) \arrow[r, "\mathcal{C}(k,Y)"'] & \mathcal{C}(A',Y) \\
\mathcal{C}(B,Y) \arrow[ur, "\mathcal{C}(\beta,Y)"'] & &
\end{tikzcd}$$

there exist a morphism $\beta' : A' \to B'$ in Ω' and a function $h : \mathbf{1} \to \mathcal{C}(B',X)$ such that the following diagram commutes.

(4.5)
$$\begin{tikzcd}
 & \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(k,X)"] \arrow[dd] & & \mathcal{C}(A',X) \arrow[dd, "\mathcal{C}(A',f)"] \\
\mathbf{1} \arrow[ur, "x"] \arrow[rr, dashed, "h"] \arrow[dd, "y"'] & & \mathcal{C}(B',X) \arrow[ur, "\mathcal{C}(\beta',X)"'] \arrow[dd] & \\
 & \mathcal{C}(A,Y) \arrow[rr] & & \mathcal{C}(A',Y) \\
\mathcal{C}(B,Y) \arrow[ur, "\mathcal{C}(\beta,Y)"] & & \mathcal{C}(B',Y) \arrow[ur, "\mathcal{C}(\beta',Y)"'] &
\end{tikzcd}$$

Let now $\mathcal{C}(\mathcal{S},f)^A_\beta$ denote the diagram resulting from the removal of the object $\mathbf{1}$ and the morphisms x and y in (4.4). Similarly, let $\mathcal{C}(\mathcal{S},f)^D_{\beta,\beta'}$ denote the result of removing $\mathbf{1}$, x, y and h from (4.5).

Proposition 4.3. *The morphism $f : X \to Y$ is simple with respect to $(\Omega, \varkappa, \Omega')$ if and only if the factorisation game (A, D, Ω, Ω'), where*

- *A_β is given by the diagram $\mathcal{C}(\mathcal{S},f)^A_\beta$ for every $\beta \in \Omega$;*
- *$D_{\beta,\beta'}$ is given by the diagram $\mathcal{C}(\mathcal{S},f)^D_{\beta,\beta'}$ for every $(\beta,\beta') \in \Omega \times \Omega'$,*

has a winning strategy.

Proof. Straightforward. □

4.2.2.6. *Division as factorisation gaming.* Let $\mathcal{C}$ be a category and consider a besom in $\mathcal{C}$ made of the following two commutative squares as well as a class Ω in $\mathcal{C}$ whose domains are all equal to B''.

$$\begin{tikzcd}
A' \arrow[r, "\theta"] \arrow[d, "\overline{\gamma}"'] & A \arrow[d, "\gamma"] \\
B' \arrow[r, "\theta'"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[r, "\gamma'"] \arrow[d, "\overline{\gamma}"'] & A'' \arrow[d, "\delta_1"] \\
B' \arrow[r, "\delta_2"'] & B''
\end{tikzcd}$$

The preceding besom, which was denoted as a triple of the form $(\Omega, \mathbf{d}, \theta)$ in Chapter 2, will here be denoted by the letter $\mathcal{B}$. It follows from the Yoneda Lemma that a morphism

$f : X \to Y$ in $\mathcal{C}$ is divisible by $\mathcal{B}$ if and only if for every pair of functions $x : \mathbf{1} \to \mathcal{C}(A, X)$ and $y : \mathbf{1} \to \mathcal{C}(B, Y)$ making the following diagram commute

(4.6)
$$\begin{tikzcd}
 & \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(\theta,X)"] \arrow[dd, "\mathcal{C}(A,f)"] & & \mathcal{C}(A',X) \arrow[dd, "\mathcal{C}(A',f)"] \\
\mathbf{1} \arrow[ur, "x"] \arrow[dd, "y"'] & & & \\
 & \mathcal{C}(A,Y) \arrow[rr, "\mathcal{C}(\theta,Y)"] & & \mathcal{C}(A',Y) \\
\mathcal{C}(B,Y) \arrow[ur, "\mathcal{C}(\gamma,Y)"] \arrow[rr, "\mathcal{C}(\theta',Y)"] & & \mathcal{C}(B',Y) \arrow[ur, "\mathcal{C}(\overline{\gamma},Y)"'] &
\end{tikzcd}$$

there exist an arrow $\beta : B'' \to A$ in Ω and a pair of functions $x' : \mathbf{1} \to \mathcal{C}(A'', X)$ and $y' : \mathbf{1} \to \mathcal{C}(B'', Y)$ making the following diagram commute in **Set**.

(4.7)
$$\begin{tikzcd}
 & \mathcal{C}(A,X) \arrow[rr, "\mathcal{C}(\theta,X)"] \arrow[dd] & & \mathcal{C}(A',X) \arrow[dd] & & \mathcal{C}(A'',X) \arrow[ll, "\mathcal{C}(\gamma',X)"'] \arrow[dd, "\mathcal{C}(A'',f)"'] \\
\mathbf{1} \arrow[ur, "x"] \arrow[dd, "y"'] \arrow[urrrrr, dashed, "x'"] \arrow[ddrrrr, dashed, "y'"] & & & & & \\
 & \mathcal{C}(A,Y) \arrow[rr] & & \mathcal{C}(A',Y) & & \mathcal{C}(A'',Y) \arrow[ll] \\
\mathcal{C}(B,Y) \arrow[ur, "\mathcal{C}(\gamma,Y)"] \arrow[rr, "\mathcal{C}(\theta',Y)"] & & \mathcal{C}(B',Y) \arrow[ur, "\mathcal{C}(\overline{\gamma},Y)"] & & \mathcal{C}(B'',Y) \arrow[ll, "\mathcal{C}(\beta\circ\delta_2,Y)"'] \arrow[ur, "\mathcal{C}(\beta\circ\delta_1,Y)"'] &
\end{tikzcd}$$

Let now $\mathcal{C}(\mathcal{B}, f)_0^A$ denote the diagram resulting from the removal of the object $\mathbf{1}$ and the morphisms x and y in (4.6). Similarly, let $\mathcal{C}(\mathcal{B}, f)_{0,\beta}^D$ denote the result of removing $\mathbf{1}$, x, y, x' and y' from (4.5).

Proposition 4.4. *The morphism $f : X \to Y$ is divisible by $\mathcal{B}$ if and only if the factorisation game $(A, D, \{0\}, \Omega)$ where*

- *A_0 is given by the diagram $\mathcal{C}(\mathcal{B}, f)_0^A$;*
- *$D_{0,\beta}$ is given by the diagram $\mathcal{C}(\mathcal{B}, f)_{0,\beta}^D$,*

has a winning strategy.

Proof. Straightforward. □

4.3. Algebraic structures on vertebrae and spines

The following section might seem a little bit slow and repetitive to the reader as it only gives a dictionary of elementary structures that will be used later on, the goal being to abstract and encode the common features and structures corresponding to the various actions coming from the concepts given in earlier chapters. The important definitions of the section are however Definition 4.20, Definition 4.24 and section 4.3.8.4. The section also provides examples that will be used as basic definitions for our structures of vertebrae.

4.3.1. Spans.

4.3.1.1. *Spans.* The notion of span appears under various forms in the literature. The following definition gives a category-like presentation of it. A *span* consists of

1) a higher class S_0, called the *left object-class*;
2) a higher class S_1, called the *right object-class*;
3) for every $\gamma \in S_0$ and $a \in S_1$, a higher class $E(\gamma, a)$, called the *hom-class*.

Such a structure will usually be referred to by the letter used to denote its hom-classes; e.g. the previous structure is a span E. The object-classes S_0 and S_1 will then be denoted as $\mathrm{Obj}_{\mathrm{L}}(E)$ and $\mathrm{Obj}_{\mathrm{R}}(E)$, respectively.

Example 4.5. Let $\mathcal{C}$ be a category. The metacategory $\mathbf{Ally}(\mathcal{C})$ defines an obvious span whose left and right object-classes are equal to the class of objects of $\mathbf{Ally}(\mathcal{C})$ and whose hom-class are given by the hom-classes of $\mathbf{Ally}(\mathcal{C})$.

Example 4.6. Let $\mathcal{C}$ be a category. For every object γ in $\mathbf{Com}(\mathcal{C})$ and $\overline{\nu}$ in $\mathbf{Ally}(\mathcal{C})$, denote by $\mathbf{Enov}(\gamma, \overline{\nu})$ the set of extended nodes of vertebrae in $\mathcal{C}$ whose preseeds are equal to γ and whose underlying nodes of vertebrae are equal to $\overline{\nu}$. This induces a span $\mathbf{Enov}(\mathcal{C})$ over the 2-classes $\mathrm{Obj}(\mathbf{Com}(\mathcal{C}))$ and $\mathrm{Obj}(\mathbf{Ally}(\mathcal{C}))$.

Example 4.7. Let $\mathcal{C}$ be a category and n be a positive integer. For every spine s in $\mathbf{Aos}(\mathcal{C}, n-1)$ and nodes of spines $\overline{\sigma}$ in $\mathbf{Anos}(\mathcal{C}, n)$ (see notations of section 4.2.1.4), denote by $\mathbf{Enos}_n(s, \overline{\sigma})$ the set of extended nodes of spines of degree n in $\mathcal{C}$ of the form $s \overset{\mathrm{EX}}{\rightsquigarrow} \overline{\sigma}$. This induces a span $\mathbf{Enos}_n(\mathcal{C})$ over the 2-classes $\mathrm{Obj}(\mathbf{Aos}(\mathcal{C}, n-1))$ and $\mathrm{Obj}(\mathbf{Anos}(\mathcal{C}, n))$.

4.3.1.2. *Morphisms of spans.* The notion of morphism given in this section is more general than the one usually appearing in the literature. Let F and E be two spans. A *morphism of spans* from F to E consists of

1) a metafunction $f_{\mathrm{L}} : \mathrm{Obj}_{\mathrm{L}}(F) \to \mathrm{Obj}_{\mathrm{L}}(E)$;
2) a metafunction $f_{\mathrm{R}} : \mathrm{Obj}_{\mathrm{R}}(F) \to \mathrm{Obj}_{\mathrm{R}}(E)$;
3) for every $\gamma \in \mathrm{Obj}_{\mathrm{L}}(F)$ and $a \in \mathrm{Obj}_{\mathrm{R}}(F)$, a metafunction as follows.

$$f_{\mathrm{H}} : F(\gamma, a) \to E(f_{\mathrm{L}}(\gamma), f_{\mathrm{R}}(a))$$

Such a morphism will later be written by $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : F \Rightarrow E$. The composition of two such morphisms is componentwise.

Example 4.8. Let $\mathcal{C}$ be a category and n be a positive integer. There is an obvious morphism of spans $\lambda^n := (\lambda_0^n, \lambda_1^n, \lambda_2^n) : \mathbf{Enos}_n(\mathcal{C}) \Rightarrow \mathbf{Enov}(\mathcal{C})$ where

1) the component λ_0^n maps any spine $P \cdot \gamma$ of degree $n-1$ in $\mathcal{C}$ to its stem γ;
2) the component λ_1^n maps any node of spines $P \cdot \Omega$ of degree n in $\mathcal{C}$ to the node of vertebrae $p_n \cdot \Omega$ associated with its head p_n;
3) the component λ_2^n maps any extended nodes of spines $P \cdot \gamma \overset{\mathrm{EX}}{\rightsquigarrow} \overline{P} \cdot \Omega$ to its underlying extended nodes of vertebrae $\gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{p}_n \cdot \Omega$.

Example 4.9. Let $\mathcal{C}$ be a category. The operation $\mathbf{ext}(\_)$ mapping an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ to its underlying extended node of vertebrae $\mathbf{ext}(\mathfrak{a}) : \gamma \rightsquigarrow \nu_*$ induces a morphism of spans from $\mathbf{Ally}(\mathcal{C})$ to $\mathbf{Enov}(\mathcal{C})$ whose component

1) f_{L} is given by the metafunction mapping a node of vertebrae to its seed;
2) f_{R} is given by the identity metafunction;
3) $f_{\mathrm{H}} : \mathbf{Ally}(\nu, \nu_*) \to E(\gamma, \nu_*)$ is given by the operation $\mathbf{ext}(\_)$.

4.3.1.3. *Fibration of spans.* This section defines the notion of fibration of spans, which is a generalisation of the notion of fibration for categories. Let F and E be two spans. A morphism of spans $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : F \Rightarrow E$ will be called a *fibration* if for every $\gamma \in \mathrm{Obj}_{\mathrm{L}}(E)$, $a \in \mathrm{Obj}_{\mathrm{R}}(E)$ and $e \in E(f_{\mathrm{L}}(\gamma), a)$, there exists $a' \in \mathrm{Obj}_{\mathrm{R}}(F)$ and $e' \in F(\gamma, a')$ such that the equalities $f_{\mathrm{R}}(a') = a$ and $f_{\mathrm{H}}(e') = e$ hold.

4.3.1.4. *Subspans.* Let E be a span. A *subspan of* E is a span F equipped with a morphism of spans $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : F \Rightarrow E$ whose components f_{L}, f_{R} and f_{H} are all inclusions of higher classes. The fact of being a subspan will later be denoted with the symbol inclusion; e.g. in the previous case $F \subseteq E$.

4.3.2. Precompasses and graphs.

4.3.2.1. *Precompasses.* A *precompass* consists of a span E and a metafunction $h : \mathrm{Obj}_{\mathrm{R}}(E) \to \mathrm{Obj}_{\mathrm{L}}(E)$, called the *hinge*. Such a structure will later be denoted by (E, h).

Example 4.10. Let $\mathcal{C}$ be a category. The span $\mathbf{Enov}(\mathcal{C})$ of Example 4.6 defines a precompass when equipped with the metafunction $\eta : \mathrm{Obj}(\mathbf{Ally}(\mathcal{C}, n)) \to \mathrm{Obj}(\mathbf{Com}(\mathcal{C}, n-1))$ mapping a node of vertebrae to its seed.

Example 4.11. Let $\mathcal{C}$ be a category, n be a positive integer. The span $\mathbf{Enos}_n(\mathcal{C})$ defined in Example 4.7 may be seen as a precompass when equipped with the metafunction η : $\mathrm{Obj}(\mathbf{Anos}(\mathcal{C}, n)) \to \mathrm{Obj}(\mathbf{Aos}(\mathcal{C}, n-1))$ mapping a node of spines to its spinal seed.

4.3.2.2. *Morphisms of precompasses.* Let (F, h_F) and (E, h_E) be two precompasses. A *morphism of precompasses* from (F, h_F) to (E, h_E) is a morphism of spans $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : F \Rightarrow E$ for which the following diagram commutes.

$$\begin{tikzcd} \mathrm{Obj}_{\mathrm{R}}(F) \arrow[r, "h_F"] \arrow[d, "f_{\mathrm{R}}"'] & \mathrm{Obj}_{\mathrm{L}}(F) \arrow[d, "f_{\mathrm{L}}"] \\ \mathrm{Obj}_{\mathrm{R}}(E) \arrow[r, "h_E"'] & \mathrm{Obj}_{\mathrm{L}}(E) \end{tikzcd}$$

Example 4.12. Let $\mathcal{C}$ be a category and n be a positive integer. The morphism of spans $\lambda^n : \mathbf{Enos}_n(\mathcal{C}) \Rightarrow \mathbf{Enov}(\mathcal{C})$ of Example 4.8 extends to a morphism of precompasses for the structures given by Example 4.10 and Example 4.11.

4.3.2.3. *Subprecompasses.* Let (E, h_E) be a precompass. A precompass (F, h_F) will be said to *define a subprecompass of* (E, h_E) if it is equipped with a morphism of precompasses inducing an inclusion of spans $F \subseteq E$.

4.3.2.4. *Graphs.* A *graph* is a precompass whose hinge is an identity. In other words, the equality $\mathrm{Obj}_{\mathrm{L}}(A) = \mathrm{Obj}_{\mathrm{L}}(A)$ holds. In this case, both object-classes will be denoted by $\mathrm{Obj}(A)$. In the sequel, a graph of the form (A, id) will be denoted by the letter A.

Example 4.13. Let $\mathcal{C}$ be a category and n be a positive integer. The categories $\mathbf{Anos}(\mathcal{C}, n)$ and $\mathbf{Ally}(\mathcal{C})$ define obvious graphs.

Example 4.14. Let us fix some category $\mathcal{C}$. An example of interest will be the graph $\mathbf{Sev}(\mathcal{C})$ whose object-class is the class of arrows of $\mathcal{C}$ and whose hom-class $\mathbf{Sev}(\gamma, \gamma')$ contains all semi-extended vertebrae whose preseeds and coseeds are equal to γ and γ'. If the class $\mathbf{Sev}(\gamma, \gamma')$ is non-empty, then the domains of γ and γ' are equal.

4.3.2.5. *Morphisms of graphs.* Let A and B be two graphs. A *morphism of graphs* from B to A is a morphism of precompasses of the form $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : (B, \mathrm{id}) \Rightarrow (A, \mathrm{id})$. The fact that the hinges are identities forces the equality $f_{\mathrm{L}} = f_{\mathrm{R}}$ to hold for every morphism of graphs. Later on, such a morphism will be denoted as a pair $(f_{\mathrm{R}}, f_{\mathrm{H}})$.

Example 4.15. Let $\mathcal{C}$ be a category and n be a positive integer. There is an obvious morphism of graphs $\kappa^n := (\kappa_1^n, \kappa_2^n) : \mathbf{Anos}(\mathcal{C}, n) \Rightarrow \mathbf{Ally}(\mathcal{C})$ mapping any alliance of nodes of spines $P \cdot \Omega$ to the node of vertebrae $p_n \cdot \Omega$ associated with its head p_n.

4.3.2.6. *Subgraphs.* Let A be a graph. A graph B will be said to *define a subgraph of A* if it is equipped with an inclusion $A \subseteq B$ of subprecompasses.

4.3.3. Compasses, magmoids and algebras.

4.3.3.1. *Compasses.* The idea behind the notion of compass is to add a composition operation to the structure of a precompass. A *compass* is a precompass (E, h) equipped, for every $\gamma \in \mathrm{Obj}_{\mathrm{L}}(E)$ and $a, b \in \mathrm{Obj}_{\mathrm{R}}(E)$, with a metafunction of the following form, which will be called a *composition.*

$$\odot : E(\gamma, a) \times E(h(a), b) \to E(\gamma, b),$$

Such a structure will usually be referred to as a triple $(E, h, \odot)$.

Example 4.16. Let $\mathcal{C}$ be a category. The precompass $(\mathbf{Enov}(\mathcal{C}), \eta)$ of Example 4.6 may be endowed with a structure of a compass $(\mathbf{Enov}(\mathcal{C}), \eta, \odot)$ by considering, for every $\gamma \in \mathrm{Obj}(\mathbf{Com}(\mathcal{C}))$ and $\nu_*, \nu_\flat \in \mathrm{Obj}(\mathbf{Ally}(\mathcal{C}))$, the operation

$$\odot : \mathbf{Enov}(\gamma, \nu_*) \times \mathbf{Enov}(\eta(\nu_*), \nu_\flat) \to \mathbf{Enov}(\gamma, \nu_\flat)$$

that maps any extended nodes of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and $\mathfrak{n}_* : \eta(\nu_*) \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ to the extended node of vertebrae $\mathfrak{n}_* \odot \mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$.

4.3.3.2. *Morphisms of compasses.* The present notion of morphism extends the notion of morphism of spans to that of morphism of compasses. Let $(F, h_F, \odot_F)$ and $(E, h_E, \odot_E)$ be two compasses. A *morphism of compasses* from $(F, h_F, \odot_F)$ to $(E, h_E, \odot_E)$ is a morphism of spans $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : F \Rightarrow E$ for which the following diagrams commute, where the latter holds for every $\gamma \in \mathrm{Obj}_{\mathrm{L}}(E)$ and $a, b \in \mathrm{Obj}_{\mathrm{R}}(E)$.

$$\begin{tikzcd}
\mathrm{Obj}_{\mathrm{R}}(F) \arrow[r, "h_F"] \arrow[d, "f_{\mathrm{R}}"'] & \mathrm{Obj}_{\mathrm{L}}(F) \arrow[d, "f_{\mathrm{L}}"] \\
\mathrm{Obj}_{\mathrm{R}}(E) \arrow[r, "h_E"'] & \mathrm{Obj}_{\mathrm{L}}(E)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\gamma, a) \times F(h_F(a), b) \arrow[r, "\odot_F"] \arrow[d, "f_{\mathrm{H}} \times f_{\mathrm{H}}"'] & F(\gamma, b) \arrow[d, "f_{\mathrm{H}}"] \\
E(\gamma, a) \times E(h_E(a), b) \arrow[r, "\odot_E"'] & E(\gamma, b)
\end{tikzcd}$$

4.3.3.3. *Subcompasses.* Let $(E, h_E, \odot_E)$ be a compass. A compass $(F, h_F, \odot_F)$ will be said to be a *subcompass of* $(E, h_E, \odot_E)$ if it is equipped with a morphism of compasses $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : (F, h_F, \odot_F) \Rightarrow (E, h_E, \odot_E)$ inducing an inclusion of spans.

4.3.3.4. *Magmoids.* A *magmoid* is a compass $(A, h, \circledcirc)$ whose hinge h is an identity. Later on, the left and right object-classes of A, which are equal, will be denoted by $\mathrm{Obj}(A)$. Such a structure will later be denoted as a pair $(A, \circledcirc)$.

Example 4.17. Any category is a magmoid for its underlying compositions. In the sequel, the metacategory $\mathbf{Ally}(\mathcal{C})$ will be an example of interest.

4.3.3.5. *Morphisms of magmoids.* Let $(A, \circledcirc_A)$ and $(B, \circledcirc_B)$ be two magmoids. A *morphism of magmoids* from $(B, \circledcirc_B)$ to $(A, \circledcirc_A)$ is a morphism of compasses of the from $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : (B, \mathrm{id}, \circledcirc_B) \Rightarrow (A, \mathrm{id}, \circledcirc_A)$. The fact that the hinges are identities forces the equality $f_{\mathrm{L}} = f_{\mathrm{R}}$ to hold for every morphism of magmoids. Later on, such a structure will be denoted as a pair $(f_{\mathrm{R}}, f_{\mathrm{H}})$.

Example 4.18. Any functor of categories induces a morphism of magmoids.

4.3.3.6. *Submagmoids.* Let $(A, \circledcirc)$ and $(B, \circledcirc')$ be two magmoids. The latter will be said to be a *submagmoid* of the former if it is equipped with a morphism of magmoids from $(A, \circledcirc)$ to $(B, \circledcirc')$ whose components are inclusions of higher classes.

Example 4.19. Let $\mathcal{C}$ be a category. Any subcategory of the category $\mathbf{Ally}(\mathcal{C})$ is a submagmoid of $\mathbf{Ally}(\mathcal{C})$. However, a submagmoid of $\mathbf{Ally}(\mathcal{C})$ is not necessarily a subcategory of $\mathbf{Ally}(\mathcal{C})$ as some identities might be missing.

4.3.3.7. *Algebras.* An *algebra* is a precompass (E, h) equipped with two metafunctions $h_0, h_1 : \mathrm{Obj}_{\mathrm{R}}(E) \to \mathrm{Obj}_{\mathrm{L}}(E)$, called the *source hinge* and the *target hinge*, respectively, such that the higher classes

$$\Sigma_0 E(\gamma, \gamma') := \sum_{h_0(a)=\gamma'} E(\gamma, a) \qquad \Sigma_1 E(\gamma, \gamma') := \sum_{h_1(a)=\gamma'} E(\gamma, a)$$

$$\Sigma_\star E(\gamma, \gamma') := \sum_{h(a)=\gamma'} E(\gamma, a)$$

defined for every pair $\gamma, \gamma' \in \mathrm{Obj}_{\mathrm{L}}(E)$, are endowed with a metafunction of the following form for every $\gamma, \gamma_*, \gamma'_\flat \in \mathrm{Obj}_{\mathrm{L}}(E)$, which will be called an *algebraic composition.*

$$\Sigma_0 E(\gamma, \gamma_*) \times \Sigma_1 E(\gamma_*, \gamma'_\flat) \to \Sigma_\star E(\gamma, \gamma'_\flat)$$

Definition 4.20 (Vertebral algebras)**.** Let $\mathcal{C}$ be a category and denote by $\eta' : \mathbf{Ally}(\mathcal{C}) \to \mathbf{Com}(\mathcal{C})$ the metafunction mapping a node of vertebrae to its coseed. A *vertebral algebra* is a subprecompass (E, h) of the precompass $(\mathbf{Enov}(\mathcal{C}), \eta')$ equipped with a structure of algebra along two hinges $h_0, h_1 : \mathrm{Obj}_{\mathrm{R}}(E) \to \mathrm{Obj}_{\mathrm{L}}(E)$ such that an algebraic composition

$$\Sigma_0 E(\gamma, \gamma_*) \times \Sigma_1 E(\gamma_*, \gamma'_\flat) \to \Sigma_\star E(\gamma, \gamma'_\flat)$$

maps any pair of extended nodes of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ and $\mathfrak{n}_* : \gamma'_* \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}_*$ (for which the relations $h_0(\overline{\nu}) = \gamma_*$ and $h_1(\overline{\nu}_*) = \gamma'_\flat$ must hold) to an extended node of vertebrae $\mathfrak{n}_\bullet : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ (where $\gamma'_\flat$ must be the coseed of $\nu_\flat$) of the form $\mathfrak{n}_\bullet = \mathfrak{b} \circledcirc \langle \mathfrak{n}_* \odot t, \mathfrak{n} \rangle$ where

- $\mathfrak{b}$ is some alliance $\nu_\dagger \rightsquigarrow \nu_\flat$ in $\mathbf{Ally}(\mathcal{C})$;
- $t : \overline{\gamma}' \rightsquigarrow \gamma_*$ is a communication in $\mathbf{Com}(\mathcal{C})$;
- $\langle \mathfrak{n}_* \odot t, \mathfrak{n} \rangle : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\dagger$ denotes an extended node of vertebrae in $\mathcal{C}$ that frames the communicating pair $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ and $\mathfrak{n}_* \odot t : \overline{\gamma}' \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}_*$.

By definition of the higher class $\Sigma_\star E(\gamma, \gamma'_\flat)$, the extended node of vertebrae $\mathfrak{n}_\bullet$ must belong to E.

4.3.4. Compasses and precompasses over graphs.

4.3.4.1. *Precompasses over graphs.* Let A be a graph. An *A-precompass* is a precompass (E, h) equipped with an inclusion of the form $\mathrm{Obj}(A) \subseteq \mathrm{Obj}_{\mathrm{R}}(E)$. Any A-precompass that is a compass will be called an *A-compass.*

Example 4.21. Let $\mathcal{C}$ be a category. The compass $(\mathbf{Enov}(\mathcal{C}), \eta)$ defines an $\mathbf{Ally}(\mathcal{C})$-compass while the precompass $(\mathbf{Enos}_n(\mathcal{C}), \eta)$ defines an $\mathbf{Anos}(\mathcal{C}, n)$-precompass for every positive integer n.

4.3.4.2. *Morphisms of precompasses over graphs.* Let A and B be two graphs, (E, h_E) be an A-precompass and (F, h_F) be a B-precompass. A *morphisms of precompass over graphs* from E to F consists of

1) a morphism of graphs $(g_{\mathrm{R}}, g_{\mathrm{H}}) : B \Rightarrow A$;

2) a morphism of precompasses $(f_{\mathrm{L}}, f_{\mathrm{R}}, f_{\mathrm{H}}) : (F, h_F) \Rightarrow (E, h_E)$ for which the following diagram commutes.

$$\begin{tikzcd} \mathrm{Obj}(B) \arrow[r, "\subseteq"] \arrow[d, "g_{\mathrm{R}}"'] & \mathrm{Obj}_R(F) \arrow[d, "f_{\mathrm{R}}"] \\ \mathrm{Obj}(A) \arrow[r, "\subseteq"'] & \mathrm{Obj}_R(E) \end{tikzcd}$$

Similarly, a *morphism from a B-compass to an A-compass* is a morphism of compasses coming along with a morphism of graphs $B \Rightarrow A$. The above structure will later be denoted as an arrow $(g, f) : (B, F, h_F) \Rightarrow (A, E, h_E)$.

Example 4.22. Let $\mathcal{C}$ be a category and n be a positive integer. The morphism of compasses $\lambda^n : \mathbf{Enos}_n(\mathcal{C}) \Rightarrow \mathbf{Enov}(\mathcal{C})$ of Example 4.12 extends to a morphism of compasses over graphs along the morphism of graphs $\kappa^n : \mathbf{Anos}(\mathcal{C}, n) \Rightarrow \mathbf{Ally}(\mathcal{C})$ defined in Example 4.15.

4.3.4.3. *Subprecompass over graphs.* Let A and B be two graphs and (E, h_E) be a A-precompass. A B-precompass (F, h_F) will be said to define a *B-subprecompass of* (E, h_E) if it is equipped with a morphism of precompasses over graphs consisting of inclusions $(F, h_F) \subseteq (E, h_E)$ and $B \subseteq A$. Similarly, a B-compass $(F, h_F, \odot_F)$ will be said to define a *B-subcompass* of an A-compass $(E, h_E, \odot_E)$ if it is equipped with a morphism of precompasses over graphs consisting of inclusions $(B, F, h_F, \odot_F) \subseteq (A, E, h_E, \odot_E)$ and $B \subseteq A$.

4.3.5. Modules over graphs.

4.3.5.1. *Right modules.* Let A be a graph. A *right A-module* is an A-compass $(E, h, \odot)$ equipped with a metafunction of the following form for every $\gamma \in \mathrm{Obj}_{\mathrm{L}}(E)$ and $a, b \in \mathrm{Obj}(A)$.

$$\circledast : E(\gamma, a) \times A(a, b) \to E(\gamma, b)$$

Such a structure will later be denoted as a triple $(E, h, \odot, \circledast)$.

Example 4.23. Let $\mathcal{C}$ be a category. The $\mathbf{Ally}(\mathcal{C})$-compass $(\mathbf{Enov}(\mathcal{C}), \eta, \odot)$ defines a right $\mathbf{Ally}(\mathcal{C})$-module $(\mathbf{Enov}(\mathcal{C}), \eta, \circledcirc)$ where an algebraic composition

$$\circledcirc : \mathbf{Enov}(\gamma, \nu_*) \times \mathbf{Ally}(\nu_*, \nu_\flat) \to \mathbf{Enov}(\gamma, \nu_\flat)$$

maps any extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and alliance of nodes of vertebrae $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ to the extended node of vertebrae $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$.

4.3.5.2. *Morphisms of right modules.* Let A and B be two graphs, $(E, h_E, \odot_E, \circledast_E)$ be a right A-module and $(F, h_F, \odot_F, \circledast_F)$ be a right B-module. A *morphism of right modules* from E to F consists of a morphism of compasses over graphs

$$(g, f) : (B, F, h_F, \odot_F) \Rightarrow (A, E, h_E, \odot_E)$$

such that the following diagram commutes for every $\gamma \in \mathrm{Obj}_L(F)$ and $a, b \in \mathrm{Obj}(B)$.

$$\begin{tikzcd} F(\gamma, a) \times B(a, b) \arrow[r, "f_{\mathrm{H}} \times g_{\mathrm{H}}"] \arrow[d, "\circledast_F"'] & E(f_{\mathrm{L}}(\gamma), g_{\mathrm{R}}(a)) \times A(g_{\mathrm{R}}(a), g_{\mathrm{R}}(b)) \arrow[d, "\circledast_E"] \\ F(\gamma, b) \arrow[r, "f_{\mathrm{H}}"'] & E(f_{\mathrm{L}}(\gamma), f_{\mathrm{R}}(b)) \end{tikzcd}$$

4.3.5.3. *Right submodules.* Let A and B be two graphs, $(E, h_E, \odot_E, \circledast_E)$ be a right A-module. A right B-module $(F, h_F, \odot_F, \circledast_F)$ will be said to *define a submodule of* $(E, h_E, \odot_E, \circledast_E)$ if it is equipped with a morphism of right modules

$$(g, f) : (B, F, h_F, \odot_F, \circledast_F) \Rightarrow (A, E, h_E, \odot_E, \circledast_E)$$

whose components are all inclusions of higher classes.

4.3.6. Substructures and prolinearity.

4.3.6.1. *Systems.* The following notion constitutes the common base to all our future structures (see Definition 4.24). Let M be a graph and (N, h) be a M-precompass. An *N-system* consists of

1) two subgraphs $A, A' \subseteq M$;
2) both an A- and A'-subprecompass $(E, h) \subseteq (N, h)$.

Such a structure will usually be denoted as a triple (A, A', E).

Definition 4.24 (System of vertebrae)**.** Let $\mathcal{C}$ be a category. A *system of vertebrae* is an $\mathbf{Enov}(\mathcal{C})$-system for the precompass structure defined in Example 4.21. A system of vertebrae only makes sense when associated with a notion of zoo. Let $\mathcal{S} := (A, A', E)$ be a system of vertebrae in $\mathcal{C}$. If $f : X \to Y$ denotes a morphism in $\mathcal{C}$, then it is called

- *i) fibration; ii) trivial fibration* in $\mathcal{S}$ if for every node of vertebrae $\nu \in \mathrm{Obj}(A')$, there exists an alliance $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ in A' for which $f : X \to Y$ is a i) fibration; ii) trivial fibration, respectively;
- *iii) intraction* in $\mathcal{S}$ if for every node of vertebrae $\nu \in \mathrm{Obj}(A)$, there exists an alliance $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ in A for which $f : X \to Y$ is an iii) intraction;
- *iv) surtraction; v) pseudofibration* in $\mathcal{S}$ if for every arrow γ in $\mathrm{Obj}_{\mathrm{L}}(E)$, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in E for which $f : X \to Y$ is a iv) surtraction; v) pseudofibration.

Later on, it will come in handy to denote a system of vertebrae as if it were equipped with its ambient category, namely as a quadruple $(\mathcal{C}, A, A', E)$. The arrows of $\mathcal{C}$ that belong to the left object-class of the span E will be called the *E-seeds*.

4.3.6.2. *Prolinear map.* Let $(M, \circledcirc)$ be a magmoid and A and K be a pair of subgraphs of M for which the equality $\mathrm{Obj}(K) = \mathrm{Obj}(A)$ holds. Note that for every triple $a, b, c \in \mathrm{Obj}(K)$, there exists a restriction of the compositions of $(M, \circledcirc)$ as follows.

$$\circledcirc : K(a,b) \times M(b,c) \hookrightarrow M(a,b) \times M(b,c) \longrightarrow M(a,c)$$

Denote by $M(b, A)$, $M(a, A)$ and $A(a, A)$ the higher classes defined by the following coproduct of higher classes, respectively.

$$\sum_{c \in \mathrm{Obj}(A)} M(b,c) \qquad \sum_{c \in \mathrm{Obj}(A)} M(a,c) \qquad \sum_{c \in \mathrm{Obj}(A)} A(a,c)$$

In particular, notice that the previous metafunction induces a map as follows.

$$\circledcirc : K(a,b) \times M(b,A) \longrightarrow M(a,A)$$

A *M-prolinear map* from K to A, denoted as an arrow $K \looparrowright A$, is given, for every $a, b \in \mathrm{Obj}(A)$, by two metafunctions $f_{a,b}$ and $L_{a,b}$ making the following diagram commute (an explicit description is given beneath the diagram).

$$\begin{tikzcd}
K(a,b) \times \mathbf{1} & K(a,b) \arrow[l, "\mathrm{id}"'] \arrow[r, dashed, "f_{a,b}"] \arrow[d, dashed, "L_{a,b}"'] & A(a,A) \arrow[d, "\subseteq"] \\
& K(a,b) \times M(b,A) \arrow[ul, "\mathrm{id} \times !"] \arrow[r, "\circledcirc"'] & M(a,A)
\end{tikzcd}$$

In other words, for every element $x \in K(a,b)$, there exists some element $c(x) \in \mathrm{Obj}(A)$ and $y(x) \in M(b, c(x))$ such that the following relations hold.

$$f_{a,b}(x) = y(x) \circledcirc x \in A(a, c(x)) \qquad L_{a,b}(x) = (x, y(x))$$

The form of the left-hand equation reminds that of a linear map when y does not depend on x, which explains the name 'pro-linear'. Such a structure will later be denoted by $(L, f) : K \looparrowright A$.

Example 4.25. Let $(M, \circledcirc, \mathrm{id})$ be a category and A be a subgraph of $(M, \circledcirc)$. For every pair $a, b \in \mathrm{Obj}(A)$ and $x \in A(a, b)$, denote by $L_{a,b}(x)$ the pair (x, id_b) in $A(a, b) \times M(b, b)$. It follows that the following diagram commutes and induces an obvious M-prolinear map $(L, \subseteq) : A \looparrowright A$.

$$
\begin{tikzcd}
A(a,b) \times \mathbf{1} & A(a,b) \times \mathbf{1} \arrow[l, "\mathrm{id}"'] \arrow[r, dashed, "\subseteq"] \arrow[d, dashed, "\mathrm{id}\times I"] & A(a,A) \arrow[d, "\subseteq"] \\
 & A(a,b) \times M(b,A) \arrow[ul, "\mathrm{id}\times !"] \arrow[r, "\circledcirc"'] & M(a,A)
\end{tikzcd}
$$

Example 4.26. Let $(M, \circledcirc, I)$ be a category and A be a subgraph of $(M, \circledcirc)$ such that for every $a \in \mathrm{Obj}(A)$, there exists $c(a) \in \mathrm{Obj}(A)$ for which $A(a, c(a))$ is non-empty. For every $a \in \mathrm{Obj}(A)$, choose some $y(a) \in A(a, c(a))$. Now, denote by A° the subgraph of $(M, \circledcirc)$

- whose object-class is $\mathrm{Obj}(A)$;
- whose hom-classes $A(a, b)$ are empty if $a \neq b$ and equal to the singleton consisting of the identity id_a otherwise.

For every pair $a, b \in \mathrm{Obj}(A)$ and $x \in A^\circ(a, b)$, the pair $(x, y(a))$ belongs to $A(a, b) \times M(b, b)$. Defining $L_{a,b} = (x, y(a))$ therefore induces a prolinear map $A^\circ \looparrowright A$.

4.3.6.3. *Prolinear modules.* Let $(M, \circledcirc)$ be a magmoid and $(N, h, \odot, \circledast)$ be a right M-module. An *N-prolinear module* consists of

1) two submagmoids $(A, \circledcirc), (A', \circledcirc) \subseteq (M, \circledcirc)$;
2) an M-prolinear map $(L, f) : A' \looparrowright A$;
3) a right A- and A'-submodule $(E, h, \odot, \circledast) \subseteq (N, h, \odot, \circledast)$;

Such a structure will later be denoted as a triple (A, A', E).

Example 4.27. Let $(M, \circledcirc, \mathrm{id})$ be a category and $(N, h, \odot, \circledast)$ be a right M-module. According to Example 4.25, any set of data consiting of

- a submagmoid $(A, \circledcirc) \subseteq (M, \circledcirc)$;
- a right A-submodule $(E, h, \odot, \circledast) \subseteq (N, h, \odot, \circledast)$,

induces an N-prolinear module (A, A, E).

Example 4.28. Later on, we will mainly consider $(\mathbf{Enov}(\mathcal{C}), \eta, \odot, \circledcirc)$-prolinear modules in regard to the module structure defined in Example 4.23. However, contrary to Example 4.27, our prolinear modules will be supposed of the form (A, A', E) where A' is not necessarily equal to A.

4.3.7. Local echelons over graphs.

4.3.7.1. *Cograded graphs.* The following notion is a generalisation of the notion of graph. Let n be a non-negative integer. A *cograded graph of rank n* consists of

1) two collections of $n + 1$ higher classes $O_0, O_1, \ldots, O_n$ and $S_0, S_1, \ldots, S_n$;
2) a pair of metafunctions $s_k, t_k : O_k \to S_k$ for every $0 \leq k \leq n$;
3) a metafunction $h_k : O_{k+1} \to O_k$ for every $0 \leq k \leq n - 1$.

$$\begin{tikzcd}
O_n \arrow[r, "h_{n-1}"] \arrow[d, shift right, "s_n"'] \arrow[d, shift left, "t_n"] & O_{n-1} \arrow[r, "h_{n-2}"] \arrow[d, shift right, "s_{n-1}"'] \arrow[d, shift left, "t_{n-1}"] & O_{n-2} \arrow[r, "h_{n-3}"] \arrow[d, shift right, "s_{n-2}"'] \arrow[d, shift left, "t_{n-2}"] & \dots \arrow[r, "h_0"] & O_0 \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "t_0"] \\
S_n & S_{n-1} & S_{n-2} & \dots & S_0
\end{tikzcd}$$

Later on, such a structure will be denoted as a pair $(s_{\_}, t_{\_})$ potentially equipped with the type $(O_{\_}, h_{\_}) \rightrightarrows S_{\_}$.

Remark 4.29. When $n = 0$, the above structure may be seen as a graph O whose object-class $\mathrm{Obj}(O)$ is given by S_0 and whose hom-classes $O(a, b)$ are given by the higher classes of elements $x \in O_n$ satisfying the equations $s_n(x) = a$ and $t_n(x) = b$.

In the sequel, for any pair $a, b \in S_k$ where $0 \leq k \leq n$, we will denote by $O_k(a, b)$ the class of elements $x \in O_n$ for which the following equations hold.

$$s_k \circ h_k \circ \dots \circ h_{n-1}(x) = a \qquad \text{and} \qquad t_k \circ h_k \circ \dots \circ h_{n-1}(x) = b$$

For every $0 \leq k \leq n-1$, the metafunction h_k will be called the *k-th hinge*.

4.3.7.2. *Morphisms of cograded graphs.* Let n be a non-negative integer, $(s_{\_}, t_{\_}) : (O_{\_}, h_{\_}) \rightrightarrows S_{\_}$ and $(s'_{\_}, t'_{\_}) : (O'_{\_}, h'_{\_}) \rightrightarrows S'_{\_}$ be two cograded graphs of rank n. A *morphism of cograded graphs* from $(s_{\_}, t_{\_})$ to $(s'_{\_}, t'_{\_})$ consists of a metafunction $u_k : O_k \to O'_k$ and a metafunction $v_k : S_k \to S'_k$ for every $0 \leq k \leq n$ making the following diagrams commute for every $0 \leq k \leq n$.

$$\begin{tikzcd}
O_k \arrow[r, "u_k"] \arrow[d, "s_k"'] & O'_k \arrow[d, "s'_k"] \\
S_k \arrow[r, "v_k"'] & S'_k
\end{tikzcd}
\qquad
\begin{tikzcd}
O_k \arrow[r, "u_k"] \arrow[d, "t_k"'] & O'_k \arrow[d, "t'_k"] \\
S_k \arrow[r, "v_k"'] & S'_k
\end{tikzcd}$$

Such a morphism will later be denoted as a pair $(u_{\_}, v_{\_}) : (s_{\_}, t_{\_}) \Rightarrow (s'_{\_}, t'_{\_})$.

4.3.7.3. *Cograded subgraphs.* Let n be a non-negative integer and $(s'_{\_}, t'_{\_})$ be a cograded graph of rank n. A cograded graph $(s_{\_}, t_{\_})$ of rank n will be said to *define a subgraph* of $(s'_{\_}, t'_{\_})$ if it is equipped with a morphism of cograded graphs $(u_{\_}, v_{\_}) : (s_{\_}, t_{\_}) \Rightarrow (s'_{\_}, t'_{\_})$ whose components u^k and v^k are inclusions of higher classes for every $0 \leq k \leq n$.

4.3.7.4. *Echelons over graphs.* Let A be a graph and n be a non-negative integer. The next definition needs to distinguish between the cases $n = 0$ and $n > 0$.

If $n > 0$, an A-precompass (E, h) will be said to be *echeloned* and *of rank n* if it is equipped with a cograded graph $(s_{\_}, t_{\_}) : (O_{\_}, h_{\_}) \rightrightarrows S_{\_}$ of rank n whose $(n-1)$-th hinge h_{n-1} is equal to the restriction of the hinge h of E along the underlying inclusion $\mathrm{Obj}(A) \subseteq \mathrm{Obj}_{\mathrm{R}}(E)$.

$$\begin{tikzcd}
\mathrm{Obj}(A) \arrow[r, "h"] \arrow[d, shift right] \arrow[d, shift left] & \mathrm{Obj}_{\mathrm{L}}(E) \arrow[r, "h_{n-2}"] \arrow[d, shift right] \arrow[d, shift left] & O_{n-2} \arrow[r, "h_{n-3}"] \arrow[d, shift right] \arrow[d, shift left] & \dots \arrow[r, "h_0"] & O_0 \arrow[d, shift right] \arrow[d, shift left] \\
S_n & S_{n-1} & S_{n-2} & \dots & S_0
\end{tikzcd}$$

If $n = 0$, an A-precompass (E, h) will be said to be *echeloned* and *of rank* 0 if it is equipped with a cograded graph $(s_0, t_0) : O_0 \rightrightarrows S_0$ of rank 0 such that s_0 is equal to the restriction of the hinge h of E along the underlying inclusion $\mathrm{Obj}(A) \subseteq \mathrm{Obj}_{\mathrm{R}}(E)$.

$$\mathrm{Obj}(A) \overset{h}{\underset{t_0}{\rightrightarrows}} \mathrm{Obj}_{\mathrm{L}}(E)$$

Such a structure will later be denoted by the data $(E, h, s_{\_}, t_{\_})$. For short, an echeloned A-precompass will be called an *A-echelon*.

Example 4.30. Let $\mathcal{C}$ be a category. The $\mathbf{Ally}(\mathcal{C})$-precompass $(\mathbf{Enov}(\mathcal{C}), \eta)$ defines an $\mathbf{Ally}(\mathcal{C})$-echelon of rank 0 whose cograded graphs is defined by the following diagram where η and η' send a node of vertebrae to its seed and coseed, respectively.

$$\begin{tikzcd} \mathrm{Obj}(\mathbf{Ally}(\mathcal{C})) \arrow[d, shift right, "\eta"'] \arrow[d, shift left, "\eta'"] \\ \mathrm{Obj}(\mathbf{Com}(\mathcal{C})) \end{tikzcd}$$

Example 4.31. Let $\mathcal{C}$ be a category and n be a positive integer. The $\mathbf{Anos}(\mathcal{C}, n)$-precompass $(\mathbf{Enos}_n(\mathcal{C}), \eta)$ defines an $\mathbf{Anos}(\mathcal{C}, n)$-echelon of rank n whose cograded graph is defined by the following diagram where all metafunctions η map a node of spines (or a spine) to its spinal seed and where η_k and υ_k map a node of spines (or a spine)with prespine $P = (p_k)$ to the seed and coseed of the prevertebra p_k, for every $0 \leq k \leq n$.

$$\begin{tikzcd} \mathrm{Obj}(\mathbf{Anos}(\mathcal{C}, n)) \arrow[r, "\eta"] \arrow[d, shift right, "\eta_n"'] \arrow[d, shift left, "\eta'_n"] & \mathrm{Obj}(\mathbf{Aos}(\mathcal{C}, n-1)) \arrow[r, "\eta"] \arrow[d, shift right, "\eta_{n-1}"'] \arrow[d, shift left, "\eta'_{n-1}"] & \mathrm{Obj}(\mathbf{Aos}(\mathcal{C}, n-2)) \arrow[r, "\eta"] \arrow[d, shift right, "\eta_{n-2}"'] \arrow[d, shift left, "\eta'_{n-2}"] & \dots \\ \mathrm{Obj}(\mathbf{Com}(\mathcal{C})) & \mathrm{Obj}(\mathbf{Com}(\mathcal{C})) & \mathrm{Obj}(\mathbf{Com}(\mathcal{C})) & \dots \end{tikzcd}$$

$$\begin{tikzcd} \dots \arrow[r, "\eta"] & \mathrm{Obj}(\mathbf{Aos}(\mathcal{C}, 1)) \arrow[r, "\eta"] \arrow[d, shift right, "\eta_1"'] \arrow[d, shift left, "\eta'_1"] & \mathrm{Obj}(\mathbf{Aos}(\mathcal{C}, 0)) \arrow[d, shift right, "\eta_0"'] \arrow[d, shift left, "\eta'_0"] \\ \dots & \mathrm{Obj}(\mathbf{Com}(\mathcal{C})) & \mathrm{Obj}(\mathbf{Com}(\mathcal{C})) \end{tikzcd}$$

4.3.7.5. *Morphisms of echelons.* Let A and B be two graphs, n be a non-negative integer, (E, h_E) and (F, h_F) be two A- and B-echelons of rank n whose cograded graphs will be denoted by the pairs $(s^E_{\_}, t^E_{\_}) : (O^E_{\_}, h^E_{\_}) \rightrightarrows S^E_{\_}$ and $(s^F_{\_}, t^F_{\_}) : (O^F_{\_}, h^F_{\_}) \rightrightarrows S^F_{\_}$, respectively. A *morphism of echelons* from (F, h_F) to (E, h_E) consists of

1) a morphism of precompasses $(g, f) : (B, F, h_F) \Rightarrow (A, E, h_E)$;
2) a morphism of cograded graphs $(u_{\_}, v_{\_}) : (s^F_{\_}, t^F_{\_}) \Rightarrow (s^E_{\_}, t^E_{\_})$ such that
   - the equations $u_n = g_{\mathrm{R}}$ and $u_{n-1} = f_{\mathrm{L}}$ hold when $n > 0$;
   - the equations $u_n = g_{\mathrm{R}}$ and $v_n = f_{\mathrm{L}}$ hold when $n = 0$;

Such a morphism will later be denoted as a 4-tuple $(g, f, u_{\_}, v_{\_})$.

4.3.7.6. *Subechelons.* Let A and B be two precompasses, n be a non-negative integer and $(E, h_E, s^E_{\_}, t^E_{\_})$ be an A-echelon of rank n. A B-echelon $(F, h_F, s^F_{\_}, t^F_{\_})$ will be said to be a *subechelon of* (E, h_E) if it is equipped with a morphism of echelon

$$(g, f, u_{\_}, v_{\_}) : (B, F, h_F, s^F_{\_}, t^F_{\_}) \Rightarrow (A, E, h_E, s^E_{\_}, t^E_{\_})$$

whose components encode inclusions of precompasses over graphs and cograded graphs.

4.3.7.7. *Local echelons over graphs.* We now arrive to the structure that will allow us to associate our vertebrae with spines of some given dimension (see section 4.1 for more explanation). The different possible dimensions, given by natural integers, leads to a decomposition of our base structure, here, captured by the notion of morphism of precompasses (see Example 4.32 for more intuition). Let A be a graph. An A-precompass (E, h) will be said to be *locally echeloned* if, for every $n \in \omega$, it is equipped with a graph A_n, an A_n-echelon (E_n, h_{E_n}) of rank n and a morphism of precompasses over graphs

$$\begin{cases} i^n := (i^n_0, i^n_1, i^n_2) : (E_n, h_{E_n}) \Rightarrow (E, h) \\ j^n := (j^n_1, j^n_2) : A_n \Rightarrow A \end{cases}$$

such that the following two collections of metafunctions are jointly surjective.

$$\{i_0^n : \mathrm{Obj}_L(E_n) \to \mathrm{Obj}_L(E)\}_{n\in\omega} \qquad \{j_1^n : \mathrm{Obj}(A_n) \to \mathrm{Obj}(A)\}_{n\in\omega}$$

Such an object will later be denoted by the symbols $[i^-, j^-](E, h)$ and will be said to *be defined under A_n-echelons* (E_n, h_{E_n}). A locally echeloned module as above will be said to be *regular* if the arrows i^n and j^n are fibrations of spans for every $n \in \omega$. For short, a locally echeloned A-precompass will be called a *local A-echelon.*

Example 4.32. For consistency, denote by $(\mathbf{Enos}_0(\mathcal{C}), \eta)$ the $\mathbf{Ally}(\mathcal{C})$-echelon of rank 0 defined by the $\mathbf{Ally}(\mathcal{C})$-precompass $(\mathbf{Enov}(\mathcal{C}), \eta)$ (see Example 4.30). The following collection of morphisms defines a local echelon $[\lambda^-, \kappa^-](\mathbf{Enov}(\mathcal{C}), \eta)$.

$$\left\{\begin{array}{l} \lambda^n : (\mathbf{Enos}_n(\mathcal{C}), \eta) \Rightarrow (\mathbf{Enov}(\mathcal{C}), \eta) \\ \kappa^n : \mathbf{Anos}(\mathcal{C}, n) \Rightarrow \mathbf{Ally}(\mathcal{C}) \end{array}\right. \qquad (\forall n \in \omega)$$

This follows from the fact that the morphisms λ^0 and κ^0 are identities and therefore makes the collections $(\lambda^n)_{n\in\omega}$ and $(\kappa^n)_{n\in\omega}$ jointly surjective. It is however not regular for a general category $\mathcal{C}$.

4.3.7.8. *Morphisms of local echelons.* Let $[i^-, j^-](E, h_E)$ be a local A-echelon defined under A_n-echelons (E_n, h_{E_n}) and $[w^-, x^-](F, h_F)$ be a local B-echelon defined under B_n-echelons (F_n, h_{F_n}). A *morphism of local echelons* from $[w^-, x^-](F, h_F)$ to $[i^-, j^-](E, h_E)$ consists of morphisms

1) of precompasses over gaphs $(g, f) : (B, F, h_F) \Rightarrow (A, E, h_E)$;
2) of echelons $(g^n, f^n) : (B_n, F_n, h_{F_n}) \Rightarrow (A_n, E_n, h_{E_n})$ for every $n \in \omega$;

making the following diagram of morphisms of precompasses over graphs commute.

$$\begin{tikzcd}
(B_n, F_n, h_{F_n}) \arrow[r, Rightarrow, "{(g^n,f^n)}"] \arrow[d, Rightarrow, "{(j^n,i^n)}"'] & (B_n, E_n, h_{E_n}) \arrow[d, Rightarrow, "{(x^n,w^n)}"] \\
(B, F, h_F) \arrow[r, Rightarrow, "{(g,f)}"'] & (A, E, h_E)
\end{tikzcd}$$

4.3.7.9. *Local subechelon.* Let $[i^-, j^-](E, h_E)$ denote a local A-echelon. A local B-echelon $[w^-, x^-](F, h_F)$ will be said to be a *local subechelon* of $[i^-, j^-](E, h_E)$ if it is equipped with a morphism of local echelons $[w^-, x^-](F, h_F) \Rightarrow [i^-, j^-](E, h_E)$ whose two compenents induce inclusions of echelons and precompasses over graphs.

4.3.8. Locally whiskered echelons.

4.3.8.1. *Derived cograded graphs.* Let n be a non-negative integer and consider a cograded graph $(s_-, t_-) : (O_-, h_-) \rightrightarrows S_-$ of rank n. The *derived graph of* (s_-, t_-) is the induced cograded graph of rank $n-1$ made of the following metafunctions.

$$\begin{tikzcd}
O_{n-1} \arrow[r, "h_{n-2}"] \arrow[d, shift right, "s_{n-1}"'] \arrow[d, shift left, "t_{n-1}"] & O_{n-2} \arrow[r, "h_{n-3}"] \arrow[d, shift right, "s_{n-2}"'] \arrow[d, shift left, "t_{n-2}"] & \dots \arrow[r, "h_0"] & O_0 \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "t_0"] \\
S_{n-1} & S_{n-2} & \dots & S_0
\end{tikzcd}$$

The above cograded graph will later be denoted by $\partial(s_-, t_-) : \partial(O_-, h_-) \rightrightarrows \partial S_-$. Note that when $n = 0$, the derived graph of (s_-, t_-) is empty.

4.3.8.2. *Whiskered cograded graphs.* Let $T_{\ltimes}$ and $T_{\rtimes}$ be two graphs and n be a non-negative integer. A $(T_{\ltimes}, T_{\rtimes})$*-whiskered cograded graph of rank* n consists of a cograded graph $(s_{\_}, t_{\_}) : (O_{\_}, h_{\_}) \rightrightarrows S_{\_}$ of rank n equipped with

1) inclusions $S_k \subseteq \mathrm{Obj}(T_{\ltimes})$ and $S_k \subseteq \mathrm{Obj}(T_{\rtimes})$ for every $0 \leq k \leq n$
2) for every $0 \leq k \leq n$ and $\gamma, \gamma_{\diamond}, \gamma', \gamma_{\bullet} \in S_k$, a metafunction of the following form, called the k*-whiskering.*

$$(\_ \ltimes \_ \rtimes \_)_k : T_{\ltimes}(\gamma, \gamma_{\diamond}) \times O_k(\gamma, \gamma') \times T_{\rtimes}(\gamma', \gamma_{\bullet}) \to O_k(\gamma_{\diamond}, \gamma_{\bullet})$$

4.3.8.3. *Whiskered echelons.* Let A, $T_{\ltimes}$ and $T_{\rtimes}$ denote three graphs and n be a non-negative integer. An A-echelon $(E, h, s_{\_}, t_{\_})$ of rank n will be said to be $(T_{\ltimes}, T_{\rtimes})$*-whiskered* if

- the cograded graph $(s_{\_}, t_{\_})$ is $(T_{\ltimes}, T_{\rtimes})$-whiskered.
- the cograded graph $\partial(s_{\_}, t_{\_})$ is $(T_{\ltimes}, T_{\rtimes})$-whiskered.

If n is zero, then the whiskering on the derivative is trivial (i.e. empty). The k-whiskering induced by the cograded graph $(s_{\_}, t_{\_})$ will be denoted by the symbols $(\_ \ltimes \_ \rtimes \_)^A_k$ for every $0 \leq k \leq n$ while the k-whiskering induced by its derivation $\partial(s_{\_}, t_{\_})$ will be denoted by $(\_ \ltimes \_ \rtimes \_)^E_k$ for every $0 \leq k \leq n - 1$.

Example 4.33. For every non-negative interger n, we will later consider some subechelon $[j^{\text{-}}, i^{\text{-}}](E, h)$ of the $\mathbf{Anos}(\mathcal{C}, n)$-echelon $(\mathbf{Enos}_n(\mathcal{C}), \eta)$ whose whiskering structure will be induced by framings of nodes of spines and framings of spines along subgraphs $T_{\ltimes}$ and $T_{\rtimes}$ of $\mathbf{Sev}(\mathcal{C})$.

4.3.8.4. *Locally whiskered echelons.* Let A, $T_{\ltimes}$ and $T_{\rtimes}$ be three graphs. A local A-echelon $[j^{\text{-}}, i^{\text{-}}](E, h)$ will be called a *locally* $(T_{\ltimes}, T_{\rtimes})$*-whiskered* A*-echelon* if the A_n-echelon (E_n, h_{E_n}) is $(T_{\ltimes}, T_{\rtimes})$-whiskered for every $n \in \omega$. A locally whiskered echelon will be said to be *regular* if so is its underlying local echelon.

Example 4.34. Later on, we will be interested in providing some local subechelon of the local $\mathbf{Ally}(\mathcal{C})$-echelon $(\mathbf{Enov}(\mathcal{C}), \eta)$ with a structure of locally whiskered echelons whose whiskerings will be of the form described in Example 4.33.

4.4. Theory of spinal categories

4.4.1. Vertebral categories.

4.4.1.1. *Vertebral categories.* A category $\mathcal{C}$ will be said to be *vertebral* if it is endowed with a $(\mathbf{Enov}(\mathcal{C}), \eta, \odot, \circledcirc)$-prolinear module (A, A', E) such that

1) **(compositions)** the span E defines a subprecompass of $(\mathbf{Enov}(\mathcal{C}), \eta')$ together with a structure of vertebral algebra (see Definition 4.20) along two metafunctions $h_0, h_1 : \mathrm{Obj}_R(E) \to \mathrm{Obj}_{\mathrm{L}}(E)$;
2) **(identities)** for every $\gamma \in \mathrm{Obj}_L(E)$, there exists a reflexive node of vertebrae $\overline{\nu} \in \mathrm{Obj}(A)$ for which the set $E(\gamma, \overline{\nu})$ is non-empty.

The above structure will later be denoted as a 4-tuple $(\mathcal{C}, A, A', E)$. Following the convention of systems of vertebrae, the arrows in $\mathbf{Obj}_{\mathrm{L}}(E)$ will be referred to as E*-seeds.*

4.4.1.2. *Zoo associated with a vertebral category.* The zoo of a vertebral category encompass the zoo of its underlying system of vertebrae (see Definition 4.24). It additionally comprises notions of 'cofibrations'. Let $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$ be a vertebral category. A morphism $f : X \to Y$ will be said to be a

- *i) fibration; ii) trivial fibration* in $\hat{\mathcal{C}}$ if for every node of vertebrae $\nu \in \mathrm{Obj}(A')$, there exists an alliance $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ in A' for which $f : X \to Y$ is a i) fibration; ii) trivial fibration, respectively;

- *iii) intraction* in $\hat{\mathcal{C}}$ if for every node of vertebrae $\nu \in \mathrm{Obj}(A)$, there exists an alliance $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ in A for which $f : X \to Y$ is an iii) intraction;
- *iv) surtraction; v) pseudofibration* in $\hat{\mathcal{C}}$ if for every E-seed γ, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in E for which $f : X \to Y$ is a iv) surtraction; v) pseudofibration.

In addition, the morphism $f : X \to Y$ will be called a

a) *cofibration* if it has the llp with respect to every trivial fibration in $\hat{\mathcal{C}}$;

b) *trivial cofibration* if it has the llp with respect to every fibration in $\hat{\mathcal{C}}$;

c) *weak equivalence* if it is both an intraction and a surtraction in $\hat{\mathcal{C}}$.

Most of the results proven in Chapter 2 extend to the case of vertebral categories. The statements of the next propositions refer to the zoo of $\hat{\mathcal{C}}$.

Proposition 4.35. *Fibrations define a coherent $\mathcal{C}$-class.*

Proof. Let $f : Y \to Z$ and $g : X \to Y$ be two fibrations in $\hat{\mathcal{C}}$. Consider a node of vertebrae $\nu \in \mathrm{Obj}(A')$. By assumption, there exists an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A' for which $f : Y \to Z$ is a fibration. Similarly, there exists an alliance $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ in A' for which $g : X \to Y$ is a fibration. It then follows from Proposition 2.40 that the composite $f \circ g$ is a fibration for the alliance $\mathfrak{a}_* \circledcirc \mathfrak{a} : \nu \rightsquigarrow \nu_\flat$. This last alliance is in A' since $(A', \circledcirc)$ is a submagmoid of $\mathbf{Ally}(\mathcal{C})$. Finally, it follows from Proposition 2.28 that the class of fibrations is a coherent $\mathcal{C}$-classes. $\square$

There is a similar result for pseudofibrations that requires a bit more explanation.

Proposition 4.36. *Pseudofibrations define a coherent $\mathcal{C}$-class.*

Proof. Consider an E-seed γ. By assumption, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ in E for which $f : Y \to Z$ is a pseudofibration in $\hat{\mathcal{C}}$. By Proposition 2.27, this is equivalent to saying that f is a pseudofibration for the underlying communication of $\mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \eta(\nu_*)$. Since $\eta(\nu_*)$ is an E-seed, there exists an extended node of vertebrae $\mathfrak{n}_* : \eta(\nu_*) \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ in E for which $g : X \to Y$ is a pseudofibration. It then follows from Proposition 2.38 that the composite $f \circ g$ is a pseudofibration for the extended node of vertebrae $\mathfrak{n}_* \odot \mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$. This last extended node of vertebrae belongs to E since $(E, \eta, \odot)$ is a subcompass of $(\mathbf{Enov}(\mathcal{C}), \eta, \odot)$. Finally, it follows from Proposition 2.28 that the class of pseudofibrations is a coherent $\mathcal{C}$-classes. $\square$

Proposition 4.37. *The classes of cofibrations and trivial cofibrations are coherent $\mathcal{C}$-classes.*

Proof. Follows from Proposition 1.34 and Remark 1.36. $\square$

Proposition 4.38. *Fibrations and trivial fibrations are preserved under pullbacks. Similarly, cofibrations and trivial cofibrations are preserved under pushouts.*

Proof. The first statement follows from Proposition 2.29. The second one follows from Proposition 1.33. $\square$

Proposition 4.39. *Fibrations, trivial fibrations, cofibrations, trivial cofibrations, surtractions, intractions and weak equivalences are stable under retracts.*

Proof. Follows from Proposition 2.30 in the case of fibrations, trivial fibrations, surtractions, intractions and weak equivalences. See section 1.2.2.2 otherwise. $\square$

Proposition 4.40. *Every trivial fibration is a fibration. Every trivial cofibration is a cofibration.*

Proof. The first fact follows from Proposition 2.32. The second fact easily follows from the first one. □

Proposition 4.41. *Every trivial fibration is an intraction.*

Proof. The proof uses the notion of prolinearity. Denote by $(L, f) : A' \looparrowright A$ the prolinear map associated with the vertebral category $\hat{\mathcal{C}}$. To prove the statement, let $f : X \to Y$ be a trivial fibration in $\hat{\mathcal{C}}$ and ν be a node of vertebrae in $\mathrm{Obj}(A)$. By definition of a prolinear map, the node of vertebrae ν must be in $\mathrm{Obj}(A')$. By assumption, there exists an alliance of node of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A' for which f is a trivial fibration. By Proposition 2.33, the morphism f is also an intraction for $\mathfrak{a}$, which however does not necessarily belong to A. Now, if the prolinear map provides the identity $L_{\nu,\nu_*}(\mathfrak{a}) = (\mathfrak{a}, y(\mathfrak{a}))$ where $y(\mathfrak{a}) \in \mathbf{Ally}(\nu_*, c(\nu))$, then we know that the composite $y(\mathfrak{a}) \circledcirc \mathfrak{a}$ is in $A(\nu, c(\nu))$. By Proposition 2.42, the morphism f is an intraction for $y(\mathfrak{a}) \circledcirc \mathfrak{a} : \nu \rightsquigarrow c(\nu)$, which proves the statement. □

Proposition 4.42. *Every fibration that is a surtraction is a pseudofibration.*

Proof. Let f be both a fibration and a surtraction in $\hat{\mathcal{C}}$ and γ be an E-seed. By assumption, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ in E for which f is a surtraction. There also exists an alliance $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ in A' for which f is a fibration. By Proposition 2.48, this implies that f is a pseudofibration for the composite $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$. The statement follows from the fact that E is an A'-submodule of $(\mathbf{Enov}(\mathcal{C}), \mathfrak{\eta}, \odot, \circledcirc)$. □

Proposition 4.43. *Let f and g be two morphisms such that $f \circ g$ exists. If f and g are intractions, then so is $f \circ g$.*

Proof. Follows from Proposition 2.41 and the fact that $(A, \circledcirc)$ is a magmoid. □

Proposition 4.44. *Let f and g be two morphisms such that $f \circ g$ exists. If $f \circ g$ is an intraction, then so is g.*

Proof. Follows from Proposition 2.34. □

Proposition 4.45. *Let f and g be two morphisms such that $f \circ g$ exists. If $f \circ g$ is a surtraction and f is an intraction, then g is a surtraction.*

Proof. Let γ be an E-seed. By assumption, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ in E for which $f \circ g$ is a surtraction. By assumption, there also exists an alliance $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ in A for which f is an intraction. By Proposition 2.49, it follows that g is a surtraction for the extended node of vertebrae $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$, which is in E by definition of an A-submodule of $(\mathbf{Enov}(\mathcal{C}), \mathfrak{\eta}, \odot, \circledcirc)$. □

Proposition 4.46. *Every isomorphism in $\mathcal{C}$ is an intraction.*

Proof. Follows from Proposition 2.35. □

Proposition 4.47. *Every pseudofibration is a surtraction.*

Proof. Let f be a pseudofibration in $\hat{\mathcal{C}}$ and γ be an E-seed. By assumption, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in E for which f is a pseudofibration. By Proposition 2.27, this also equivalent to saying that f is a pseudofibration for the underlying communication $\mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \mathfrak{\eta}(\overline{\nu})$. By definition of a vertebral category, there exists a reflexive extended node of vertebrae $\mathfrak{n}_* : \mathfrak{\eta}(\overline{\nu}) \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ in E. Proposition 2.28 shows that any identity in $\mathcal{C}$ is a pseudofibration for $\mathfrak{n}_*$. It then follows from Proposition 2.43 that f is a pseudofibration for the composite $\mathfrak{n}_* \odot \mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$, which is in E by definition of a subcompass of $(\mathbf{Enov}(\mathcal{C}), \eta, \odot)$. Finally, since the extended node of vertebrae $\mathfrak{n}_* \odot \mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ is reflexive, Proposition 2.50 implies that the morphism f is a surtraction for $\mathfrak{n}_* \odot \mathbf{com}(\mathfrak{n}) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$. □

Proposition 4.48. *Every trivial fibration is a pseudofibration.*

Proof. Let f be a trivial fibration in $\hat{\mathcal{C}}$ and γ be an E-seed. By assumption on $\hat{\mathcal{C}}$, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ in E where ν_* is in $\mathrm{Obj}(A')$. By hypothesis on f, there must exist an alliance of nodes of vertebrae $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ in A' for which f is a trivial fibration. It follows from Proposition 2.31 that f is a pseudofibration for $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$, so that Proposition 2.44 implies that f is a pseudofibration for the composite $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\flat$, which is in E by definition of an A'-submodule of $(\mathbf{Enov}(\mathcal{C}), \eta, \odot, \circledcirc)$. □

We easily deduce that

Theorem 4.49. *A trivial fibration is both a fibration and a weak equivalence. On the other hand, a fibration that is a weak equivalence is a pseudofibration.*

Proof. One direction follows from Proposition 4.40, Proposition 4.41 and Proposition 4.47 together with Proposition 4.48. The other direction is nothing but Proposition 4.42. □

Lemma 4.50. *Let $i : X \to Y$ be a morphism in $\mathcal{C}$. If there exists an intraction $r : Y \to X$ in $\mathcal{C}$ such that $r \circ i = \mathrm{id}_X$, then i a weak equivalence.*

Proof. By Proposition 4.44 and Proposition 4.46, the morphism i is an intraction in the vertebral category $\hat{\mathcal{C}}$. To prove it is a surtraction, consider an E-seed γ. By definition of a vertebral category, there exists a reflexive node of vertebrae ν_* in $\mathrm{Obj}(A)$ and an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ in E. Because f is an intraction in $\hat{\mathcal{C}}$, there exists an alliance $\mathfrak{a}_* : \nu_* \rightsquigarrow \nu_\flat$ in A for which f is an intraction. Since ν_* is reflexive, the alliance $\mathfrak{a}_*$ is coreflexive and Lemma 2.52 implies that the morphism i is a surtraction for $\mathfrak{a}_*$. It follows from Proposition 2.46 that f is a surtraction for the composite $\mathfrak{a}_* \circledcirc \mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\flat$, which is in E by definition of an A-submodule of $(\mathbf{Enov}(\mathcal{C}), \eta, \odot, \circledcirc)$. □

The previous lemma implies – and is even equivalent to – the next proposition.

Proposition 4.51. *Every isomorphism in $\mathcal{C}$ is a surtraction.*

Proof. Follows from Proposition 4.50. □

Proposition 4.52. *Let f and g be two morphisms such that $f \circ g$ exists. If f and g are surtractions, then so is $f \circ g$.*

Proof. Let $f : Y \to Z$ and $g : X \to Y$ be two surtractions in $\hat{\mathcal{C}}$ and γ be an E-seed. By assumption, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ for which f is a surtraction. Similarly, since $h_0(\nu_*)$ is an E-seed, there exists an extended node of vertebrae $\mathfrak{n}_* : h_0(\nu_*) \overset{\text{ex}}{\rightsquigarrow} \nu_\flat$ for which g is a surtraction. Since the pair $(\mathfrak{n}, \mathfrak{n}_*)$ belongs to the product $\Sigma_0 E(\gamma, h_0(\nu_*)) \times \Sigma_1 E(h_0(\nu_*), h_1(\nu_\flat))$, the structure of vertebral algebra of E implies that there exists an extended node of vertebrae $\mathfrak{n}_\bullet : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$ in E of the form $\mathfrak{b} \circledcirc \langle \mathfrak{n}_* \odot t, \mathfrak{n} \rangle$ where

- $\mathfrak{b}$ is some alliance in $\mathbf{Ally}(\mathcal{C})$;
- t is a communication in $\mathbf{Com}(\mathcal{C})$ via which $\mathfrak{n}$ and $\mathfrak{n}_*$ communicate;
- and $\langle \mathfrak{n}_* \odot t, \mathfrak{n} \rangle : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$ denotes an extended node of vertebrae in $\mathcal{C}$ that frames the communicating pair $\mathfrak{n}$ and $\mathfrak{n}_* \odot t$.

By Proposition 2.46, the morphism g must be a surtraction for $\mathfrak{n}_* \odot t$. Proposition 2.58 then implies that $f \circ g$ is a surtraction for $\langle \mathfrak{n}_* \odot t, \mathfrak{n} \rangle$. Finally, Proposition 2.46 shows that $f \circ g$ is a surtraction for $\mathfrak{n}_\bullet$, which is in E. □

Theorem 4.53. *Let f,g and h be morphisms such that the composite $f \circ g \circ h$ exists in $\mathcal{C}$. If $f \circ g$ and $g \circ h$ are weak equivalences, then g is an intraction and both h and $f \circ g \circ h$ are weak equivalences.*

Proof. If the composite morphisms $f \circ g$ and $g \circ h$ are intractions, then so are g, h and $f \circ g \circ h$ by Proposition 4.44 and Proposition 4.43. When $g \circ h$ is also a surtraction, the morphism h is a surtraction by Proposition 4.45. Now, if $f \circ g$ is in addition a surtraction, then $f \circ g \circ h$ is a surtraction by Proposition 4.52. □

4.4.2. Refined and discrete vertebral categories. The following section is more a discussion about properties with which vertebral categories may be endowed than a section contributing to the theoretical construct initiated by the previous sections. The only property that will turn out to have a theoretical use will be that of refinement (see Chapter 5).

4.4.2.1. *Refined vertebral categories.* Let $\mathcal{C}$ be a category. A system of vertebrae in $\mathcal{C}$ will be said to be *refined* if all its fibrations that are weak equivalences are trivial fibrations. Similarly, a vertebral category will be said to be *refined* if so is its underlying system of vertebrae.

Theorem 4.54. *In a refined vertebral category, a morphism is a trivial fibration if and only if it is a fibration and a weak equivalence.*

Proof. Follows from Theorem 4.49. □

In the sequel, the morphisms of a vertebral category that are both fibrations and weak equivalences will be called *acyclic fibrations*.

Proposition 4.55. *In a refined vertebral category, trivial fibrations form a $\mathcal{C}$-coherent class.*

Proof. Follows from Theorem 4.54 and the fact that acyclic fibrations form a $\mathcal{C}$-coherent class by Proposition 4.35, Proposition 4.46, Proposition 4.52 and Proposition 4.51. □

A system of vertebrae in $\mathcal{C}$ will be said to be *strongly refined* if all its pseudofibrations are trivial fibrations. Similarly, a vertebral category will be said to be *strongly refined* if so is its underlying system of vertebrae.

Remark 4.56. In any strongly refined vertebral category, the class of pseudofibrations and the class of trivial fibrations are equal to the class of acyclic fibrations by Proposition 4.48 and Theorem 4.49.

Remark 4.57. Every strongly refined vertebral category is refined by Remark 4.56.

4.4.2.2. *Saturation.* This section generalises the usual notion of saturation appearing in abstract homotopy theory. Let $\mathcal{C}$ be a category and consider two commutative squares of the following form, where the left one will be referred to as $\mathbf{s} : \gamma' \Rightarrow \gamma$ in $\mathcal{C}^2$ while the right one will be referred to as an arrow $\mathbf{z} : x \Rightarrow y$ in $\mathcal{C}^2$.

$$\begin{tikzcd} A' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & A \arrow[d, "\gamma"] \\ B' \arrow[r, "\theta'"'] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} A' \arrow[r, "\theta"] \arrow[d, "x"'] & A \arrow[d, "y"] \\ C' \arrow[r, "\varkappa"'] & C \end{tikzcd}$$

Forming the pushout (in $\mathcal{C}^2$) of the preceding right commutative square along the left one with respect to their common arrow θ provides the following left commutative cube. A little arrangement on the top of the cube then provides the commutative cube on the right.

$$
\begin{tikzcd}
A' \arrow[rr, "x"] \arrow[dr, "\theta"] \arrow[dd, "\gamma'"'] & & C' \arrow[dr, "\varkappa"] \arrow[dd, "\beta'"' near start] & \\
& A \arrow[rr, "y" near start] \arrow[dd, "\gamma"' near start] \arrow[dr, phantom, "\ulcorner"] & & C \arrow[dd, "\beta"] \\
B' \arrow[rr, "\delta'"' near end] \arrow[dr, "\theta'"'] & & D' \arrow[dr, dashed, "\varkappa'"] & \\
& B \arrow[rr, "\delta"'] & & D
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A' \arrow[rr, "x"] \arrow[dr, "\theta"] \arrow[dd, "\gamma'"'] & & C' \arrow[dr, equal] \arrow[dd, "\beta'"' near start] & \\
& A \arrow[rr, "y" near start] \arrow[dd, "\gamma"' near start] \arrow[dr, phantom, "\ulcorner"] & & C \arrow[dd, "\beta\circ\varkappa"] \\
B' \arrow[rr, "\delta'"' near end] \arrow[dr, "\theta'"'] & & D' \arrow[dr, dashed, "\varkappa'"] & \\
& B \arrow[rr, "\delta"'] & & D
\end{tikzcd}
$$

The left face of the previous right commutative cube will be denoted as an arrow $P_{\mathbf{z}}(\mathbf{s}) : \beta' \Rightarrow \beta \circ \varkappa$ in $\mathcal{C}^2$. Note that this commutative square is the biased square of the left face of the first cube.

Remark 4.58. By Remark 2.15 (and Proposition 2.14), if a morphism in $\mathcal{C}$ has the rlp with respect to $\mathbf{s}$, so does it with respect to $P_{\mathbf{z}}(\mathbf{s})$.

Remark 4.59. Recall that right lifting properties are stable with respect to vertical pasting of biased squares. More specifically, it follows from Proposition 2.5 that if a morphism in $\mathcal{C}$ has the rlp with respect to the following leftmost two commutative squares, then it has the rlp with respect to the vertical pasting of the two squares given on the right.

$$
\begin{tikzcd}
A' \arrow[r, equal] \arrow[d, "\delta'"'] & A \arrow[d, "\delta\circ\theta"] \\
B' \arrow[r, "\theta_*"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B' \arrow[r, equal] \arrow[d, "\beta'"'] & B' \arrow[d, "\beta\circ\theta_*"] \\
C' \arrow[r, "\theta_\dagger"'] & C
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A' \arrow[r, equal] \arrow[d, "\beta'\circ\delta'"'] & A \arrow[d, "\beta\circ\delta\circ\theta"] \\
C' \arrow[r, "\theta_\dagger"'] & C
\end{tikzcd}
$$

Let $\mathcal{S}$ and $\mathcal{Z}$ be two classes of arrows in $\mathcal{C}^2$. Denote by $\mathcal{S}_{\mathcal{Z}}$ the smallest class containing $\mathcal{S}$ that is stable under

- the operation $\mathbf{s} \mapsto P_{\mathbf{z}}(\mathbf{s})$ for any arrow $\mathbf{z}$ in $\mathcal{Z}$ for which $P_{\mathbf{z}}(\mathbf{s})$ makes sense;
- the vertical pasting of biased squares.

The *saturation of* $\mathcal{S}$ *along* $\mathcal{Z}$ is the class obtained after the removal of all the non-biased squares of $\mathcal{S}_{\mathcal{Z}}$. The saturation of $\mathcal{S}$ along $\mathcal{Z}$ will later be denoted by $\mathrm{Sat}_{\mathcal{Z}}(\mathcal{S})$. If the class $\mathcal{Z}$ consists of all the commutative squares in $\mathcal{C}$, then the saturation of $\mathcal{S}$ along $\mathcal{Z}$ will later be denoted by $\mathrm{Sat}(\mathcal{S})$. The foregoing definition forces all the squares of $\mathrm{Sat}_{\mathcal{Z}}(\mathcal{S})$ to be biased squares.

Proposition 4.60. *If a morphism $f : X \to Y$ in $\mathcal{C}$ has the rlp with respect to the commutative squares of $\mathcal{S}$, then it has the rlp with respect to $\mathrm{Sat}_{\mathcal{Z}}(\mathcal{S})$ for any class $\mathcal{Z}$ of arrows in $\mathcal{C}^2$.*

Proof. Follows from Remark 4.59 and Remark 4.58. □

Now, let $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$ be a vertebral category. An E-seed γ will be said to be *canonical* if any pair of extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ and $\mathfrak{n}_\flat : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ in E satisfies the following equation.

$$\mathbf{seed}(\mathfrak{n}_*) = \mathbf{seed}(\mathfrak{n}_\flat)$$

Now, denote by $\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}})$ the class containing the commutative squares $\mathbf{seed}(\mathfrak{n})$ for every extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ whose preseed γ is canonical. Similarly, denote by $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}})$ the class containing all the commutative biased squares $\mathbf{bste}(\mathfrak{a})$ where $\mathfrak{a}$ runs over the alliances of A'. The next proposition uses that fact observed in Example 4.9 that the operation $\mathbf{ext}(\_)$ defines a morphism of spans from $\mathbf{Ally}(\mathcal{C})$ to $\mathbf{Enov}(\mathcal{C})$.

Proposition 4.61. *If the inclusion $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}}) \subseteq \mathrm{Sat}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$ holds and the operation $\mathbf{ext}(\_)$ induces a fibration of spans from A' to E, then $\hat{\mathcal{C}}$ is strongly refined.*

Proof. Let $f : X \to Y$ be a pseudofibration in $\hat{\mathcal{C}}$. We are first going to show that f has the rlp with respect to any commutative square in $\mathrm{Sat}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$. Let $\mathbf{seed}(\mathfrak{n})$ be a commutative square in $\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}})$ and γ be the preseed of the extended node of vertebrae $\mathfrak{n}$. Since γ is an E-seed and f is a pseudofibration, there exists an extended node of vertebrae $\mathfrak{n}_* : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\flat$ for which f is a pseudofibration. This means that f has the rlp with respect to $\mathbf{seed}(\mathfrak{n}_*)$. Because γ must be canonical, the morphism f must have the rlp with respect to $\mathbf{seed}(\mathfrak{n})$ too. It finally follows from Proposition 4.60 that f has the rlp with respect to any square in $\mathrm{Sat}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$. Note that, by assumption, this implies that f has the rlp with respect to any biased square $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}})$.

We are now going to show the statement; i.e. f is a trivial fibration in $\hat{\mathcal{C}}$. Let ν be a node of vertebrae in $\mathrm{Obj}(A')$ whose seed will be denoted by γ. By assumption, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ for which f is a pseudofibration. Since $\mathbf{ext}(\_)$ induces a fibration of spans $A' \Rightarrow E$, there must exist an alliance $\mathfrak{a} : \nu \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in A' such that the equality $\mathbf{ext}(\mathfrak{a}) = \mathfrak{n}$ holds. By Proposition 2.26, the morphism is a pseudofibration for $\mathfrak{a}$. In other words, it has the rlp with respect to $\mathbf{seed}(\mathfrak{a})$. On the other hand, the above discussion showed that f had the rlp with respect to $\mathbf{bste}(\mathfrak{b})$ for every alliance $\mathfrak{b}$ in A'. These two rlp proves that f is a trivial fibration for $\mathfrak{a}$, which finishes the proof. □

Remark 4.62. Other operations could now be used to augment the image of the operation $\mathrm{Sat}\_(\_)$. For instance, transfinite compositions would constitute a good candidate. In this case, a generalisation of Proposition 5.9 would extend the previous proposition.

4.4.2.3. *Discrete vertebral categories.* Let $\mathcal{C}$ be a category. A system of vertebrae (A, A', E) in $\mathcal{C}$ will be said to be *discrete* if

1) all the nodes of vertebrae in $\mathrm{Obj}_{\mathrm{R}}(E)$ are singleton, i.e. vertebrae;
2) all the alliances in A and A' are identity alliances, i.e. vertebrae;
3) all the extended nodes of vertebrae in E are trivial, i.e. vertebrae.

Similarly, a vertebral category $(\mathcal{C}, A, A', E)$ will be said to be *discrete* if its underlying system of vertebrae is discrete. Because the equality $\mathrm{Obj}(A) = \mathrm{Obj}(A')$ holds in the case of vertebral categories (see section 4.3.6.2), the two magmoids A and A' are then equal.

Remark 4.63 (Saturation). In any discrete vertebral category, the class $\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}})$ corresponds to the class of identity commutative squares between two copies of the same E-seeds γ (see below) while the class $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}})$ corresponds to the class of identity commutative squares between two copies of the same stems β in A (or A').

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, equal] \arrow[d, "\gamma"'] & \mathbb{S} \arrow[d, "\gamma"] \\
\mathbb{D}_2 \arrow[r, equal] & \mathbb{D}_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{S}' \arrow[r, equal] \arrow[d, "\beta"'] & \mathbb{S}' \arrow[d, "\beta"] \\
\mathbb{D}' \arrow[r, equal] & \mathbb{D}'
\end{tikzcd}
$$

In addition, the operation $\mathbf{ext}(\_)$ induces a trivial morphism of spans from A' to E mapping any vertebra v in A to the vertebra v in E and may be identified with the inclusion $\mathrm{Obj}(A') \subseteq \mathrm{Obj}_{\mathrm{R}}(E)$. The operation $\mathbf{ext}(\_)$ therefore induces a fibration of spans $A' \Rightarrow E$. Note that, in the case where the inclusion $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}}) \subseteq \mathrm{Sat}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$ holds, the conditions of Proposition 4.61 are all satisfied.

Let $\hat{\mathcal{C}} = (\mathcal{C}, A, A', E)$ be a discrete vertebral category such that for every E-seed γ, there exists a vertebra v in A whose base is of the form $\|\gamma, \gamma\|$. In addition, let $\mathcal{Z}$ be the class of all identity commutative squares in $\mathcal{C}$ (identified with arrows in $\mathcal{C}$) and $\mathrm{Psh}_{\mathcal{Z}}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$ denote the smallest class containing $\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}})$ that is stable under

- the operation $\mathbf{s} \mapsto P_{\mathbf{z}}(\mathbf{s})$ for any arrow $\mathbf{z}$ in $\mathcal{Z}$ for which $P_{\mathbf{z}}(\mathbf{s})$ make sense;
- coproduct of arrows in $\mathcal{C}$, i.e. $\{\gamma_i : \mathbb{A}_i \to \mathbb{B}_i\}_{i\in I} \mapsto \coprod_{i\in I} \gamma_i$;

Proposition 4.64. *If the above assumptions are satisfied and the class of arrows $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}})$ is included in $\mathrm{Psh}_{\mathcal{Z}}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$, then any surtraction in $\hat{\mathcal{C}}$ is a weak equivalence.*

Proof. Let $f : X \to Y$ be a surtraction in $\hat{\mathcal{C}}$. We are going to show that f is simple with respect to any stem in $\mathcal{B}\mathbf{ste}(\hat{\mathcal{C}})$ and is hence an intraction in $\hat{\mathcal{C}}$. Because for every E-seed γ, there exists a vertebra v in E whose base is of the form $\|\gamma,\gamma\|$, it follows from Proposition 2.9 that any $f : X \to Y$ is simple with respect to any E-seed. It is therefore simple with respect to any arrow in $\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}})$. We are now going to show that simplicity is stable with respect to the operation $P_{\mathbf{z}}(\_)$ for any admissible $\mathbf{z}$ in $\mathcal{Z}$ and coproducts. For some arrow $z : \mathbb{A} \to \mathbb{A}'$, consider the following leftmost commutative square where $P_z(\gamma) : \mathbb{A} \to \mathbb{A}'$ is the pushout of an arrow $\gamma : \mathbb{A} \to \mathbb{B}$ along z. Using the underlying pushout square and the fact that f is simple with respect to γ leads to the existence of a lift $h : \mathbb{B} \to X$ making the top triangle of the following middle diagram commute. Reforming the pushout of γ along z then produces a canonical arrow $h' : \mathbb{B}' \to X$ making the corresponding rightmost triangle commute and proves that f is simple with respect to $P_z(\gamma) : \mathbb{A} \to \mathbb{A}'$.

$$
\begin{tikzcd}
\mathbb{A}' \arrow[r, "x"] \arrow[d, "P_z(\gamma)"'] & X \arrow[d, "f"] \\
\mathbb{B}' \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{A} \arrow[r, "z"] \arrow[d, "\gamma"'] & \mathbb{A}' \arrow[r, "x"] & X \arrow[d, dotted, "f"] \\
\mathbb{B} \arrow[rru, dashed, "h"'] \arrow[r, dotted] & \mathbb{B}' \arrow[r, dotted, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{A}' \arrow[r, "x"] \arrow[d, "P_z(\gamma)"'] & X \\
\mathbb{B}' \arrow[ru, dashed, "h'"'] &
\end{tikzcd}
$$

Similarly, it is not hard to show that if f is simple with respect to every arrow contained in a collection of arrows $\{\gamma_i\}_{i\in I}$ in $\mathcal{C}$, then it is simple with respect to the coproduct of all these arrows in $\mathcal{C}$. Finally, the above discussion shows that f is simple with respect to any arrow in $\mathrm{Psh}_{\mathcal{Z}}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$. It is therefore simple with respect to the stems of A and hence is an intraction in $\hat{\mathcal{C}}$. □

Unfortunately, simplicity is not stable with respect to compositions of arrows in general, this prevents one from extending the above proposition to the saturation $\mathrm{Sat}_{\mathcal{Z}}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$.

Remark 4.65 (Construction)**.** Saturation suggests how to generate strongly refined discrete vertebral categories. For instance, it is always possible to start with an arrow $\gamma : \mathbb{S} \to \mathbb{D}$ and generate the leftmost diagram, below. In an ideal case, we may assume that the corresponding middle pushout exists, which leads to the existence of a boundary contraction $u : \mathbb{S}' \to \mathbb{D}$ for the induced prevertebra $\|\gamma,\gamma\|$ on the right.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathbb{D} \arrow[d, equal] \\
\mathbb{D} \arrow[r, equal] & \mathbb{D}
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathbb{D} \arrow[d, "\delta_1"] \\
\mathbb{D} \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}'
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathbb{D} \arrow[d, "\delta_1"] \arrow[rd, bend left, "\mathrm{id}_{\mathbb{D}}"] & \\
\mathbb{D} \arrow[r, "\delta_2"] \arrow[ur, phantom, "\ulcorner"] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{D}}"'] & \mathbb{S}' \arrow[r, dashed, "u"] & \mathbb{D}
\end{tikzcd}
$$

By using usual methods (i.e. small object argument), it is often possible to factorise $u : \mathbb{S}' \to \mathbb{D}$ into two arrows $\alpha \circ \beta$ where the arrow $\beta : \mathbb{S}' \to \mathbb{D}'$ belongs to the saturated class $\mathrm{Sat}(\mathcal{S}\mathbf{eed}(\hat{\mathcal{C}}))$. The vertebra $\|\gamma,\gamma'\| \cdot \beta$ then defines a reflexive vertebra with homotopy contraction $\alpha : \mathbb{D}' \to \mathbb{D}$. After generating as many reflexive vertebrae as one wants, one may use similar methods to create the framings of those reflexive vertebrae (see section 2.4.2.4). Interestingly, Proposition 2.56 ensures that the framings are also reflexive. In the end, we obtain a strongly refined discrete vertebral category wherein all the vertebrae are reflexive. This type of algorithmic process will be the key of Chapter 6 to provide Grothendieck's ∞-groupoids with a vertebral category – or in fact a spinal category (see section 4.4.4).

4.4.2.4. *Epi-correction for discrete vertebral categories.* This section defines an operation making vertebral categories more likely to be refined (at least in practice). Let $\mathcal{C}$ be a category. A vertebra $v := \|\gamma,\gamma'\| \cdot \beta$ in $\mathcal{C}$ will be said to be *rectifiable* if its stem $\beta : \mathbb{S}' \to \mathbb{D}'$ is an epimorphism in $\mathcal{C}$. In this case, the definition of an epimorphism implies that the prevertebra $\|\beta,\beta\|$ is well-defined and of the form $\mathbb{S}' \multimap (\mathrm{id}_{\mathbb{D}'}, \mathrm{id}_{\mathbb{D}'})$. The trivial vertebra $\|\beta,\beta\| \cdot \mathrm{id}_{\mathbb{D}'}$ that results from this will be referred to as the *rectification of* v and denoted as $\mathrm{Rec}(v)$.

Proposition 4.66. *Let* v *be a rectifiable vertebra. A morphism is a surtraction for* $\mathrm{Rec}(v)$ *if and only if it is an intraction for* v.

Proof. We shall denote v by $\|\gamma,\gamma'\| \cdot \beta$. Let $f : X \to Y$ be a surtraction for $\mathrm{Rec}(v)$. To show that f is an intraction for v, start with the left commutative square, below. Because f is a surtraction for $\mathrm{Rec}(v)$, there exists an arrow $h : \mathbb{D}' \to X$ making the corresponding middle diagram commute, but note that this diagram is equivalent to the triangle on the right.

$$
\begin{tikzcd}
\mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "\beta"] \arrow[d, "\beta"'] \arrow[rr, bend left, "x"] & \mathbb{D}' \arrow[r, dashed, "h"'] \arrow[d, equal] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r, equal] \arrow[rr, bend right, "y"'] & \mathbb{D}' \arrow[r, "y"] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & {} \\
\mathbb{D}' \arrow[ur, "h"'] &
\end{tikzcd}
$$

On the other hand, let $f : X \to Y$ be an intraction for v. To show that f is an surtraction for $\mathrm{Rec}(v)$, start with the following leftmost commutative square. Because f is an intraction for v, there exists a semi-lift $h : \mathbb{D}' \to X$ making the corresponding middle diagram commute. But this diagram is equivalent to giving the rightmost one since the leftmost two diagrams imply the equality $y \circ \beta = f \circ h \circ \beta$ and hence $y = f \circ h$.

$$
\begin{tikzcd}
\mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r, "y"'] & Y
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & {} \\
\mathbb{D}' \arrow[ur, dashed, "h"'] &
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "\beta"] \arrow[d, "\beta"'] \arrow[rr, bend left, "x"] & \mathbb{D}' \arrow[r, "h"'] \arrow[d, equal] & X \arrow[d, "f"] \\
\mathbb{D}' \arrow[r, equal] \arrow[rr, bend right, "y"'] & \mathbb{D}' \arrow[r, "y"] & Y
\end{tikzcd}
$$

□

Proposition 4.67. *Let* v *be a rectifiable vertebra. The vertebra* $\mathrm{Rec}(v)$ *is reflexive and communicates with itself. It also frames two copies of itself.*

Proof. The reflexivity and communication is obvious. Since β is an epimorphism, checking that $\mathrm{Rec}(v)$ frames two copies of itself boils down to observing that the equality $\mathrm{id}_{\mathbb{D}'} = \mathrm{id}_{\mathbb{D}'}$ is true. □

Let $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$ be a discrete system of vertebrae. An *E-rectifiable vertebra* is a rectifiable vertebra in E whose stem is not an E-seed. A system of vertebrae $(\mathcal{C}, A, A', E)$ will be said to be *epi-correctible* if it is discrete and satisfies the property that if two E-rectifiable vertebrae v and v_* in E have same stems, then v and v_* are equal.

Let $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$ be an epi-correctible system of vertebrae. We shall call the *epi-correction* of $\hat{\mathcal{C}}$ the discrete system of vertebrae $\mathrm{Epic}(\hat{\mathcal{C}})$ that consisting of $\hat{\mathcal{C}}$ to which is added

- the stems of the E-rectifiable vertebrae in its object-classes;
- the rectifications of the E-rectifiable vertebrae in its hom-classes.

A vertebral category will be said to be *epi-correctible* if its underlying system of vertebrae is.

Proposition 4.68. *Let $\hat{\mathcal{C}}$ be an epi-correctible vertebral category. The epi-correction* $\mathrm{Epic}(\hat{\mathcal{C}})$ *is a vertebral category. In addition, a morphism is a weak equivalence in $\hat{\mathcal{C}}$ if and only if it is one in* $\mathrm{Epic}(\hat{\mathcal{C}})$.

Proof. Denote $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$. If we extend the source and target hinges of $(E, \mathfrak{y})$ via a trivial mapping rule $\|\gamma,\gamma\| \cdot \mathrm{id}_{\mathbb{S}'} \mapsto \beta$, then the first statement follows from Proposition 4.67 as any other framing is prevented by the fact that if two E-rectifiable vertebrae v and v_* have same stems, then v and v_* are equal and the fact that the stem of any E-rectifiable vertebra is not an E-seed. The second statement regarding weak equivalences follows from Proposition 4.66. □

4.4.3. Whiskering bundles for extended nodes of vertebrae. This section transforms the extended nodes of vertebrae of a vertebral category into semi-extended nodes of vertebrae. If all the extended nodes of vertebrae of the vertebral category are already semi-extended, then the operation of this section is trivial. The derivation of an extended nodes of vertebrae into a semi-extended one may be related to the process of differentiation in analysis where the idea is to obtain the closest description of a function in terms of the most elementary ones, namely the linear maps.

4.4.3.1. *Differentiable nodes of vertebrae.* Let $\mathcal{C}$ be a category. The next notion, which is that of differentiability for extended nodes of vertebrae, may be seen as the ability of those to generate a semi-extended node of vertebrae when pushing out their diskads along their spherical transitions. Specifically, an extended node of vertebrae $\mathfrak{n} := (\varkappa, \varrho) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in $\mathcal{C}$ will be said to be *differentiable* if

1) every vertebra $\|\overline{\gamma}, \overline{\gamma}'\| \cdot \overline{\beta}$ in $\overline{\nu}$ is equipped with three pushouts in $\mathcal{C}$ of the following form (where the leftmost two do not depend on $\overline{\beta}$);

$$\begin{tikzcd} \overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[d, "\overline{\gamma}"'] & \mathbb{S} \arrow[d, dashed, "\gamma_\star"] \\ \overline{\mathbb{D}}_2 \arrow[r, dashed, "\varrho_\star"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}_2^\star \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[d, "\overline{\gamma}'"'] & \mathbb{S} \arrow[d, dashed, "\gamma'_\star"] \\ \overline{\mathbb{D}}_1 \arrow[r, dashed, "\varrho'_\star"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}_1^\star \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[d, "\overline{\beta}\circ\overline{\delta}_2\circ\overline{\gamma}"'] & \mathbb{S} \arrow[d, dashed, "e^\star_{\overline{\beta}}"] \\ \overline{\mathbb{D}}' \arrow[r, dashed, "u^\star_{\overline{\beta}}"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}'_\star \end{tikzcd}$$

2) the pair of arrows $\gamma_\star$ and $\gamma'_\star$ is equipped with a structure of prevertebra $\|\gamma_\star, \gamma'_\star\| : \mathbb{S} \multimap \mathbb{S}'_\star$ in $\mathcal{C}$.

Because the left vertical arrow of the rightmost square of item 1) is also equal to the composite $\overline{\beta} \circ \overline{\delta}_1 \circ \overline{\gamma}'$, the universality of the leftmost two pushouts of item 1) implies that there must exist two canonical arrows $b_1 : \mathbb{D}_1^\star \to \mathbb{D}'_\star$ and $b_2 : \mathbb{D}_2^\star \to \mathbb{D}'_\star$ making the following diagrams commute.

$$\begin{tikzcd} \overline{\mathbb{S}} \arrow[r, "\overline{\gamma}'"] \arrow[d, "\varkappa"'] & \overline{\mathbb{D}}_1 \arrow[r, "\overline{\beta}\circ\overline{\delta}_1"] \arrow[d, "\varrho'_\star"] & \overline{\mathbb{D}}' \arrow[d, "u^\star_{\overline{\beta}}"] \\ \mathbb{S} \arrow[r, "\gamma'_\star"] \arrow[rr, bend right, "e^\star_{\overline{\beta}}"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}_1^\star \arrow[r, dashed, "b_1"] & \mathbb{D}'_\star \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbb{S}} \arrow[r, "\overline{\gamma}"] \arrow[d, "\varkappa"'] & \overline{\mathbb{D}}_2 \arrow[r, "\overline{\beta}\circ\overline{\delta}_2"] \arrow[d, "\varrho_\star"] & \overline{\mathbb{D}}' \arrow[d, "u^\star_{\overline{\beta}}"] \\ \mathbb{S} \arrow[r, "\gamma_\star"] \arrow[rr, bend right, "e^\star_{\overline{\beta}}"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}_2^\star \arrow[r, dashed, "b_2"] & \mathbb{D}'_\star \end{tikzcd}$$

The fact that the bottom arrow of the two previous commutative diagrams is equal provides the following left commutative square. Then, the structure of prevertebra $\|\gamma_\star, \gamma'_\star\| : \mathbb{S} \multimap \mathbb{S}'_\star$

leads to the existence a canonical morphism $\varphi(\overline{\beta}) : \mathbb{S}'_\star \to \mathbb{D}'_\star$ making the following right diagram commute.

(4.8)
$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'_\star"] \arrow[d, "\gamma_\star"'] & \mathbb{D}_1^\star \arrow[d, "b_1"] \\
\mathbb{D}_2^\star \arrow[r, "b_2"'] & \mathbb{D}'_\star
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'_\star"] \arrow[d, "\gamma_\star"'] & \mathbb{D}_1^\star \arrow[d, "\delta_1^\star"] \arrow[dr, bend left, "b_1"] & \\
\mathbb{D}_2^\star \arrow[r, "\delta_2^\star"] \arrow[rr, bend right, "b_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}'_\star \arrow[r, dashed, "\varphi(\overline{\beta})"] & \mathbb{D}'_\star
\end{tikzcd}
$$

Denote by $\Omega_\star$ the class of arrows $\varphi(\overline{\beta}) : \mathbb{S}'_\star \to \mathbb{D}'_\star$ for every stem $\overline{\beta}$ of $\overline{\Omega}$. The mapping $\overline{\beta} \mapsto \varphi(\overline{\beta})$ defines a metafunction $\varphi : \overline{\Omega} \to \Omega_\star$. Denote by $\nu_\star$ the node of vertebrae $\|\gamma_\star, \gamma'_\star\| \cdot \Omega_\star$, which contains all the vertebrae of the form given on the right-hand side of (4.8) after removing the arrows b_1 and b_2. Denoting by $\varkappa'_\star$ the canonical arrow induced by the following leftmost universal problem then leads to an alliance of nodes of vertebrae of the form given on the right. The commutative squares in brackets are, for their part, induced by universality using b_1 and b_2.

$$
\begin{tikzcd}
& \overline{\mathbb{S}} \arrow[rr, "\varkappa_\star"] \arrow[dl, "\overline{\gamma}"'] \arrow[dd, "\overline{\gamma}"] & & \mathbb{S} \arrow[dl, "\gamma'_\star"'] \arrow[dd, "\gamma_\star"] \\
\overline{\mathbb{D}}_1 \arrow[rr, "\varrho'_\star"] \arrow[dd, "\overline{\delta}_1"'] & & \mathbb{D}_1^\star \arrow[dd, "\delta_1^\star"] & \\
& \overline{\mathbb{D}}_2 \arrow[rr, "\varrho_\star"] \arrow[dl, "\overline{\delta}_2"'] & & \mathbb{D}_2^\star \arrow[dl, "\delta_2^\star"] \\
\overline{\mathbb{S}}' \arrow[rr, dashed, "\varkappa'_\star"] & & \mathbb{S}'_\star &
\end{tikzcd}
$$

$$
\mathfrak{a}_\star := (\varkappa, \varrho_\star, \varrho'_\star, \varkappa'_\star, \varphi, u^\star) : \nu_\star \rightsquigarrow \overline{\nu}
$$

$$
\left(
\begin{tikzcd}
\overline{\mathbb{S}}' \arrow[r, dashed, "\varkappa'_\star"] \arrow[d, "\overline{\beta}"'] & \mathbb{S}' \arrow[d, "\varphi(\overline{\beta})"] \\
\overline{\mathbb{D}}' \arrow[r, "u^\star_{\overline{\beta}}"'] & \mathbb{D}'_\star
\end{tikzcd}
\right)
$$

Now, recall that the extended node of vertebrae $\mathfrak{n} := (\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ is equipped with the leftmost commutative diagram, below, which implies the existence of a unique arrow $\varrho_\flat$: $\mathbb{D}_2^\star \to \mathbb{D}_2$ making the succeeding right diagram commute.

$$
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[d, "\overline{\gamma}"'] & \mathbb{S} \arrow[d, "\gamma"] \\
\overline{\mathbb{D}}_2 \arrow[r, "\varrho"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}_2
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[d, "\overline{\gamma}"'] & \mathbb{S} \arrow[d, dashed, "\gamma_\star"] \arrow[dr, bend left, "\gamma"] & \\
\overline{\mathbb{D}}_2 \arrow[r, dashed, "\varrho_\star"'] \arrow[rr, bend right, "\varrho"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D}_2^\star \arrow[r, dashed, "\varrho_\flat"] & \mathbb{D}_2
\end{tikzcd}
$$

The canonical arrow $\varrho_\flat : \mathbb{D}_2^\star \to \mathbb{D}_2$ then defines a semi-extended node of vertebrae $\mathfrak{n}_\star$: $(\mathrm{id}_\mathbb{S}, \varrho_\flat) : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\star$. Interestingly, note that composing $\mathfrak{a}_\star$ with $\mathfrak{n}_\star$ gives back the extended node of vertebrae $\mathfrak{n}$ as shown in the following equations.

$$
\begin{aligned}
\mathfrak{a}_\star \circledcirc \mathfrak{n}_\star &= (\varkappa, \varrho_\star, \varrho'_\star, \varkappa'_\star, \varphi, u^\star) \circledcirc (\mathrm{id}_\mathbb{S}, \varrho_\flat) \\
&= (\varkappa, \varrho_\flat \circ \varrho_\star) \\
&= (\varkappa, \varrho) = \mathfrak{n}
\end{aligned}
$$

The semi-extended node of vertebrae $\mathfrak{n}_\star : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_\star$ will later be called the *derivative of* $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ and denoted as $d\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} d\overline{\nu}$.

Remark 4.69. If $\mathfrak{n}$ is a semi-extended node of vertebrae, then $d\mathfrak{n} = \mathfrak{n}$ and $d\overline{\nu} = \overline{\nu}$.

Proposition 4.70. *A morphism in $\mathcal{C}$ is a surtraction for $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ if and only if it is a surtraction for $d\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} d\overline{\nu}$.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. The notations for the extended node of vertebrae $d\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} d\overline{\nu}$ will be the same as those used for the above discussion. Suppose that f is a surtraction for $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ and let us prove that it is a surtraction for $d\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} d\overline{\nu}$. Consider a commutative diagram of the form given below on the right. Because f is a surtraction for $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$, there exists a stem $\overline{\beta} : \overline{\mathbb{S}}' \to \overline{\mathbb{D}}'$ in $\overline{\Omega}$ and two arrows $x' : \overline{\mathbb{D}}_1 \to X$ and $y' : \overline{\mathbb{D}}' \to Y$ making the following right diagram commute.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
\mathbb{D}_2 \arrow[r, "y"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & \overline{\mathbb{S}} \arrow[r, "\varkappa"] \arrow[dl, "\overline{\gamma}'"'] \arrow[d, "\overline{\gamma}"] & \mathbb{S} \arrow[r, "u"] \arrow[d, "\gamma"] & X \arrow[d, "f"] \\
\overline{\mathbb{D}}_1 \arrow[r, dashed, no head] \arrow[d, "\overline{\beta}\circ\overline{\delta}_1"'] \arrow[urrr, dashed, bend right, "x'"] & \overline{\mathbb{D}}_2 \arrow[r, "\varrho"'] \arrow[dl, "\overline{\beta}\circ\overline{\delta}_2"] & \mathbb{D}_2 \arrow[r, "v"'] & Y \\
\overline{\mathbb{D}}' \arrow[urrr, dashed, bend right, "y'"'] & & &
\end{tikzcd}
$$

Pushing out the diskad of the vertebra $\|\overline{\gamma}, \overline{\gamma}'\| \cdot \overline{\beta}$ along $\varkappa : \overline{\mathbb{S}} \to \mathbb{S}$ then makes the diskad of the associated vertebra $\|\gamma_\star, \gamma'_\star\| \cdot \varphi_{\overline{\beta}}$ appear as follows.

$$
\begin{tikzcd}
 & \overline{\mathbb{S}} \arrow[rr, "\varkappa"] \arrow[dl, "\overline{\gamma}'"'] \arrow[dd, "\overline{\gamma}"] & & \mathbb{S} \arrow[rr, "u"] \arrow[dl, dashed, "\gamma'_\star"'] \arrow[dd, dashed, "\gamma_\star"] \arrow[ddr, "\gamma"] & & X \arrow[dd, "f"] \\
 & & \mathbb{D}^\star_1 \arrow[urrr, dotted] \arrow[dd, dashed] & & & \\
\overline{\mathbb{D}}_1 \arrow[urr, dashed] \arrow[uurrrrr, bend right, "x'"] \arrow[dd, "\overline{\beta}\circ\overline{\delta}_1"'] & \overline{\mathbb{D}}_2 \arrow[rr, dashed] \arrow[ddl, "\overline{\beta}\circ\overline{\delta}_2"] & & \mathbb{D}^\star_2 \arrow[r, dashed, "\varrho_\flat"] \arrow[dl, dashed] & \mathbb{D}_2 \arrow[r, "v"] & Y \\
 & & \mathbb{D}'_\star \arrow[urrr, dotted] & & & \\
\overline{\mathbb{D}}' \arrow[urr, dashed] \arrow[uurrrrr, bend right, "y'"] & & & & &
\end{tikzcd}
$$

The existence of dotted canonical arrows $\mathbb{D}^\star_1 \to X$ and $\mathbb{D}'_\star \to Y$ as shown above then proves that $f : X \to Y$ is divisible by the underlying besom of $\mathfrak{n}_\star : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\star$. In other words, the morphism $f : X \to Y$ is a surtraction for $d\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} d\overline{\nu}$. Conversely, since the equation $\mathfrak{n} = \mathfrak{a}_\star \circledcirc d\mathfrak{n}$ holds, it follows from Proposition 2.46 that if f is a surtraction for $d\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} d\overline{\nu}$, then it is a surtraction for $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$. $\square$

4.4.3.2. *Semi-alliances of nodes of vertebrae.* Later on, the term *semi-alliance of nodes of vertebrae* will be used for any alliance of nodes of vertebrae whose spherical transition is an identity, namely an alliance of the form $\mathfrak{b} := (\mathrm{id}, \varrho, \varrho', \varkappa', \varphi, u)$.

Remark 4.71. Composing a semi-alliance $\mathfrak{b} : \overline{\nu} \rightsquigarrow \nu_\dagger$ in $\mathbf{Ally}(\mathcal{C})$ with a semi-extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ produces a semi-extended nodes of vertebrae of the form $\mathfrak{b} \circledcirc \mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\dagger$.

4.4.3.3. *Whiskering bundles.* The intuition behind the notion of whiskering bunble is the same as that of tangent bundle for differentiation in differential geometry. Let $\mathcal{C}$ be a category and E be a subspan of the span $\mathbf{Enov}(\mathcal{C})$ in $\mathcal{C}$. Consider some subclass $S \subseteq \mathrm{Obj}_{\mathrm{L}}(E)$. A *whiskering bundle of E above S* consists of a subgraph T of $\mathbf{Sev}(\mathcal{C})$ such that

1) **(base)** the inclusion $S \subseteq \mathrm{Obj}(T)$ holds;
2) **(smoothness)** for every arrow $\gamma \in S$, node of vertebrae $\overline{\nu} \in \mathrm{Obj}_{\mathrm{R}}(E)$ and extended node of vertebrae $\mathfrak{n} \in E(\gamma, \overline{\nu})$, the derivative $d\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} d\overline{\nu}$ exists;

3) **(tangent vectors)** there exists a semi-alliance $\mathfrak{b} : d\overline{\nu} \rightsquigarrow \nu_\dagger$ in $\mathbf{Ally}(\mathcal{C})$ such that the coseed $\eta'(\nu_\dagger)$ belongs to S and every semi-extended vertebra contained in $\mathfrak{b} \circledcirc d\mathfrak{n}$ belongs to the class $T(\gamma, \eta'(\nu_\dagger))$.

Proposition 4.70 implies the following proposition, which states that surtractions are preserved from the span E to its associated whiskering bundle.

Proposition 4.72. *If a morphism in $\mathcal{C}$ is a surtraction for some extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ in E where $\gamma \in S$, then there exists a semi-extended vertebra $\mathfrak{v}_\dagger : \gamma \overset{\text{ex}}{\rightsquigarrow} v_\dagger$ in T, where $\eta'(v_\dagger) \in S$, for which it is a surtraction.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ that is a surtraction for some extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ in E where $\gamma \in S$. Because T is a whiskering bundle of E over S, the derivative $d\mathfrak{n}$ exists. By Proposition 4.70, f is a surtraction for $d\mathfrak{n}$. By assumption on T, we know that there exists a semi-alliance $\mathfrak{b} : \overline{\nu} \rightsquigarrow \nu_\dagger$ in $\mathbf{Ally}(\mathcal{C})$ such that the coseed $\eta'(\nu_\dagger)$ belongs to S and every extended vertebra contained in $b \circledcirc d\mathfrak{n}$ belongs to the class $T(\gamma, \eta'(\nu_\dagger))$. By Proposition 2.46, the morphism $f : X \to Y$ is also a surtraction for the composite $\mathfrak{b} \circledcirc d\mathfrak{n}$. In particular, this means that there exists a semi-extended vertebra $\mathfrak{v}_\dagger : \gamma \overset{\text{ex}}{\rightsquigarrow} v_\dagger$ in the senov $\mathfrak{b} \circledcirc d\mathfrak{n}$ whose underlying besom divises f. This means that f is a surtraction for the sev $\mathfrak{v}_\dagger : \gamma \overset{\text{ex}}{\rightsquigarrow} v_\dagger$. Finally, since the equality $\eta'(\nu_\dagger) = \eta'(v_\dagger)$ holds, the relation $\eta'(\nu_\dagger) \in S$ implies the relation $\eta'(v_\dagger) \in S$. $\square$

4.4.4. Spinal categories.

4.4.4.1. *Spinal categories.* A category $\mathcal{C}$ will be said to be *spinal* if it is endowed with a subgraph $A \subseteq \mathbf{Ally}(\mathcal{C})$, a pair of subgraphs $T_\ltimes, T_\rtimes \subseteq \mathbf{Sev}(\mathcal{C})$ and a regular locally $(T_\ltimes, T_\rtimes)$-whiskered A-subechelon $[\pi^-, \tau^-](E, \eta) \subseteq [\lambda^-, \kappa^-](\mathbf{Enov}(\mathcal{C}), \eta)$ such that

1) **(zoology)** the triple $(\mathcal{C}, A, E)$ is equipped with a structure of vertebral category, which will later be denoted by $\hat{\mathcal{C}}$;
2) **(local projectivity)** the local A-echelon (E, η) is defined under A_n-echelons (E_n, η) such that, for every $n \in \omega$, the node of spines in $\mathrm{Obj}(A_n)$ and spines in $\mathrm{Obj}_{\mathrm{L}}(E_n)$ are projective with respect to every surtraction of the vertebral category $\hat{\mathcal{C}}$. The cograded graph of (E_n, μ_n) will be denoted by $(s^n_-, t^n_-) : (O^n_-, h^n_-) \rightrightarrows S^n_-$ for every $n \in \omega$;
3) **(whiskering)** The subgraphs $T_\ltimes$ and $T_\rtimes$ are whiskering bundles of E above the class S^n_k for every pair of integers k and n satisfying the inequalities $0 \leq k \leq n$;

Remark 4.73. Recall that for every pair of integers k and n such that $0 \leq q \leq n$ and pair of arrows $\gamma, \gamma' \in S^n_q$, the object-class $O^n_q(\gamma, \gamma')$ is a 2-class of nodes of spines $(p_k) \cdot \Omega$ of degree n in $\mathrm{Obj}(A_n)$ whose prevertebra p_q is of the form $\|\gamma, \gamma'\|$.

5) **(framing i)** for every pair of integers q and n such that $0 \leq q \leq n$ and 3-tuple of the form
$$(\mathfrak{v}_\diamond, \sigma, \mathfrak{v}_\bullet) \in T_\ltimes(\gamma, \gamma'_\diamond) \times O^n_q(\gamma, \gamma') \times T_\rtimes(\gamma', \gamma'_\bullet),$$
the T-whiskering $(\mathfrak{v}_\diamond \ltimes \sigma \rtimes \mathfrak{v}_\bullet)^A_q$ is a node of spines that simply frames the node of spines σ along the pair of sev's $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ at rank q;
6) **(conjugation i)** every 3-tuple of the form
$$(\mathfrak{v}_\diamond, \sigma, \mathfrak{v}_\bullet) \in T_\ltimes(\gamma, \gamma'_\diamond) \times O^n_k(\gamma, \gamma') \times T_\rtimes(\gamma', \gamma'_\bullet)$$
where $0 \leq q \leq n$ and alliance of nodes of spines in A_n of the form
$$\mathfrak{a} : (\mathfrak{v}_\diamond \ltimes \sigma \rtimes \mathfrak{v}_\bullet)^A_q \rightsquigarrow \widehat{\sigma}$$

is associated with a structure of convergent conjugation of nodes of spines $(\sigma, \mathfrak{a}, \overline{\sigma})$ in $\mathcal{C}$ such that the canonical alliance of nodes of spines $\mathfrak{all}_0(\sigma, \mathfrak{a}, \overline{\sigma})$ defined in section 3.3.8.1 – which must be of the from $\sigma \rightsquigarrow \sigma_\sharp$ – belongs to A_n.

Remark 4.74. Recall that for every pair of integers q and n such that $0 \leq q \leq n-1$ and pair of arrows $\gamma, \gamma' \in S_q^n$, the object-class $\partial O_q^n(\gamma, \gamma')$ is a class of spines $(p_k) \cdot \beta$ of degree $n-1$ in $\mathrm{Obj}_{\mathrm{L}}(E_n)$ whose prevertebra p_q is of the form $\|\gamma, \gamma'\|$.

7) **(framing ii)** for every pair of integers q and n such that $0 \leq q \leq n-1$ and 3-tuple of the form

$$(\mathfrak{v}_\diamond, s, \mathfrak{v}_\bullet) \in T_\ltimes(\gamma, \gamma'_\diamond) \times \partial O_q^n(\gamma, \gamma') \times T_\rtimes(\gamma', \gamma'_\bullet),$$

the T-whiskering $(\mathfrak{v}_\diamond \ltimes s \rtimes \mathfrak{v}_\bullet)_q^A$ is a spine that simply frames the spine s along the pair of sev's $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$ at rank q;

8) **(conjugation ii)** every T-whiskering 3-tuple

$$(\mathfrak{v}_\diamond, s, \mathfrak{v}_\bullet) \in T_\ltimes(\gamma, \gamma'_\diamond) \times \partial O_k^n(\gamma, \gamma') \times T_\rtimes(\gamma', \gamma'_\bullet)$$

where $0 \leq q \leq n-1$ and extended nodes of spines in E_n of the form

$$\varsigma : (\mathfrak{v}_\diamond \ltimes s \rtimes \mathfrak{v}_\bullet)_q^A \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}$$

is associated with a structure of convergent extended conjugation of nodes of spines $(s, \varsigma, \overline{\sigma})$ in $\mathcal{C}$ such that the closure of the conjugation – which must be of the form $s \overset{\mathrm{EX}}{\rightsquigarrow} \sigma_\sharp$ – belongs to E_n.

Such a structure will later be denoted as a 4-tuple $(\mathcal{C}, A, E, T)$ where the symbol T will stand for the pair of graphs $(T_\ltimes, T_\rtimes)$.

4.4.4.2. *Zoo associated with a spinal category.* The zoo of a spinal category is, by definition, the same as that of its underlying vertebral category. The next proposition will concern a spinal category of the same form as that defined in section 4.4.4.1.

Proposition 4.75. *Let f and g be two morphisms such that $f \circ g$ exists. If g is a surtraction and $f \circ g$ is an intraction, then f is an intraction.*

Proof. Let ν be a node of vertebrae in A. We are going to show that there exists an alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \overline{\nu}$ in A for which f is an intraction. By assumption, there exists a non-negative integer n and a node of spines σ_0 in $\mathrm{Obj}(A_n)$ such that the identity $\tau^n(\sigma_0) = \nu$ holds. Suppose that σ_0 is of the form $P_0 \cdot \Omega_0$ with $P_0 = (p_k^0)$ and $p_k^0 = \|\gamma_k^0, \gamma_k'^0\|$. By definition of the cograded graph (s_-^n, t_-^n), the arrows γ_0^0 and $\gamma_0'^0$ must belong to S_0^n. It follows from item 3) of section 4.4.4.1 and Proposition 4.72 that there must exist two semi-extended vertebrae $\mathfrak{v}_\diamond^0 : \gamma_0^0 \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond^0$ and $\mathfrak{v}_\bullet^0 : \gamma_0'^0 \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet^0$ in T for which g is a surtraction and such that the coseeds $\eta'(v_\diamond^0)$ and $\eta'(v_\diamond^0)$ are in S_0^n. The T-whiskering operation

$$(\_ \ltimes \_ \rtimes \_)_0^A : T_\ltimes(\gamma_0^0, \eta'(v_\diamond^0)) \times O_0^n(\gamma_0^0, \gamma_0'^0) \times T_\rtimes(\gamma_0'^0, \eta'(v_\bullet^0)) \to O_0^n(\eta'(v_\diamond^0), \eta'(v_\bullet^0))$$

then provides a node of spines $\sigma_1 := (\mathfrak{v}_\diamond^0 \ltimes \sigma_0 \rtimes \mathfrak{v}_\bullet^0)_0^A$ that simply frames the node of spines σ_0 along $\mathfrak{v}_\diamond^0$ and $\mathfrak{v}_\bullet^0$. We may now repeat the above operation by induction as follows. Let σ_i be a node of spines of the form $P_i \cdot \Omega_i$ with $P_i = (p_k^i)$ and $p_k^i = \|\gamma_k^i, \gamma_k'^i\|$. By item 3) of section 4.4.4.1 and Proposition 4.72, there exist two semi-extended vertebrae $\mathfrak{v}_\diamond^i : \gamma_i^i \overset{\mathrm{ex}}{\rightsquigarrow} v_\diamond^i$ and $\mathfrak{v}_\bullet^i : \gamma_i'^i \overset{\mathrm{ex}}{\rightsquigarrow} v_\bullet^i$ for which g is a surtraction and such that the coseeds $\eta'(v_\diamond^i)$ and $\eta'(v_\diamond^i)$ are in S_i^n. The T-whiskering operation

$$(\_ \ltimes \_ \rtimes \_)_i^A : T_\ltimes(\gamma_i^i, \eta'(v_\diamond^i)) \times O_i^n(\gamma_i^i, \gamma_i'^i) \times T_\rtimes(\gamma_i'^i, \eta'(v_\bullet^i)) \to O_i^n(\eta'(v_\diamond^i), \eta'(v_\bullet^i))$$

then provides a node of spines $\sigma_{i+1} := (\mathfrak{v}_\diamond^i \ltimes \sigma_i \rtimes \mathfrak{v}_\bullet^i)_i^A$ that simply frames the node of spines σ_i along $\mathfrak{v}_\diamond^i$ and $\mathfrak{v}_\bullet^i$. Let this operation be repeated until $i = n$. We thus obtain a sequence of simple framings of nodes of spines as follows.

$$(\sigma_0, \mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet) \quad \triangleright^V_0 \quad (\sigma_1, \mathfrak{v}^1_\diamond, \mathfrak{v}^1_\bullet) \quad \triangleright^V_1 \cdots \triangleright^V_{n-1} \quad (\sigma_n, \mathfrak{v}^n_\diamond, \mathfrak{v}^n_\bullet) \quad \triangleright^V_n \quad \sigma_{n+1}$$

By construction, the morphism g is a surtraction for its tubular shell. Also, by definition, the node of spines σ_{n+1} is in $\mathrm{Obj}(A_n)$, which implies that $\tau^n(\sigma_{n+1})$ is a node of vertebrae in A. By assumption on $f \circ g$, there exists an alliance of nodes of vertebrae $\mathfrak{a}_{n+1} : \tau^n(\sigma_{n+1}) \rightsquigarrow \widehat{\nu}_{n+1}$ in A for which $f \circ g$ is an intraction. Because the local A-echelon E is regular, the alliance $\mathfrak{a}_{n+1} : \tau^n(\sigma_{n+1}) \rightsquigarrow \widehat{\nu}_{n+1}$ may be lifted to an alliance of nodes of spines $\mathfrak{b}_{n+1} : \sigma_{n+1} \rightsquigarrow \widehat{\sigma}_{n+1}$ in A_n along $\tau^n : A_n \to A$. By definition, the morphism $f \circ g$ is an intraction for the alliance of nodes of spines $\mathfrak{b}_{n+1}$. By item 6) of section 4.4.4.1, the existence of such an alliance implies the existence of a convergent conjugation of nodes of spines $\chi_n := (\sigma_n, \mathfrak{b}_{n+1}, \overline{\sigma}_n)$ in $\mathcal{C}$ such that the canonical alliance of nodes of spines $\mathfrak{b}_n := \mathfrak{all}_0(\chi_n) : \sigma_n \rightsquigarrow \widehat{\sigma}_n$ belongs to A_n. Because we again obtain an alliance $\mathfrak{b}_n : \sigma_n \rightsquigarrow \widehat{\sigma}_n$ in A_n, we may keep doing the above process so that we obtain a sequence of convergent conjugations χ_0, χ_1, …, χ_n where χ_i is an i-conjugation of the form

$$(\sigma_i, \mathfrak{b}_{i+1}, \overline{\sigma}_i)$$

wherein $\mathfrak{b}_{i+1}$ is of the form $\sigma_{i+1} \rightsquigarrow \widehat{\sigma}_{i+1}$ for every $0 \leq i \leq n$ and $\mathfrak{b}_i = \mathfrak{all}_0(\chi_i)$ for every $1 \leq i \leq n+1$. By item 2) of section 4.4.4.1, the node of spines σ_0 is g-projective, so it follows from Theorem 3.106 that f is an intraction for the alliance $\mathfrak{b}_0 := \mathfrak{all}_0(\chi_0) : \sigma_0 \rightsquigarrow \widehat{\sigma}_0$, which is in A_n. This is also equivalent to saying that f is an intraction for the alliance of nodes of vertebrae $\tau^n(\mathfrak{b}_0) : \nu \rightsquigarrow \tau^n(\widehat{\sigma}_0)$, which belongs to A. □

Proposition 4.76. *Let f and g be two morphisms such that $f \circ g$ exists. If g is a surtraction and $f \circ g$ is an surtraction, then f is a surtraction.*

Proof. Let γ be an E-seed. We are going to show that there exists an extended nodes of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in E for which f is an surtraction. By assumption, there exists a non-negative integer n and an element s_0 in $\mathrm{Obj}_{\mathrm{L}}(E_n)$ such that $\pi^n(s_0) = \gamma$. If $n = 0$, then s_0 is an object of $\mathbf{Com}(\mathcal{C})$ and equal to the arrow γ itself. It follows from item 2) of section 4.4.4.1 that the domain of γ is g-projective. Because the composite $f \circ g$ is a surtraction, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \overline{\nu}$ in E for which $f \circ g$ is a surtraction. In particular, the extended node of vertebrae $\mathfrak{n}$ is g-projective. It then follows from Proposition 3.24 that f is a surtraction for $\mathfrak{n}$. The above reasoning shows that f is a surtraction in $\hat{\mathcal{C}}$. Now, suppose that $n > 0$. In this case, the lifting $\pi^n(s_0) = \gamma$ provides a spine s_0 of degree $n-1$ of the form $P_0 \cdot \beta_0$ with $P_0 = (p^0_k)$ and $p^0_k = \|\gamma^0_k, \gamma'^0_k\|$. By definition of the cograded graph $\partial(s^n_{\_}, t^n_{\_})$, the arrows γ^0_0 and γ'^0_0 must belong to S^n_0. It follows from item 3) of section 4.4.4.1 and Proposition 4.72 that there must exist two semi-extended vertebrae $\mathfrak{v}^0_\diamond : \gamma^0_0 \overset{\mathrm{ex}}{\rightsquigarrow} v^0_\diamond$ and $\mathfrak{v}^0_\bullet : \gamma'^0_0 \overset{\mathrm{ex}}{\rightsquigarrow} v^0_\bullet$ in T for which g is a surtraction and such that the coseeds $\eta'(v^0_\diamond)$ and $\eta'(v^0_\diamond)$ are in S^n_0. The T-whiskering operation

$$(\_ \ltimes \_ \rtimes \_)^E_0 : T(\gamma^0_0, \eta'(v^0_\diamond)) \times \partial O^n_0(\gamma^0_0, \gamma'^0_0) \times T(\gamma'^0_0, \eta'(v^0_\bullet)) \to O^n_0(\eta'(v^0_\diamond), \eta'(v^0_\bullet))$$

then provides a spine $s_1 := (\mathfrak{v}^0_\diamond \ltimes s_0 \rtimes \mathfrak{v}^0_\bullet)^E_0$ that simply frames the spine s_0 along $\mathfrak{v}^0_\diamond$ and $\mathfrak{v}^0_\bullet$. We may now repeat the above operation by induction as follows. Let s_i be a spine of the form $P_i \cdot \Omega_i$ with $P_i = (p^i_k)$ and $p^i_k = \|\gamma^i_k, \gamma'^i_k\|$. By item 3) of section 4.4.4.1 and Proposition 4.72, there exist two semi-extended vertebrae $\mathfrak{v}^i_\diamond : \gamma^i_i \overset{\mathrm{ex}}{\rightsquigarrow} v^i_\diamond$ and $\mathfrak{v}^i_\bullet : \gamma'^i_i \overset{\mathrm{ex}}{\rightsquigarrow} v^i_\bullet$ for which g is a surtraction and such that the coseeds $\eta'(v^i_\diamond)$ and $\eta'(v^i_\diamond)$ are in S^n_i. The T-whiskering operation

$$(\_ \ltimes \_ \rtimes \_)^E_i : T(\gamma^i_i, \eta'(v^i_\diamond)) \times \partial O^n_i(\gamma^i_i, \gamma'^i_i) \times T(\gamma'^i_i, \eta'(v^i_\bullet)) \to O^n_i(\eta'(v^i_\diamond), \eta'(v^i_\bullet))$$

then provides a spine $s_{i+1} := (\mathfrak{v}^i_\diamond \ltimes s_i \rtimes \mathfrak{v}^i_\bullet)^E_i$ that simply frames the node of spines s_i along $\mathfrak{v}^i_\diamond$ and $\mathfrak{v}^i_\bullet$. Let this operation be repeated until $i = n$. We thus obtain a sequence of simple framings of nodes of spines as follows.

$$(s_0, \mathfrak{v}^0_\diamond, \mathfrak{v}^0_\bullet) \quad \triangleright^V_0 \quad (s_1, \mathfrak{v}^1_\diamond, \mathfrak{v}^1_\bullet) \quad \triangleright^V_1 \cdots \triangleright^V_{n-1} \quad (s_n, \mathfrak{v}^n_\diamond, \mathfrak{v}^n_\bullet) \quad \triangleright^V_n \quad s_{n+1}$$

By construction, the morphism g is a surtraction for its tubular shell. Also, by definition, the node of spines s_{n+1} is in $\mathrm{Obj}_{\mathrm{L}}(E_n)$, which implies that $\pi^n(s_{n+1})$ is an element in $\mathrm{Obj}_{\mathrm{L}}(E)$, namely an object of $\mathbf{Com}(\mathcal{C})$. It follows that there exists an extended node of vertebrae $\mathfrak{n}_{n+1} : \pi^n(s_{n+1}) \overset{\mathrm{ex}}{\rightsquigarrow} \widehat{\nu}_{n+1}$ in E for which $f \circ g$ is an intraction. Because the local A-echelon E is regular, the extended node of vertebrae $\mathfrak{n}_{n+1} : \pi^n(s_{n+1}) \overset{\mathrm{ex}}{\rightsquigarrow} \widehat{\nu}_{n+1}$ may be lifted to an extended node of spines $\varsigma_{n+1} : s_{n+1} \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}_{n+1}$ in E_n along $\pi^n : E_n \to E$. By definition, the morphism $f \circ g$ is an intraction for the extended node of spines ς_{n+1}. By item 8) of section 4.4.4.1, the existence of such a ς_{n+1} implies the existence of a convergent extended conjugation of nodes of spines $\chi_n := (s_n, \varsigma_{n+1}, \overline{\sigma}_n)$ in $\mathcal{C}$ such that the closure of the underlying chaining of the convergent conjugation χ_n, say $\varsigma_n : s_n \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}_n$, belongs to E_n. Because we again obtain an extended node of spines $\varsigma_n : s_n \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}_n$ in E_n, we may keep doing the above process so that we obtain a sequence of convergent extended conjugations $\chi_0, \chi_1, \dots, \chi_n$ where χ_i is an i-conjugation of the form

$$(s_i, \varsigma_{i+1}, \overline{\sigma}_i)$$

wherein ς_{i+1} is of the form $s_{i+1} \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}_{i+1}$ for every $0 \leq i \leq n$ and ς_i is the closure of the underlying chaining of the convergent conjugation χ_i for every $1 \leq i \leq n+1$. By item 2) of section 4.4.4.1, the spine s_0 is g-projective, so it follows from Theorem 3.111 that f is a surtraction for the extended node of spines $\varsigma_0 : s_0 \overset{\mathrm{EX}}{\rightsquigarrow} \widehat{\sigma}_0$, which is in E_n. This is also equivalent to saying that f is an surtraction for the extended nodes of vertebrae $\pi^n(\varsigma_0) : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \pi^n(\widehat{\sigma}_0)$, which belongs to E. □

Theorem 4.77 (two-out-of-six property)**.** *The class of weak equivalences satisfies the two-out-of-six property, that is to say: Let f, g and h be morphisms such that the composite $f \circ g \circ h$ exists in $\mathcal{C}$. If $f \circ g$ and $g \circ h$ are weak equivalences, then f, g, h and $f \circ g \circ h$ are weak equivalences.*

Proof. A part of the theorem is proven by Theorem 4.53. It remains to prove that g is a surtraction and f is a weak equivalence. Since h is a surtraction and $g \circ h$ is a surtraction, it follows from Proposition 4.76 that g is a surtraction. Finally, since $f \circ g$ is a weak equivalence and g is a surtraction, it follows from Proposition 4.75 and Proposition 4.76 that f is a weak equivalence. □

4.4.4.3. *Refined spinal categories.* A spinal category will be said to be *refined* (resp. *strongly refined*) if its underlying vertebral category is. It follows from the previous sections that the following properties hold in any refined spinal category.

- S0 Weak equivalences, fibrations and cofibrations form coherent $\mathcal{C}$-classes;
- S1 Let f, g and h be morphisms such that the composite $f \circ g \circ h$ exists in $\mathcal{C}$. If $f \circ g$ and $g \circ h$ are weak equivalences, then so are f, g, h and $f \circ g \circ h$;
- S2 Weak equivalences, fibrations and cofibrations are stable under retracts;
- S3 Every cofibration has the llp with respect every acyclic fibration;

Axiom S0 is provided by Proposition 4.37 and Proposition 4.35. Axiom S1 is nothing but Theorem 4.77. Axiom S2 is given by Proposition 4.39 and axiom S3 comes from the refinement hypothesis and Proposition 4.54.

4.4.4.4. *Discrete spinal categories.* A spinal category will be said to be *discrete* if its underlying vertebral category is discrete.

4.4.5. Functors of systems of vertebrae.

4.4.5.1. *Vertebrae as functors.* Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and F a functor $\mathcal{C} \to \mathcal{D}$. In Chapter 2, a vertebra v in $\mathcal{C}$ could be seen as a model of the sketch $\mathtt{Vert}$ in $\mathcal{C}$, that is to say in terms of a functor $v : \mathtt{Vert} \to \mathcal{C}$ preserving the chosen colimits of $\mathtt{Vert}$. It therefore makes sense to consider the composite functor $F \circ v : \mathtt{Vert} \to \mathcal{D}$, which defines a vertebra in $\mathcal{D}$ if it preserves the chosen colimit of the sketch $\mathtt{Vert}$. The functor will be said to *send the vertebra v to $\mathcal{D}$* if the functor $F \circ v$ defines a vertebrae in $\mathcal{D}$. The resulting vertebra in $\mathcal{D}$ will then be denoted by $F(v)$. Let ν be a node of vertebrae in $\mathcal{C}$. The functor $F : \mathcal{C} \to \mathcal{D}$ will be said to *send the node of vertebrae ν to $\mathcal{D}$* if it sends every vertebra contained in ν to $\mathcal{D}$. The class made of the image vertebrae $F \circ v$ for every $v \in \nu$ then forms a node of vertebrae in $\mathcal{D}$, which will be denoted as $F(\nu)$.

Proposition 4.78. *Let ν be a node of vertebrae in $\mathcal{C}$ that is sent to $\mathcal{D}$ via the functor F. If ν is reflexive, then so is $F(\nu)$.*

Proof. Note that the definition of a reflexive node of vertebrae (see Chapter 2) only requires the commutativity of certain diagrams, which is preserved by functoriality. $\square$

Let $\mathfrak{a} := (\varkappa, \varrho, \varrho', \varkappa', \phi, u) : \nu \rightsquigarrow \overline{\nu}$ be an alliance of nodes of vertebrae in $\mathcal{C}$ whose domain and codomain are sent to $\mathcal{D}$ by the functor F. The functor $F : \mathcal{C} \to \mathcal{D}$ will be said to *send the alliance $\mathfrak{a}$ to $\mathcal{D}$* if the mapping $F(\beta) \mapsto F(\phi(\beta))$ defines a metafunction from the class of stems of $F(\nu)$ to that of $F(\overline{\nu})$. In this case, the resulting metafunction will be denoted by ϕ_F. By functoriality, it follows that if F sends the alliance $\mathfrak{a}$ to $\mathcal{D}$, then the following alliance of node of vertebrae exists in $\mathcal{D}$.

$$(F(\varkappa), F(\varrho), F(\varrho'), F(\varkappa'), \phi_F, u) : F(\nu) \rightsquigarrow F(\overline{\nu})$$

This alliance in $\mathcal{D}$ will later be denoted as $F(\mathfrak{a})$.

Remark 4.79. In the above situation, the identities $\phi_F(F(\beta)) = F(\phi(\beta))$ and $F(\phi_F(\beta')) = \phi(F^{-1}(\beta'))$ hold for every stem β of ν and β' of $F(\nu)$, respectively.

Similarly, let $\mathfrak{n} := (\varkappa, \varrho) : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ be an extended node of vertebrae in $\mathcal{C}$. The functor F will be said to *send $\mathfrak{n}$ to $\mathcal{D}$* if it sends $\overline{\nu}$ to $\mathcal{D}$. In this case, the functor F gives rise to an extended node of vertebrae $(F(\varkappa), F(\varrho)) : F(\gamma) \overset{\text{ex}}{\rightsquigarrow} F(\overline{\nu})$ in $\mathcal{D}$. This extended node of vertebrae in $\mathcal{D}$ will be denoted by $F(\mathfrak{n})$.

4.4.5.2. *Functors of systems of vertebrae.* Let $\mathcal{C}$ and $\mathcal{D}$ be two categories equipped with systems of vertebrae $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$ and $\hat{\mathcal{D}} := (\mathcal{D}, B, B', F)$. A *functor of systems of vertebrae* from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$ is a functor of categories $G : \mathcal{C} \to \mathcal{D}$ that sends

1) all arrows in $\mathrm{Obj}_{\mathrm{L}}(E)$ to arrows in $\mathrm{Obj}_{\mathrm{L}}(F)$ via a metafunction

$$[G]_{\mathrm{L}} : \left(\begin{array}{lll} \mathrm{Obj}_{\mathrm{L}}(E) & \to & \mathrm{Obj}_{\mathrm{L}}(F) \\ \gamma & \mapsto & G(\gamma) \end{array}\right);$$

2) all nodes of vertebrae in $\mathrm{Obj}_{\mathrm{R}}(E)$ to $\mathrm{Obj}_{\mathrm{R}}(F)$ via a metafunction

$$[G]_{\mathrm{R}} : \left(\begin{array}{lll} \mathrm{Obj}_{\mathrm{R}}(E) & \to & \mathrm{Obj}_{\mathrm{R}}(F) \\ \nu & \mapsto & G(\nu) \end{array}\right);$$

3) all extended nodes of vertebrae in E to F via functions

$$[G]_{\mathrm{H}} : \left(\begin{array}{lll} E(\nu, \overline{\nu}) & \to & F(G(\gamma), G(\overline{\nu})) \\ \mathfrak{n} & \mapsto & G(\mathfrak{n}) \end{array}\right).$$

4) all nodes of vertebrae in $\mathrm{Obj}(A)$ to $\mathrm{Obj}(B)$ via

$$\langle G \rangle_{\mathrm{R}} : \left(\begin{array}{lll} \mathrm{Obj}(A) & \to & \mathrm{Obj}(B) \\ \nu & \mapsto & G(\nu) \end{array}\right);$$

5) all alliances of nodes of vertebrae in A to B functions

$$\langle G\rangle_{\mathrm{H}} : \left(\begin{array}{lll} A(\nu,\overline{\nu}) & \to & B(G(\nu),G(\overline{\nu})) \\ \mathfrak{a} & \mapsto & G(\mathfrak{a}) \end{array}\right);$$

6) all nodes of vertebrae in $\mathrm{Obj}(A')$ to $\mathrm{Obj}(B')$ a metafunction

$$\langle G\rangle'_{\mathrm{R}} : \left(\begin{array}{lll} \mathrm{Obj}(A') & \to & \mathrm{Obj}(B') \\ \nu & \mapsto & G(\nu) \end{array}\right);$$

7) all alliances of nodes of vertebrae in A' to B' via functions

$$\langle G\rangle'_{\mathrm{H}} : \left(\begin{array}{lll} A'(\nu,\overline{\nu}) & \to & B'(G(\nu),G(\overline{\nu})) \\ \mathfrak{a} & \mapsto & G(\mathfrak{a}) \end{array}\right);$$

such that the triple $([G]_{\mathrm{L}},[G]_{\mathrm{R}},[G]_{\mathrm{H}})$ equipped with the pairs $(\langle G\rangle_{\mathrm{R}},\langle G\rangle_{\mathrm{H}})$ and $(\langle G\rangle'_{\mathrm{R}},\langle G\rangle'_{\mathrm{H}})$ define two morphisms of precompasses over graphs $(A,E,\mathfrak{y}) \Rightarrow (B,F,\mathfrak{y})$ and $(A',E,\mathfrak{y}) \Rightarrow (B',F,\mathfrak{y})$, respectively. A functor of systems of vertebrae such as above will be said to be

- *smooth* if its components $[G]_{\mathrm{L}}$, $\langle G\rangle_{\mathrm{R}}$ and $\langle G\rangle'_{\mathrm{R}}$ are injective;
- *0-regular* if its components $[G]_{\mathrm{L}}$, $\langle G\rangle_{\mathrm{R}}$ and $\langle G\rangle'_{\mathrm{R}}$ are surjective;
- *1-regular* if the morphisms of spans $([G]_{\mathrm{L}},[G]_{\mathrm{R}},[G]_{\mathrm{H}}) : E \Rightarrow F$, $(\langle G\rangle_{\mathrm{R}},\langle G\rangle_{\mathrm{H}}) : A \Rightarrow B$ and $(\langle G\rangle'_{\mathrm{R}},\langle G\rangle'_{\mathrm{H}}) : A' \Rightarrow B'$ are fibrations of spans.
- *pseudo-1-regular* if the morphisms of spans $(\langle G\rangle_{\mathrm{R}},\langle G\rangle_{\mathrm{H}}) : A \Rightarrow B$ and $(\langle G\rangle'_{\mathrm{R}},\langle G\rangle'_{\mathrm{H}}) : A' \Rightarrow B'$ are fibrations of spans.

4.4.5.3. *Transfers.* Let $\hat{\mathcal{C}} := (\mathcal{C},A,A',E)$ be a system of vertebrae and $\mathcal{D}$ be some other category. A *transfer of structure from $\hat{\mathcal{C}}$ to $\mathcal{D}$* is a functor of categories $F : \mathcal{C} \to \mathcal{D}$ that sends

1) all nodes of vertebrae of A, A' and E to $\mathcal{D}$;
2) all alliances of nodes of vertebrae of A and A' to $\mathcal{D}$;
3) all extended nodes of vertebrae of E to $\mathcal{D}$.

A transfer $F : \mathcal{C} \to \mathcal{D}$ as defined above induces a structure of system of vertebrae for $\mathcal{D}$ given by the image of F in $\mathcal{D}$. Precisely, the system of vertebrae is given by a triple $(F(A),F(A'),F(E))$ where

i) $F(A)$ is the subgraph of $\mathbf{Ally}(\mathcal{D})$ whose object-class is the image of $\mathrm{Obj}(A)$ via F and whose hom-classes

$$F(A)(F(\nu),F(\nu_*))$$

are the unions of the images of all hom-classes of $A(\nu',\nu'_*)$ where ν' and ν'_* are those nodes of vertebrae sent to $F(\nu)$ and $F(\nu_*)$ via F, respectively;

ii) $F(A')$ is the subgraph of $\mathbf{Ally}(\mathcal{D})$ defined as above, but with respect to A'.

iii) $F(E)$ is the subspan of $\mathbf{Enov}(\mathcal{D})$ whose left and right object-classes are the images of $\mathrm{Obj}_{\mathrm{L}}(E)$ and $\mathrm{Obj}_{\mathrm{R}}(E)$ via F and whose hom-classes

$$F(E)(F(\gamma),F(\nu_*))$$

are the unions of the images of all hom-classes of $E(\gamma',\nu'_*)$ such that γ' and ν'_* are sent to $F(\gamma)$ and $F(\nu_*)$ via F, respectively;

The preceding system of vertebrae category will be denoted as $F(\hat{\mathcal{C}})$.

Proposition 4.80. *Let $\hat{\mathcal{C}} := (\mathcal{C},A,A',E)$ be a system of vertebrae. Suppose that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a transfer of structure from $\hat{\mathcal{C}}$ to $\mathcal{D}$. The functor F induces a 0-regular functor of systems from $\hat{\mathcal{C}}$ to $F(\hat{\mathcal{C}})$. If this functor is smooth, then it is 1-regular. If $\hat{\mathcal{C}}$ is discrete, then F is pseudo-1-regular.*

Proof. The fact that F induces a functor of systems of vertebrae is straightforward. To see that F is 0-regular, it suffices to observe that any $F(E)$-seed in $F(\hat{\mathcal{C}})$ is of the form $F(\gamma)$ where γ is an E-seed in $\hat{\mathcal{C}}$ by definition. Similarly, any node of vertebrae $F(E)$ is of the form $F(\nu)$ where ν is a node of vertebrae in E.

Now, suppose that F is smooth. For every E-seed γ and extended node of vertebrae in $\mathfrak{n} : F(\gamma) \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}$ in $F(E)$, the definition of $F(\hat{\mathcal{C}})$ implies that there exists an extended node of vertebrae $\mathfrak{n}_* : \gamma_* \overset{\text{ex}}{\rightsquigarrow} \overline{\nu}_*$ in E such that the equality $F(\mathfrak{n}_*) = \mathfrak{n}$ holds. By smoothness, the seeds γ and γ_* must be equal, which shows that $([G]_{\mathrm{L}}, [G]_{\mathrm{R}}, [G]_{\mathrm{H}}) : E \Rightarrow F$ is a fibration of spans. Similarly, for every node of vertebrae ν and alliance of nodes of vertebrae in $\mathfrak{a} : F(\nu) \rightsquigarrow \overline{\nu}$ in $F(A)$, the definition of $F(\hat{\mathcal{C}})$ implies that there exists an alliance of nodes of vertebrae $\mathfrak{a}_* : \nu_* \rightsquigarrow \overline{\nu}_*$ in A such that the equality $F(\mathfrak{a}_*) = \mathfrak{a}$ holds. By smoothness, the seeds ν and ν_* must be equal. A similar argument for the graphs A' and $F(A')$ finally shows that the functor $F : \mathcal{C} \to \mathcal{D}$ is 1-regular.

If $\hat{\mathcal{C}}$ is discrete, then the alliances of A and A' are identities. The pseudo-1-regularity is therefore straightforward. □

Example 4.81. Let $(\mathcal{C}, A, E, T)$ be a discrete spinal category whose underlying system of vertebrae will be denoted by $\hat{\mathcal{C}}$ and D be a small category for which the functor $\mathrm{Lan}_d : \mathcal{C} \to \mathcal{C}^D$ exists for some object d in D (see Example 1.22). As a left adjoint, the functor Lan_d preserves all colimits of $\mathcal{C}$ and *a fortiori* all vertebrae in $\mathcal{C}$. Because $(\mathcal{C}, A, E, T)$ is discrete, the functor $\mathrm{Lan}_d : \mathcal{C} \to \mathcal{C}^D$ induces a transfer of structure from $\hat{\mathcal{C}}$ to $\mathcal{C}^D$. The functor Lan_d therefore provides a system of vertebrae $\mathrm{Lan}_d(\hat{\mathcal{C}})$ for the category $\mathcal{C}^D$. Proposition 4.80 then states that Lan_d induces a 0-regular functor of systems of vertebrae from $\hat{\mathcal{C}}$ to $\mathrm{Lan}_d(\hat{\mathcal{C}})$. Because $\hat{\mathcal{C}}$ is discrete, the functor Lan_d is pseudo-1-regular.

Example 4.82. Let $\mathtt{K}$ be a colimit sketch. Even if $\mathtt{K}$ does not have a structure of a spinal category, it may happen that the free completion $\mathtt{K}^{\triangledown}$ of $\mathtt{K}$ has one, say given by $(\mathtt{K}^{\triangledown}, A, E, T)$, whose vertebrae are defined over pushouts produced by either the free cocompletion or the chosen colimits of $\mathtt{K}$. Denote by $\hat{\mathtt{K}}^{\triangledown}$ the underlying system of vertebrae of $(\mathtt{K}^{\triangledown}, A, E, T)$. As a functor preserving the chosen colimits of $\mathtt{K}$ and those resulting from the free cocompletion, the free extension $\mathtt{K}[\_] : \mathtt{K}^{\triangledown} \to \mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$ (see Example 1.20) defines a transfer of structure from $\hat{\mathtt{K}}^{\triangledown}$ to $\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$. Proposition 4.80 then states that $\mathtt{K}[\_]$ induces a 0-regular functor of systems of vertebrae from $\hat{\mathtt{K}}^{\triangledown}$ to $\mathtt{K}[\hat{\mathtt{K}}^{\triangledown}]$. It also follows from the Yoneda Lemma that the functor $\mathtt{K}[\_]$ is smooth (by fully faithfulness of $X \mapsto \mathtt{K}(\_, X)$ and definition of the free completion $\mathtt{K}^{\triangledown}$) and hence 1-regular by Proposition 4.80.

4.4.5.4. *Contravariant action on vertebrae.* Let $\mathcal{C}$ be a category, $\mathcal{E}$ be a metacategory and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ be a functor. Any vertebra v in $\mathcal{C}$ of the form given below on the left is sent via F to an object in $\mathcal{E}^{\mathtt{Vert}^{\mathrm{op}}}$ of the form given on the right.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] & \\
\mathbb{D}_2 \arrow[r, "\delta_2"'] & \mathbb{S}' \arrow[r, "\beta"'] \arrow[ul, phantom, "\ulcorner", very near start] & \mathbb{D}'
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
F(\mathbb{S}) & F(\mathbb{D}_1) \arrow[l, "F(\gamma')"'] & \\
F(\mathbb{D}_2) \arrow[u, "F(\gamma')"] & F(\mathbb{S}') \arrow[l, "F(\delta_2)"] \arrow[u, "F(\delta_1)"'] & F(\mathbb{D}') \arrow[l, "F(\beta)"']
\end{tikzcd}
$$

The above right diagram will be denoted as $F(\ell v)$ where ℓ means that v is first sent to $\mathcal{C}^{\mathrm{op}}$ by reversing the direction of the arrows. Similarly, any alliance of vertebrae $\mathfrak{a} : v \rightsquigarrow \overline{v}$ in $\mathcal{C}$ leads to a morphism $F(\ell\mathfrak{a}) : F(\ell v) \Rightarrow F(\ell\overline{v})$ in $\mathcal{E}^{\mathtt{Vert}^{\mathrm{op}}}$. Note that the order of source and target are not reversed as the components of an alliance are defined in $\mathcal{C}^{\mathrm{op}}$. In the same way, any extended vertebra $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \overline{v}$ in $\mathcal{C}$ is sent to an obvious diagram, but this diagram does not live in $\mathcal{E}^{\mathtt{Vert}^{\mathrm{op}}}$. More specifically, the diagram consists of the diagram

$F(\ell\mathbf{seed}(\mathfrak{n})) : F(\ell\gamma) \Rightarrow F(\ell\overline{\gamma})$, which lives in the arrow category of $\mathcal{E}$, as well as the diagram $F(\ell\overline{v})$, which lives in $\mathcal{E}^{\mathtt{Vert}^{\mathrm{op}}}$.

4.4.5.5. *Congruent presheaves over alliances of nodes of vertebrae.* Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $\mathcal{E}$ be a metacategory and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ and $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ two functors. Consider two alliances of nodes of vertebrae $\mathfrak{a} : p \cdot \Omega \rightsquigarrow p_* \cdot \Omega_*$ and $\mathfrak{b} : p_\flat \cdot \Omega_\flat \rightsquigarrow p_\dagger \cdot \Omega_\dagger$ in $\mathcal{D}$ and $\mathcal{C}$, respectively. The respective components of $\mathfrak{a}$ and $\mathfrak{b}$ for $\beta \in \Omega$ and $\beta_\flat \in \Omega_\flat$ will be denoted by $\mathfrak{a}_\beta : p \cdot \beta \rightsquigarrow p_* \cdot \varphi(\beta)$ and $\mathfrak{b}_{\beta_\flat} : p_\flat \cdot \beta_\flat \rightsquigarrow p_\dagger \cdot \varphi_\flat(\beta_\flat)$ where $\varphi : \Omega \to \Omega_*$ and $\varphi : \Omega_\flat \to \Omega_\dagger$ are the respective metafunctions associated with $\mathfrak{a}$ and $\mathfrak{b}$. The pair (F, G) will be said to be *congruent over the pair* $(\mathfrak{a}, \mathfrak{b})$ if it is equipped with

1) two surjective metafunctions $\psi_* : \Omega_* \to \Omega_\dagger$ and $\psi : \Omega \to \Omega_\flat$ making the following diagram commute;

$$\begin{tikzcd} \Omega \arrow[r, "\psi"] \arrow[d, "\varphi"'] & \Omega_\flat \arrow[d, "\varphi_\flat"] \\ \Omega_* \arrow[r, "\psi_*"] & \Omega_\dagger \end{tikzcd}$$

2) for every $\beta \in \Omega$ and $\beta_* \in \Omega_*$, two isomorphisms in $[\mathtt{Vert}^{\mathrm{op}}, \mathcal{E}]$ of the form

$$F(\ell p \cdot \beta) \cong G(\ell p_\flat \cdot \psi(\beta)) \qquad F(\ell p_* \cdot \beta_*) \cong G(\ell p_\dagger \cdot \psi_*(\beta_*))$$

making the following diagram commute for every stem $\beta \in \Omega$.

$$\begin{tikzcd} F(\ell p \cdot \beta) \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "F(\ell\mathfrak{a}_\beta)"'] & G(\ell p_\flat \cdot \psi(\beta)) \arrow[d, Rightarrow, "G(\ell\mathfrak{b}_{\psi(\beta)})"] \\ F(\ell p_* \cdot \varphi(\beta)) \arrow[r, Rightarrow, "\cong"'] & G(\ell p_\dagger \cdot (\psi_* \circ \varphi(\beta))) \end{tikzcd}$$

Such a congruence will later be denoted as $(\psi, \psi_*) \vdash F(\mathfrak{a}) \equiv G(\mathfrak{b})$ and said to lie in the category $\mathcal{E}$. It will usually not be needed to give names to the isomorphisms required in item 2).

4.4.5.6. *Congruent presheaves over extended nodes of vertebrae.* Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $\mathcal{E}$ be a metacategory and $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ and $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ two functors. Consider two extended nodes of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \Omega_*$ and $\mathfrak{m} : \gamma_\flat \overset{\mathrm{ex}}{\rightsquigarrow} p_\dagger \cdot \Omega_\dagger$ in $\mathcal{D}$ and $\mathcal{C}$, respectively. The respective components of $\mathfrak{n}$ and $\mathfrak{m}$ for $\beta_* \in \Omega_*$ and $\beta_\dagger \in \Omega_\dagger$ will be denoted by $\mathfrak{n}_\beta : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} p_* \cdot \beta_*$ and $\mathfrak{m}_{\beta_\dagger} : \gamma_\flat \overset{\mathrm{ex}}{\rightsquigarrow} p_\dagger \cdot \beta_\dagger$. The pair (F, G) will be said to be *congruent over the pair* $(\mathfrak{n}, \mathfrak{m})$ if it is equipped with

1) a surjective metafunction $\psi_* : \Omega_* \to \Omega_\dagger$;

2) for every $\beta_* \in \Omega_*$, an isomorphism in $[\mathtt{Vert}^{\mathrm{op}}, \mathcal{E}]$ as given below on the left such that its restriction on the seeds (i.e. $F(\ell\gamma_*) \cong G(\ell\gamma_\dagger)$) comes along with an isomorphism $F(\ell\gamma) \cong F(\ell\gamma_\flat)$ making the right diagram commute in $\mathcal{E}^2$.

$$F(\ell p_* \cdot \beta_*) \cong G(\ell p_\dagger \cdot \psi_*(\beta_*)) \qquad \Rightarrow \qquad \begin{tikzcd} F(\ell\gamma) \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "F(\ell\mathbf{seed}(\mathfrak{n}_\beta))"'] & G(\ell\gamma_\flat) \arrow[d, Rightarrow, "G(\ell\mathbf{seed}(\mathfrak{m}_{\psi(\beta)}))"] \\ F(\ell\gamma_*) \arrow[r, Rightarrow, "\cong"'] & G(\ell\gamma_\dagger) \end{tikzcd}$$

Such a congruence will later be denoted as $\psi_* \vdash F(\mathfrak{n}) \equiv G(\mathfrak{m})$ and said to lie in the category $\mathcal{E}$. It will usually not be needed to give names to the isomorphisms required in item 2).

4.4.5.7. *Covertebral functors.* Despite being long, the proofs of the results of the present section are quite automatic (and all very similar). Let $\hat{\mathcal{C}} := (\mathcal{C}, A, A', E)$ and $\hat{\mathcal{D}} := (\mathcal{D}, B, B', F)$ be two systems of vertebrae. A *covertebral functor* from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$ consists of a functor of categories $U : \mathcal{C} \to \mathcal{D}$ such that

1) for every node of vertebrae ν in A (resp. A'), there exists a node of vertebrae in $\nu_\flat$ in B (resp. B') such that for every alliance of nodes of vertebrae $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B (resp. B'), there exists an alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A (resp. A') for which there exists a congruence of presheaves in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ over the pair $(\mathfrak{b}, \mathfrak{a})$ as follows;
$$(\psi, \psi_*) \vdash \mathcal{D}(\mathfrak{b}, U(\_)) \equiv \mathcal{C}(\mathfrak{a}, \_)$$
2) for every E-seed γ, there exists an F-seed $\gamma_\flat$ such that for every extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ in E, there exist an extended node of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\dagger$ in F for which there exists a congruence of presheaves in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ over the pair $(\mathfrak{m}, \mathfrak{n})$ as follows;
$$\psi_* \vdash \mathcal{D}(\mathfrak{m}, U(\_)) \equiv \mathcal{C}(\mathfrak{n}, \_)$$

A *pseudo-covertebral functor* from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$ is a functor of categories $U : \mathcal{C} \to \mathcal{D}$ that only satisfies item 1).

Proposition 4.83. *A (pseudo-)covertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ reflects fibrations and trivial fibrations.*

Proof. Because the proof is very similar between the two classes of morphisms, only the case of fibrations will be discussed. Let $f : X \to Y$ be a morphism such that $U(f)$ is a fibration in the system of vertebrae $\hat{\mathcal{D}}$ and consider a node of vertebrae ν in A'. The goal is to find an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A' for which f is a fibration. By assumption on U, there exists a node of vertebrae $\nu_\flat$ in B' satisfying property 1) relative to ν. Since $U(f)$ is a fibration in $\hat{\mathcal{D}}$, there exists an alliance $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B' for which $U(f)$ is a fibration. Because U satisfies item 1), there must exist an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A' such that the following isomorphism of diagrams in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ holds for every $\beta_\flat \in \Omega_\flat$ (see section 4.4.5.5).
$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(\_)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, \_)$$
If the naturality of the earlier isomorphism is applied on $f : X \to Y$, this isomorphism leads to the following isomorphism of diagrams in **Set** for every $\beta_\flat \in \Omega_\flat$ (think of an isomorphism in dimension 4).
$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(f)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, f) \tag{4.9}$$
A restriction of isomorphism (4.9) on certain and obvious subcategories of its category of definition leads to the following pair of isomorphisms of diagrams (see definitions of section 4.2.2.4).
$$\begin{cases} \mathcal{D}(\mathbf{triv}(\mathfrak{b}_{\beta_\flat}), U(f))_0^A \cong \mathcal{C}(\mathbf{triv}(\mathfrak{a}_{\psi(\beta_\flat)}), f)_0^A \\ \mathcal{D}(\mathbf{triv}(\mathfrak{b}_{\beta_\flat}), U(f))_0^D \cong \mathcal{C}(\mathbf{triv}(\mathfrak{a}_{\psi(\beta_\flat)}), f)_0^D \end{cases} \tag{4.10}$$
If we denote by $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ and $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ the factorisation games defined by Proposition 4.2 for the respective commutative squares $\mathbf{triv}(\mathfrak{a}_{\psi(\beta_\flat)})$ and $\mathbf{triv}(\mathfrak{b}_{\beta_\flat})$, then the isomorphisms of (4.10) define an equivalence of factorisation games from $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ to $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ (see section 4.2.2.3). We now have all the arguments to prove that $f : X \to Y$ is a fibration for the alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_\flat$. First, because $U(f)$ is a fibration for $\mathfrak{b}$, Proposition 4.2 and the definition of $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ imply that the factorisation game $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ has a winning strategy. The above equivalence of factorisation games and Proposition 4.1 then implies that the factorisation game $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ also has a winning strategy. Finally, Proposition 4.2 and the definition of $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$

imply that the morphism $f : X \to Y$ is a fibration for $\mathfrak{a}_{\psi(\beta_\flat)}$. Since the metafunction ψ is surjective, it follows that the morphism $f : X \to Y$ is a fibration for $\mathfrak{a}$. The above discussion thus proves that f is fibration in $\hat{\mathcal{C}}$. $\square$

Proposition 4.84. *A (pseudo-)covertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ reflects intractions.*

Proof. Let $f : X \to Y$ be a morphism such that $U(f)$ is an intraction in the system of vertebrae $\hat{\mathcal{D}}$ and consider a node of vertebrae ν in A. The goal is to find an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ for which f is an intraction. By assumption on U, there exists a node of vertebrae $\nu_\flat$ in B satisfying property 1) relative to ν. Since $U(f)$ is an intraction in $\hat{\mathcal{D}}$, there exists an alliance $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B for which $U(f)$ is an intraction. Because U satisfies item 1), there must exist an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A such that the following isomorphisms of diagrams in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ hold for every $\beta_\flat \in \Omega$ and $\beta_\dagger \in \Omega_\dagger$ (see section 4.4.5.5).

$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(\_)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, \_) \qquad \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(\_)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), \_)$$

If the naturality of the earlier isomorphisms is applied on $f : X \to Y$, these isomorphisms lead to the following isomorphisms of diagrams in **Set** for every $\beta_\flat \in \Omega_\flat$ and $\beta_\dagger \in \Omega_\dagger$ (think of isomorphisms in dimension 4 and 3).

$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(f)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, f) \qquad \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(f)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), f) \tag{4.11}$$

If we denote by $\mathcal{S}_\mathfrak{a}$ and $\mathcal{S}_\mathfrak{a}$ the respective scales of $\mathfrak{b}$ and $\mathfrak{a}$, a restriction of the isomorphisms of (4.11) on certain and obvious subcategories of their categories of definition leads to the following pair of isomorphisms of diagrams for every $\beta_\flat \in \Omega_\flat$ and $\beta_\dagger \in \Omega_\dagger$ (see definitions of section 4.2.2.5)

$$\left\{ \begin{array}{l} \mathcal{D}(\mathcal{S}_\mathfrak{b}, f)^A_{\beta_\flat} \cong \mathcal{C}(\mathcal{S}_\mathfrak{a}, f)^A_{\psi(\beta_\flat)} \\ \mathcal{D}(\mathcal{S}_\mathfrak{b}, f)^D_{\beta_\flat,\beta_\dagger} \cong \mathcal{C}(\mathcal{S}_\mathfrak{a}, f)^D_{\psi(\beta_\flat),\psi_*(\beta_\dagger)} \end{array} \right. \tag{4.12}$$

If we denote by $(A^\mathfrak{a}, D^\mathfrak{a}, \Omega, \Omega_*)$ and $(A^\mathfrak{b}, D^\mathfrak{b}, \Omega_\flat, \Omega_\dagger)$ the factorisation games defined by Proposition 4.3 for the respective scales $\mathcal{S}_\mathfrak{a}$ and $\mathcal{S}_\mathfrak{b}$, then the isomorphisms of (4.12) define an equivalence of factorisation games from $(A^\mathfrak{b}, D^\mathfrak{b}, \Omega_\flat, \Omega_\dagger)$ to $(A^\mathfrak{a}, D^\mathfrak{a}, \Omega, \Omega_*)$ (see section 4.2.2.3). We now have all the arguments to prove that $f : X \to Y$ is an intraction for the alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_\flat$. First, because $U(f)$ is an intraction for $\mathfrak{b}$, Proposition 4.3 and the definition of $(A^\mathfrak{b}, D^\mathfrak{b}, \Omega_\flat, \Omega_\dagger)$ imply that the factorisation game $(A^\mathfrak{b}, D^\mathfrak{b}, \Omega_\flat, \Omega_\dagger)$ has a winning strategy. The above equivalence of factorisation games and Proposition 4.1 then implies that the factorisation game $(A^\mathfrak{a}, D^\mathfrak{a}, \Omega, \Omega_*)$ also has a winning strategy. Finally, Proposition 4.3 and the definition of $(A^\mathfrak{a}, D^\mathfrak{a}, \Omega, \Omega_*)$ imply that the morphism $f : X \to Y$ is simple with respect to $\mathcal{S}_\mathfrak{a}$ and is hence an intraction for $\mathfrak{a}$. The above discussion thus proves that f is intraction in $\hat{\mathcal{C}}$. $\square$

Proposition 4.85. *A covertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ reflects surtractions and pseudofibrations.*

Proof. The proof for pseudofibrations is very similar to that of Proposition 4.83. Let $f : X \to Y$ be a morphism such that $U(f)$ is a surtraction in the system of vertebrae $\hat{\mathcal{D}}$ and consider an E-seed γ. The goal is to find an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ for which f is a surtraction. By assumption on U, there exists an F-seed $\gamma_\flat$ satisfying property 2) relative to γ. Since $U(f)$ is a surtraction in $\hat{\mathcal{D}}$, there exists an extended node of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\mathrm{ex}}{\rightsquigarrow} \nu_\dagger$ in F for which $U(f)$ is a surtraction. Because U satisfies item 2), there must exist an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\mathrm{ex}}{\rightsquigarrow} \nu_*$ in E such that the following isomorphisms of diagrams in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ hold for every $\beta_\dagger \in \Omega_\dagger$ where the bottom one is compatible with the top one over the seeds γ_* and $\gamma_\dagger$ (see section 4.4.5.6).

$$\left\{ \begin{array}{l} \mathcal{D}(\ell\mathbf{seed}(\mathfrak{m}_{\beta_\dagger}), U(\_)) \cong \mathcal{C}(\ell\mathbf{seed}(\mathfrak{n}_{\psi_*(\beta_\dagger)}), \_) \\ \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(\_)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), \_) \end{array} \right.$$

If the naturality of the earlier isomorphisms is applied on $f : X \to Y$, these isomorphisms lead to the following isomorphisms of diagrams in **Set** for every $\beta_\dagger \in \Omega_\dagger$ (think of isomorphisms in dimension 4 and 3).

$$(4.13) \qquad \begin{cases} \mathcal{D}(\ell\mathbf{seed}(\mathfrak{m}_{\beta_\dagger}), U(f)) \cong \mathcal{C}(\ell\mathbf{seed}(\mathfrak{n}_{\psi_*(\beta_\dagger)}), f) \\ \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(f)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), f) \end{cases}$$

If we denote by $\mathcal{B}_\mathfrak{n}$ and $\mathcal{B}_\mathfrak{n}$ the respective besoms of $\mathfrak{m}$ and $\mathfrak{n}$, a restriction of the isomorphisms of (4.13) on certain and obvious subcategories of their categories of definition leads to the following pair of isomorphisms of diagrams for every $\beta_\dagger \in \Omega_\dagger$ (see definitions of section 4.2.2.6)

$$(4.14) \qquad \begin{cases} \mathcal{D}(\mathcal{B}_\mathfrak{m}, f)_0^A \cong \mathcal{C}(\mathcal{B}_\mathfrak{n}, f)_0^A \\ \mathcal{D}(\mathcal{B}_\mathfrak{m}, f)_{0,\beta_\dagger}^D \cong \mathcal{C}(\mathcal{B}_\mathfrak{n}, f)_{0,\psi_*(\beta_\dagger)}^D \end{cases}$$

If we denote by $(A^\mathfrak{n}, D^\mathfrak{n}, \{0\}, \Omega_*)$ and $(A^\mathfrak{m}, D^\mathfrak{m}, \{0\}, \Omega_\dagger)$ the factorisation games defined by Proposition 4.4 for the respective besoms $\mathcal{B}_\mathfrak{n}$ and $\mathcal{B}_\mathfrak{m}$, then the isomorphisms of (4.14) define an equivalence of factorisation games from $(A^\mathfrak{m}, D^\mathfrak{m}, \{0\}, \Omega_\dagger)$ to $(A^\mathfrak{n}, D^\mathfrak{n}, \{0\}, \Omega_*)$ (see section 4.2.2.3). We now have all the arguments to prove that $f : X \to Y$ is a surtraction for the alliance of nodes of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$. First, because $U(f)$ is a surtraction for $\mathfrak{m}$, Proposition 4.4 and the definition of $(A^\mathfrak{m}, D^\mathfrak{m}, \{0\}, \Omega_\dagger)$ imply that the factorisation game $(A^\mathfrak{m}, D^\mathfrak{m}, \{0\}, \Omega_\dagger)$ has a winning strategy. The above equivalence of factorisation games and Proposition 4.1 then implies that the factorisation game $(A^\mathfrak{n}, D^\mathfrak{n}, \{0\}, \Omega_*)$ also has a winning strategy. Finally, Proposition 4.4 and the definition of $(A^\mathfrak{n}, D^\mathfrak{n}, \{0\}, \Omega_*)$ imply that the morphism $f : X \to Y$ is divisible with respect to the besom $\mathcal{B}_\mathfrak{n}$ and is hence a surtraction for $\mathfrak{n}$. The above discussion thus proves that f is surtraction in $\hat{\mathcal{C}}$. $\square$

An dual version of the previous gives the following. An *opcovertebral functor* from the system of vertebrae $\hat{\mathcal{C}}$ to the system of vertebrae $\hat{\mathcal{D}}$ consists of a functor of categories $U : \mathcal{C} \to \mathcal{D}$ such that

3) for every node of vertebrae $\nu_\flat$ in B (resp. B'), there exists a node of vertebrae in ν in A (resp. A') such that for every alliance of nodes of vertebrae $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A (resp. A'), there exists an alliance of nodes of vertebrae $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B (resp. B') for which there exists a congruence of presheaves in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ over the pair $(\mathfrak{b}, \mathfrak{a})$ of the following form;
$$(\psi, \psi_*) \vdash \mathcal{D}(\mathfrak{b}, U(\_)) \equiv \mathcal{C}(\mathfrak{a}, \_)$$
4) for every F-seed $\gamma_\flat$, there exists an E-seed γ such that for every extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ in E, there exist an extended node of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$ in F for which there exists a congruence of presheaves in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ over the pair $(\mathfrak{m}, \mathfrak{n})$ of the following form;
$$\psi_* \vdash \mathcal{D}(\mathfrak{m}, U(\_)) \equiv \mathcal{C}(\mathfrak{n}, \_)$$

A *pseudo-opcovertebral functor* from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$ is a functor of categories $U : \mathcal{C} \to \mathcal{D}$ that only satisfies item 1). The proof of the following propositions are the logical dual of the above propositions given in the case of covertebral functors.

Proposition 4.86. *An (pseudo-)opcovertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ preserves fibrations and trivial fibrations.*

Proof. Because the proof is very similar between the two classes of morphisms, only the case of fibrations will be discussed. Let $f : X \to Y$ be a fibration in the system of vertebrae $\hat{\mathcal{C}}$ and consider a node of vertebrae $\nu_\flat$ in B'. The goal is to find an alliance $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ for which $U(f)$ is a fibration. By assumption on U, there exists a node of vertebrae ν in A' satisfying property 3) relative to $\nu_\flat$. Since $U(f)$ is a fibration in $\hat{\mathcal{C}}$, there exists an alliance

$\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A' for which f is a fibration. Because U satisfies item 3), there must exist an alliance $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B' such that the following isomorphism of diagrams in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ holds for every $\beta_\flat \in \Omega_\flat$ (see section 4.4.5.5).

$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(\_)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, \_)$$

If the naturality of the earlier isomorphism is applied on $f : X \to Y$, this isomorphism leads to the following isomorphism of diagrams in **Set** for every $\beta_\flat \in \Omega_\flat$ (think of an isomorphism in dimension 4).

$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(f)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, f) \tag{4.15}$$

A restriction of isomorphism (4.15) on certain and obvious subcategories of its category of definition leads to the following pair of isomorphisms of diagrams (see definitions of section 4.2.2.4).

$$\begin{cases} \mathcal{D}(\mathbf{triv}(\mathfrak{b}_{\beta_\flat}), U(f))_0^A \cong \mathcal{C}(\mathbf{triv}(\mathfrak{a}_{\psi(\beta_\flat)}), f)_0^A \\ \mathcal{D}(\mathbf{triv}(\mathfrak{b}_{\beta_\flat}), U(f))_0^D \cong \mathcal{C}(\mathbf{triv}(\mathfrak{a}_{\psi(\beta_\flat)}), f)_0^D \end{cases} \tag{4.16}$$

If we denote by $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ and $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ the factorisation games defined by Proposition 4.2 for the respective commutative squares $\mathbf{triv}(\mathfrak{a}_{\psi(\beta_\flat)})$ and $\mathbf{triv}(\mathfrak{b}_{\beta_\flat})$, then the isomorphisms of (4.16) define an equivalence of factorisation games from $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ to $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ (see section 4.2.2.3). We now have all the arguments to prove that $U(f) : U(X) \to U(Y)$ is a fibration for the alliance of nodes of vertebrae $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_*$. First, because f is a fibration for $\mathfrak{a}$, Proposition 4.2 and the definition of $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ imply that the factorisation game $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \{0\}, \{0\})$ has a winning strategy. The above equivalence of factorisation games and Proposition 4.1 then implies that the factorisation game $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ also has a winning strategy. Finally, Proposition 4.2 and the definition of $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \{0\}, \{0\})$ imply that the morphism $U(f) : U(X) \to U(Y)$ is a fibration for $\mathfrak{b}_{\beta_\flat}$. The above discussion thus proves that $U(f)$ is fibration in $\hat{\mathcal{D}}$. □

Proposition 4.87. *An (pseudo-)opcovertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ preserves intractions.*

Proof. Let $f : X \to Y$ be an intraction in the system of vertebrae $\hat{\mathcal{C}}$ and consider a node of vertebrae $\nu_\flat$ in B. The goal is to find an alliance $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ for which $U(f)$ is an intraction. By assumption on U, there exists a node of vertebrae ν in A satisfying property 3) relative to $\nu_\flat$. Since f is an intraction in $\hat{\mathcal{C}}$, there exists an alliance $\mathfrak{a} : \nu \rightsquigarrow \nu_*$ in A for which f is an intraction. Because U satisfies item 3), there must exist an alliance $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B such that the following isomorphisms of diagrams in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ hold for every $\beta_\flat \in \Omega$ and $\beta_\dagger \in \Omega_\dagger$ (see section 4.4.5.5).

$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(\_)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, \_) \qquad \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(\_)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), \_)$$

If the naturality of the earlier isomorphisms is applied on $f : X \to Y$, these isomorphisms lead to the following isomorphisms of diagrams in **Set** for every $\beta_\flat \in \Omega_\flat$ and $\beta_\dagger \in \Omega_\dagger$ (think of isomorphisms in dimension 4 and 3).

$$\mathcal{D}(\ell\mathfrak{b}_{\beta_\flat}, U(f)) \cong \mathcal{C}(\ell\mathfrak{a}_{\psi(\beta_\flat)}, f) \qquad \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(f)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), f) \tag{4.17}$$

If we denote by $\mathcal{S}_{\mathfrak{a}}$ and $\mathcal{S}_{\mathfrak{a}}$ the respective scales of $\mathfrak{b}$ and $\mathfrak{a}$, a restriction of the isomorphisms of (4.17) on certain and obvious subcategories of their categories of definition leads to the next pair of isomorphisms of diagrams for every $\beta_\flat \in \Omega_\flat$ and $\beta_\dagger \in \Omega_\dagger$ (see definitions of section 4.2.2.5)

$$\begin{cases} \mathcal{D}(\mathcal{S}_{\mathfrak{b}}, f)_{\beta_\flat}^A \cong \mathcal{C}(\mathcal{S}_{\mathfrak{a}}, f)_{\psi(\beta_\flat)}^A \\ \mathcal{D}(\mathcal{S}_{\mathfrak{b}}, f)_{\beta_\flat, \beta_\dagger}^D \cong \mathcal{C}(\mathcal{S}_{\mathfrak{a}}, f)_{\psi(\beta_\flat), \psi_*(\beta_\dagger)}^D \end{cases} \tag{4.18}$$

If we denote by $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \Omega, \Omega_*)$ and $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \Omega_\flat, \Omega_\dagger)$ the factorisation games defined by Proposition 4.3 for the respective scales $\mathcal{S}_{\mathfrak{a}}$ and $\mathcal{S}_{\mathfrak{b}}$, then the isomorphisms of (4.18) define an equivalence of factorisation games from $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \Omega_\flat, \Omega_\dagger)$ to $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \Omega, \Omega_*)$ (see section 4.2.2.3). We now have all the arguments to prove that $U(f) : U(X) \to U(Y)$ is an intraction for the alliance of nodes of vertebrae $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$. First, because f is an intraction for $\mathfrak{a}$, Proposition 4.3 and the definition of $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \Omega, \Omega_*)$ imply that the factorisation game $(A^{\mathfrak{a}}, D^{\mathfrak{a}}, \Omega, \Omega_*)$ has a winning strategy. The above equivalence of factorisation games and Proposition 4.1 then implies that the factorisation game $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \Omega_\flat, \Omega_\dagger)$ also has a winning strategy. Finally, Proposition 4.3 and the definition of $(A^{\mathfrak{b}}, D^{\mathfrak{b}}, \Omega_\flat, \Omega_\dagger)$ imply that the morphism $U(f) : U(X) \to U(Y)$ is simple with respect to $\mathcal{S}_{\mathfrak{b}}$ and is hence an intraction for $\mathfrak{b}$. The above discussion thus proves that $U(f)$ is intraction in $\hat{\mathcal{D}}$. $\square$

Proposition 4.88. *An opcovertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ preserves surtractions and pseudofibrations.*

Proof. The proof for pseudofibrations is very similar to Proposition 4.86. Let $f : X \to Y$ be a surtraction in the system of vertebrae $\hat{\mathcal{C}}$ and consider an F-seed $\gamma_\flat$. The goal is to find an extended node of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$ for which $U(f)$ is a surtraction. By assumption on U, there exists an E-seed γ satisfying property 4) relative to $\gamma_\flat$. Since f is a surtraction in $\hat{\mathcal{C}}$, there exists an extended node of vertebrae $\mathfrak{n} : \gamma \overset{\text{ex}}{\rightsquigarrow} \nu_*$ in A for which f is a surtraction. Because U satisfies item 4), there must exist an extended node of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$ in E such that the following isomorphisms of diagrams in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ hold for every $\beta_\dagger \in \Omega_\dagger$ where the bottom one is compatible with the top one over the seeds γ_* and $\gamma_\dagger$ (see section 4.4.5.6).

$$\begin{cases} \mathcal{D}(\ell\mathbf{seed}(\mathfrak{m}_{\beta_\dagger}), U(\_)) \cong \mathcal{C}(\ell\mathbf{seed}(\mathfrak{n}_{\psi_*(\beta_\dagger)}), \_) \\ \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(\_)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), \_) \end{cases}$$

If the naturality of the earlier isomorphisms is applied on $f : X \to Y$, these isomorphisms lead to the following isomorphisms of diagrams in **Set** for every $\beta_\dagger \in \Omega_\dagger$ (think of isomorphisms in dimension 4 and 3).

$$\begin{cases} \mathcal{D}(\ell\mathbf{seed}(\mathfrak{m}_{\beta_\dagger}), U(f)) \cong \mathcal{C}(\ell\mathbf{seed}(\mathfrak{n}_{\psi_*(\beta_\dagger)}), f) \\ \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(f)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), f) \end{cases} \tag{4.19}$$

If we denote by $\mathcal{B}_{\mathfrak{n}}$ and $\mathcal{B}_{\mathfrak{n}}$ the respective besoms of $\mathfrak{m}$ and $\mathfrak{n}$, a restriction of the isomorphisms of (4.19) on certain and obvious subcategories of their categories of definition leads to the following pair of isomorphisms of diagrams for every $\beta_\dagger \in \Omega_\dagger$ (see definitions of section 4.2.2.6)

$$\begin{cases} \mathcal{D}(\mathcal{B}_{\mathfrak{m}}, f)_0^A \cong \mathcal{C}(\mathcal{B}_{\mathfrak{n}}, f)_0^A \\ \mathcal{D}(\mathcal{B}_{\mathfrak{m}}, f)_{0,\beta_\dagger}^D \cong \mathcal{C}(\mathcal{B}_{\mathfrak{n}}, f)_{0,\psi_*(\beta_\dagger)}^D \end{cases} \tag{4.20}$$

If we denote by $(A^{\mathfrak{n}}, D^{\mathfrak{n}}, \{0\}, \Omega_*)$ and $(A^{\mathfrak{m}}, D^{\mathfrak{m}}, \{0\}, \Omega_\dagger)$ the factorisation games defined by Proposition 4.4 for the respective besoms $\mathcal{B}_{\mathfrak{n}}$ and $\mathcal{B}_{\mathfrak{m}}$, then the isomorphisms of (4.20) define an equivalence of factorisation games from $(A^{\mathfrak{m}}, D^{\mathfrak{m}}, \{0\}, \Omega_\dagger)$ to $(A^{\mathfrak{n}}, D^{\mathfrak{n}}, \{0\}, \Omega_*)$ (see section 4.2.2.3). We now have all the arguments to prove that $U(f) : U(X) \to U(Y)$ is a surtraction for the alliance of nodes of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$. First, because f is a surtraction for $\mathfrak{n}$, Proposition 4.4 and the definition of $(A^{\mathfrak{n}}, D^{\mathfrak{n}}, \{0\}, \Omega_*)$ imply that the factorisation game $(A^{\mathfrak{n}}, D^{\mathfrak{n}}, \{0\}, \Omega_*)$ has a winning strategy. The above equivalence of factorisation games and Proposition 4.1 then implies that the factorisation game $(A^{\mathfrak{m}}, D^{\mathfrak{m}}, \{0\}, \Omega_\dagger)$ also has a winning strategy. Finally, Proposition 4.4 and the definition of $(A^{\mathfrak{m}}, D^{\mathfrak{m}}, \{0\}, \Omega_\dagger)$ imply that the morphism $U(f) : U(X) \to U(Y)$ is divisible with respect to the besom $\mathcal{B}_{\mathfrak{m}}$ and is hence a surtraction for $\mathfrak{m}$. The above discussion thus proves that $U(f)$ is surtraction in $\hat{\mathcal{D}}$. $\square$

Any functor that is both covertebral and opcovertebral will be called a *bicovertebral functor*. This type of functor come in handy when reflecting properties involving different items of a same zoo.

Proposition 4.89. *A bicovertebral functor $U : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ reflects (strong) refinement.*

Proof. We shall prove that a bicovertebral functor reflects strong refinement. The statements about refinement being very similar. Suppose that $\hat{\mathcal{D}}$ is strongly refined and consider a pseudofibration $f : X \to Y$ in $\hat{\mathcal{C}}$. Because U is opcovertebral, Proposition 4.86 implies that the morphism $U(f) : U(X) \to U(Y)$ is a pseudofibation in $\hat{\mathcal{D}}$. By strong refinement, $U(f)$ is a trivial fibration. Now, using the fact that U is covertebral, Proposition 4.83 implies that f is a trivial fibration, which shows that $\hat{\mathcal{C}}$ is strongly refined. $\square$

The next statement is a generalisation of the Crans' Transfer Theorem stated in [**9**, Theorem 3.3] at the level of systems of vertebrae.

Proposition 4.90. *Suppose to be given three systems of vertebrae $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{S}}$, two functors of systems of vertebrae $G : \hat{\mathcal{S}} \to \hat{\mathcal{C}}$ and $H : \hat{\mathcal{S}} \to \hat{\mathcal{D}}$ and a functor $U : \mathcal{C} \to \mathcal{D}$ for which there is a natural isomorphism in Z and X as follows.*

$$\mathcal{D}(H(Z), U(X)) \cong \mathcal{C}(G(Z), X) \tag{4.21}$$

1) *If H is 1-regular and G is 0-regular, then U covertebral;*
2) *If H is 0-regular and G is 1-regular, then U opcovertebral;*
3) *If H is pseudo-1-regular and G is 0-regular, then U pseudo-covertebral;*
4) *If H is 0-regular and G is pseudo-1-regular, then U pseudo-opcovertebral;*

Proof. Consider the notations $\hat{\mathcal{C}} = (\mathcal{C}, A, A', E)$, $\hat{\mathcal{D}} = (\mathcal{D}, B, B', F)$ and $\hat{\mathcal{S}} = (\mathcal{S}, I, I', J)$. Let us prove item 1) and item 3). To do so, suppose given a node of vertebrae ν in A. Because G is 0-regular, there exists an alliance ν' in I such that the equality $\nu = G(\nu')$ holds. Now, define $\nu_\flat := H(\nu')$ in B. If there exists an alliance of nodes of vertebrae $\mathfrak{b} : \nu_\flat \rightsquigarrow \nu_\dagger$ in B, then because H is 1-regular, there exists an alliance of nodes of vertebrae $\mathfrak{b}' : \nu' \rightsquigarrow \nu'_\dagger$ in I such that $H(\mathfrak{b}') = \mathfrak{b}$. Denote by $\mathfrak{a} : \nu_\flat \rightsquigarrow \nu_*$ the image $G(\mathfrak{b}')$ in A. Now, for every stem $\beta_\flat \in \Omega_\flat$, choose a stem $\beta' \in \Omega'$ such that the equality $H(\beta') = \beta_\flat$ holds (which is possible by definition of $\nu_\flat$) and define the metafunction $\psi : \Omega_\flat \to \Omega$ mapping $\beta_\flat$ to $G(\beta')$. Similarly, for every stem $\beta_\dagger \in \Omega_\dagger$, choose a stem $\beta'_\dagger \in \Omega'_\dagger$ such that the equality $H(\beta'_\dagger) = \beta_\dagger$ holds (which is possible by definition of $\nu'_\dagger$) and define the metafunction $\psi_* : \Omega_\dagger \to \Omega_*$ mapping $\beta_\dagger$ to $G(\beta'_\dagger)$. It follows from the definition of the image of a node of vertebrae via a functor of systems of vertebrae (see section 4.4.5.1) that ψ and ψ_* are surjective. Denote by $\varphi_\flat$, φ and φ' the metafunctions associated with the alliances $\mathfrak{b}$, $\mathfrak{a}$ and $\mathfrak{b}'$, respectively. Because the equalities

$$\varphi(G(H^{-1}(\beta_\flat))) = G(\varphi'(H^{-1}(\beta_\flat))) = G(H^{-1}(\varphi_\flat(\beta_\flat)))$$

hold (see Remark 4.79), it is easy to check that following diagram commutes.

$$\begin{tikzcd} \Omega_\flat \arrow[r, "\psi"] \arrow[d, "\varphi_\flat"'] & \Omega \arrow[d, "\varphi"] \\ \Omega_\dagger \arrow[r, "\psi_*"] & \Omega_* \end{tikzcd} \tag{4.22}$$

By replacing the object Z with the vertebrae $p' \cdot \beta'$ and $p'_\dagger \cdot \beta'_\dagger$ in isomorphism (4.21) and using the definition of ψ and ψ_*, we deduce that the following isomorphisms holds for every $\beta_\flat \in \Omega_\flat$ and $\beta_\dagger \in \Omega_\dagger$.

$$\mathcal{D}(\ell p_\flat \cdot \beta_\flat, U(X)) \cong \mathcal{C}(\ell p \cdot \psi(\beta_\flat), X) \qquad \mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(X)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), X)$$

It follows from diagram (4.22) and isomorphism (4.21) that the two previous isomorphisms make the following square commute for every stem $\beta_\flat \in \Omega_\flat$.

$$\begin{tikzcd}
\mathcal{D}(\ell p_\flat \cdot \beta_\flat, U(X)) \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\mathcal{D}(\ell \mathfrak{b}_{\beta_\flat}, U(X))"'] & \mathcal{C}(\ell p \cdot \psi(\beta_\flat), X) \arrow[d, Rightarrow, "\mathcal{C}(\ell \mathfrak{a}_{\psi(\beta_\flat)}, X)"] \\
\mathcal{D}(\ell p_\dagger \cdot \varphi_\flat(\beta_\flat), U(X)) \arrow[r, Rightarrow, "\cong"'] & \mathcal{C}(\ell p_* \cdot (\psi_* \circ \varphi(\beta_\flat)), X)
\end{tikzcd}$$

In other words, the following congruence holds, which proves one of the three congruences involved by item 1).

$$(\psi, \psi_*) \vdash \mathcal{D}(\mathfrak{b}, U(\_)) \equiv \mathcal{C}(\mathfrak{a}, \_)$$

Since the proof of the second congruence copies the above reasoning with respect to A', B' and I', item 3) is proven and there only remains to prove the last congruence involved by item 1). To finish the proof of item 1), suppose given an E-seed γ. Because G is 0-regular, there exists a J-seed γ' such that the equality $\gamma = G(\gamma')$ holds. Now, define $\gamma_\flat := H(\gamma')$ in F. If there exists an alliance of nodes of vertebrae $\mathfrak{m} : \gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_\dagger$ in F, then because H is 1-regular, there exists an extended node of vertebrae $\mathfrak{m}' : \gamma' \overset{\text{ex}}{\rightsquigarrow} \nu'_\dagger$ in J such that $H(\mathfrak{m}') = \mathfrak{m}$. Denote by $\mathfrak{n} : \gamma_\flat \overset{\text{ex}}{\rightsquigarrow} \nu_*$ the image $G(\mathfrak{m}')$ in E. Now, for every stem $\beta_\dagger \in \Omega_\dagger$, choose a stem $\beta'_\dagger \in \Omega'_\dagger$ such that the equality $H(\beta'_\dagger) = \beta_\dagger$ holds (which is possible by definition of $\nu'_\dagger$) and define the metafunction $\psi_* : \Omega_\dagger \to \Omega_*$ mapping $\beta_\dagger$ to $G(\beta'_\dagger)$. It is straightforward to check that ψ_* is surjective. By replacing the object Z with the vertebra $p'_\dagger \cdot \beta'_\dagger$ in isomorphism (4.21), we deduce an isomorphism of the following form, which holds for every $\beta_\dagger \in \Omega_\dagger$.

$$\mathcal{D}(\ell p_\dagger \cdot \beta_\dagger, U(X)) \cong \mathcal{C}(\ell p_* \cdot \psi_*(\beta_\dagger), X)$$

By construction, the restriction of this isomorphism on the seeds makes the following diagram commute for every stem $\beta_\flat \in \Omega_\flat$.

$$\begin{tikzcd}
\mathcal{D}(\ell \gamma_\flat, U(X)) \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\mathcal{D}(\ell \mathbf{seed}(\mathfrak{n}_{\beta_\flat}), U(X))"'] & \mathcal{C}(\ell \gamma, X) \arrow[d, Rightarrow, "\mathcal{C}(\ell \mathbf{seed}(\mathfrak{m}_{\psi_*(\beta_\flat)}), X)"] \\
\mathcal{D}(\ell \gamma_\dagger, U(X)) \arrow[r, Rightarrow, "\cong"'] & \mathcal{C}(\ell \gamma_*, X)
\end{tikzcd}$$

In other words, the following congruence holds, which proves the other half of item 1).

$$\psi_* \vdash \mathcal{D}(\mathfrak{m}, U(\_)) \equiv \mathcal{C}(\mathfrak{n}, \_)$$

Because the proof of item 1) and item 3) did not depend on the object $U(X)$ or X, it is easy to see that a dual reasoning leads to the proof of item 2) and item 4). $\square$

Example 4.91. Let $(\mathcal{C}, A, E, T)$ be a discrete spinal category whose underlying system of vertebra will be denoted by $\hat{\mathcal{C}}$ and D be a small category. If the functor $\mathrm{Lan}_d : \mathcal{C} \to \mathcal{C}^D$ exists for some object d in D, then the functor $\Delta_D : \mathcal{C} \to \mathcal{C}^D$ defines a opcovertebral functor from $\hat{\mathcal{C}}$ to $\mathrm{Lan}_d(\hat{\mathcal{C}})$. To be more specific, the adjunction $\mathrm{Lan}_d \dashv \nabla_d$ first provides the following natural isomorphisms in Z and Y.

$$\mathcal{C}^D(\mathrm{Lan}_d(Z), \Delta_D(Y)) \cong \mathcal{C}(Z, \nabla_d \circ \Delta_D(Y)) = \mathcal{C}(Z, Y)$$

It then follows from Example 4.81 and Proposition 4.90 that the functor $\Delta_D : \mathcal{C} \to \mathcal{C}^D$ defines a opcovertebral and pseudo-covertebral functor from $\hat{\mathcal{C}}$ to $\mathrm{Lan}_d(\hat{\mathcal{C}})$.

Example 4.92. Let $\mathtt{K}$ be a colimit sketch, $\mathtt{K}^\triangledown$ be a free cocompletion of $\mathtt{K}$ equipped with a system of vertebra $\hat{\mathtt{K}}^\triangledown$ and let $\mathtt{K}[\hat{\mathtt{K}}^\triangledown]$ denote the resulting system of vertebrae for $\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$ as defined in Example 4.82. Let now $\mathcal{C}$ be a cocomplete category and $(\mathcal{C}, A, E, T)$ be a spinal category of system of vertebrae $\hat{\mathcal{C}}$ for which there exists a functor $i : \mathtt{K} \to \mathcal{C}$ such that the free extension $i^\triangledown : \mathtt{K}^\triangledown \to \mathcal{C}$ induces a 0- and 1-regular funtor of system of vertebrae.

We are going to show that the following functor induces a bivertebral functor from $\hat{\mathcal{C}}$ to $\mathtt{K}[\hat{\mathtt{K}}^{\triangledown}]$.

$$\mathcal{C}(i(\_),\_) : \mathcal{C} \to \mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$$

However, before showing this, notice that any object s in $\mathtt{K}^{\triangledown}$ may be expressed either as an object of $\mathtt{K}$ or a colimit of $\mathtt{K}$ over some functor of the form $x : I \to \mathtt{K}$. Because any object of $\mathtt{K}$ is a colimit of $\mathtt{K}$, the definition of the functor $\mathtt{K}[\_]$ then states that the object $\mathtt{K}[s]$ is a colimit in $\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$ of the following form where s is equal to the colimit $\mathrm{col}_k\, x_k$ in $\mathtt{K}$.

$$\mathtt{K}[s] \cong \mathrm{col}_k\, \mathtt{K}(\_, x_k)$$

Now, by Proposition 4.90 and the fact that $\mathtt{K}[\_]$ is 0- and 1-regular (see Example 4.82), the following series of natural isomorphisms in s and Y shows that $\mathcal{C}(i(\_),\_)$ is a bivertebral functor in the case where the functor $i^{\triangledown}$ is 0- and 1-regular.

$$\begin{aligned}
\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})(\mathtt{K}[s], \mathcal{C}(i(_), Y)) &\cong \mathbf{Mod}(\mathtt{K}^{\mathrm{op}})(\mathrm{col}_k\, \mathtt{K}(_, x_k), \mathcal{C}(i(_), Y)) \\
&\cong \lim_k \mathbf{Mod}(\mathtt{K}^{\mathrm{op}})(\mathtt{K}(_, x_k), \mathcal{C}(i(_), Y)) \\
&\cong \lim_k \mathcal{C}(i(x_k), Y) \\
&\cong \mathcal{C}(\mathrm{col}_k\, i(x_k), Y) \\
&\cong \mathcal{C}(i^{\triangledown}(\mathrm{col}_k\, x_k), Y) \\
&\cong \mathcal{C}(i^{\triangledown}(s), Y).
\end{aligned}$$

Proposition 4.93 (Crans' transfer)**.** *Suppose that a functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $U : \mathcal{D} \to \mathcal{C}$. The functor F induces a transfer of structure from any discrete system of vertebrae $\hat{\mathcal{C}}$ to $\mathcal{D}$ while the functor U induces a covertebral functor $U : F(\hat{\mathcal{C}}) \to \hat{\mathcal{C}}$. If F turns out to be a 1-regular (resp. pseudo-1-regular) functor, then U is opcovertebral (resp. pseudo-opcovertebral).*

Proof. Since any left adjoint preserves colimits, the functor F sends any vertebra in $\mathcal{C}$ to $\mathcal{D}$ and thereby induces a transfer of structure from any discrete system of vertebrae $\hat{\mathcal{C}}$ to $\mathcal{D}$. The statement regarding the functor U is a consequence of Proposition 4.90 as the adjunction $F \dashv U$ provides an isomorphism $\mathcal{D}(Y, U(X)) \cong \mathcal{C}(F(Y), X)$ natural in X and Y and F is 0-regular from $F(\hat{\mathcal{C}})$ to $\hat{\mathcal{C}}$ by Proposition 4.80. The last statement also follows from Proposition 4.90. $\square$

Example 4.94. Let $(\mathcal{C}, A, E, T)$ be a discrete spinal category with system of vertebrae $\hat{\mathcal{C}}$, D be a small category and suppose that the functor $\mathrm{Lan}_d : \mathcal{C} \to \mathcal{C}^D$ exists for some object d in D. It follows from Proposition 4.93 and Example 4.81 that the right adjoint $\nabla_d : \mathcal{C}^D \to \mathcal{C}$ induces a covertebral and pseudo-opcovertebral functor from $\mathrm{Lan}_d(\hat{\mathcal{C}})$ to $\hat{\mathcal{C}}$.

4.4.5.8. *Covertebral equivalences.* Let $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ be two systems of vertebrae. A *covertebral equivalence* from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$ consists of a bicovertebral functor $U : \mathcal{C} \to \mathcal{D}$ and a left adjoint $F \dashv U$ such that all the components of the unit $\eta_{(\_)} : \mathrm{id}_{\mathcal{D}}(\_) \Rightarrow UF(\_)$ are weak equivalences in $\hat{\mathcal{D}}$.

Proposition 4.95. *Suppose that $\hat{\mathcal{D}}$ is equipped with a structure of spinal category. A morphism $f : X \to U(Y)$ is a weak equivalence in $\hat{\mathcal{D}}$ if and only if the morphism $\varepsilon_Y \circ F(f) : F(X) \to Y$ is a weak equivalence in $\hat{\mathcal{C}}$.*

Proof. Consider the following equation given by the universal property of the adjunction $F \dashv U$.

$$f = U(\varepsilon_Y \circ F(f)) \circ \eta_X$$

If f is a weak equivalence, then Theorem 4.77 implies that $U(\varepsilon_Y \circ F(f))$ is a weak equivalence since η_X is also a weak equivalence. Finally, because U covertebral, it reflects weak equivalences (see Proposition 4.84 and Proposition 4.85), which implies that $\varepsilon_Y \circ F(f)$ is a weak equivalence. Conversely, if $\varepsilon_Y \circ F(f)$ is a weak equivalence, then so is $U(\varepsilon_Y \circ F(f))$ since U is opcovertebral (see Proposition 4.87 and Proposition 4.88). Theorem 4.77 then implies that $f = U(\varepsilon_Y \circ F(f)) \circ \eta_X$ is a weak equivalence. □

Example 4.96. Let $\mathtt{K}$ be a colimit sketch, $\mathcal{C}$ be a category and $i : \mathtt{K} \to \mathcal{C}$ be as defined in Example 4.92. When $\mathcal{C}$ is cocomplete, the bicovertebral functor

$$\mathcal{C}(i(\_), \_) : \mathcal{C} \to \mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$$

has a left adjoint that is given by the following calculation.

$$\begin{aligned}
\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})\Big(X(_), \mathcal{C}(i(_), Y)\Big) &\cong \mathbf{Nat}\Big(X(_), \mathcal{C}(i(_), Y)\Big) \\
&\cong \int_{a\in\mathtt{K}} \mathbf{Set}\Big(X(a), \mathcal{C}(i(a), Y)\Big) \\
&\cong \int_{a\in\mathtt{K}} \mathcal{C}\big(X(a) \otimes i(a), Y\big) \\
&\cong \mathcal{C}\Big(\int^{a\in\mathtt{K}} X(a) \otimes i(a), Y\Big)
\end{aligned}$$

If the underlying system of vertebrae of $\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$ is equipped with a spinal structure and each component of the unit of the above adjunction is a weak equivalence in $\mathbf{Mod}(\mathtt{K}^{\mathrm{op}})$, then the functor $\mathcal{C}(i(\_), \_)$ defines a covertebral equivalence.

4.5. Everyday examples of vertebral and spinal categories

4.5.1. Set and higher category theory.

4.5.1.1. *Sets.* The category **Set** has an obvious discrete spinal category made of the following reflexive vertebra.

$$\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, "\delta_1"] & \\
\mathbf{1} \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbf{1}+\mathbf{1} \arrow[r, "\beta"] & \mathbf{1},
\end{tikzcd}$$

The structure of vertebral algebra follows from the ability of the above vertebra to frame two copies of itself. The convergent conjugations are made of obvious diagrammatic constructions using the arrows displayed above. It is worth noting that the spinal category made of the above vertebra is not refined and the proof of Theorem 4.54 actually uses Lemma 2.64. However, it may be turned into a strongly refined spinal category by considering the epi-correction of its vertebral category, which is given by the following vertebra.

$$\begin{tikzcd}
\mathbf{1}+\mathbf{1} \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, "!"] & \\
\mathbf{1} \arrow[r, "!"'] \arrow[ur, phantom, "\ulcorner"] & \mathbf{1} \arrow[r, equal] & \mathbf{1},
\end{tikzcd}$$

The last vertebra is equipped with the structure of a spine of degree 2 when endowed with the base of the very first vertebra (as explained in Example 1.9).

4.5.1.2. *Higher categories.* The case of 1-categories, 2-categories as well as that of strict ω-categories are the natural extensions of that of **Set** and have already been discussed in the introduction. The spines and vertebra presented there are reflexive and admit convergent conjugations for natural framing operations. The case of weak ω-groupoids and categories is discussed in detailed in Chapter 6.

4.5.1.3. *Closed model structures.* In section 1.1.2.3, a set of vertebrae was associated with a closed model structure. Using the technics described in Example 3.4.2 with respect to the weak factorisation system for the cofibrations and the acyclic fibrations, it is possible to show that these vertebrae (see vertebra (1.7)) induce a vertebral category and even a spinal category by taking the 'extension' of vertebra (1.7) to a spine as follows.

$$\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, equal] & U \arrow[d, equal] & & & \\
\emptyset \arrow[r, "!"'] & U \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & V \arrow[d, "\delta_1^\gamma"] & & \\
 & V \arrow[r, "\delta_2^\gamma"'] \arrow[ur, phantom, "\ulcorner"] & V \cup_U V \arrow[r, "\beta"] \arrow[rr, dotted, bend right, "u"'] & I(\gamma) \arrow[r, dotted, "u'"] & V
\end{tikzcd}$$

4.5.2. Algebraic topology and abstract algebra.

4.5.2.1. *Topological spaces.* The category of topological spaces **Top** is equipped with a structure of strongly refined discrete spinal category made of the following reflexive vertebrae for every $n \in \omega$.

$$V_n := \quad \begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}^n \arrow[d, "\delta_1^n"] & \\
\mathbb{D}^n \arrow[r, "\delta_2^n"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}^n \arrow[r, "\gamma_{n+1}"'] & \mathbb{D}^{n+1}
\end{tikzcd}$$

The case of this category is further discussed in Chapter 6.

4.5.2.2. *Simplicial sets.* The category of simplicial sets **sSet** is equipped with an obvious structure of spinal category arising from the class of reflexive vertebrae $\mathcal{E}$ defined in Example 2.4.2.4 and discussed in both Example 2.4.2.4 and Example 3.4.2.

4.5.2.3. *Chain complexes.* Let R be a ring. The category of non-negatively graded chain complexes $\mathbf{Ch}_R$ is equipped with a structure of spinal category arising from the class containing the following reflexive nodes of vertebrae for every $n \in \omega$ and $\delta \in \{0, 1\}$ (see section 2.4.3.4 for the notations).

$$\begin{tikzcd}
0 \arrow[r, "!"] \arrow[d, "!"'] & D_n \arrow[d] & \\
D_n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S_n \arrow[r, "\beta_n(\delta)"] & D_n(\delta)
\end{tikzcd}$$

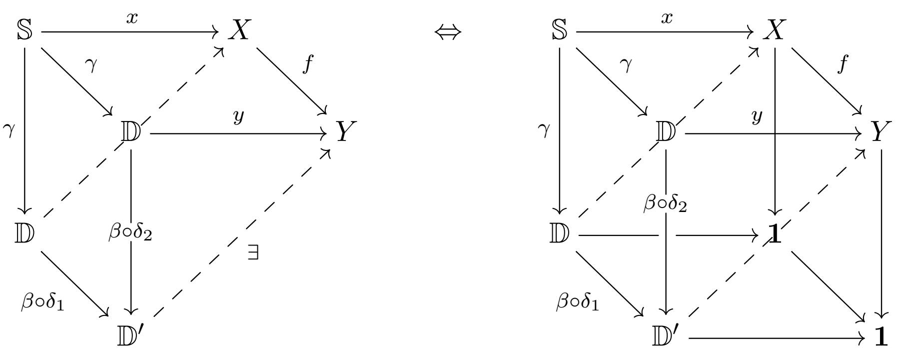
This spinal category may be shown to be refined by Lemma 2.64 (see section 2.4.3.4 for more details). It is however not strongly refined.

Chapter 5

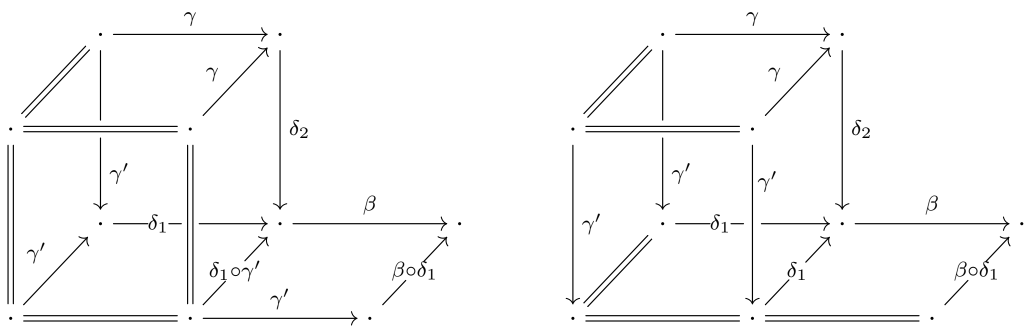
Construct of Homotopy Theories

5.1. Introduction

We saw in Chapter 1 as well as in Chapter 4 via the notions of saturation and epi-correction that weak equivalences could be characterised by surtractions. Specifically, in the case of a discrete system of vertebrae $(\mathcal{C}, A, A', E)$, a weak equivalence is an object $f : X \to Y$ in $\mathcal{C}^2$ that has the right lifting property with respect to all the diskads of the vertebrae in $\mathrm{Obj}_{\mathrm{L}}(E)$.

(5.1)


From this characterisation of weak equivalences in terms of right lifting property, it is tempting to try to see there is a vertebra in $\mathcal{C}^2$ for which the weak equivalences are the *fibrant objects* for this vertebra, namely the object f of $\mathcal{C}^2$ such that the canonical arrow $f \Rightarrow \mathbf{1}$ is a fibration for that vertebra – provided that $\mathcal{C}^2$ has a terminal object $\mathbf{1}$. The answer may be given by all sorts of vertebrae. There are however two canonical vertebrae that only use the data given by the initial vertebra in $\mathcal{C}$ (see diagrams below).



The problem with such vertebrae is that if the vertebra $v := \|\gamma, \gamma'\| \cdot \beta$ is reflexive, then the above-displayed ones are not necessarily so. We may only transport the reflexive structure of v when considering the previous diagrams for the dual vertebra v^{rv}. In this case, only the vertebra on the right will be reflexive. The left vertebra is reflexive when the seed γ of v (or coseed of v^{rv}) is a split monomorphism, which never happens in practice. This therefore prevents the construction of a vertebral category in $\mathcal{C}^2$ by using the preceding pair[1]. A first idea would be to remove the vertebra on the left, but this would prevent the definition of a source hinge for a vertebral algebra[2]. The solution is to construct these vertebrae in $\mathcal{C}^\omega$ – instead of $\mathcal{C}^2$ – to shift the structure of the left vertebra so that the seeds of the vertebrae resulting from the right one run over the seeds of the vertebrae resulting from the left one, allowing these to not be reflexive. The information spreading beyond $\mathcal{C}^2$ within the category $\mathcal{C}^\omega$ will later be recovered by the notion of *cohesion* (see section 5.3.2.5).

The goal of section 5.3 will be to formalise the previous construction and extract from ‘good’ vertebral categories in $\mathcal{C}$ a structure of vertebral category in $\mathcal{C}^\omega$ where the fibrant objects will be the sequences of arrows whose successive compositions from 0 to any ordinal in ω are weak equivalences. This property will be used to characterise the general notions of sheaves, models for a sketch, spectra and even flabby sheaves (for sheaf cohomology) as fibrant objects in $\mathcal{C}^\omega$ by embedding the descent condition expressed in $\mathcal{C}^2$ into the category $\mathcal{C}^\omega$.

$$X \xrightarrow{f} Y \qquad \mapsto \qquad X \xrightarrow{f} Y = Y = Y = Y = \dots$$

Such a characterisation will require us to generalise the notions of Grothendieck’s pretopology and sketch to that of *croquis* (see section 5.3.1.2). The embedding $\mathcal{C}^2 \hookrightarrow \mathcal{C}^\omega$ will enable us to fetch the homotopical properties living in $\mathcal{C}^\omega$ to the categories of presheaves, prespectra and so on and see the sheaves, spectra and so on in terms of fibrant objects (see section 5.5). The proposed method obviously paves the way for the characterisation of weaker objects such as stacks, (∞, n)-stacks and even strong stacks (see [**31**]). This prospect is briefly discussed in section 5.3.2.9 in comparison to the model structures of [**30**] and the systems of fibrant objects of [**37**].

The transport of spinal structures from $\mathcal{C}$ to the functor category $\mathcal{C}^\omega$ is more subtle and is not treated. Its treatment is somewhat cumbersome due to the need of the notion of alliance (see Chapter 1 at the end of section 1.1.2.4) and would thus require its own chapter. Future work will aim at writing up the whole structure in detail.

Lastly, the right lifting property of (5.1) suggests that we could use Quillen’s small object argument [**38**] to construct sheafification, stackification, spectrumification functors or even Godement replacements. Such an argument would require us to produce a functor $F : D \to \mathcal{C}$ such that some of the morphisms that it induces in $\mathcal{C}^2$ (i.e. the morphisms for the descent conditions) satisfy a right lifting property with respect to the diskads of a certain set of vertebrae. In this respect, section 5.4 will provide a generalisation of the small object argument processing a morphism of functors (i.e. the unit of the reflection) at two different levels:

- a ‘usual’ small object argument in $\mathcal{C}^D$;
- a generalised small object argument in $\mathcal{C}^2$ conditioning the usual one;

[1] By definition of a vertebral category, because the seeds of the vertebrae are different, they both need to be reflexive.

[2] The need of the right vertebra is substantial since there is no possible communication between the seed and coseed of two copies of the left vertebra, at least when one only requires the use of the data provided by the initial vertebrae in $\mathcal{C}$.

The idea behind the new argument is to start with a (collection of) commutative cube as the one given below on the left, wherein the symbol $\mathbf{D}$ stands for the descent data of a certain descent condition. Then, applying a 'functorial' small object argument on its back face produces a functor $N_1 : D \to \mathcal{C}$ and an arrow $\pi_1 : \mathbb{D}_2 \to N_1(U)$ factorising the left diagram into the diagram of non-dotted arrows given on the right.

$$
\begin{tikzcd}
\mathbb{S} \arrow[rr] \arrow[dr, "\gamma'"] \arrow[dd, "\gamma"'] & & F(U) \arrow[dr] \arrow[dd] & \\
& \mathbb{D}_1 \arrow[rr] \arrow[dd, "\beta\circ\delta_1"] & & \mathbf{D}(F) \arrow[dd] \\
\mathbb{D}_2 \arrow[rr] \arrow[dr, "\beta\circ\delta_2"'] & & G(U) \arrow[dr] & \\
& \mathbb{D}' \arrow[rr, "y"] & & \mathbf{D}(G)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \arrow[rr] \arrow[dr, "\gamma'"] \arrow[ddd, "\gamma"'] & & F(U) \arrow[dr] \arrow[d] & \\
& \mathbb{D}_1 \arrow[rr] \arrow[ddd, "\beta\circ\delta_1"] & N_1(U) \arrow[d] \arrow[dr] & \mathbf{D}(F) \arrow[d] \\
& & & \\
\mathbb{D}_2 \arrow[uur, "\pi_1"] \arrow[rr] \arrow[dr, "\beta\circ\delta_2"'] & & G(U) \arrow[dr] & \mathbf{D}(N_1) \arrow[d] \\
& \mathbb{D}' \arrow[rr, "y"] \arrow[urr, dotted, "?"] & & \mathbf{D}(G)
\end{tikzcd}
$$

The problem is that the usual small object argument does not provide the dotted filler of the preceding right diagram as the descent data $\mathbf{D}(\_)$ does not generally commute with pushouts. To obtain such a lift, we will need to force the next step of the small object argument to make it appear. Later on, this forcing will take the form of a *modifier* (see section 5.4.4.2) that will influence the 'functorial' small object argument in the second step.

Specifically, we will use the pushout of the vertebra that appears in the previous right commutative cuboid (see the following left cuboid) to form a collection of commutative squares of the form given below on the right, where ε represents a natural transformation coming along with $\mathbf{D}(\_)$. These squares exactly represent the 'local obstruction at V' preventing the descent data from commuting with pushouts.

$$
\begin{tikzcd}
& F(U) \arrow[rr] & & \mathbf{D}(F) \arrow[d] \\
\mathbb{S} \arrow[ur] \arrow[rr, "\gamma'"] \arrow[dd, "\gamma"'] & & \mathbb{D}_1 \arrow[ur] \arrow[dl] \arrow[dd, "\beta\circ\delta_1"] & \mathbf{D}(N_1) \arrow[d] \\
& \mathbb{S}' \arrow[urr, dashed, "h"] \arrow[dr, "\beta"] & & \mathbf{D}(G) \\
\mathbb{D}_2 \arrow[ur] \arrow[rr, "\beta\circ\delta_2"] & & \mathbb{D}' \arrow[ur, "y"] &
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "h"] \arrow[d, "\beta"'] & \mathbf{D}(N_1) \arrow[r, "\varepsilon_V"] \arrow[d, dotted] & N_1(V) \arrow[d] \\
\mathbb{D}' \arrow[r, "y"] & \mathbf{D}(G) \arrow[r, "\varepsilon_V"'] & G(V)
\end{tikzcd}
$$

Putting these 'obstruction squares' in the set of squares to be factorised in the next step of the 'usual' small object argument will allow us to obtain a commutative diagram as given below on the left. Inserting this diagram in the preceding left cuboid will then allow us to form the wanted lift up to shifting of the pushout N_1 to that of the second step (as shown below, on the right).

$$
\begin{tikzcd}
\mathbb{S}' \arrow[rr] \arrow[dd, "\beta"'] & & \mathbf{D}(N_1) \arrow[dl] \arrow[dd] \\
& \mathbf{D}(N_2) \arrow[dr] & \\
\mathbb{D}' \arrow[ur, dashed, "\mathbf{D}(\varpi_1)"] \arrow[rr, "y"] & & \mathbf{D}(G)
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
& F(U) \arrow[rr] & & \mathbf{D}(F) \arrow[d] \\
\mathbb{S} \arrow[ur] \arrow[rr, "\gamma'"] \arrow[dd, "\gamma"'] & & \mathbb{D}_1 \arrow[ur] \arrow[dl] \arrow[dd, "\beta\circ\delta_1"] & \mathbf{D}(N_2) \arrow[d] \\
& \mathbb{S}' \arrow[dr, "\beta"] & & \mathbf{D}(G) \\
\mathbb{D}_2 \arrow[ur] \arrow[rr, "\beta\circ\delta_2"] & & \mathbb{D}' \arrow[ur, "y"] \arrow[uur, dashed, "\mathbf{D}(\varpi_1)"] &
\end{tikzcd}
$$

Continuing this process to a certain limit ordinal will finally enable us to produce a factorisation of the form $F \Rightarrow N \Rightarrow G$ in $\mathcal{C}^D$ where the right-hand arrow will satisfy some fibrational property in $\mathcal{C}^2$ while the left-hand arrow will satisfy componentwise left lifting properties in $\mathcal{C}$ (see Theorem 5.83). In the case where G is a terminal object in $\mathcal{C}^D$, the functor $N : D \to \mathcal{C}$ will satisfy the wanted descent conditions. The universal properties associated with the arrow $F \Rightarrow N$ may be unravelled when further specifying the properties satisfied by our vertebrae[3].

In terms of interpretation, forming the pushout N_1 corresponds to the process of adding 'gluings' to F while adding the obstruction square to the next step of the argument corresponds to the process of quotienting the gluings that have been added too many times in N_1. Future work will aim at using this algorithm to obtain a combinatorial description of the reflection associated with the category of Grothendieck's ∞-groupoids and thus characterise the colimits living there.

The inductive process that produces the factorisation $F \Rightarrow N \Rightarrow G$ is more general than that used in the usual small object argument. Of course, it recovers the usual argument by taking degenerate vertebrae in the algorithm, but it will need to be generalised in various aspects. First, the usual notion of 'small object' will not be sufficient and the notion of *convergent functor* (see section 5.2.1.1) will be used instead. In addition, the usual pushouts will need to be replaced with weaker pushouts. This type of information will be carried by the notion of *narrative* (see section 5.4.2.1). Finally, we will introduce the notion of *constructor* (see section 5.4.3.2) to formally describe the algorithm described above. The notion of constructor will then give rise to a pair of narratives that will recover the preceding construction.

We will finish the chapter (see section 5.5) by briefly discussing the construction of homotopy theories for sheaves, spectra and so on via the notions of *vertebral* and *spinal theories*. The consideration of the preceding small object argument will provide us with the usual factorisation axioms in order to define model categories and categories of fibrant objects. This section will be the appropriate place to discuss the construction of model categories in detail.

5.2. Preparation

5.2.1. Functors with properties.

[3]This is reminiscent of some combinatorial difficulty appearing in the usual small object argument when proving that one of arrow must be an acyclic cofibration

5.2.1.1. *Convergent functors.* Let κ be a limit ordinal in $\mathbf{O}$ and $\mathcal{C}$ be a category. Denote by ι_κ the inclusion functor $\mathbf{O}(\kappa) \hookrightarrow \mathbf{O}(\kappa+1)$. For any class of objects of $\mathcal{C}$, say $\mathcal{G}$, a functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}$ will be said to be *$\mathcal{G}$-convergent in $\mathcal{C}$* if for every object $\mathbb{D}$ in $\mathcal{G}$, the canonical function

$$\mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}(\mathbb{D}, F \circ \iota_\kappa(\_)) \to \mathcal{C}(\mathbb{D}, F(\kappa))$$

is an isomorphism in **Set**. If the class $\mathcal{G}$ turns out to be a singleton $\{\mathbb{D}\}$, the functor will more explicitly be said to be $\mathbb{D}$-convergent.

Let now $\mathtt{T}$ and $\mathtt{S}$ denote two small categories and $G : \mathtt{T} \to \mathcal{C}$ be a functor. A functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}^{\mathtt{S}}$ will be said to be *uniformly G-convergent in $\mathcal{C}$* if for every object s in $\mathtt{S}$ and object t in $\mathtt{T}$, the canonical function

$$\mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}(G(t), F \circ \iota_\kappa(\_)(s)) \to \mathcal{C}(G(t), F(\kappa)(s))$$

is an isomorphism in **Set**. In other words, the evaluation of F at an object s in $\mathtt{S}$ is $\{G(t) \mid t \in \mathrm{Obj}(\mathtt{T})\}$-convergent.

Lemma 5.1. *Let $\mathtt{T}$ and $\mathtt{S}$ be two small categories such that $|\mathtt{T}| \leq \kappa$ and $\mathcal{C}$ be a category. Let $G : \mathtt{T} \to \mathcal{C}$ be a functor and consider a uniformly G-convergent functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}^{\mathtt{S}}$ in $\mathcal{C}$. For any pair of functors $g : \mathcal{C}^{\mathtt{S}} \to \mathcal{C}^{\mathtt{T}}$ and $\hat{g} : \mathbf{Set}^{\mathtt{S}} \to \mathbf{Set}^{\mathtt{T}}$ satisfying a natural isomorphism (in the variables X, Y and t) of the form*

$$\mathcal{C}\Big(X, g(Y)(t)\Big) \cong \hat{g}\Big(\mathcal{C}(X, Y(\_))\Big)(t) \tag{5.2}$$

where $\hat{g} : \mathbf{Set}^{\mathtt{S}} \to \mathbf{Set}^{\mathtt{T}}$ commutes with colimits over $\mathbf{O}(\kappa)$, the composite functor $g \circ F : \mathbf{O}(\kappa+1) \to \mathcal{C}^{\mathtt{T}}$ is G-convergent in $\mathcal{C}^{\mathtt{T}}$.

Proof. The following series of natural isomorphisms proves the statement. The combinatorial subtlety of the theorem resides in the use of Proposition 1.24. Remark 1.25 is implicitly used from one line to another.

$$\begin{aligned}
\mathcal{C}^{\mathtt{T}}(G, g \circ F(\kappa)) &\cong \int_{t \in \mathtt{T}} \mathcal{C}\big(G(t), g \circ F(\kappa)(t)\big) && \text{(Example 1.23)}\\
&\cong \int_{t \in \mathtt{T}} \hat{g}\Big(\mathcal{C}\big(G(t), F(\kappa)(_)\big)\Big)(t) && \text{(Equation (5.2))}\\
&\cong \int_{t \in \mathtt{T}} \hat{g}\Big(\mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}\big(G(t), F(\iota_\kappa(_))(_)\big)\Big)(t) && \text{(Uniform conv.)}\\
&\cong \int_{t \in \mathtt{T}} \mathrm{col}_{\mathbf{O}(\kappa)}\hat{g}\Big(\mathcal{C}\big(G(t), F(\iota_\kappa(_))(_)\big)\Big)(t) && \text{(Hyp. on } \hat{g})\\
&\cong \mathrm{col}_{\mathbf{O}(\kappa)} \int_{t \in \mathtt{T}} \hat{g}\Big(\mathcal{C}\big(G(t), F(\iota_\kappa(_))(_)\big)\Big)(t) && \text{(Prop. 1.24 \& 1.30)}\\
&\cong \mathrm{col}_{\mathbf{O}(\kappa)} \int_{t \in \mathtt{T}} \mathcal{C}\big(G(t), g(F \circ \iota_\kappa(_))(t)\big) && \text{(Equation (5.2))}\\
&\cong \mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}^{\mathtt{T}}(G, g \circ F \circ \iota_\kappa(_)) && \text{(Example 1.23)}
\end{aligned}$$

This last isomorphism shows that $g \circ F$ is G-convergent in $\mathcal{C}^{\mathtt{T}}$. □

The next remarks show different uses of Lemma 5.1. In particular, the remarks give a first taste of the type of mathematical tool that will be used later on.

Remark 5.2. Lemma 5.1 can be used to show that *if a functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}^{\mathtt{T}}$ is uniformly G-convergent in $\mathcal{C}$ and the inequality $|\mathtt{T}| \leq \kappa$ holds, then the functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}^{\mathtt{T}}$ is G-convergent in $\mathcal{C}^{\mathtt{T}}$*. Precisely, this implication follows from Lemma 5.1 by taking the set $\mathtt{S}$ to be $\mathtt{T}$, the functor g to be the identity $\mathrm{id}_{\mathcal{C}^{\mathtt{T}}}$ and the functor $\hat{g}$ to be the identity $\mathrm{id}_{\mathbf{Set}^{\mathtt{T}}}$. The assumptions of Lemma 5.1 are obviously satisfied in this case.

Remark 5.3. It follows from Lemma 5.1 that *if a functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}$ is uniformly G-convergent in $\mathcal{C}$, then $F : \mathbf{O}(\kappa+1) \to \mathcal{C}$ is $\mathrm{col}_{\mathtt{T}}(G)$-convergent in $\mathcal{C}$.* Specifically, this follows from the fact that $\Delta_{\mathtt{T}}$ commutes with hom-sets (see section 1.2.1.38) and the following series of isomorphisms.

$$\begin{aligned}
\mathcal{C}(\mathrm{col}_{\mathtt{T}}G, F(\kappa)) &\cong \mathcal{C}^{\mathtt{T}}(G, \Delta_{\mathtt{T}}(F(\kappa))) && \text{(adjointness)}\\
&\cong \mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}^{\mathtt{T}}(G, \Delta_{\mathtt{T}} \circ F \circ \iota_{\kappa}(_)) && \text{(Lemma 5.1)}\\
&\cong \mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}(\mathrm{col}_{\mathtt{T}}G, F \circ \iota_{\kappa}(_)) && \text{(adjointness)}
\end{aligned}$$

Remark 5.4. For any small category A satisfying the inequality $|A| \leq \kappa$ such that the adjunction $\lim_A \dashv \Delta_A$ holds in $\mathcal{C}$, Lemma 5.1 may be applied to the choices $\mathtt{T} := \mathbf{1}$, $\mathtt{S} := A$, $g := \lim_A : \mathcal{C}^A \to \mathcal{C}$ and $\hat{g} := \lim_A : \mathbf{Set}^A \to \mathbf{Set}$. Equation (5.2) is then proven at the beginning of section 1.2.1.38. The fact that $\hat{g}(\_) : \mathbf{Set}^A \to \mathbf{Set}$ commutes with colimits over $\mathbf{O}(\kappa)$ follows from Proposition 1.30. In this case, Lemma 5.1 states that *for every uniformly $\mathbb{D}$-convergent functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}^A$ in $\mathcal{C}$, where $\mathbb{D}$ is an object of $\mathcal{C}$, the following isomorphism holds.*

$$\mathcal{C}(\mathbb{D}, \lim_A F(\kappa)) \cong \mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}(\mathbb{D}, \lim_A \circ F \circ \iota_{\kappa}(\_))$$

Remark 5.5. Consider a small category A satisfying the inequality $|A| \leq \kappa$ such that the adjunction $\lim_A \dashv \Delta_A$ holds in $\mathcal{C}$. Denote by η the unit of the adjunction $\lim_A \dashv \Delta_A$. By section 1.2.1.19, the unit η may be seen as a functor $\mathcal{C} \to \mathcal{C}^2$ mapping an object X to η_X. Lemma 5.1 may be applied to the choices $\mathtt{T} := \mathbf{2}$, $\mathtt{S} := \mathbf{1}$, $g := \eta : \mathcal{C} \to \mathcal{C}^2$ (the unit η in $\mathcal{C}$) and $\hat{g} := \eta : \mathbf{Set} \to \mathbf{Set}^2$ (the unit η in $\mathbf{Set}$). Equation (5.2) is then proven at the end of section 1.2.1.38. The fact that $\hat{g} : \mathbf{Set} \to \mathbf{Set}^{\mathtt{T}}$ commutes with colimits over $\mathbf{O}(\kappa)$ follows from Proposition 1.31. In this case, Lemma 5.1 states that *for every uniformly γ-convergent functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}$ in $\mathcal{C}$, where $\gamma : \mathbb{S} \to \mathbb{D}$ is an object of $\mathcal{C}^2$, the following isomorphism holds.*

$$\mathcal{C}^2(\gamma, \eta_{F(\kappa)}) \cong \mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}^2(\gamma, \eta \circ F \circ \iota_{\kappa}(\_))$$

Remark 5.6. Remark 5.4 and Remark 5.5 may be extended as follows. First, consider a small category A satisfying the inequality $|A| \leq \kappa$ such that the adjunction $\lim_A \dashv \Delta_A$ holds in $\mathcal{C}$. Denote by η the unit of the adjunction $\lim_A \dashv \Delta_A$. Then, consider a small category D as well as a cone $r : \Delta_A(d) \Rightarrow U$ in D^A where d is an object of D and U is a functor $A \to D$. Now, take $\mathtt{T}$ to be $\mathbf{2}$, $\mathtt{S}$ to be D, $g : \mathcal{C}^D \to \mathcal{C}^2$ to be the obvious functor mapping an object $P : D \to \mathcal{C}$ of $\mathcal{C}^D$ to the arrow

$$P(d) \xrightarrow{\eta_{P(d)}} \lim_A \Delta_A(P(d)) \xrightarrow{\lim_A Pr} \lim_A P \circ U$$

and $\hat{g} : \mathbf{Set}^D \to \mathbf{Set}^2$ to be the version of g when $\mathcal{C}$ is taken to be $\mathbf{Set}$. Equation (5.2) is proven from the results of section 1.2.1.38. The fact that $\hat{g} : \mathbf{Set}^D \to \mathbf{Set}^{\mathtt{T}}$ commutes with colimits over $\mathbf{O}(\kappa)$ follows from Proposition 1.31 and Proposition 1.30. In this case, Lemma 5.1 states that *for every uniformly γ-convergent functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}^D$ in $\mathcal{C}$, where $\gamma : \mathbb{S} \to \mathbb{D}$ is an object of $\mathcal{C}^2$, the following isomorphism holds.*

$$\mathcal{C}^2(\gamma, g(F(\kappa))) \cong \mathrm{col}_{\mathbf{O}(\kappa)}\mathcal{C}^2(\gamma, g \circ F \circ \iota_{\kappa}(\_))$$

Remark 5.7. Remark 5.6 may further be extended. First, consider a small category A satisfying the inequality $|A| \leq \kappa$ such that the adjunction $\lim_A \dashv \Delta_A$ holds in $\mathcal{C}$. Denote by η the unit of the adjunction $\lim_A \dashv \Delta_A$. Then, consider a small category D as well as a cone $r : \Delta_A(d) \Rightarrow U$ in D^A where d is an object of D and U is a functor $A \to D$. Now, take $\mathtt{T}$ to be $\mathbf{2}$, $\mathtt{S}$ to be $\mathbf{2} \times D$, $g : (\mathcal{C}^D)^2 \to \mathcal{C}^2$ to be the obvious functor mapping any morphism

$\theta : P \Rightarrow P'$ of $\mathcal{C}^D$ to the arrow

$$\begin{tikzcd} P(d) \arrow[r, "\eta_{P(d)}"] & \lim_A \Delta_A(P(d)) \arrow[r, "\lim_A Pr"] \arrow[rr, bend left, dashed, "\lim_A(\theta U \star Pr)"] & \lim_A P \circ U \arrow[r, "\lim_A \theta U"] & \lim_A P' \circ U \end{tikzcd}$$

and $\hat{g} : (\mathbf{Set}^D)^2 \to \mathbf{Set}^2$ to be the version of g when $\mathcal{C}$ is taken to be $\mathbf{Set}$. Equation (5.2) is again proven from the results of section 1.2.1.38. The fact that $\hat{g} : (\mathbf{Set}^D)^2 \to \mathbf{Set}^{\mathrm{T}}$ commutes with colimits over $\mathbf{O}(\kappa)$ follows from Proposition 1.31 and Proposition 1.30. In this case, Lemma 5.1 states that *for every uniformly γ-convergent functor $F : \mathbf{O}(\kappa+1) \to (\mathcal{C}^D)^2$ in $\mathcal{C}$, where $\gamma : \mathbb{S} \to \mathbb{D}$ is an object of $\mathcal{C}^2$, the following isomorphism holds.*

$$\mathcal{C}^2(\gamma, g(F(\kappa))) \cong \mathrm{col}_{\mathbf{O}(\kappa)} \mathcal{C}^2(\gamma, g \circ F \circ \iota_\kappa(\_))$$

Example 5.8. Let κ be a limit ordinal in $\mathbf{O}$ and $\mathtt{A}$ be a colimit sketch whose chosen colimits are defined above diagrams of the form $x : I \to \mathtt{A}$ where I has cardinality less than or equal to κ. This last assumption implies that any colimit over the category $\mathbf{O}(\kappa)$ in $\mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$ is computed, componentwise, in $\mathbf{Set}$ (see Proposition 1.30). It then follows from the Yoneda Lemma that any functor $F : \mathbf{O}(\kappa+1) \to \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$ that satisfies the equation

$$F(\kappa) \cong \mathrm{col}_{\mathbf{O}(\kappa)} F \circ \iota_\kappa$$

is uniformly convergent in $\mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$ with respect to the functor $\mathtt{A}(\_,\_) : \mathtt{A} \to \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})$. More specifically, the Yoneda Lemma implies the following series of isomorphisms for every object a in $\mathtt{A}$.

$$\begin{aligned} \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})(\mathtt{A}(\_,a), \mathrm{col}_{i \in \mathbf{O}(\kappa)} F_i(\_)) &\cong \big(\mathrm{col}_{i \in \mathbf{O}(\kappa)} F_i(\_)\big)(a) \\ &\cong \mathrm{col}_{i \in \mathbf{O}(\kappa)} F_i(a) \\ &\cong \mathrm{col}_{i \in \mathbf{O}(\kappa)} \mathbf{Mod}(\mathtt{A}^{\mathrm{op}})(\mathtt{A}(\_,a), F_i(\_)) \end{aligned}$$

5.2.1.2. *Sequential functors.* Let κ be some ordinal in $\mathbf{O}$ and $\mathcal{C}$ be a category admitting for every limit ordinal α in $\mathbf{O}(\kappa+1)$ colimits over the ordinal category $\mathbf{O}(\alpha)$. For any limit ordinal α in $\mathbf{O}(\kappa+1)$, denote by ι_α the restriction functor $\mathbf{O}(\alpha) \hookrightarrow \mathbf{O}(\kappa+1)$. A functor $F : \mathbf{O}(\kappa+1) \to \mathcal{C}$ will be said to be *sequential* if for any limit ordinal α in $\mathbf{O}(\kappa+1)$, the object $F(\alpha)$ may be identified with a colimit of the functor $F \circ \iota_\alpha$ such that the collection of morphisms

$$\{ \quad F(k < \alpha) : F(k) \to F(\alpha) \quad \}_{k \in \alpha}$$

that induces a natural transformation of type

$$F \circ \iota_\alpha \Rightarrow \Delta_{\mathbf{O}(\alpha)}(F(\alpha)) \qquad \text{or} \qquad F \circ \iota_\alpha \Rightarrow \Delta_{\mathbf{O}(\alpha)}(\mathrm{col}_{\mathbf{O}(\alpha)} F \circ \iota_\alpha)$$

over $\mathbf{O}(\alpha)$ corresponds to the unit of the adjunction $\mathrm{col}_{\mathbf{O}(\alpha)} \vdash \Delta_{\mathbf{O}(\alpha)}$ evaluated at the composite functor $F \circ \iota_\alpha$.

The next proposition is a well-known result.

Proposition 5.9. *If a morphism $f : X \to Y$ has the rlp with respect to every arrow $F(k < k+1)$ for every $k \in \kappa$, then f has the rlp with respect to the arrow $F(0 < k)$ for every $k \in \kappa+1$.*

Proof. Follows from a transfinite induction using Proposition 1.34. $\square$

5.2.1.3. *Coretractive functors.* Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $U : \mathcal{C} \to \mathcal{D}$ will be said to be *coretractive* if it is equipped with a retraction $R : \mathcal{D} \to \mathcal{C}$.

5.2.2. Universal shiftings.

5.2.2.1. *Universal shifting and cylindanias.* Let A_0 and A_1 be two small categories and $\mathcal{C}$ be a complete category. Recall that any functor $a : A_1 \to A_0$ induces a pre-composition functor whose rules on objects and morphisms are defined as follows.

$$\_ \circ a : \left[\begin{array}{ccc} \mathcal{C}^{A_0} & \to & \mathcal{C}^{A_1} \\ F & \mapsto & F \circ a \\ \theta : F \Rightarrow G & \mapsto & \theta a : Fa \Rightarrow Ga \end{array}\right]$$

Because $\mathcal{C}$ is complete, the following composition of functor exists.

$$\begin{tikzcd} \mathcal{C}^{A_0} \arrow[r, "\_\circ a"] \arrow[rr, bend right, "\lim_a"'] & \mathcal{C}^{A_1} \arrow[r, "\lim_{A_1}"] & \mathcal{C} \end{tikzcd}$$

The short notation $\lim_a$ will later be conventional. It follows from the structure of the adjunctions

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Delta_{A_0}"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{A_0} \arrow[l, shift left, "\lim_{A_0}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Delta_{A_1}"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{A_1} \arrow[l, shift left, "\lim_{A_1}"] \end{tikzcd}$$

that any object F in $\mathcal{C}^{A_0}$ may be associated with the following canonical arrows.

$$\begin{array}{rcccl} \text{(counit)} & \Delta_{A_0}\lim_{A_0} F & \Rightarrow & F & \text{in } \mathcal{C}^{A_0} \\ \hline \text{(post-comp. with } a\text{)} & \Delta_{A_1}\lim_{A_0} F & \Rightarrow & F \circ a & \text{in } \mathcal{C}^{A_1} \\ \hline \text{(unit)} & \lim_{A_0} F & \to & \lim_{A_1} F \circ a & \text{in } \mathcal{C} \end{array}$$

In particular, the last arrow induces a natural transformation

$$u_a : \lim_{A_0} \Rightarrow \lim_a$$

valued in $\mathcal{C}$ over $\mathcal{C}^{A_0}$. This arrow will later be referred to as a *universal shifting*.

Remark 5.10. By universality, it is not hard to show that for any commutative diagram of functors of the form

(5.3)
$$\begin{tikzcd} A_1 \arrow[r, "a"] & A_0 \\ A_1' \arrow[u, "F_1"] \arrow[r, "a'"] & A_0' \arrow[u, "F_0"'] \end{tikzcd}$$

the following diagram commutes in $\mathcal{C}$ for any functor $P : A_0 \to \mathcal{C}$.

$$\begin{tikzcd} \lim_{A_0} P \arrow[r, "u_a(P)"] \arrow[d, "u_{F_0}(P)"'] & \lim_{A_1} P \circ a \arrow[d, "u_{F_1}(P\circ a)"] \\ \lim_{A_0'} P \circ F_0 \arrow[r, "u_{a'}(P\circ F_0)"'] & \lim_{A_1'} P \circ a \circ F_1 \end{tikzcd}$$

Note that the above commutative square shows a semblance of functoriality with respect to diagram (5.3).

In regard to the previous remark, the rest of the section addresses the question whether there exists a category on which the natural transformation $u_a(\_)$ may somehow be seen as a functor in $a : A_0 \to A_1$. The answer is positive when one considers the following category. Fix a complete category $\mathcal{C}$ and denote by $\mathbf{Cylia}(\mathcal{C})$ the category whose objects are pairs (a, P) where a is a functor in $\mathbf{Cat}(1)$ and P is a functor composable with a as follows

$$A_1 \xrightarrow{a} A_0 \xrightarrow{P} \mathcal{C}$$

and whose morphisms from $A_1 \xrightarrow{a} A_0 \xrightarrow{P} \mathcal{C}$ to $A_1' \xrightarrow{a'} A_0' \xrightarrow{P'} \mathcal{C}$ are triples (F_0, F_1, t) where $F_0 : A_0' \to A_0$ and $F_1 : A_1' \to A_1$ are functors in $\mathbf{Cat}(1)$ such that the equality $a \circ F_1 = F_0 \circ a'$

holds and t is a natural transformation of the form $P \circ F_0 \Rightarrow P'$. These data provide the following diagram.

$$\begin{tikzcd} A_1 \arrow[r, "a"] & A_0 \arrow[r, "P"] \arrow[dr, phantom, "\Downarrow t"] & \mathcal{C} \arrow[d, equal] \\ A_1' \arrow[u, "F_1"] \arrow[r, "a'"] & A_0' \arrow[u, "F_0"] \arrow[r, "P'"] & \mathcal{C} \end{tikzcd}$$

The composition of the morphisms is the obvious one, that is to say that given by the following compositions of functors and natural transformations.

$$(F_0', F_1', t') \circ (F_0, F_1, t) = (F_0 \circ F_0', F_1 \circ F_1', t' \circ (tF_0))$$

The object of $\mathbf{Cylia}(\mathcal{C})$ will be called *cylindania*[4]. Now, it is not hard to see that there is a metafunction from the objects of $\mathbf{Cylia}(\mathcal{C})$ to the objects of $\mathcal{C}^2$ mapping a pair (a, P) to the arrow

$$u_a(P) : \lim_{A_0}(P) \to \lim_a(P)$$

and an arrow $(F_0, F_1, t) : (a, P) \Rightarrow (a', P')$ to the outer commutative square of the following commutative diagram.

$$\begin{tikzcd} \lim_{A_0} P \arrow[r, "u_a(P)"] \arrow[d, "u_{F_0}(P)"'] & \lim_{A_1} P \circ a \arrow[d, "u_{F_1}(P\circ a)"] \\ \lim_{A_0'} P \circ F_0 \arrow[r, dashed, "u_{a'}(P\circ F_0)"] \arrow[d, "\lim_{A_0'} t"'] & \lim_{A_1'} P \circ a \circ F_1 \arrow[d, "\lim_{A_1'} ta'"] \\ \lim_{A_0'} P' \arrow[r, "u_{a'}(P')"'] & \lim_{A_1'} P' \circ a' \end{tikzcd}$$

The top square commutes by Remark 5.10 and the bottom square commutes by naturality of $u_{a'}$. The functorality of the mapping $(a, P) \mapsto u_a(P)$ then follows from the naturality and universality of the morphisms 'u' on both sides.

5.2.2.2. *Universal shifting for colimits.* As in the case of limits, colimits admit universal shiftings. In this section, some hypotheses and notations might be different from that given in section 5.2.2.1 due to the different treatment given to limits and colimits in the sequel. Let $\mathtt{S}$ and $\mathtt{T}$ be two small categories and $\mathcal{C}$ be a category that admits colimits over $\mathtt{S}$ and $\mathtt{T}$. As in section 5.2.2.1, define, for every functor $i : \mathtt{T} \to \mathtt{S}$, the following composite functor.

$$\begin{tikzcd} \mathcal{C}^{\mathtt{S}} \arrow[r, "\_\circ i"] \arrow[rr, bend right, "\mathrm{col}_i"'] & \mathcal{C}^{\mathtt{T}} \arrow[r, "\mathrm{col}_{\mathtt{T}}"] & \mathcal{C} \end{tikzcd}$$

The short notation col_i will later be conventional. It then follows from the structure of the adjunctions

$$\begin{tikzcd} \mathcal{C}^{\mathtt{S}} \arrow[r, shift left, "\mathrm{col}_{\mathtt{S}}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "\Delta_{\mathtt{S}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}^{\mathtt{T}} \arrow[r, shift left, "\mathrm{col}_{\mathtt{T}}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "\Delta_{\mathtt{T}}"] \end{tikzcd}$$

that any object F in $\mathcal{C}^{\mathtt{S}}$ may be associated with the following canonical arrows.

$$\begin{array}{rccl} \text{(unit)} & F \Rightarrow \Delta_{\mathtt{S}}\mathrm{col}_{\mathtt{S}}\,F & & \text{in } \mathcal{C}^{\mathtt{S}} \\ \hline \text{(post-comp. with } i) & F \circ i \Rightarrow \Delta_{\mathtt{T}}\mathrm{col}_{\mathtt{S}}\,F & & \text{in } \mathcal{C}^{\mathtt{T}} \\ \hline \text{(counit)} & \mathrm{col}_{\mathtt{T}}\,F \circ i \to \mathrm{col}_{\mathtt{S}}\,F & & \text{in } \mathcal{C} \end{array}$$

In particular, the last arrow induces a natural transformation $\xi_i : \mathrm{col}_i \Rightarrow \mathrm{col}_{\mathtt{S}}$ valued in $\mathcal{C}$ over $\mathcal{C}^{\mathtt{S}}$. This arrow will later be referred to as a *universal shifting*.

[4]Name for a half cylinder.

5.3. System of models for a croquis

5.3.1. Presheaves, prespectra and premodels.

5.3.1.1. *Cylinders.* Let $\mathcal{D}$ be a category. A *face* in the category $\mathcal{D}$ consists of a small category A and a functor $d : A \to \mathcal{D}$ in $\mathbf{Cat}(1)$. A *cylinder* (or *morphism of faces*) from a face $d_0 : A_0 \to \mathcal{D}$ to a face $d_1 : A_1 \to \mathcal{D}$ consists of a functor $a : A_1 \to A_0$ and a natural transformation $t : d_0 \circ a \Rightarrow d_1$ in $\mathcal{D}$. A cylinder $(a, t) : d_0 \Rightarrow d_1$ may thus be represented by the following picture.

$$\begin{tikzcd} A_0 \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & A_1 \arrow[l, "a"'] \arrow[d, "d_1"] \\ \mathcal{D} \arrow[r, equal] & \mathcal{D} \end{tikzcd}$$

The composition of cylinders is the obvious one, namely for two morphisms $(a_0, t_0) : d_0 \Rightarrow d_1$ and $(a_1, t_1) : d_1 \Rightarrow d_2$, the composition $(a_1, t_1) \circ (a_0, t_0) : d_0 \Rightarrow d_2$ is given by the following pair of compositions.

$$A_0 \xleftarrow{a_1} A_1 \xleftarrow{a_0} A_2 \qquad\qquad d_0 \circ a_0 \circ a_1 \xRightarrow{t_0 a_1} d_1 \circ a_1 \xRightarrow{t_1} d_2$$

The category of faces and cylinders will be denoted by $\mathbf{Cyl}(\mathcal{D})$.

Remark 5.11. This remark is made for future use and concerns the form of the morphisms in the arrow category of $\mathbf{Cyl}(\mathcal{D})$. Explicitly, a morphism of the form $(a, t) \Rrightarrow (a', t')$ in $\mathbf{Cyl}(\mathcal{D})^2$ consists of an equality of pasting of two cells as follows.

$$\begin{tikzcd} & A_0 \arrow[dd, "d_0"] \arrow[ddrr, phantom, "\Searrow t"] & & A_1 \arrow[ll, "a"'] \arrow[dd, "d_1"] \\ A'_0 \arrow[ur, "F_0"] \arrow[dd, "d'_0"'] \arrow[r, phantom, "v_0 \Downarrow"] & & & \\ & \mathcal{D} \arrow[rr, equal] \arrow[dr, phantom, "="] & & \mathcal{D} \\ \mathcal{D} \arrow[ur, equal] \arrow[rr, equal] & & \mathcal{D} \arrow[ur, equal] & \end{tikzcd} \quad = \quad \begin{tikzcd} & A_0 & & A_1 \arrow[ll, "a"'] \arrow[dd, "d_1"] \\ A'_0 \arrow[ur, "F_0"] \arrow[dd, "d'_0"'] \arrow[ddrr, phantom, "\Searrow t'"] & & A'_1 \arrow[ll, "a'"'] \arrow[ur, "F_1"] \arrow[dd, "d'_1"] \arrow[r, phantom, "v_1 \Downarrow"] & \\ & & & \mathcal{D} \\ \mathcal{D} \arrow[rr, equal] & & \mathcal{D} \arrow[ur, equal] & \end{tikzcd}$$

More concisely, this means that the following diagrams commute.

$$\begin{tikzcd} A_0 & A_1 \arrow[l, "a"'] \\ A'_0 \arrow[u, "F_0"] & A'_0 \arrow[l, "a'"] \arrow[u, "F_1"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} d_0 F_0 a' \arrow[r, Rightarrow, "t F_1"] \arrow[d, Rightarrow, "v_0 a'"'] & d_1 F_1 \arrow[d, Rightarrow, "v_1"] \\ d'_0 a' \arrow[r, Rightarrow, "t'"'] & d'_1 \end{tikzcd}$$

Remark 5.12. Let D be a small category. Remark 5.11 shows that there exists an obvious functor $\mathbf{Cyl}(\mathcal{D})^2 \to \mathbf{Cylia}(\mathcal{D})$ whose mapping on the object is defined as follows.

$$\begin{tikzcd} A_0 \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & A_1 \arrow[l, "a"'] \arrow[d, "d_1"] \\ \mathcal{D} \arrow[r, equal] & \mathcal{D} \end{tikzcd} \quad \mapsto \quad A_1 \xrightarrow{a} A_0 \xrightarrow{d_0} \mathcal{D}$$

Remark 5.13. The category of faces and cylinders induce a metafunctor $\mathbf{Cyl}(\_)$ that maps any category $\mathcal{D}$ to the category $\mathbf{Cyl}(\mathcal{D})$ and any functor $P : \mathcal{D} \to \mathcal{C}$ to the functor $\mathbf{Cyl}(P)$:

$\mathbf{Cyl}(\mathcal{D}) \to \mathbf{Cyl}(\mathcal{C})$ that post-composes faces and cylinders in $\mathcal{D}$ with the functor P.

$$\begin{tikzcd} A_0 \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & A_1 \arrow[l, "a"'] \arrow[d, "d_1"] \\ \mathcal{D} \arrow[r, equal] & \mathcal{D} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A_0 \arrow[d, "P\circ d_0"'] \arrow[dr, phantom, "\Searrow Pt"] & A_1 \arrow[l, "a"'] \arrow[d, "P\circ d_1"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

By section 1.2.1.20, the functor $\mathbf{Cyl}(P)$ obviously extends to a functor $\mathbf{Cyl}(\mathcal{D})^2 \to \mathbf{Cyl}(\mathcal{C})^2$.

Remark 5.14. Denote by $\mathbf{1}$ a terminal object of $\mathbf{Cat}(1)$. There is an obvious functor $I_{\mathbf{1}} : \mathcal{D} \to \mathbf{Cyl}(\mathcal{D})$ mapping an object d of $\mathcal{D}$ to the following cylinder in $\mathcal{D}$.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "d"'] \arrow[dr, phantom, "\Searrow \mathrm{id}_d"] & \mathbf{1} \arrow[l, "!"'] \arrow[d, "d"] \\ \mathcal{D} \arrow[r, equal] & \mathcal{D} \end{tikzcd}$$

Note that the functor $I_{\mathbf{1}} : \mathcal{D} \to \mathbf{Cyl}(\mathcal{D})$ is injective on objects, which implies that its image is a category.

5.3.1.2. *Croquis.* Let $\mathcal{D}$ be a category. A *croquis category* or *croquis*[5] in $\mathcal{D}$ consists of a small subcategory $K \subseteq \mathbf{Cyl}(\mathcal{D})^2$ and a functor $T : \mathcal{D} \to \mathcal{D}$ equipped with two functors $I_{\mathbf{1}} : \mathcal{D} \to K$ and $T \cdot \_ : K \to K$ making the following diagrams commute.

$$\begin{tikzcd} & K \arrow[d, "\subseteq"] \\ \mathcal{D} \arrow[ur, dashed, "I_{\mathbf{1}}"] \arrow[r, "I_{\mathbf{1}}"'] & \mathbf{Cyl}(\mathcal{D})^2 \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "T\cdot\_"] \arrow[d, "\subseteq"'] & K \arrow[d, "\subseteq"] \\ \mathbf{Cyl}(D)^2 \arrow[r, "\mathbf{Cyl}(T)"'] & \mathbf{Cyl}(\mathcal{D})^2 \end{tikzcd}$$

Such a structure will be denoted by a triple $(\mathcal{D}, K, T)$. When the ambient category $\mathcal{D}$ is obvious, the previous croquis will often be referred to as a pair (K, T). Note that if the functor $T : \mathcal{D} \to \mathcal{D}$ is the identity, then so is the functor $T \cdot \_ : K \to K$. In this case, the croquis will be denoted by $(\mathcal{D}, K)$ or K only.

Example 5.15 (Arrow categories)**.** Let D be a small category and $T : D \to D$ be some given endofunctor. The arrow category D^2 may be seen as a croquis category via the functor mapping an arrow $t : d_0 \to d_1$ in D to the cylinder of the following form.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & \mathbf{1} \arrow[l, "!"'] \arrow[d, "d_1"] \\ D \arrow[r, equal] & D \end{tikzcd}$$

When seen as a subcategory of $\mathbf{Cyl}(D)^2$, the category D^2 will be denoted by $\mathbf{Cr}(D, T)$. It is straightforward to check that $I_{\mathbf{1}} : D \to \mathbf{Cyl}(D)^2$ lifts to a functor $I_{\mathbf{1}} : D \to \mathbf{Cr}(D, T)$. Similarly, the endofunctor $T : D \to D$ induces an endofunctor $T \cdot \_ : \mathbf{Cr}(D, T) \to \mathbf{Cr}(D, T)$.

Example 5.16 (Spectra)**.** Let $\mathbb{N}$ denote the discrete subcategory of $\mathbf{O}(\omega)$ containing all its objects and $\mathtt{succ} : \mathbb{N} \to \mathbb{N}$ be the successor operation $n \mapsto n + 1$. Denote by $I_{\mathbf{1}}(\mathbb{N})$ the image of the functor $I_{\mathbf{1}} : \mathbb{N} \to \mathbf{Cyl}(\mathbb{N})^2$ (see Remark 5.14). The triple $(\mathbb{N}, I_{\mathbf{1}}(\mathbb{N}), \mathtt{succ})$ then defines a croquis in $\mathbb{N}$. This croquis will later be used to characterise spectra.

[5] The word 'croquis' is another word for 'sketch' (from French).

Example 5.17 (Sketches)**.** Let $\mathtt{S}$ be a limit sketch. A chosen cone in $\mathtt{S}$ is a natural transformation of the form $t : \Delta_A(u) \Rightarrow v(\_)$ on some small category A, which may also be seen as a cylinder as follows.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "u"'] \arrow[dr, phantom, "\Searrow t"] & A \arrow[l, "!"'] \arrow[d, "v"] \\ \mathtt{S} \arrow[r, equal] & \mathtt{S} \end{tikzcd}$$

The limit sketch $\mathtt{S}$ may thus be associated with the full subcateogry $K_{\mathtt{S}} \subseteq \mathbf{Cyl}(D)^2$ whose objects are given by the cylinder of the above form. By definition of a sketch, the functor $I_{\mathbf{1}} : \mathtt{S} \to \mathbf{Cyl}(\mathtt{S})^2$ lifts to $K_{\mathtt{S}}$, which shows that $K_{\mathtt{S}}$ defines a croquis in $\mathtt{S}$.

Example 5.18 (Grothendieck's pretopologies)**.** Let J denote a Grothendieck's pretopology on a small (opposite) category D^{op}. A covering family $S = \{v_i \to u\}_{i\in A}$ in J_u may be defined as a natural transformation of the form $t : \Delta_A(u) \Rightarrow v(\_)$ in D over A. If one denotes by $A' \to D^{\mathrm{op}}/d$ the stabilisation of S (see Remark 1.21), this cone gives rise to another cone $t' : \Delta_{A'}(u) \Rightarrow v'(\_)$ in D over A', which may be seen as a cylinder as follows.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "u"'] \arrow[dr, phantom, "\Searrow t"] & A' \arrow[l, "!"'] \arrow[d, "v"] \\ D \arrow[r, equal] & D \end{tikzcd}$$

The Grothendieck's pretopology J may be associated with the full subcateogry $K_J \subseteq \mathbf{Cyl}(D)^2$ whose objects are given by the cylinders of the preceding form. In particular, the identity axiom of a Grothendieck's pretopology forces the functor $I_{\mathbf{1}} : D \to \mathbf{Cyl}(D)^2$ to lift to K_J. This shows that the pair (D, K_J) is a croquis.

Example 5.19 (Flabby pretopologies)**.** Let J denote a Grothendieck's pretopology on a small (opposite) category D^{op}. The croquis that will later give rise to flabby sheaves and the Godement's resolution is the union of the two croquis (D, K_J) and $\mathbf{Cr}(D, \mathrm{id}_D)$. Precisely, this croquis consists of the union of the two subcategories K_J and D^2 in $\mathbf{Cyl}(D)^2$ equipped with the identity functor $\mathrm{id} : K_J \cup D^2 \to K_J \cup D^2$ and the obvious functor $I_{\mathbf{1}} : D \to K_J \cup D^2$.

5.3.1.3. *Conical croquis.* Let $\mathcal{D}$ be a category and (K, T) be a croquis in $\mathcal{D}$. A cylinder in K will be said to be *conical* if the face encoding its domain is a terminal category. In other words, the cylinder c has the following form.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & A \arrow[l, "!"'] \arrow[d, "d_1"] \\ \mathtt{S} \arrow[r, equal] & \mathtt{S} \end{tikzcd}$$

The face d_0 of the cylinder c thus picks out a particular object d_0 in $\mathcal{D}$. This object will be called the *peak* of c and denoted by $\mathtt{peak}_K(c)$. A croquis (K, T) in $\mathcal{D}$ will be said to be *conical* if all its cylinders in K are conical. In this case, the operation $\mathtt{peak}_K(\_)$ induces a (truncation) functor from K to D.

5.3.1.4. *Cardinality of a croquis.* Let $\mathcal{D}$ be a category and consider a croquis (K, T) in $\mathcal{D}$. An *elementary shape* in $(\mathcal{D}, K, T)$ is a small category A such that there exists a cylinder c in K of the following form.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & A \arrow[l, "!"'] \arrow[d, "d_1"] \\ \mathcal{D} \arrow[r, equal] & \mathcal{D} \end{tikzcd}$$

Because K is a small category, the class of elementary shapes of $(\mathcal{D}, K, T)$ is a set, which will be denoted by $\mathrm{Es}(K)$. The *cardinality of a croquis* $(\mathcal{D}, K, T)$ is then given by the cardinal of the coproduct of the small categories in $\mathrm{Es}(K)$.

$$|(K,T)| := |\coprod_{A \in \mathrm{Es}(K)} A|$$

5.3.1.5. *Domain, codomain and top functors for croquis.* Let $\mathcal{D}$ be a small category and (K, s) be a croquis in $\mathcal{D}$. The restriction of the domain and codomain functors $\mathbf{Cyl}(\mathcal{D})^2 \to \mathbf{Cyl}(\mathcal{D})$ to K will later be denoted by dom_K and cod_K, respectively. Similarly, consider the functor of arrow categories

$$\mathrm{top} : \mathbf{Cyl}(\mathcal{D})^2 \to (\mathbf{Cat}(1)^{\mathrm{op}})^2$$

induced by the functor $\mathbf{Cyl}(\mathcal{D}) \to \mathbf{Cat}(1)^{\mathrm{op}}$ that maps a face $d : A \to \mathcal{D}$ to the small category A and any cylinder of the following form to the functor $a : A_1 \to A_0$.

$$\begin{tikzcd}
A_0 \arrow[d, "d_0"'] \arrow[dr, phantom, "\Searrow t"] & A_1 \arrow[l, "a"'] \arrow[d, "d_1"] \\
\mathcal{D} \arrow[r, equal] & \mathcal{D}
\end{tikzcd}$$

We will denote by top_K the restriction of top to K.

5.3.1.6. *Premodels of a croquis.* Let D be a small category, (K, T) be a croquis in D and $\mathcal{C}$ be a category. For any endofunctor $R : \mathcal{C} \to \mathcal{C}$, an *$R$-premodel of* (K, T) *in* $\mathcal{C}$ consists a functor $P : D \to \mathcal{C}$ that is equipped with a functor $E : K \to \mathbf{Cyl}(\mathcal{C})^2$ making the following diagrams commute.

$$\begin{tikzcd}
\mathbf{Cyl}(D) \arrow[r, "\mathbf{Cyl}(P)"] & \mathbf{Cyl}(\mathcal{C}) \\
K \arrow[u, "\mathrm{dom}_K"] \arrow[r, "E"] \arrow[d, "\mathrm{cod}_K^n"'] & \mathbf{Cyl}(\mathcal{C})^2 \arrow[u, "\mathrm{dom}"'] \arrow[d, "\mathrm{cod}"] \\
\mathbf{Cyl}(D) \arrow[r, "\mathbf{Cyl}(R \circ P \circ T)"'] & \mathbf{Cyl}(\mathcal{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Cyl}(D) \arrow[r, equal] & \mathbf{Cyl}(\mathcal{C}) \\
K \arrow[u, "\mathrm{top}_K"] \arrow[r, "E"] & \mathbf{Cyl}(\mathcal{C})^2 \arrow[u, "\mathrm{top}"']
\end{tikzcd}$$

In other words, the functor E will map any morphism of cylinders $(F, v) : (a, t) \Rrightarrow (a', t')$ in K, say of the form given in Remark 5.11, to a morphism in cylinder $\mathbf{Cyl}(\mathcal{C})^2$ as follows.

$$\begin{tikzcd}
 & A_0 \arrow[dd, "Pd_0"] \arrow[rr, phantom, "\Searrow E(t)" near end, yshift=-4ex] & & A_1 \arrow[ll, "a"'] \arrow[dd, "d_1"] \\
A'_0 \arrow[ur, "F_0"] \arrow[dd, "Pd'_0"'] \arrow[r, phantom, "Pv_0 \Downarrow"] & & & \\
 & \mathcal{C} \arrow[rr, equal] \arrow[dl, equal] \arrow[r, phantom, "=" below, yshift=-4ex] & & \mathcal{C} \arrow[dl, equal] \\
\mathcal{C} \arrow[rr, equal] & & \mathcal{C} &
\end{tikzcd}
=
\begin{tikzcd}
 & A_0 \arrow[dr, phantom, "="] & & A_1 \arrow[ll, "a"'] \arrow[dd, "RPTd_1"] \\
A'_0 \arrow[ur, "F_0"] \arrow[dd, "Pd'_0"'] \arrow[dr, phantom, "\Searrow E(t')"] & & A'_1 \arrow[ll, "a'"'] \arrow[ur, "F_1"'] \arrow[dd, "RPTd'_1"] \arrow[r, phantom, "RPTv_1 \Downarrow"] & \\
 & & & \mathcal{C} \\
\mathcal{C} \arrow[rr, equal] & & \mathcal{C} \arrow[ur, equal] &
\end{tikzcd}$$

The above equality of diagrams is equivalent to saying the following diagram commutes in $\mathcal{C}^{A'_1}$.

(5.4)
$$
\begin{tikzcd}
Pd_0F_0a' \arrow[r, Rightarrow, "E(t)F_1"] \arrow[d, Rightarrow, "Pv_0a'"'] & RPTd_1F_1 \arrow[d, Rightarrow, "RPTv_1"] \\
Pd'_0a' \arrow[r, Rightarrow, "E(t')"'] & RPTd'_1
\end{tikzcd}
$$

A *morphism of R-premodels of* (K,T) from (P,E) to (P',E) is given by any morphism $\alpha : P \Rightarrow P'$ in $\mathcal{C}^D$ that induces a natural transformation $[\alpha] : E \Rightarrow E'$ such that the evaluation of $[\alpha]$ at a cylinder $(a,t) : d_0 \Rightarrow d_1$ in K is given by a morphism in $\mathbf{Cyl}(\mathcal{C})^2$ of the following form.

(5.5)
$$
\begin{tikzcd}
 & A_0 \arrow[dl, equal] \arrow[dd, "d_0"] & & A_1 \arrow[ll, "a"'] \arrow[dd, "d_1"] \\
A_0 \arrow[dd, "d_0"'] & & & \\
 & D \arrow[dl, equal] \arrow[dd, "P"'] & \arrow[l, phantom, "\Downarrow E(t)"] & D \arrow[dd, "R\circ P\circ T"] \\
D \arrow[dd, "P'"'] & \arrow[l, phantom, "\Downarrow\alpha"] & & \\
 & \mathcal{C} \arrow[dl, equal] \arrow[rr, equal] & & \mathcal{C} \arrow[dl, equal] \\
\mathcal{C} \arrow[rr, equal] & & \mathcal{C} &
\end{tikzcd}
=
\begin{tikzcd}
 & A_0 \arrow[dl, equal] & & A_1 \arrow[ll, "a"'] \arrow[dl, equal] \arrow[dd, "d_1"] \\
A_0 \arrow[dd, "d_0"'] & & A_1 \arrow[ll, "a"'] \arrow[dd, "d_1"'] & \\
 & & & D \arrow[dl, equal] \arrow[dd, "R\circ P\circ T"] \\
D \arrow[dd, "P"'] & \arrow[r, phantom, "\Downarrow E'(t)"] & D \arrow[dd, "R\circ P'\circ T"'] & \arrow[l, phantom, "\Downarrow R\alpha T"] \\
 & & & \mathcal{C} \arrow[dl, equal] \\
\mathcal{C} \arrow[rr, equal] & & \mathcal{C} &
\end{tikzcd}
$$

The above equality of diagrams is equivalent to saying the following diagram commutes in $\mathcal{C}^{A'_1}$.

(5.6)
$$
\begin{tikzcd}
Pd_0a \arrow[r, Rightarrow, "E(t)"] \arrow[d, Rightarrow, "\alpha d_0 a"'] & RPTd_1 \arrow[d, Rightarrow, "R\alpha T d_1"] \\
P'd_0a \arrow[r, Rightarrow, "E'(t)"'] & RP'Td_1
\end{tikzcd}
$$

The category whose objects are R-premodels of K in $\mathcal{C}$ and whose morphisms are the morphisms of premodels between them will be denoted by $\mathbf{Pm}_{\mathcal{C}}(K,R,T)$.

5.3.1.7. *Natural premodels.* Let D be a small category and $\mathcal{C}$ be a category. For any pair of endofunctors $R : \mathcal{C} \to \mathcal{C}$ and $T : D \to D$, denote by $\mathbf{Np}_{\mathcal{C}}(D,R,T)$ the category whose objects are pairs (P,e) where P is a functor of type $D \to \mathcal{C}$ and e is a natural transformation $P \Rightarrow RPT$ and whose morphisms, say of the form $(P,e) \Rightarrow (P',e')$, are natural transformations $\alpha : P \Rightarrow P'$ for which the following diagram commutes.

(5.7)
$$
\begin{tikzcd}
P \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "e"'] & P' \arrow[d, Rightarrow, "e'"] \\
R\circ P\circ T \arrow[r, Rightarrow, "R\alpha T"'] & R\circ P'\circ T
\end{tikzcd}
$$

The goal of this section is to show that there exists a coretractive inclusion of the following form.

$$U : \mathbf{Np}_{\mathcal{C}}(D,R,T) \hookrightarrow \mathbf{Pm}_{\mathcal{C}}(K,R,T)$$

The functor U is defined as follows. Any object (P, e) in $\mathbf{Np}_{\mathcal{C}}(D, R, T)$ is mapped to an R-premodel $(P, [e])$ where the functor $[e] : K \to \mathbf{Cyl}(\mathcal{C})^2$ is the obvious functor post-composing a cylinder in K with the natural transformation $e : P \Rightarrow R \circ P$ as follows.

$$\begin{tikzcd} A_0 \arrow[d, "d_0"'] \arrow[dr, phantom, "\searrow t"] & A_1 \arrow[l, "a_0"'] \arrow[d, "d_1"] \\ D \arrow[r, equal] & D \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A_0 \arrow[d, "d_0"'] \arrow[dr, phantom, "\searrow t"] & A_1 \arrow[l, "a_0"'] \arrow[d, "d_1"] \\ D \arrow[r, equal] \arrow[d, "P"'] \arrow[dr, phantom, "\searrow e"] & D \arrow[d, "R \circ P \circ T"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

It is then easy to induce the mapping on the morphisms since every commutative diagram of the form (5.7) may be used to define a commutative diagram of the form (5.5) with respect to the above mapping on objects.

The retraction $V : \mathbf{Pm}_{\mathcal{C}}(K, R, T) \to \mathbf{Np}_{\mathcal{C}}(D, R, T)$ associated with U is defined as follows. An R-premodel (P, E) is mapped to the pair $(P, E \circ I_{\mathbf{1}})$ where the composite $E \circ I_{\mathbf{1}}$ maps an object d of D to a cylinder as follows.

$$\begin{tikzcd} \mathbf{1} \arrow[r, equal] \arrow[d, "P(d)"'] \arrow[dr, phantom, "E_0(\mathrm{id}_d) \searrow"] & \mathbf{1} \arrow[d, "RPT(d)"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Such a cylinder provides a morphism $E(\mathrm{id}_d) : P(d) \Rightarrow RPT(d)$ in $\mathcal{C}$ that is natural in d by functoriality of E. In other words, the composite $E \circ I_{\mathbf{1}} : D \to \mathbf{Cyl}(\mathcal{C})^2$ encodes a natural transformation of the form $P \Rightarrow R \circ P \circ T$. The mapping on the morphisms is the obvious one since diagram (5.6) shows that any morphism $\alpha : (P, E) \Rightarrow (P', E')$ of R-premodels makes the following diagram commute for every object d of D.

$$\begin{tikzcd} P(d) \arrow[r, Rightarrow, "E_0(\mathrm{id}_d)"] \arrow[d, Rightarrow, "\alpha_d"'] & R \circ P \circ T(d) \arrow[d, Rightarrow, "R\alpha_{T(d)}"] \\ P'(d) \arrow[r, Rightarrow, "E'_0(\mathrm{id}_d)"'] & R \circ P' \circ T(d) \end{tikzcd}$$

Example 5.20 (Functors)**.** The category of functors from a small category D to a category $\mathcal{C}$ corresponds to the full subcategory of $\mathbf{Np}_{\mathcal{C}}(D, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_D)$ whose objects (P, e) are such that the natural transformation $e : P \Rightarrow P$ is an identity.

Example 5.21 (Presheaves)**.** The category of presheaves over a category D^{op} corresponds to the full subcategory of $\mathbf{Np}_{\mathbf{Set}}(D, \mathrm{id}_{\mathbf{Set}}, \mathrm{id}_D)$ whose objects (P, e) are such that the natural transformation $e : P \Rightarrow P$ is an identity.

Example 5.22 (Prespectra)**.** If $\Omega : \mathbf{pTop} \to \mathbf{pTop}$ denotes the loop space functor on the category of pointed topological spaces and $\mathtt{succ}$ denotes the successor operation $n \mapsto n + 1$ on $\mathbb{N}$, then the category of prespectra is exactly $\mathbf{Np}_{\mathbf{Top}}(\mathbb{N}, \Omega, \mathtt{succ})$.

5.3.1.8. *Categories of premodels.* Let D be a small category, (K, T) be a croquis in D and $\mathcal{C}$ be a category. For any given endofunctor $R : \mathcal{C} \to \mathcal{C}$, a *category of R-premodels over* (K, T) is a subcategory of the category $\mathbf{Pm}_{\mathcal{C}}(K, R, T)$.

Example 5.23. Prespectra, functors and presheaves on a site are examples of such categories (see Example 5.22, Example 5.20 and Example 5.21)

5.3.1.9. *Categories of premodels to functor categories.* Let D be a small category, (K,T) be a croquis in D and $\mathcal{C}$ be a complete category. For any given endofunctor $R : \mathcal{C} \to \mathcal{C}$ in $\mathcal{C}$, consider a subcategory $\mathcal{P} \subseteq \mathbf{Pm}_{\mathcal{C}}(K,R,T)$. The goal of this section is to define a functor $\mathcal{G}\mathrm{l}^K : \mathcal{P} \to (\mathcal{C}^2)^K$ (where $\mathcal{G}\mathrm{l}$ stands for the word 'gluing').

For every object (P,E) in $\mathcal{P}$, the object $\mathcal{G}\mathrm{l}^K(P,E)$ in $(\mathcal{C}^2)^K$ is given by the functor $K \to \mathcal{C}^2$ that maps any cylinder in K of the form given on the left to an arrow in $\mathcal{C}$ of the form given on the right.

$$\begin{tikzcd} A_0 \arrow[d, "d_0"'] & A_1 \arrow[l, "a_0"'] \arrow[d, "d_1"] \arrow[dl, Rightarrow, "t"'] \\ D \arrow[r, equal] & D \end{tikzcd} \quad \mapsto \quad \lim_{A_0} Pd_0 \xrightarrow{u_a(Pd_0)} \lim_{A_1} Pd_0a \xrightarrow{\lim_{A_1} E(t)} \lim_{A_1} RPTd_1$$

The mapping on the arrows of K is deduced from those of the functors u_a, E, $\mathbf{Cyl}(P)$ and $\mathbf{Cyl}(R \circ P \circ T)$. Specifically, a morphism of cylinders $(F,u) : (a,t) \Rrightarrow (a',t')$ of the form given in Remark 5.11 will be mapped to the following commutative diagram.

$$\begin{tikzcd} \lim_{A_0} Pd_0 \arrow[r, "u_a(Pd_0)"] \arrow[d, "u_{F_0}(Pd_0)"] & \lim_{A_1} Pd_0a \arrow[r, "\lim_{A_1} E(t)"] \arrow[d, "u_{F_1}(Pd_0a)"] & \lim_{A_1} RPTd_1 \arrow[d, "u_{F_1}(RPTd_1)"] \\ \lim_{A_0'} Pd_0F_0 \arrow[r, "u_{a'}(Pd_0F_0)"] \arrow[d, "\lim_{A_0'} Pv_0"] & \lim_{A_1'} Pd_0F_0a' \arrow[r, "\lim_{A_1'} E(t)F_1"] \arrow[d, "\lim_{A_1'} Pv_0a'"] & \lim_{A_1'} RPTd_1F_1 \arrow[d, "\lim_{A_1'} RPTv_1"] \\ \lim_{A_0'} Pd_0' \arrow[r, "u_{a'}(Pd_0)"] & \lim_{A_1'} Pd_0a' \arrow[r, "\lim_{A_1'} E(t')"] & \lim_{A_1'} RPTd_1' \end{tikzcd}$$

The commutativity of the top-left square follows from Remark 5.10, that of the top-right squares follows from naturality of u_{F_1}, that of the bottom-left square follows from the naturality of the universal shifting $u_{a_0'}$ and that of the bottom-right square follows from diagram (5.4).

For every morphism $\alpha : (P,E) \Rightarrow (P',E')$, the arrow

$$\mathcal{G}\mathrm{l}^K(\alpha) : \mathcal{G}\mathrm{l}^K(P,E) \Rightarrow \mathcal{G}\mathrm{l}^K(P',E')$$

is given by a natural transformation in $\mathcal{C}^2$ over K whose components above a cylinder $(a,t) : d_0 \Rightarrow d_1$ in K is encoded by the following morphism in $\mathcal{C}^2$.

$$\begin{tikzcd} \lim_{A_0} Pd_0 \arrow[r, "u_a(Pd_0)"] \arrow[d, "\lim_{A_0} \alpha d_0"] & \lim_{A_1} Pd_0a \arrow[r, "\lim_{A_1} E(t)"] \arrow[d, "\lim_{A_1} \alpha d_0 a"] & \lim_{A_1} RPTd_1 \arrow[d, "\lim_{A_1} R\alpha Td_1"] \\ \lim_{A_0} P'd_0 \arrow[r, "u_a(P'd_0)"] & \lim_{A_1} P'd_0a \arrow[r, "\lim_{A_1} E'(t)"] & \lim_{A_1} RP'Td_1 \end{tikzcd}$$

The commutativity of the left square follows from naturality of u_a while that of the right square follows from diagram (5.6).

Proposition 5.24. *In the case where the category $\mathcal{P}$ is equal to $\mathcal{C}^D$, the functor $\mathcal{G}\mathrm{l}^K$ is coretractive.*

Proof. Let us show that the functor $\mathcal{G}\mathrm{l}^K : \mathcal{P} \to (\mathcal{C}^2)^K$ admits a retraction $(\mathcal{C}^2)^K \to \mathcal{P}$ when $\mathcal{P}$ is equal to $\mathcal{C}^D$. This follows from the fact that for every object G in $(\mathcal{C}^2)^K$ provides a functor $\mathrm{dom}_{\mathcal{C}} \circ G \circ \mathbf{I_1} : D \to \mathcal{C}$. The statement follows from the fact that for every premodel (P,e) in $\mathcal{C}^D$, the composite functor

$$D \xrightarrow{\mathbf{I_1}} K \xrightarrow{\mathcal{G}\mathrm{l}^K(P,[e])} \mathcal{C}^2 \xrightarrow{\mathrm{dom}_{\mathcal{C}}} \mathcal{C}$$

may be identified with the functor $P : D \to \mathcal{C}$. This shows that the functor $\mathcal{G}l^K : \mathcal{P} \to (\mathcal{C}^2)^K$ admits a retraction $(\mathcal{C}^2)^K \to \mathcal{P}$ when $\mathcal{P}$ is equal to $\mathcal{C}^D$. □

5.3.2. System of models for a croquis.

5.3.2.1. *Combinatorics for omega-functors.* Let $\mathcal{C}$ be a category and n be some non-negative integer. For every finite sequence of $n+1$ arrows $\{f_i : X_i \to X_{i+1}\}_{0\leq i\leq n}$ in $\mathcal{C}$ and every increasing[6] sequence of $n+1$ integers $\{x_i\}_{0\leq i\leq n}$, we shall denote by

$$^{[x_0]}f_0^{[x_1]}f_1 \dots {}^{[x_n]}f_n \tag{5.8}$$

the functor $\mathbf{O}(\omega) \to \mathcal{C}$ mapping the arrow $x_i \to x_i + 1$ to the arrow $f_i : X_i \to X_{i+1}$ and all the other arrows to identities on the objects $X_0, X_1, \dots X_n$ and X_{n+1} in the obvious way. Even though the mapping is clearly not defined at the arrow $x_i \to x_i + 1$ where $x_i \leq -1$, the notation of (5.8) will still be relevant for such values. For the sake of convenience, it will later come in handy to denote formula (5.8) as follows when $x_0 = x_1 - 1$.

$$^{[\cdot]}f_0^{[x_1]}f_1 \dots {}^{[x_n]}f_n$$

A morphism in $\mathcal{C}^\omega$ of the form

$$^{[x_0]}f_0^{[x_1]}f_1 \dots {}^{[x_n]}f_n \Rightarrow {}^{[x_0]}f_0^{[x_1]}f_1 \dots {}^{[x_n]}f_n$$

is always determined by an arrow $e : X_0 \to Y_0$ and a sequence of arrows $\{e_i : X_{i+1} \to X_{i+1}\}_{0\leq i\leq n}$ making the following diagrams commute.

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "e"'] & X_1 \arrow[d, "e_0"] \\ Y_0 \arrow[r, "g_0"] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "e_{i-1}"'] & X_{i+1} \arrow[d, "e_i"] \\ Y_i \arrow[r, "g_i"] & Y_{i+1} \end{tikzcd}$$

Such a morphism will later be denoted as a sequence of the form given below on the left, if e is not an identity and of the form given on the right otherwise.

$$^{|\cdot}e^{|x_0}e_0{}^{|x_1}e_1 \dots {}^{|x_n}e_n \qquad {}^{|x_0}e_0{}^{|x_1}e_1 \dots {}^{|x_n}e_n \tag{5.9}$$

Of course, the composition for such a notation is componentwise. A very important point to remember about the previous notations is that morphisms as in (5.9) are equal to the following morphisms for every $x_{n+1} > x_n$, respectively.

$$^{|\cdot}e^{|x_0}e_0{}^{|x_1}e_1 \dots {}^{|x_n}e_n{}^{|x_{n+1}}e_n \qquad {}^{|x_0}e_0{}^{|x_1}e_1 \dots {}^{|x_n}e_n{}^{|x_{n+1}}e_n$$

In order to make the above notations less cumbersome, we shall also forget the symbols of composition as often as possible; .e.g an object $^{[s]}\gamma^{[r]}\beta \circ \delta_2$ for some $s < r$ will instead be denoted as $^{[s]}\gamma^{[r]}\beta\delta_2$ while the composition rule will be written in the form $(^{|x_0}e_0{}^{|x_1}e_1) \circ (^{|x_0}h_0{}^{|x_1}h_1) = {}^{|x_0}e_0h_0{}^{|x_1}e_1h_0$. We shall also denote by 1 any identity morphism used in the notation (5.9).

The previous language will be applied in the context of vertebrae in section 5.3.2.4, but before before doing so, we will need to introduce some terminology, which justifies the next two sections (i.e. section 5.3.2.2 and section 5.3.2.3) whose content is quite unrelated to the present section.

5.3.2.2. *Semi-communications.* Let $\mathcal{C}$ be a category. A *semi-communication* in $\mathcal{C}$ is a communication $(\varkappa, \varrho) : \gamma \rightsquigarrow \gamma'$ living in the category $\mathbf{Com}(\mathcal{C})$ whose spherical transition $\varkappa$ is an identity morphism of $\mathcal{C}$.

[6] In the strict sense.

5.3.2.3. *Semi-direct vertebral algebras and categories.* Let $\mathcal{C}$ be a category. A vertebral algebra (E,η) defined along source and target hinges $h_0, h_1 : \mathrm{Obj}_{\mathrm{R}}(E) \to \mathrm{Obj}_{\mathrm{L}}(E)$ will be said to be *semi-direct* if it involves algebraic operations

$$\Sigma_0 E(\gamma,\gamma_*) \times \Sigma_1 E(\gamma_*,\gamma'_\flat) \to \Sigma_\star E(\gamma,\gamma'_\flat)$$

mapping any pair of extended nodes of vertebrae $\mathfrak{n} : \gamma \rightsquigarrow \overline{\nu}$ and $\mathfrak{n}_* : \gamma_* \rightsquigarrow \overline{\nu}_*$ to an extended node of vertebrae of the form $\mathfrak{n}_\bullet : \gamma \rightsquigarrow \overline{\nu}_\bullet$ where $\mathfrak{n}_\bullet$ frames the pair $\mathfrak{n}$ and $\mathfrak{n}_*$ along a given semi-communication $(\mathrm{id},\varrho) : \overline{\gamma}' \rightsquigarrow \gamma_*$ between $\mathfrak{n}$ and $\mathfrak{n}_*$.

Remark 5.25. Because, in the previous case, the E-seed $h_1(\mathfrak{n}_*)$ is determined by the coseed of $\overline{\nu}_*$, the algebra structure of (E,η) is equivalent to only specifying the source hinge h_0 and a mapping

$$(\mathfrak{n},\mathfrak{n}_*) \mapsto (\mathrm{id},\varrho(\mathfrak{n},\mathfrak{n}_*)) : \eta(\mathfrak{n}) \rightsquigarrow h_0(\mathfrak{n}) \tag{5.10}$$

defined for any pair of extended nodes of vertebrae $(\mathfrak{n},\mathfrak{n}_*)$ such that the seed $\eta(\mathfrak{n}_*)$ equals $h_0(\mathfrak{n})$ and the communication allows a framing of $\mathfrak{n}$ and $\mathfrak{n}_*$. The mapping given in (5.10) will later be referred to as an *extensive communication* for (E,η).

In the sequel, a vertebral category whose underlying vertebral algebra is semi-direct will also be said to be *semi-direct*.

5.3.2.4. *Transport of vertebrae towards omega-functors.* Let $\mathcal{C}$ be a category and v be a vertebra in $\mathcal{C}$ of the form given below on the left. For every pair s and r of non-negative integers satisfying the inequality $s < r$, it is not hard to see that the following righthand diagram defines a vertebra in $\mathcal{C}^\omega$ wherein the morphism 'id' stands for the obvious identity morphisms.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \\
\mathbb{D}_2 \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}' \arrow[r, "\beta"'] & \mathbb{D}'
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathrm{id} \arrow[r, Rightarrow, "{}^{|s}1^{|r}\gamma'"] \arrow[d, Rightarrow, "{}^{|s}\gamma^{|r}\gamma"'] & {}^{[r]}\gamma' \arrow[d, Rightarrow, "{}^{|s}\gamma^{|r}\delta_1"] \\
{}^{[s]}\gamma \arrow[r, Rightarrow, "{}^{|s}1^{|r}\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & {}^{[s]}\gamma^{[r]}\delta_2 \arrow[r, Rightarrow, "{}^{|r}\beta"'] & {}^{[s]}\gamma^{[r]}\beta\delta_2
\end{tikzcd} \tag{5.11}$$

The right vertebra will be denoted by $\mathbf{T}^r_s(v)$ for every pair of integers s and r (i.e. non necessarily non-negative) satisfying the inequality $s < r$.

Proposition 5.26. *If v is reflexive with reflexive transition $\lambda : \mathbb{D}_1 \to \mathbb{D}_2$ and homotopic contraction $\alpha : \mathbb{D}' \to \mathbb{D}_2$, then so is $\mathbf{T}^r_s(v)$ with reflexive transition ${}^{|s}\gamma^{|r}\lambda : {}^{[r]}\gamma' \Rightarrow {}^{[s]}\gamma$ and homotopic contraction ${}^{|r}\alpha : {}^{[s]}\gamma^{[r]}\beta\delta_2 \Rightarrow {}^{[s]}\gamma$.*

Proof. The proof follows from a straightforward verification of the definitions. The reflexivity of the base of $\mathbf{T}^r_s(v)$ is given by the equation $({}^{|s}\gamma^{|r}\lambda) \circ ({}^{|s}1^{|r}\gamma') = {}^{|s}\gamma^{|r}\gamma$ while the reflexive structure of $\mathbf{T}^r_s(v)$ follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\mathrm{id} \arrow[r, Rightarrow, "{}^{|s}1^{|r}\gamma'"] \arrow[d, Rightarrow, "{}^{|s}\gamma^{|r}\gamma"'] & {}^{[r]}\gamma' \arrow[d, Rightarrow, "{}^{|s}\gamma^{|r}\delta_1"] \arrow[drr, Rightarrow, bend left, "{}^{|s}\gamma^{|r}\lambda"] \\
{}^{[s]}\gamma \arrow[r, Rightarrow, "{}^{|s}1^{|r}\delta_2"] \arrow[ur, phantom, "\ulcorner"] \arrow[rrr, Rightarrow, bend right, "{}^{|s}1^{|r}1"'] & {}^{[s]}\gamma^{[r]}\delta_2 \arrow[r, Rightarrow, "{}^{|r}\beta"] & {}^{[s]}\gamma^{[r]}\beta\delta_2 \arrow[r, Rightarrow, "{}^{|r}\alpha"] & {}^{[s]}\gamma
\end{tikzcd}$$

□

It is possible to extend the definition of $\mathbf{T}^r_s(v)$ to the case where $s = r$. For any integer r, the extension $\mathbf{T}^r_r(v)$ is given by the following right vertebra, which is not reflexive but does preserve the structure of communication and framing.

$$(5.12)\qquad \begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}' \arrow[r, "\beta"'] & \mathbb{D}' \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \mathrm{id} \arrow[r, Rightarrow, "{}^{|r}\gamma'"] \arrow[d, Rightarrow, "{}^{|r}\gamma"'] & {}^{[r]}\gamma' \arrow[d, Rightarrow, "{}^{|r}\delta_1"] \\ {}^{[r]}\gamma \arrow[r, Rightarrow, "{}^{|r}\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & {}^{[r]}\delta_2\gamma \arrow[r, Rightarrow, "{}^{|\cdot}\gamma^{|r}\beta"'] & {}^{[\cdot]}\gamma^{[r]}\beta\delta_2 \end{tikzcd}$$

Proposition 5.27. *If two vertebrae v and v_*, seen as extended vertebrae $\gamma \overset{\text{ex}}{\rightsquigarrow} v$ and $\gamma_* \overset{\text{ex}}{\rightsquigarrow} v_*$, communicate via a semi-communication $(\mathrm{id}, \varrho) : \gamma' \rightsquigarrow \gamma_*$, then so do $\mathbf{T}^r_s(v)$ and $\mathbf{T}^n_q(v_*)$ for every quadruple of non-negative integers $s \leq r \leq q \leq n$ via the following semi-communications with respect to the underlying conditions.*

$$\underbrace{({}^{|r}1, {}^{|r}\gamma'^{|q}\varrho) : {}^{|r}\gamma' \rightsquigarrow {}^{|q}\gamma_*}_{\textit{when } r<q} \qquad \underbrace{({}^{|r}1, {}^{|r}\varrho) : {}^{|r}\gamma' \rightsquigarrow {}^{|r}\gamma_*}_{\textit{when } r=q}$$

Proof. It suffices to notice that following left diagram commutes when $r < q$ while the right one commutes for any non-negative integer r.

$$\begin{tikzcd} \mathrm{id} \arrow[r, equal, "{}^{|r}1^{|q}1"] \arrow[d, Rightarrow, "{}^{|r}1^{|q}\gamma_*"'] & \mathrm{id} \arrow[d, Rightarrow, "{}^{|r}\gamma'"] \\ {}^{[q]}\gamma_* \arrow[r, Rightarrow, "{}^{|r}\gamma'^{|q}\varrho"'] & {}^{[r]}\gamma' \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id} \arrow[r, equal, "{}^{|r}1"] \arrow[d, Rightarrow, "{}^{|r}\gamma_*"'] & \mathrm{id} \arrow[d, Rightarrow, "{}^{|r}\gamma'"] \\ {}^{[r]}\gamma_* \arrow[r, Rightarrow, "{}^{|r}\varrho"'] & {}^{[r]}\gamma' \end{tikzcd}$$

The preceding squares indeed define semi-communications. □

Although the next proposition is proven for particular cases of the inequalities $s \leq r \leq q \leq n$, its statement may be extended to all cases. However, only the cases that are listed will turn out to be necessary to the purposes of this thesis.

Proposition 5.28. *If a vertebra $v_\bullet$ frames a pair of vertebrae v and v_* that communicate via a semi-communication $(\mathrm{id}, \varrho) : \gamma' \rightsquigarrow \gamma_*$, then the vertebra*

1) *$\mathbf{T}^n_s(v_\bullet)$ frames the induced communicating pair $\mathbf{T}^r_s(v)$ and $\mathbf{T}^n_q(v_*)$ for non-negative integers satisfying the inequalities $s < r \leq q < n$ or the relations $r = q = n$ and $s = q - 1$;*
2) *$\mathbf{T}^n_{r-1}(v_\bullet)$ frames the induced communicating pair $\mathbf{T}^r_s(v)$ and $\mathbf{T}^n_q(v_*)$ for non-negative integers satisfying the inequalities $s = r \leq q < n$ or the equalities $s = r = q = n$;*

Proof. Denote $v := \|\gamma, \gamma\| \cdot \beta$ and $v_* := \|\gamma_*, \gamma'_*\| \cdot \beta_*$ so that the framing vertebra $v_\bullet$ must be of the form $\|\gamma, \gamma'_*\| \cdot \beta_\bullet$. The structure of framing for the communicating pair $v : \gamma \overset{\text{ex}}{\rightsquigarrow} v$ and $v_* \odot (\mathrm{id}, \varrho) : \gamma' \overset{\text{ex}}{\rightsquigarrow} v_*$ will be supposed to be given by the following left pushout square as well as a cooperadic transition $\eta : \mathbb{D}'_\bullet \to \mathbb{E}$ (making the right diagram commute).

$$\begin{tikzcd} \mathbb{D}^*_2 \arrow[r, "\beta\circ\delta_1\circ\varrho"] \arrow[d, "\beta_*\circ\delta^*_2"'] & \mathbb{D}' \arrow[d, "\varepsilon_1"] \\ \mathbb{D}'_* \arrow[r, "\varepsilon_2"'] \arrow[ur, phantom, "\lrcorner"] & \mathbb{E} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}'_\bullet \arrow[dr, "\eta\circ\beta_\bullet"] & \mathbb{D}^*_1 \arrow[l, "\delta^\bullet_1"'] \arrow[d, "\varepsilon_2\circ\beta_*\circ\delta^*_1"] \\ \mathbb{D}_2 \arrow[u, "\delta^\bullet_2"] \arrow[r, "\varepsilon_1\circ\beta\circ\delta_2"'] & \mathbb{E} \end{tikzcd}$$

Throughout the proof, the quantity x will stand for the integer s when $s < r$ and $r - 1$ when $s = r$. Proposition 5.27 shows that the vertebrae $\mathbf{T}^r_s(v)$ and $\mathbf{T}^n_q(v_*)$ communicate for every set of integers $s \leq r \leq q \leq n$. In the case where $s \leq r < q < n$, the discal transition given by Proposition 5.27 is of the form ${}^{|r}\gamma'^{|q}\varrho$. It then takes a few line of calculation to check that the framing structure is given by the following left pushout square in $\mathcal{C}^{\omega}$ while the cooperadic transition is given by the right vertical morphism.

$$\begin{tikzcd}
{}^{[q]}\gamma_* \arrow[r, Rightarrow, "{{}^{|x}\gamma^{|r}\beta\delta_1\gamma'^{|q}\beta\delta_1\varrho^{|n}\beta\delta_1\varrho}"] \arrow[d, Rightarrow, "{{}^{|x}1^{|n}\beta_*\delta_2^*}"'] & {}^{[x]}\gamma^{[r]}\beta\delta_2 \arrow[d, Rightarrow, "{{}^{|x}1^{|n}\varepsilon_1}"] & {}^{[x]}\gamma^{[n]}\beta_\bullet\delta_2^\bullet \arrow[d, Rightarrow, "{{}^{|r}\beta\delta_2{}^{|n}\eta}"] \\
{}^{[q]}\gamma_*{}^{[n]}\beta_*\delta_2^* \arrow[r, Rightarrow, "{{}^{|x}\gamma^{|r}\beta\delta_2\gamma^{|q}\beta\delta_1\varrho^{|n}\varepsilon_2}"'] & {}^{[x]}\gamma^{[r]}\beta\delta_2{}^{[n]}\varepsilon_1 \arrow[ul, phantom, "\ulcorner", very near start] & {}^{[x]}\gamma^{[r]}\beta\delta_2{}^{[n]}\varepsilon_1
\end{tikzcd}$$

In the case where $s \leq r = q < n$, the discal transition given by Proposition 5.27 is of the form ${}^{|q}\varrho$. It then takes a few line of calculation to check that the framing structure is given by the following left pushout square in $\mathcal{C}^{\omega}$ while the cooperadic transition is given by the right vertical morphism.

$$\begin{tikzcd}
{}^{[q]}\gamma_* \arrow[r, Rightarrow, "{{}^{|x}\gamma^{|q}\beta\delta_1\varrho^{|n}\beta\delta_1\varrho}"] \arrow[d, Rightarrow, "{{}^{|x}1^{|n}\beta_*\delta_2^*}"'] & {}^{[x]}\gamma^{[q]}\beta\delta_2 \arrow[d, Rightarrow, "{{}^{|x}1^{|n}\varepsilon_1}"] & {}^{[x]}\gamma^{[n]}\beta_\bullet\delta_2^\bullet \arrow[d, Rightarrow, "{{}^{|q}\beta\delta_2{}^{|n}\eta}"] \\
{}^{[q]}\gamma_*{}^{[n]}\beta_*\delta_2^* \arrow[r, Rightarrow, "{{}^{|x}\gamma^{|q}\beta\delta_1\varrho^{|n}\varepsilon_2}"'] & {}^{[x]}\gamma^{[q]}\beta\delta_2{}^{[n]}\varepsilon_1 \arrow[ul, phantom, "\ulcorner", very near start] & {}^{[x]}\gamma^{[q]}\beta\delta_2{}^{[n]}\varepsilon_1
\end{tikzcd}$$

In the case where $r = q = n$ and s is equal to either $q-1$ or q, the discal transition given by Proposition 5.27 is of the form ${}^{|q}\varrho$. It then takes a few line of calculation to check that the framing structure is given by the following left pushout square in $\mathcal{C}^{\omega}$ while the cooperadic transition is given by the right vertical morphism.

$$\begin{tikzcd}
{}^{[q]}\gamma_* \arrow[r, Rightarrow, "{{}^{|\cdot}\gamma^{|q}\beta\delta_1\varrho}"] \arrow[d, Rightarrow, "{{}^{|\cdot}1^{|q}\beta_*\delta_2^*}"'] & {}^{[\cdot]}\gamma^{[q]}\beta\delta_2 \arrow[d, Rightarrow, "{{}^{|\cdot}\beta\delta_2{}^{|q}\varepsilon_1}"] & {}^{[\cdot]}\gamma^{[q]}\beta_\bullet\delta_2^\bullet \arrow[d, Rightarrow, "{{}^{|\cdot}\beta\delta_2{}^{|q}\eta}"] \\
{}^{[q]}\beta_*\delta_2^*\gamma_* \arrow[r, Rightarrow, "{{}^{|\cdot}\beta\delta_1\varrho^{|q}\varepsilon_2}"'] & {}^{[\cdot]}\beta\delta_2\gamma^{[q]}\varepsilon_1 \arrow[ul, phantom, "\ulcorner", very near start] & {}^{[\cdot]}\beta\delta_2\gamma^{[q]}\varepsilon_1
\end{tikzcd}$$

Finally, this covers all the cases of the statement. $\square$

5.3.2.5. *Cohesive set of vertebrae.* Let $\mathcal{C}$ be a category and $\mathtt{V}$ be a set of vertebrae in $\mathcal{C}$. Denote by $A(\mathtt{V})$ the subgraph of $\mathbf{Ally}(\mathcal{C})$ (see section 2.3.2.1) whose object-class is equal to $\mathtt{V}$ and whose alliances are identities in $\mathbf{Ally}(\mathcal{C})$. Similarly, denote by $E(\mathtt{V})$ the subspan of $\mathbf{Enov}(\mathcal{C})$ whose left object-class contains the seed of the vertebrae in $\mathtt{V}$, whose right object-class is equal to $\mathtt{V}$ and whose extended nodes of vertebrae are given by the vertebrae of $\mathtt{V}$.

Proposition 5.29. *The triple $(A(\mathtt{V}), A(\mathtt{V}), E(\mathtt{V}))$ defines a $\mathbf{Enov}(\mathcal{C})$-prolinear module whose prolinear map is given by the identity $A(\mathtt{V}) \looparrowright A(\mathtt{V})$ (see Example 4.25).*

Proof. It is easy to check that $A(\mathtt{V})$ defines a submagmoid of $\mathbf{Ally}(\mathcal{C})$; $E(\mathtt{V})$ defines a sub-precompass of $(\mathbf{Enov}(\mathcal{C}), \eta, \odot)$ and the trivial action $(v, \mathrm{id}_v) \mapsto v$ gives a structure of right A-submodule of $(\mathbf{Enov}(\mathcal{C}), \eta, \odot, \circledcirc)$ to $E(\mathtt{V})$. $\square$

This structure will be referred to as the *modular structure of* $\mathtt{V}$. A set $\mathtt{V}$ of vertebrae in $\mathcal{C}$ will be said to be *cohesive* if

1) the seeds and coseeds of the vertebrae in $\mathtt{V}$ are not identities in $\mathcal{C}$;
2) it is equipped with a function $\psi : \mathtt{V} \to \mathtt{V}$, called *cohesion*, such that for every vertebra $v \in \mathtt{V}$,
   - v^{rv} communicates with $(\psi(v))^{\mathrm{rv}}$ via a semi-communication t_v;
   - v^{rv} frames the communicating pair v^{rv} and $(\psi(v))^{\mathrm{rv}} \odot t_v$;

3) the modular structure of $\mathtt{V}$ is equipped with a structure of a semi-direct vertebral category;

Now, for every cohesive set $\mathtt{V}$ of vertebrae in $\mathcal{C}$, define the following set of vertebrae in the category $\mathcal{C}^{\omega}$, which is going to be the set of vertebrae defining the homotopy theory associated with $\mathcal{C}^{\omega}$.

$$\mathtt{V}^{(\omega)} := \{\mathbf{T}_s^r(v) \mid v \in \mathtt{V} \text{ and } s = r = 0 \text{ or } -1 \leq s < r\}$$

Proposition 5.30. *For every seed g of a vertebra in $\mathtt{V}^{(\omega)}$, there exists a unique pair (x, γ) such that $g = {}^{|x}\gamma$ where γ is an $E(\mathtt{V})$-seed. Similarly, every vertebra $\mathbf{v}$ in $\mathtt{V}^{(\omega)}$, there exists a unique triple (s, r, v) such that $\mathbf{v} = \mathbf{T}_s^r(v)$.*

Proof. For both statements, the existence follows from the definitions. To show the uniqueness of the statement regarding the seeds, suppose that an equality of the form ${}^{|x}\gamma = {}^{|y}\gamma$ holds. It is not hard to see that if the inequality $x \neq y$ holds, then the previous equality forces the identities $\gamma = \gamma' = \mathrm{id}$, which is impossible since $\mathtt{V}$ is cohesive. Now, suppose that an equality of the form $\mathbf{T}_s^r(v) = \mathbf{T}_q^n(v_*)$ holds for any pair of vertebrae $v = \|\gamma, \gamma\| \cdot \beta$ and $v_* = \|\gamma_*, \gamma'_*\| \cdot \beta_*$. If the inequality $s \neq q$ holds, then the equality involved at the level of the seeds implies the identities $\gamma = \gamma_* = \mathrm{id}$, which is impossible. Similarly, the equality involved at the level of the coseeds makes the inequality $r \neq n$ impossible. In other words, the identities $s = q$ and $r = n$ hold, which force the equalities $\gamma = \gamma_*$, $\gamma' = \gamma'_*$ and $\beta = \beta_*$ and hence $v = v_*$. $\square$

Proposition 5.31. *For every cohesive set of vertebrae $\mathtt{V}$ in $\mathcal{C}$, the modular structure of $\mathtt{V}^{(\omega)}$ induces a semi-direct vertebral category in $\mathcal{C}^{\omega}$.*

Proof. Recall that a vertebral category is a vertebral algebra satisfying some reflexivity condition (see section 4.4.1.1, p. 174). Let us first prove the structure of vertebral algebra structure. Following Remark 5.25, suppose that the precompass $(E(\mathtt{V}), \mathfrak{y})$ has a structre of algebra determined by its source hinge $h_0 : \mathrm{Obj}_{\mathrm{R}}(E(\mathtt{V})) \to \mathrm{Obj}_{\mathrm{L}}(E(\mathtt{V}))$ and an extensive communication $(v, v_*) \mapsto (\mathrm{id}, \varrho(v, v_*))$. By Proposition 5.30, it makes sense to define a metafunction

$$h_0^{\mathbf{T}} : \mathrm{Obj}_{\mathrm{R}}(E(\mathtt{V}^{(\omega)})) \to \mathrm{Obj}_{\mathrm{L}}(E(\mathtt{V}^{(\omega)}))$$

with a mapping rule of the form $\mathbf{T}_s^r(v) \mapsto {}^{|x}h_0(v)$ for any choice $x \geq r$. It also follows from Proposition 5.30 that if $h_0^{\mathbf{T}}(\mathbf{T}_s^r(v))$ is equal to the seed of a vertebra $\mathbf{T}_q^n(v_*)$ in $\mathtt{V}^{(\omega)}$, then the equality $x = q$ holds and the $E(\mathtt{V})$-seed $h_0(v)$ is equal to the seed of v_*. Because $E(\mathtt{V})$ is a semi-direct vertebral algebra, there must hence exist a vertebra $v_\bullet$ framing the pair v and v_* via a semi-communication $(\mathrm{id}, \varrho(v, v_*)) : \gamma' \rightsquigarrow \gamma_*$. Let us use this framing to define the framing of $\mathbf{T}_s^r(v)$ and $\mathbf{T}_q^n(v_*)$. First, Proposition 5.27 implies that the extensive communication of $\mathtt{V}^{(\omega)}$ may be defined as follows.

$$(\mathbf{T}_s^r(v), \mathbf{T}_q^n(v_*)) \;\mapsto \begin{cases} ({}^{|r}1, {}^{|r}\gamma'^{|q}\varrho(v, v_*)) & \text{when } r < q = x; \\ ({}^{|r}1, {}^{|r}\varrho(v, v_*)) & \text{when } r = q = x; \end{cases}$$

By definition of $\mathtt{V}^{(\omega)}$, the only possible quadruples of integers $s \leq r \leq q \leq n$ for the pair $(\mathbf{T}_s^r(v), \mathbf{T}_q^n(v_*))$ must be the following.

$$(-1, 0, 0, 0) \quad (-1, 0, 0 < n) \quad (-1, 0 < q < n) \quad (0, 0, 0, 0)$$

$$(0, 0, 0 < n) \quad (0, 0 < q < n) \quad (1 < r \leq q < n)$$

This exactly covers the cases of Proposition 5.28. This means that the vertebra $\mathbf{T}_x^n(v_\bullet)$ frames the communicating pair $\mathbf{T}_s^r(v)$ and $\mathbf{T}_q^n(v_*)$ where the quantity x stands for $r - 1$ when $s = r$ and s when $s < r$. It only remains to check that $\mathbf{T}_x^n(v_\bullet)$ belongs to $\mathtt{V}^{(\omega)}$. If the equality $s = r$ holds, then s and r must be zero, which implies that $(x, n) = (-1, n)$ with $n > -1$. Otherwise, it is straightforward. This shows that the modular structure of $\mathtt{V}^{(\omega)}$ has a vertebral algebra structure.

Let us now prove the reflexive structure. By Proposition 5.30, any $E(\mathtt{V})$-seed g is of the form ${}^{|s}\gamma$ – where the inequality $s \geq -1$ must hold. Because $E(\mathtt{V})$ defines a vertebral category, there exists a reflexive vertebra $\gamma \rightsquigarrow v$ in $\mathtt{V}$. The $E(\mathtt{V})$-seed g is therefore the seed of any vertebra of the form $\mathbf{T}^r_s(v)$, which, by Proposition 5.26, is reflexive in the case where $s < r$. $\square$

In the sequel, for any cohesive set $\mathtt{V}$ of vertebrae in $\mathcal{C}$ and every vertebra v in $\mathtt{V}$, a vertebra of the form $\mathbf{T}^r_s(v)$ in $\mathtt{V}^{(\omega)}$ will formally be denoted as follows.

$$\begin{tikzcd}
\mathbf{I}(v) \arrow[r, Rightarrow, "\gamma'"] \arrow[d, Rightarrow, "\gamma"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbf{D}^r(v) \arrow[d, Rightarrow, "\delta_1"] & \\
\mathbf{D}_s(v) \arrow[r, Rightarrow, "\delta_2"'] & \mathbf{S}^r_s(v) \arrow[r, Rightarrow, "\beta"'] & \mathbf{D}^r_s(v)
\end{tikzcd}$$

5.3.2.6. *Fibrant objects.* Let $\mathcal{C}$ be a category and $\mathtt{V}$ be a cohesive set of vertebrae in $\mathcal{C}$. An object X in $\mathcal{C}^\omega$ will be said to be $\mathtt{V}$-*fibrant* if for every vertebra $\mathbf{T}^r_s(v)$ in $\mathtt{V}^{(\omega)}$ and morphism of the form $\mathbf{D}^r(v) \Rightarrow X$ in $\mathcal{C}^\omega$, there exists a morphism $h : \mathbf{D}^r_s(v) \Rightarrow X$ making the following diagram commute.

$$\begin{tikzcd}
\mathbf{D}^r(v) \arrow[r, Rightarrow, "x"] \arrow[d, Rightarrow, "\beta\circ\delta_1"'] & X \\
\mathbf{D}^r_s(v) \arrow[ur, Rightarrow, dashed, "h"'] &
\end{tikzcd}$$

If the category $\mathcal{C}^\omega$ has a terminal object $\mathbf{1}$, this is equivalent to saying that the canonical arrow $X \Rightarrow \mathbf{1}$ is a fibration for every vertebra in $\mathtt{V}^{(\omega)}$ and hence a fibration for its structure of vertebral category.

In the following two propositions, the image of the arrow $n \to n+1$ via the object $X : \mathbf{O}(\omega) \to \mathcal{C}$ will be denoted as an arrow $i_n : X_n \to X_{n+1}$ for every $n \in \omega$.

Proposition 5.32. *If X is $\mathtt{V}$-fibrant, then the image of the arrow $0 \to n$ via the functor X is a surtraction for the dual v^{rv} of every vertebra v in $\mathtt{V}$ for every $n \geq 1$.*

$$X_0 \xrightarrow{i_{n-1}\circ\cdots\circ i_0} X_n$$

Proof. Let $v^{\mathrm{rv}} := \|\gamma', \gamma\| \cdot \beta$ be the dual of a vertebra in $\mathtt{V}$ and denote by $f_n : X_0 \to X_n$ the arrow displayed in the above statement for every $n \geq 1$. To show that f_n is a surtraction for v^{rv}, consider a commutative square as given below on the left, for some integer $n \geq 1$. The diagram on the right-hand side of the implication is the same diagram that has been reflected about the diagonal top-left to bottom-right.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma'"'] & X_0 \arrow[d, "f_n"] & & \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "x"'] & \mathbb{D}_1 \arrow[d, "y"] \\
\mathbb{D}_1 \arrow[r, "y"'] & X_n & \Rightarrow & X_0 \arrow[r, "f_n"'] & X_n
\end{tikzcd} \tag{5.13}$$

If we decompose the morphism f_n into its composite i_k for every $0 \leq k \leq n-1$, the above right diagram may be turned into the following one, which exactly corresponds to giving a morphism $a : \mathbf{D}^{n-1}(v) \Rightarrow X$ in $\mathcal{C}^{\omega}$.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "="] \arrow[d, "x"'] & \mathbb{S} \arrow[r, "="] \arrow[d, "i_0\circ x"'] & \mathbb{S} \arrow[r, "="] \arrow[d, "i_1\circ i_0\circ x"'] & \dots \arrow[r, "="] & \mathbb{S} \arrow[r, "\gamma'"] \arrow[d] & \mathbb{D}_1 \arrow[r, "="] \arrow[d, "y"'] & \dots \\
X_0 \arrow[r, "i_0"'] & X_1 \arrow[r, "i_1"'] & X_2 \arrow[r, "i_2"'] & \dots \arrow[r, "i_{n-2}"'] & X_{n-1} \arrow[r, "i_{n-1}"'] & X_n \arrow[r] & \dots
\end{tikzcd}$$

Since the object X is a $\mathtt{V}$-fibrant, their exists a lift $h : \mathbf{D}_0^{n-1}(v) \Rightarrow X$ making the left diagram, below, commute in $\mathcal{C}^{\omega}$. When $n = 1$, this factorisation provides the succeeding right factorisation in $\mathcal{C}$ when evaluated above the inequality $0 < 1$ in $\mathbf{O}(\omega)$. This exactly says that f_0 is divisible by the underlying besom of the vertebra v^{rv}, which means that f_0 is a surtraction for v^{rv}.

$$\begin{tikzcd}
\mathbf{D}^{n-1}(v) \arrow[r, Rightarrow, "x"] \arrow[d, Rightarrow, "\beta\circ\delta_1"'] & X \\
\mathbf{D}_0^{n-1}(v) \arrow[ur, Rightarrow, dashed, "h"'] &
\end{tikzcd}
\qquad \Rightarrow \qquad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] \arrow[rr, bend left, "x"] & \mathbb{D}_2 \arrow[r, "h_0"] \arrow[d, "\beta\circ\delta_2"] & X_0 \arrow[d, "f_1"] \\
\mathbb{D}_1 \arrow[r, "\beta\circ\delta_1"] \arrow[rr, bend right, "y"'] & \mathbb{D}' \arrow[r, "h_1"] & X_1
\end{tikzcd}$$

In the case where $n > 1$, we are going to use an inductive argument. Suppose that the inequality $n > 1$ holds and the statement of the proposition is true for $n - 1$. The above left factorisation provides the following commutative diagram.

(5.14)
$$\begin{tikzcd}
\mathbb{S} \arrow[r, "="] \arrow[d, "="] \arrow[dd, bend right, "x"'] & \mathbb{S} \arrow[r, "="] \arrow[d, "\gamma"'] & \mathbb{S} \arrow[r, "="] \arrow[d, "\gamma"'] & \dots \arrow[r, "="] & \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[r, "="] \arrow[d, "\beta\circ\delta_1"'] \arrow[dd, bend left, "y"] & \dots \\
\mathbb{S} \arrow[r, "\gamma"] \arrow[d, "h_0"] & \mathbb{D}_2 \arrow[r, "="] \arrow[d, "h_1"'] & \mathbb{D}_2 \arrow[r, "="] \arrow[d, "h_2"'] & \dots \arrow[r, "="] & \mathbb{D}_2 \arrow[r, "\beta\circ\delta_2"] \arrow[d, "h_{n-1}"'] & \mathbb{D}' \arrow[r, "="] \arrow[d, "h_n"'] & \dots \\
X_0 \arrow[r, "i_0"'] & X_1 \arrow[r, "i_1"'] & X_2 \arrow[r] & \dots \arrow[r, "i_{n-2}"'] & X_{n-1} \arrow[r, "i_{n-1}"'] & X_n \arrow[r] & \dots
\end{tikzcd}$$

Before continuing, recall that the cohesion $\psi : \mathtt{V} \to \mathtt{V}$ is defined so that the vertebra $v^{\mathrm{rv}} = \|\gamma', \gamma\| \cdot \beta$ communicates with the image $(\psi(v))^{\mathrm{rv}}$, say of the form $\|\gamma'_*, \gamma_*\| \cdot \beta_*$, via a communication $(\mathrm{id}, \varrho) : \gamma \rightsquigarrow \gamma'_*$ and v^{rv} frames the communication. In particular, this last point forces the equality $\gamma_* = \gamma$. Below is given from left to right the structure of communication, the pushout of the framing and the commutative diagram associated with the following cooperadic transition $\eta : \mathbb{D}' \to \mathbb{E}$.

(5.15)
$$\begin{tikzcd}
\mathbb{S}_* \arrow[r, equal] \arrow[d, "\gamma'_*"'] & \mathbb{S} \arrow[d, "\gamma"] \\
\mathbb{D}_2^* \arrow[r, "\varrho"'] & \mathbb{D}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_2^* \arrow[r, "\beta\circ\delta_2\circ\varrho"] \arrow[d, "\beta_*\circ\delta_1^*"'] & \mathbb{D}' \arrow[d, "\varepsilon_1"] \\
\mathbb{D}'_* \arrow[r, "\varepsilon_2"'] & \mathbb{E} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}' \arrow[dr, "\eta\circ\beta"] & \mathbb{D}_2^* \arrow[l, "\delta_2"'] \arrow[d, "\varepsilon_2\circ\beta_*\circ\delta_2^*"] \\
\mathbb{D}_1 \arrow[u, "\delta_1"] \arrow[r, "\varepsilon_1\circ\beta\circ\delta_1"'] & \mathbb{E}
\end{tikzcd}$$

Now, if we come back to the proof, diagram (5.14) leads to the following left commutative diagram. Using the factorisation $\gamma = \varrho \circ \gamma'_*$ of diagram (5.15) and the fact that $f_{n-1} : X_0 \to X_{n-1}$ is a surtraction for $(\psi(v))^{\mathrm{rv}}$ (by induction), there exist two arrows $x' : \mathbb{D}_2 \to X_0$ and

$h : \mathbb{D}' \to X_{n-1}$ factorising this latest diagram into the right commutative diagram.

$$
\begin{tikzcd}
\mathbb{S} \arrow[rr, "x"] \arrow[dr, "\gamma"] \arrow[dd, "\gamma'"'] & & X_0 \arrow[d, "f_{n-1}"] \arrow[dd, bend left, "f_n"] \\
& \mathbb{D}_2 \arrow[r, "h_1"] \arrow[d, "\beta\circ\delta_2"] & X_{n-1} \arrow[d, "i_{n-1}"] \\
\mathbb{D}_1 \arrow[r, "\beta\circ\delta_1"] \arrow[rr, bend right, "y"'] & \mathbb{D}' \arrow[r, "h_n"] & X_n
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[rr, "\gamma"] \arrow[rrr, bend left, "x"] \arrow[dr, "\gamma'_*"] \arrow[dd, "\gamma'"'] & & \mathbb{D}_2 \arrow[r, "x'"] \arrow[d, "\beta_*\circ\delta_2^*"] & X_0 \arrow[d, "f_{n-1}"] \\
& \mathbb{D}_1^* \arrow[r, "\beta_*\circ\delta_1^*"] \arrow[rr, bend right, "h_1\circ\varrho"] \arrow[d, "\beta\circ\delta_2\circ\varrho"] & \mathbb{D}'_* \arrow[r, "h'"] & X_{n-1} \arrow[d, "i_{n-1}"] \\
\mathbb{D}_1 \arrow[r, "\beta\circ\delta_1"] \arrow[rrr, bend right, "y"'] & \mathbb{D}' \arrow[rr, "h_n"] & & X_n
\end{tikzcd}
$$

If follows from the structure of framing given in (5.15) that we may form the pushout $\mathbb{E}$ in the above right diagram. This implies the existence of a canonical arrow $h' : \mathbb{E} \to X_n$ making the following left diagram commute. Using the diagrammatic relations of the left diagram of (5.15) then leads to the corresponding right commutative diagram.

$$
\begin{tikzcd}
\mathbb{S} \arrow[rr, "\gamma"] \arrow[rrr, bend left, "x"] \arrow[dr, "\gamma'_*"] \arrow[dd, "\gamma'"'] & & \mathbb{D}_2 \arrow[r, "x'"] \arrow[d, "\beta_*\circ\delta_2^*"] & X_0 \arrow[d, "f_{n-1}"] \\
& \mathbb{D}_1^* \arrow[r, "\beta_*\circ\delta_1^*"] \arrow[d, "\beta\circ\delta_2\circ\varrho"] & \mathbb{D}'_* \arrow[r, "h'"] \arrow[d, dashed, "\varepsilon_2"] & X_{n-1} \arrow[d, "i_{n-1}"] \\
\mathbb{D}_1 \arrow[r, "\beta\circ\delta_1"] \arrow[rrr, bend right=40, "y"'] & \mathbb{D}' \arrow[r, dashed, "\varepsilon_1"] \arrow[rr, bend right, "h_n"] & \mathbb{E} \arrow[r, dashed, "h"] & X_n
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r, "\gamma"] \arrow[rr, bend left, "x"] \arrow[dd, "\gamma'"'] & \mathbb{D}_2 \arrow[r] \arrow[d, "\beta\circ\delta_2"] & X_0 \arrow[d, "f_{n-1}"] \arrow[dd, bend left, "f_n"] \\
& \mathbb{D}' \arrow[dr, "h\circ\eta\circ\beta"] & X_{n-1} \arrow[d, "i_{n-1}"] \\
\mathbb{D}_1 \arrow[ur, "\beta\circ\delta_1"] \arrow[rr, bend right, "y"'] & & X_n
\end{tikzcd}
$$

This last diagram exactly proves that $f_n : X_0 \to X_n$ is a surtraction for v^{rv}, which proves the statement by induction. $\square$

Proposition 5.33. *If X is an object in $\mathcal{C}^\omega$ whose image above $0 \to n$ is a surtraction for the dual v^{rv} of every vertebra v in $\mathtt{V}$ for every $n \geq 1$, then X is $\mathtt{V}$-fibrant.*

$$X_0 \xrightarrow{i_{n-1}\circ\cdots\circ i_0} X_n$$

Proof. Let $v^{\mathrm{rv}} := \|\gamma', \gamma\| \cdot \beta$ be the dual of a vertebra in $\mathtt{V}$ and denote by $f_n : X_0 \to X_n$ the arrow displayed in the statement for every $n \geq 1$. Let us show that X is $\mathtt{V}$-fibrant. First, note that any morphism $x : \mathbf{D}^r(v) \Rightarrow X$ has the following form.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "="] \arrow[d, "x_0"'] & \mathbb{S} \arrow[r, "="] \arrow[d, "x_1"'] & \mathbb{S} \arrow[r, "="] \arrow[d, "x_2"'] & \dots \arrow[r, "="] & \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "x_r"'] & \mathbb{D}_1 \arrow[r, "="] \arrow[d, "x_{r+1}"'] & \dots \\
X_0 \arrow[r, "i_0"'] & X_1 \arrow[r, "i_1"'] & X_2 \arrow[r, "i_2"'] & \dots \arrow[r, "i_{r-1}"'] & X_r \arrow[r, "i_r"'] & X_{r+1} \arrow[r] & \dots
\end{tikzcd}
\tag{5.16}
$$

This commutative diagram leads to the following left commutative square in $\mathcal{C}$. Now, the fact that $f_{r+1} : X_0 \to X_{r+1}$ is a surtraction for v^{rv} implies that there exist two morphisms $x'_0 : \mathbb{D}_2 \to X_0$ and $x'_{r+1} : \mathbb{D}' \to X_{r+1}$ in $\mathcal{C}$ factorising this commutative square into the following right commutative diagram.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "x_0"] \arrow[d, "\gamma'"'] & X_0 \arrow[d, "f_{r+1}"] \\ \mathbb{D}_1 \arrow[r, "x_{r+1}"'] & X_{r+1} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S} \arrow[r, "\gamma"] \arrow[rr, bend left, "x_0"] \arrow[d, "\gamma'"'] & \mathbb{D}_2 \arrow[r, "x'_0"] \arrow[d, "\beta\circ\delta_2"] & X_0 \arrow[d, "f_{r+1}"] \\ \mathbb{D}_1 \arrow[r, "\beta\circ\delta_1"] \arrow[rr, bend right, "x_{r+1}"'] & \mathbb{D}' \arrow[r, "x'_{r+1}"] & X_{r+1} \end{tikzcd}$$

By using the diagrammatic relations involved in diagram (5.16), the above rightmost factorization may be rewritten in terms of the following diagram for every $s \leq r$ (modulo some obvious truncation when $s < 0 \leq r$).

$$\begin{tikzcd} \mathbb{S} \arrow[r, "="] \arrow[d, equal] \arrow[dd, bend right, "x_0"'] & \dots \arrow[r, "="] & \mathbb{S} \arrow[r, "="] \arrow[d, equal] & \mathbb{S} \arrow[r, "="] \arrow[d, "\gamma"] \arrow[dd, bend left, "x_{l+1}"] & \dots \arrow[r, "="] & \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"] & \mathbb{D}_1 \arrow[r, "="] \arrow[d, "\beta\circ\delta_1"] \arrow[dd, bend left, "x_{r+1}"] & \dots \\ \mathbb{S} \arrow[r, "="] \arrow[d, "x'_0"] & \dots \arrow[r, "="] & \mathbb{S} \arrow[r, "\gamma"] \arrow[d, "f_l\circ x'_0"] & \mathbb{D}_2 \arrow[r, "="] \arrow[d, "f_{l+1}\circ x'_0"] & \dots \arrow[r, "="] & \mathbb{D}_2 \arrow[r, "\beta\circ\delta_2"] \arrow[d, "f_r\circ x'_0"] & \mathbb{D}' \arrow[r, "="] \arrow[d, "x'_{r+1}"] & \dots \\ X_0 \arrow[r, "i_0"'] & \dots \arrow[r, "i_{l-1}"'] & X_l \arrow[r, "i_l"'] & X_{l+1} \arrow[r] & \dots \arrow[r, "i_{r-1}"'] & X_r \arrow[r, "i_r"'] & X_{r+1} \arrow[r] & \dots \end{tikzcd}$$

This last commutative diagram clearly exposes a factorization of the arrow $x : \mathbf{D}^r(v) \Rightarrow X$ (defined in (5.16)) in terms of the trivial stem $\beta \circ \delta_1 : \mathbf{D}^r(v) \Rightarrow \mathbf{D}^r_s(v)$ and an arrow $x' : \mathbf{D}^r_s(v) \Rightarrow X$ in $\mathcal{C}^\omega$. This therefore proves the statement. $\square$

The previous two propositions show the following theorem.

Theorem 5.34. *Let $\mathcal{C}$ be a category and $\mathcal{V}$ be a cohesive set of vertebrae in $\mathcal{C}$. An object X of $\mathcal{C}^\omega$ is $\mathtt{V}$-fibrant if and only if its image above the arrow $0 \to n$ is a surtraction for the dual v^{rv} of every vertebra v in $\mathtt{V}$ for every $n \geq 1$.*

5.3.2.7. *Portfolios of vertebrae.* Let $\mathcal{C}$ be a category and K be a small category. A *portfolio* of vertebrae in $\mathcal{C}$ over K consists, for every object c in K, of a set of vertebrae $\mathtt{V}_c$ in $\mathcal{C}$. Such a collection will usually be denoted as $\mathtt{V}$. A portfolio $\mathtt{V}$ in $\mathcal{C}$ over K will be said to be *cohesive* when all its components $\mathtt{V}_c$ are cohesive sets of vertebrae in $\mathcal{C}$.

5.3.2.8. *Systems of models.* Let D be a small category, (K,T) be a conical croquis in D and $\mathcal{C}$ be a complete category. The terminal object of $\mathcal{C}$ will be denoted by $\mathbf{1}$. For every endofunctor $R : \mathcal{C} \to \mathcal{C}$, a *system of R-models* in $\mathcal{C}$ consists of

1) a cohesive portfolio $\mathtt{V}$ of vertebrae in $\mathcal{C}$ over K;
2) a category of R-premodels $\mathcal{P}$ for K;

Such a structure will be denoted as a quadruple $(\mathtt{V}, \mathcal{P}, K, T)$. Now, denote by $\mathcal{G}\mathrm{l}_0^K(P,E)$ the functor resulting from the composition of $\mathcal{G}\mathrm{l}^K(P,E) : K \to \mathcal{C}^2$ with the obvious inclusion $\mathcal{C}^2 \hookrightarrow \mathcal{C}^\omega$, which maps an arrow $a : X_0 \to X_1$ to the sequence of arrows consisting of $a : X_0 \to X_1$ followed by identities on X_1. An *R-model* for $(\mathtt{V}, \mathcal{P}, K, T)$ is an R-premodel (P,E) in $\mathcal{P}$ such that for every cylinder c in K, the canonical arrow

$$\mathcal{G}\mathrm{l}_0^K(P,E)(c) \Rightarrow \mathbf{1}$$

is a fibration in the vertebral category of $\mathtt{V}_c^{(\omega)}$ (see Proposition 5.31).

Remark 5.35. The fact that the portfolio of vertebrae $\mathtt{V}$ is cohesive will be used in section 5.5 to equip the category $\mathcal{P}$ with a homotopy theory (coming along with notions of fibration, cofibration and weak equivalence) where the models will exactly be the fibrant objects.

Theorem 5.36. *An R-premodel (P, E) in $\mathcal{P}$ is an R-model in $(\mathtt{V}, \mathcal{P}, K, T)$ if and only if the following morphism is a surtraction for the dual of every vertebra in $\mathtt{V}_c^{(\omega)}$ for every cylinder c of the form $(!, t) : \mathtt{peak}_K(c) \Rightarrow d_1$.*

$$\mathcal{G}\mathrm{l}_0^K(P, E)(c) : P(\mathtt{peak}_K(c)) \to \lim_{d_1} RPT$$

Proof. Follows from Theorem 5.34. □

Example 5.37 (Spectra)**.** Define the system of models consisting of the category of prespectra $\mathbf{Np}_{\mathbf{pTop}}(\mathbb{N}, \Omega, \mathtt{succ})$, the croquis $(\mathbb{N}, I_{\mathbf{1}}(\mathbb{N}), \mathtt{succ})$ and, for every cylinder c in $I_{\mathbf{1}}(\mathbb{N})$, the set consisting of the vertebrae of pointed spaces defined in section 2.4.2.3 together with what could be seen as their generalised epi-corrections, namely the following vertebrae

- where the object $\mathbb{S}^{n-1}/\partial$ is the quotient of the $(n-1)$-sphere by itself,
- where the object $\mathbb{D}^n/\partial$ is the quotient of the n-disc by its boundary,
- where the object $\mathbb{S}^n/\partial$ is the quotient of the n-sphere by its equator
- where the object $\mathbb{D}^{n+1}/\partial$ is the quotient of the $(n+1)$-disc by its equator,
- where the object $\mathbb{D}^{n+1}/h_n$ is the quotient of the $(n+1)$-disc by one of its hemispheres,
- where the maps between the different objects are induced by the obvious inclusions.

$$\begin{tikzcd} \mathbb{S}^{n-1}/\partial \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}^n/\partial \arrow[d, "\delta_1^n"] & \\ \mathbb{D}^n/\partial \arrow[r, "\delta_2^n"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathbb{S}^n/\partial \arrow[r, "\beta_n"'] & \mathbb{D}^{n+1}/\partial \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}^n/\partial \arrow[r, "\gamma'_n"] \arrow[d, "\gamma'_n"'] & \mathbb{D}^{n+1}/h_n \arrow[d, "\delta_1^n"] & \\ \mathbb{D}^{n+1}/h_n \arrow[r, "\delta_2^n"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathbb{D}^{n+1}/\partial \arrow[r, equal] & \mathbb{D}^{n+1}/\partial \end{tikzcd} \tag{5.17}$$

This set is equipped with an obvious cohesive structure where the cohesion is given by the identity function. The Ω-models for such a system correspond to the Ω-spectra.

Example 5.38 (Models for a sketch)**.** For every limit sketch $\mathtt{S}$, define the system of models consisting of the functor category $\mathcal{C}^{\mathtt{S}} \subseteq \mathbf{Np}_{\mathbf{Set}}(\mathtt{S}, \mathrm{id}_{\mathbf{Set}}, \mathrm{id}_D)$, the croquis $K_{\mathtt{S}}$ (see Example 5.17) and, for every cylinder c in $K_{\mathtt{S}}$, the set consisting of the canonical vertebra of **Set** (see section 2.4.1.1) and its epi-correction.

$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, "\delta_1"] & \\ \mathbf{1} \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathbf{1}+\mathbf{1} \arrow[r, "\beta"] & \mathbf{1}, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}+\mathbf{1} \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, equal] & \\ \mathbf{1} \arrow[r, equal] \arrow[ur, phantom, "\ulcorner" near end] & \mathbf{1} \arrow[r, equal] & \mathbf{1}, \end{tikzcd}$$

This set is equipped with an obvious cohesive structure where the cohesion is given by the identity function. The $\mathrm{id}_{\mathbf{Set}}$-models for such a system correspond to the models for the sketch $\mathtt{S}$.

Example 5.39 (Sheaves)**.** For every site (D^{op}, J), define the system of models consisting of the functor category $\mathcal{C}^D \subseteq \mathbf{Np}_{\mathbf{Set}}(D, \mathrm{id}_{\mathbf{Set}}, \mathrm{id}_D)$, the croquis K_J and, for every cylinder c in K_J, the set consisting of the canonical vertebra of **Set** (see section 2.4.1.1) and its epi-correction.

$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, "\delta_1"] & \\ \mathbf{1} \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner" near end] & \mathbf{1}+\mathbf{1} \arrow[r, "\beta"] & \mathbf{1}, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}+\mathbf{1} \arrow[r, "!"] \arrow[d, "!"'] & \mathbf{1} \arrow[d, equal] & \\ \mathbf{1} \arrow[r, equal] \arrow[ur, phantom, "\ulcorner" near end] & \mathbf{1} \arrow[r, equal] & \mathbf{1}, \end{tikzcd}$$

The $\mathrm{id}_{\mathbf{Set}}$-models for such a system correspond to the sheaves over (D^{op}, J).

Example 5.40 (Flabby sheaves)**.** For every site (D^{op}, J), define the system of models consisting of the functor category $\mathcal{C}^D \subseteq \mathbf{Np}_{\mathbf{Set}}(D, \mathrm{id}_{\mathbf{Set}}, \mathrm{id}_D)$, the croquis $K_J \cup D^2$ defined in Example 5.19 and

i) for every cylinder c in K_J, the set consisting of the canonical vertebra of **Set** (see section 2.4.1.1) and its epi-correction;

ii) for every cylinder c in D^2, the set consisting of the canonical vertebra of **Set** only.

The $\mathrm{id}_{\mathbf{Set}}$-models $F : D \to \mathbf{Set}$ for such a system correspond to the sheaves over (D^{op}, J) whose morphisms $F(U) \to F(V)$ over an arrow $U \to V$ in D are surjective, namely the flabby sheaves over (D^{op}, J).

5.3.2.9. *Towards other models.* Of course, the list of examples previously given is not exhaustive and do not consider the possible enrichement.

For instance, one could define the fibrant objects of the Jardine's model structure [**30**] by considering simplicial presheaves over a croquis containing coverings as well as hypercoverings (see [**29**]). The involved vertebrae would be those defined in section 2.4.2.4. Regarding the homotopy theory mentioned in Remark 5.35, one could recover the weak equivalences of simplicial presheaves of [**13**] from those of section 5.5 by taking the 'local configuration' (see section 5.5) to be the functor $\mathrm{peak}_K : K \to D$.

In the spirit of [**37**], another prospect would be the generalisation of the notion of vertebra to that of 2-vertebra to characterise stacks valued in $\mathbf{Cat}(1)$ as fibrant objects. The weak equivalences of *ibid* are required to be locally essentially surjective and fully faithful. Such a definition would be obtained from the definitions of section 5.5 by taking a 'local configuration' managing the local essential surjectivity on one side and the fully faithfulness on the other side[7].

Before discussing the homotopy theories of section 5.5, it is necessary to discuss the construction of weak factorisation systems and fibrant replacements. This is precisely the goal of section 5.4.

5.4. From narratives to combinatorial categories

5.4.1. Lifting systems and tomes.

5.4.1.1. *Numbered categories and compatibility.* In the sequel, the term *numbered category* will be used for any pair $(\mathcal{C}, \kappa)$ where $\mathcal{C}$ is a category and κ is a limit ordinal. A small category $\mathtt{T}$ will be said to be *compatible* with $(\mathcal{C}, \kappa)$ if

1) the category $\mathcal{C}$ admits colimits over $\mathtt{T}$;

2) the inequality $|\mathtt{T}| \leq \kappa$ holds.

By extension, a functor of small categories $i : \mathtt{T} \to \mathtt{S}$ will be said to be *compatible* with a numbered category $(\mathcal{C}, \kappa)$ if its domain $\mathtt{T}$ is compatible with $(\mathcal{C}, \kappa)$.

5.4.1.2. *Lifting systems.* This section defines in formal terms what will later be seen as a set of generating cofibrations for the small object argument. Let $(\mathcal{C}, \kappa)$ be a numbered category. A *lifting system in* $(\mathcal{C}, \kappa)$ consists of

1) a small category $\mathtt{S}$ as well as a functor $\varphi : \mathtt{S} \to \mathcal{C}^2$;

2) a set J of functors in $\mathbf{Cat}(1)$ that are compatible with $(\mathcal{C}, \kappa)$ and whose codomains are equal to $\mathtt{S}$.

Such a lifting system will later be denoted by $(J, \varphi) : \mathtt{S} \to \mathcal{C}^2$.

[7] Specifically, the local configuration would be the coproduct of the functor $\mathrm{peak}_K : K \to D$ with the identity functor $\mathrm{id}_K : K \to K$. The functor peak_K would serve the encoding of the local essential surjectivity while the functor id_K would serve the encoding of the fully faithfulness.

5.4.1.3. *Lifting properties.* This section generalises the notion of right lifting property with respect to an arrow defined in Chapter 2. On the other hand, the notion of left lifting property will remain the same. Let $(\mathcal{C},\kappa)$ be a numbered category and $(J,\varphi) : \mathtt{S} \to \mathcal{C}^2$ be a lifting system in $(\mathcal{C},\kappa)$. The image of any object s in $\mathtt{S}$ via φ will be denoted by $\varphi(s) : A(s) \to B(s)$. A morphism $f : X \to Y$ in $\mathcal{C}$ will be said to have the *right lifting property with respect to the system* $(J,\varphi) : \mathtt{S} \to \mathcal{C}^2$ if for any functor $i : \mathtt{T} \to \mathtt{S}$ in J, the morphism $f : X \to Y$ has the right lifting property with respect to the next arrow in $\mathcal{C}$.

$$\mathrm{col}_{\mathtt{T}}(\varphi \circ i) : \mathrm{col}_{\mathtt{T}}(A \circ i) \to \mathrm{col}_{\mathtt{T}}(B \circ i)$$

Example 5.41. If J is the set of functors of the form $\mathbf{1} \to \mathtt{S}$ picking out the objects of $\mathtt{S}$, then having the right lifting property with respect to $(J,\varphi) : \mathtt{S} \to \mathcal{C}^2$ is equivalent to having the right lifting property with respect to every object in the image of φ (i.e. arrow in $\mathcal{C}$).

In the sequel, the class of morphisms of $\mathcal{C}$ that have the right lifting property with respect to a lifting system $(J,\varphi) : \mathtt{S} \to \mathcal{C}^2$ (as defined above) will be denoted by $\mathbf{rlp}_\kappa(J,\varphi)$. By definition, the following equality holds.

$$\mathbf{rlp}_\kappa(J,\varphi) = \mathbf{rlp}(\{\mathrm{col}_{\mathtt{T}}(\varphi \circ i) \mid \forall i : \mathtt{T} \to \mathtt{S} \text{ in } J\}) \tag{5.18}$$

5.4.1.4. *Overcategories.* Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. Recall that the *category over* X, denoted by $\mathcal{C}/X$, is the category whose objects are morphisms of $\mathcal{C}$ of the form $f : A \to X$ and whose morphisms are given by commutative squares as follows.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{5.19}$$

One may see the category $\mathcal{C}/X$ as a subcategory of $\mathcal{C}^2$. From this point of view, there is an obvious *domain functor* $\partial : \mathcal{C}/X \to \mathcal{C}$ mapping an arrow $f : A \to X$ to the object A in $\mathcal{C}$ and a morphism as in diagram (5.19) to the arrow $h : A \to B$.

Remark 5.42. Let X be an object in some category $\mathcal{C}$ and let $\mathtt{T}$ be a small category. Any functor of the form $F : \mathtt{T} \to \mathcal{C}/X$ may be seen as a natural transformation in $\mathcal{C}$ over $\mathtt{T}$ of the form $h : \partial \circ F \Rightarrow \Delta_{\mathtt{T}}(X)$ where $\Delta_{\mathtt{T}}$ is the functor that maps an object X to the constant functor $\mathtt{T} \to \mathcal{C}$ whose unique value on objects is X.

Note that the mapping $X \mapsto \mathcal{C}/X$ is functorial in X. To be more specific, the post-composition of any morphism $u : X \to Y$ in $\mathcal{C}$ induces a functor $\mathcal{C}/u : \mathcal{C}/X \to \mathcal{C}/Y$ mapping an object $f : A \to X$ to an object $u \circ f : A \to Y$. It is not hard to see that for any pair of composable morphism $u : X \to Y$ and $v : Y \to Z$, the following relation holds.

$$\mathcal{C}/v \circ \mathcal{C}/u = \mathcal{C}/(v \circ u)$$

Let $h : \mathcal{A} \to \mathcal{C}$ be a functor. It will come in handy to denote the metafunctor on $\mathcal{A}$ satisfying the following mapping rule by $\mathcal{C}\downarrow_{\circ} h$.

$$\begin{array}{ccc} X & \mapsto & \mathcal{C}/h(X) \\ (u : X \to Y) & \mapsto & \mathcal{C}/h(u) \end{array}$$

Finally, for every object X in $\mathcal{C}$ and functor $M : \mathcal{C} \to \mathcal{B}$, we will denote by the same letter M the obvious functor $\mathcal{C}/X \to \mathcal{B}/M(X)$ whose mapping rule on objects is of the following form.

$$f : A \to X \quad \mapsto \quad M(f) : M(A) \to M(X)$$

Note that this functor is compatible with the domain functors ∂ relative to $\mathcal{C}$ and $\mathcal{B}$ in the sense that the following diagram commutes.

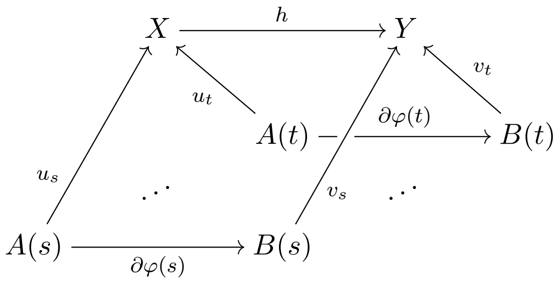
$$\begin{tikzcd}
\mathcal{C}/X \arrow[r, "M"] \arrow[d, "\partial"'] & \mathcal{B}/M(X) \arrow[d, "\partial"] \\
\mathcal{C} \arrow[r, "M"] & \mathcal{B}
\end{tikzcd}$$

Remark 5.43. In the sequel, the preceding definitions will be applied in the case of arrow categories. More specifically, we will be interested in functors of the form $\mathcal{C}^2\!\downarrow_{\circ} h$ where h denotes some functor $\mathcal{A} \to \mathcal{C}^2$. Remark 5.42 then states that for any small category $\mathtt{T}$ and object ϑ in $\mathcal{A}$, a functor of the form $F : \mathtt{T} \to \mathcal{C}^2/h(\vartheta)$ may be seen as a natural transformation in $\mathcal{C}$ over $\mathtt{T}$ of the following form where $\Delta_{\mathtt{T}}$ goes from $\mathcal{C}^2$ to the functor category $(\mathcal{C}^2)^{\mathtt{T}}$.

$$h : \partial F \Rightarrow \Delta_{\mathtt{T}} \circ h(\vartheta)$$

5.4.1.5. *Tomes.* Let $\mathcal{C}$ be a category. A *tome* in $\mathcal{C}$ is a triple consisting of a morphism $h : X \to Y$ in $\mathcal{C}$, a small category $\mathtt{S}$ on which $\mathcal{C}$ admits all colimits and a functor $\varphi : \mathtt{S} \to \mathcal{C}^2/h$. According to Remark 5.42 applied to the arrow category $\mathcal{C}^2$, a way of seeing a tome in $\mathcal{C}$ is in the form of a cocone in $\mathcal{C}^2$ over the functor $\partial\varphi : \mathtt{S} \to \mathcal{C}^2$.

$$\begin{tikzcd}
\Delta_{\mathtt{S}}(h) \\
\partial\varphi \arrow[u, Rightarrow, "(u,v)"']
\end{tikzcd}$$



Because $\mathcal{C}$ has all colimits over $\mathtt{S}$, the earlier cocone provides an arrow $\mathrm{col}_{\mathtt{S}}\partial\varphi \Rightarrow h$ in $\mathcal{C}^2$ that is obtained by using the counit of the adjunction $\mathrm{col}_{\mathtt{S}} \dashv \Delta_{\mathtt{S}}$. This may be presented by the following diagram, which will be referred to as the *content of* $(h, \mathtt{S}, \varphi)$.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{S}}A \arrow[r, "\mathrm{col}_{\mathtt{S}}u"] \arrow[d, "\mathrm{col}_{\mathtt{S}}\partial\varphi"'] & X \arrow[d, "h"] \\
\mathrm{col}_{\mathtt{S}}B \arrow[r, "\mathrm{col}_{\mathtt{S}}v"'] & Y
\end{tikzcd}$$

Note that for any functor $i : \mathtt{T} \to \mathtt{S}$, we may paste the universal shifting induced by i on the content of $(f, \mathtt{S}, \varphi)$ as follows.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{S}}(A \circ i) \arrow[r, "\xi_i(A)"] \arrow[d, "\mathrm{col}_{\mathtt{S}}\partial\varphi\circ i"'] & \mathrm{col}_{\mathtt{S}}A \arrow[r, "\mathrm{col}_{\mathtt{S}}u"] \arrow[d, "\mathrm{col}_{\mathtt{S}}\partial\varphi"'] & X \arrow[d, "h"] \\
\mathrm{col}_{\mathtt{S}}(B \circ i) \arrow[r, "\xi_i(B)"] & \mathrm{col}_{\mathtt{S}}B \arrow[r, "\mathrm{col}_{\mathtt{S}}v"] & Y
\end{tikzcd} \tag{5.20}$$

Such a construction will later play a central role and be referred to as the *content of* $(f, \mathtt{S}, \varphi)$ *along* $i : \mathtt{T} \to \mathtt{S}$. Now, a *loose morphism of tomes* from $\mathbf{T}_0 := (h_0, \mathtt{S}_0, \varphi_0)$ to $\mathbf{T}_1 := (h_1, \mathtt{S}_1, \varphi_1)$ is given by a morphism $(x, y) : h_0 \Rightarrow h_1$ in $\mathcal{C}^2$ (see the left-hand square, below). A *(regular) morphism of tomes* $\mathbf{T}_0 \Rightarrow \mathbf{T}_1$ is given by a morphism $(x, y) : h_0 \Rightarrow h_1$ in $\mathcal{C}^2$ and a functor

$\sigma : \mathtt{S}_0 \to \mathtt{S}_1$ making the corresponding right diagram commute.

$$\begin{tikzcd} X_0 \arrow[r, "x"] \arrow[d, "h_0"'] & X_1 \arrow[d, "h_1"] \\ Y_0 \arrow[r, "y"] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} \mathtt{S}_0 \arrow[r, "\sigma"] \arrow[d, "\varphi_0"'] & \mathtt{S}_1 \arrow[d, "\varphi_1"] \\ \mathcal{C}^2/h_0 \arrow[r, "\mathcal{C}^2/(x,y)"'] & \mathcal{C}^2/h_1 \end{tikzcd}$$

The arrow associated with loose morphisms will be denoted as $\mathbf{T}_0 \overset{\star}{\Rightarrow} \mathbf{T}_1$. The category whose objects are tomes in $\mathcal{C}$ and whose arrows are morphisms (resp. loose morphisms) of tomes will be denoted by $\mathbf{Tome}(\mathcal{C})$ (resp. $\mathbf{Ltom}(\mathcal{C})$). For a fixed object Q in $\mathcal{C}$, the subcategory of $\mathbf{Ltom}(\mathcal{C})$ containing all the objects and restricted to the loose morphisms $(x, y) : \mathbf{T}_0 \overset{\star}{\Rightarrow} \mathbf{T}_1$ are such that $Y_0 = Y_1 = Q$ and $y = \mathrm{id}_Q$ will be denoted by $\mathbf{Ltom}(Q, \mathcal{C})$.

5.4.2. Oeuvres and narratives.

5.4.2.1. *Oeuvres and narratives.* Let $(\mathcal{C}, \kappa)$ be a numbered category and Q be an object in $\mathcal{C}$. An *oeuvre of theme* Q in $(\mathcal{C}, \kappa)$ is a functor $\mathfrak{O} : \mathbf{O}(\kappa + 1) \to \mathbf{Ltom}(\mathcal{C})$ factorising through $\mathbf{Ltom}(Q, \mathcal{C})$ as follows.

$$\begin{tikzcd} & \mathbf{Ltom}(Q,\mathcal{C}) \arrow[d, "\subseteq"] \\ \mathbf{O}(\kappa+1) \arrow[ur, dashed, "\mathfrak{O}'"] \arrow[r, "\mathfrak{O}"'] & \mathbf{Ltom}(\mathcal{C}) \end{tikzcd}$$

The rest of the section fixes some conventional notations for such a structure and finishes with the notion of narrative. The image of an inequality $k < l$ in $\mathbf{O}(\kappa + 1)$ via an oeuvre $\mathfrak{O}$ will be denoted as follows.

$$(\chi_k^l, \mathrm{id}_Q) : (h_k, \mathtt{S}_k, \varphi_k) \overset{\star}{\Rightarrow} (h_l, \mathtt{S}_l, \varphi_l)$$

For convenience, when l is successor of k in $\mathbf{O}(\kappa + 1)$, the notations χ_k^l will be shortened to χ_k. For every object k in $\mathbf{O}(\kappa + 1)$, the morphism h_k will be denoted as an arrow $G_k \to Q$ while the image of the composite functor $\partial\varphi_k : \mathtt{S}_k \to \mathcal{C}^2$ at an object s in $\mathtt{S}_k$ will be denoted as $\varphi_k(s) : \mathbf{A}_k(s) \to \mathbf{B}_k(s)$. Now, a *narrative of theme* Q in $(\mathcal{C}, \kappa)$ is an oeuvre $\mathfrak{O} : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$ of theme Q equipped with

1) (*events*) a set J_k of functors in $\mathbf{Cat}(1)$ that are compatible with $(\mathcal{C}, \kappa)$ and whose codomains are equal to $\mathtt{S}_k$ for every ordinal $k \in \kappa$;
2) (*transitions*) a factorisation as follows for every ordinal $k \in \kappa$;

$$\begin{tikzcd} G_{k+1} \arrow[r, "\lambda_k"'] \arrow[d, "h_{k+1}"'] \arrow[rr, bend left, "\chi_{k+1}"] & N_k \arrow[r, "\rho_k"'] \arrow[d, "\alpha_k"] & G_{k+2} \arrow[d, "h_{k+2}"] \\ Q \arrow[r, equal] & Q \arrow[r, equal] & Q \end{tikzcd}$$

3) (*point-of-view*) for every functor $i : \mathtt{T} \to \mathtt{S}_k$ in the set J_k, a morphism $\pi_k^i : \mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \to N_k$ factorising the content of $(h_k, \mathtt{S}_k, \varphi_k)$ along $i : \mathtt{T} \to \mathtt{S}_k$ into a commutative

diagram as follows for every ordinal $k \in \kappa$.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(\mathbf{A}_k \circ i) \arrow[rr, "\mathrm{col}_{\mathtt{S}_k} u \circ \xi_i(\mathbf{A}_k)"] \arrow[dd, "\mathrm{col}_{\mathtt{T}} \partial \varphi_k \circ i"'] & & G_k \arrow[dl, "\lambda_k \circ \chi_k"'] \arrow[dd, "h_k"] \\
& N_k \arrow[dr, "\alpha_k"] & \\
\mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \arrow[ur, "\pi_k^i"] \arrow[rr, "\mathrm{col}_{\mathtt{S}_k} v \circ \xi_i(\mathbf{B}_k)"'] & & Q
\end{tikzcd}$$

For every $k \in \kappa$, the set J_k and morphism $\pi_k^i : \mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \to N_k$ will later be referred to as *set of events at rank k* and *point of view at rank k along i*, respectively. The morphisms λ_k and ρ_k will be called *pre-transition* and *post-transition morphisms at rank k*, respectively. The factorisation of χ_k that they induce will be referred to as the *transition factorisation*. Finally, the functor induced by the sequence of arrows $\chi_k^l : G_k \to G_l$ for every inequality $k < l$ in $\mathbf{O}(\kappa+1)$ will be denoted as $G : \mathbf{O}(\kappa+1) \to \mathcal{C}$ and called the *context functor*. Note that any oeuvre and *a fortiori* any narrative as defined above provides a factorisation in $\mathcal{C}$ as follows.

(5.21)
$$\begin{tikzcd}
G_0 \arrow[r, "\chi_0^\kappa"] \arrow[rr, bend right, "h_0"'] & G_\kappa \arrow[r, "h_\kappa"] & Q
\end{tikzcd}$$

5.4.2.2. *Small object argument.* Let $(\mathcal{C}, \kappa)$ be a numbered category, Q be an object in $\mathcal{C}$ and $\mathfrak{O} : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$ be a narrative of theme Q with the notations of section 5.4.2.1. A lifting system $(J, \varphi) : \mathtt{S} \to \mathcal{C}^2$ in $(\mathcal{C}, \kappa)$ will be said to *agree with* the narrative $\mathfrak{O}$ if for every ordinal $k \in \kappa$, every functor $i : \mathtt{T} \to \mathtt{S}$ in J and every functor $\psi : \mathtt{T} \to \mathcal{C}^2/h_k$ making the following left diagram commute, there exists a functor $i' : \mathtt{T} \to \mathtt{S}_k$ in J_k making the corresponding right diagram commute.

$$\begin{tikzcd}
\mathtt{T} \arrow[r, "\varphi \circ i"] \arrow[dr, "\psi"'] & \mathcal{C}^2 \\
& \mathcal{C}^2/h_k \arrow[u, "\partial"']
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathtt{T} \arrow[r, "\varphi \circ i"] \arrow[dr, "\psi"] \arrow[d, dashed, "i'"'] & \mathcal{C}^2 \\
\mathtt{S}_k \arrow[r, "\varphi_k"'] & \mathcal{C}^2/h_k \arrow[u, "\partial"']
\end{tikzcd}$$

Proposition 5.44. *Let $(J, \varphi) : \mathtt{S} \to \mathcal{C}^2$ be a lifting system in $(\mathcal{C}, \kappa)$ agreeing with the narrative $\mathfrak{O}$. If the context functor $G : \mathbf{O}(\kappa+1) \to \mathcal{C}$ is uniformly* $(\mathrm{dom} \circ \varphi)$*-convergent in $\mathcal{C}$, then the morphism $h_\kappa : G_\kappa \to Q$ appearing in diagram (5.21) has the rlp with respect to the lifting system $(J, \varphi) : \mathtt{S} \to \mathcal{C}^2$.*

Proof. For every object in $s \in \mathtt{S}$, the image $\varphi(s)$ will be denoted as an arrow $A(s) \to B(s)$. The goal of the proof is to show that the morphism $h_\kappa : G_\kappa \to Q$ is in $\mathbf{rlp}_\kappa(J, \varphi)$. Let $i : \mathtt{T} \to \mathtt{S}$ be a functor in J and consider a commutative square as follows.

(5.22)
$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(A \circ i) \arrow[r, "x"] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\varphi \circ i)"'] & G_\kappa \arrow[d, "h_\kappa"] \\
\mathrm{col}_{\mathtt{T}}(B \circ i) \arrow[r, "y"'] & Q
\end{tikzcd}$$

By assumption, the functor $G : \mathbf{O}(\kappa+1) \to \mathcal{C}$ is uniformly $(\mathrm{dom} \circ \varphi)$-convergent in $\mathcal{C}$. It follows from Remark 5.3 and the fact that κ is limit that there exist an ordinal $k \in \kappa$ and an arrow $x' : A \to G_k$ making the following diagram commute.

(5.23)
$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(A \circ i) \arrow[r, dashed, "x'"'] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\varphi\circ i)"'] \arrow[rrrr, bend left, "x"] & G_k \arrow[r, "\chi_k"] \arrow[d, "h_k"] & G_{k+1} \arrow[r, "\chi_{k+1}"] \arrow[d, "h_{k+1}"] & G_{k+2} \arrow[r, "\chi_{k+2}^{\kappa}"'] \arrow[d, "h_{k+2}"] & G_\kappa \arrow[d, "h_\kappa"] \\
\mathrm{col}_{\mathtt{T}}(B \circ i) \arrow[r, "y"'] & Q \arrow[r, equal] & Q \arrow[r, equal] & Q \arrow[r, equal] & Q
\end{tikzcd}$$

Note that an application of the universal property of the adjunction $\mathrm{col}_{\mathtt{T}} \dashv \Delta_{\mathtt{T}}$ on the leftmost commutative square of diagram (5.23) provides a commutative square of arrows in $\mathcal{C}^{\mathtt{T}}$ as follows (where η denotes the unit of $\mathrm{col}_{\mathtt{T}} \dashv \Delta_{\mathtt{T}}$).

(5.24)
$$\begin{tikzcd}
A \circ i(_) \arrow[r, "\Delta_{\mathtt{T}}(x')\circ\eta_{A\circ i}"] \arrow[d, "\varphi\circ i(_)"'] & \Delta_{\mathtt{T}}\big(G_k\big) \arrow[d, "\Delta_{\mathtt{T}} h_k"] \\
B \circ i(_) \arrow[r, "\Delta_{\mathtt{T}}(y)\circ\eta_{B\circ i}"'] & \Delta_{\mathtt{T}}\big(Q\big)
\end{tikzcd}$$

According to Remark 5.43, diagram (5.24) induces a functor $\psi : \mathtt{T} \to \mathcal{C}^2/h_k$, which makes the following leftmost diagram commute.

$$\begin{tikzcd}
\mathtt{T} \arrow[r, "\varphi\circ i"] \arrow[dr, "\psi"'] & \mathcal{C}^2 \\
& \mathcal{C}^2/h_k \arrow[u, "\partial"']
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathtt{T} \arrow[r, "\varphi\circ i"] \arrow[dr, "\psi"'] \arrow[d, dashed, "i'"'] & \mathcal{C}^2 \\
\mathtt{S}_k \arrow[r, "\varphi_k"'] & \mathcal{C}^2/h_k \arrow[u, "\partial"']
\end{tikzcd}$$

Because the lifting system (J, φ) agrees with the narrative $\mathfrak{O}$, there must exists a functor $i' : \mathtt{T} \to S_k$ making the preceding right diagram commute. Now, the equation $\psi = \varphi_k \circ i'$ says that the content of $(h_k, \mathtt{S}_k, \varphi_k)$ along $i' : \mathtt{T} \to \mathtt{S}_k$ exactly corresponds to the content of the tome induced by the triple $(h_k, \mathtt{T}, \psi)$. By definition of $\psi : \mathtt{T} \to \mathcal{C}^2/h_k$, the latter content is the left commutative square of diagram (5.23). It then follows from the point-of-view structure along $i' : \mathtt{T} \to \mathtt{S}_k$ associated with $(h_k, \mathtt{S}_k, \varphi_k)$ that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(A \circ i) \arrow[rr, "x'"] \arrow[dd, "\mathrm{col}_{\mathtt{T}}(\varphi\circ i)"'] & & G_k \arrow[dl, "\lambda_k\circ\chi_k"'] \arrow[dd, "h_k"] \\
& N_k \arrow[dr, "\alpha_k"] & \\
\mathrm{col}_{\mathtt{T}}(B \circ i) \arrow[ur, "\pi_k^{i'}"] \arrow[rr, "y"'] & & Q
\end{tikzcd}$$

Inserting the relations of the above commutative diagram into diagram (5.23) and using the structure of transition factorisation associated with the narrative $\mathfrak{O}$ at rank k provides the following commutative diagram.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(A \circ i) \arrow[r, "x'"] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\varphi\circ i)"'] \arrow[rrrrr, bend left, "u"] & G_k \arrow[r, "\chi_k"] & G_{k+1} \arrow[r, "\lambda_k"] & N_k \arrow[r, "\rho_k"] \arrow[d, "\alpha_k"] & G_{k+2} \arrow[r] \arrow[d, "h_{k+2}"] & G_\kappa \arrow[d, "h_\kappa"] \\
\mathrm{col}_{\mathtt{T}}(B \circ i) \arrow[r, "y"'] \arrow[urrr, "\pi_k^{i'}"] & Q \arrow[r, equal] & Q \arrow[r, equal] & Q \arrow[r, equal] & Q \arrow[r, equal] & Q
\end{tikzcd}$$

The above commutative diagram defines a lift for diagram (5.22). □

5.4.2.3. *Strict narratives.* Let $(\mathcal{C},\kappa)$ be a numbered category and Q be an object in $\mathcal{C}$. For any narrative $\mathfrak{O} : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$ of theme Q, the set of events J_k gives a collection of functors that induces a coproduct functor as shown by the following implication.

$$\begin{tikzcd} & \mathtt{S}_k & \\ \underbrace{\mathtt{T} \arrow[ur, "j"] & \dots & \mathtt{T}' \arrow[ul, "j'"']}_{\in J_k} \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \mathtt{S}_k \\ \coprod_{j\in J_k}\mathrm{dom}(j) \arrow[u, "\coprod_{j\in J_k} j"'] \end{tikzcd}$$

The preceding right functor will be called the *total functor of* J_k and denoted as an arrow $\tau_k : \mathtt{R}_k \to \mathtt{S}_k$. By definition, for every functor $i : \mathtt{T} \to \mathtt{S}_k$, there is a canonical arrow $\kappa_i : \mathtt{T} \to \mathtt{R}_k$ such that the composite $\tau_k \circ \kappa_i : \mathtt{T} \to \mathtt{S}_k$ is equal to the functor $i : \mathtt{T} \to \mathtt{S}_k$ itself. It follows that, when $\mathcal{C}$ admits coproducts over $\mathtt{R}_k$, pasting the content of the tome $(h_k, \mathtt{S}_k, \varphi_k)$ along $\tau_k : \mathtt{R}_k \to \mathtt{S}_k$ with the universal shifting induced by $\kappa_i : \mathtt{T} \to \mathtt{S}_k$ gives back the content of $(h_k, \mathtt{S}_k, \varphi_k)$ along $i : \mathtt{T} \to \mathtt{S}_k$.

$$\begin{tikzcd} \mathrm{col}_{\mathtt{T}}(\mathbf{A}_k \circ i) \arrow[r, "\xi_{\kappa_i}(\mathbf{A}_k\circ\tau_k)"] \arrow[d, "\mathrm{col}_{\mathtt{T}}\partial\varphi_k\circ i"'] & \mathrm{col}_{\mathtt{R}_k}(\mathbf{A}_k \circ \tau_k) \arrow[r, "\xi_{\tau_k}(\mathbf{A}_k)"] \arrow[d, "\mathrm{col}_{\mathtt{R}_k}\partial\varphi_k\circ\tau_k"'] & \mathrm{col}_{\mathtt{S}}\mathbf{A}_k \arrow[r, "\mathrm{col}_{\mathtt{S}}u"] \arrow[d, "\mathrm{col}_{\mathtt{S}}\partial\varphi_k"'] & X \arrow[d, "h"] \\ \mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \arrow[r, "\xi_{\kappa_i}(\mathbf{B}_k\circ\tau_k)"] & \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}_k \circ \tau_k) \arrow[r, "\xi_{\tau_k}(\mathbf{B}_k)"] & \mathrm{col}_{\mathtt{T}}\mathbf{B}_k \arrow[r, "\mathrm{col}_{\mathtt{S}}v"] & Y \end{tikzcd} \tag{5.25}$$

The notations of the above commutative diagram will be used in the next definition. A narrative $\mathfrak{O} : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$ of theme Q will be said to be *strict* in $\mathcal{C}$ if

1) the category $\mathcal{C}$ admits coproducts over $\mathtt{R}_k$ for every $k \in \kappa$;
2) for every ordinal $k \in \kappa$, the morphism $\lambda_k : G_{k+1} \to N_k$ is an identity;
3) it is equipped with a morphism $\pi_k : \mathrm{col}_{\mathtt{R}_k}(B \circ \tau_k) \to G_{k+1}$ factorising the content of $(h_k, \mathtt{S}_k, \varphi_k)$ along $\tau_k : \mathtt{R}_k \to \mathtt{S}_k$ into a pushout as follows

$$\begin{tikzcd} \mathrm{col}_{\mathtt{R}_k}(\mathbf{A}_k \circ \tau_k) \arrow[rr, "\mathrm{col}_{\mathtt{S}_k}u\circ\xi_{\tau_k}(\mathbf{A}_k)"] \arrow[dd, "\mathrm{col}_{\mathtt{R}_k}(\partial\varphi_k\circ\tau_k)"'] & & G_k \arrow[dl, "\chi_k"'] \arrow[dd, "h_k"] \\ & N_k \arrow[dr, "\alpha_k"] \arrow[ul, phantom, "\ulcorner"] & \\ \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}_k \circ \tau_k) \arrow[ur, "\pi_k"] \arrow[rr, "\mathrm{col}_{\mathtt{S}_k}v\circ\xi_{\tau_k}(\mathbf{B}_k)"'] & & Q \end{tikzcd}$$

such that for every functor $i : \mathtt{T} \to \mathtt{S}_k$ in J_k, the following composite arrow is equal to the point of view $\pi_k^i : \mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \to N_k$ along $i : \mathtt{T} \to \mathtt{S}_k$;

$$\mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \xrightarrow{\xi_{\kappa_i}(\mathbf{B}_k\circ\tau_k)} \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}_k \circ \tau_k) \xrightarrow{\pi_k} N_k$$

4) the context functor $G : \mathbf{O}(\kappa+1) \to \mathcal{C}$ is sequential (see section 5.2.1.2).

For convenience, we will later denote by $\upsilon_k(j)$ and $\overline{\upsilon}_k(j)$ the composite arrows $\mathrm{col}_{\mathtt{K}}u \circ \xi_j(\mathbf{A}_k)$ and $\mathrm{col}_{\mathtt{K}}v \circ \xi_j(\mathbf{B}_k)$ resulting from the shifting of the content of a narrative along any functor $j : \mathtt{K} \to \mathtt{S}_k$, respectively.

Proposition 5.45. *If a morphism $f : X \to Y$ has the rlp with respect to the lifting system defined by the pair $(J_k, \partial\varphi_k) : \mathtt{S}_k \to \mathcal{C}^2$ for every $k \in \kappa$, then it has the rlp with respect to the morphism $\chi_0^\kappa : G_0 \to G_\kappa$ of diagram (5.21)*

Proof. Let $f : X \to Y$ be a morphism that has the rlp with respect to the lifting system $(J_k, \partial\varphi_k) : \mathtt{S}_k \to \mathcal{C}^2$ for every $k \in \kappa$. This means that it has the rlp with respect to the following arrow in $\mathcal{C}$ for every functor $i : \mathtt{T} \to \mathtt{S}_k$ in J_k.

$$\mathrm{col}_{\mathtt{S}_k}(\partial\varphi_k \circ i) : \mathrm{col}_{\mathtt{S}_k}(\mathbf{A}_k \circ i) \to \mathrm{col}_{\mathtt{S}_k}(\mathbf{B}_k \circ i)$$

It directly follows that f has the rlp with respect to the coproduct of these arrows over J_k – for some fixed $k \in \kappa$ – which may be identified to the arrow $\mathrm{col}_{\mathtt{S}_k}(\partial\varphi_k \circ \tau_k)$ up to isomorphism as shown below.

$$\begin{aligned}
\coprod_{i \in J_k} \mathrm{col}_{\mathtt{S}_k}(\partial\varphi_k \circ i) &\cong \mathrm{col}_{\mathtt{S}_k}(\coprod_{i \in J_k} \partial\varphi_k \circ i) && \text{(colimits commute)}\\
&\cong \mathrm{col}_{\mathtt{S}_k}(\partial\varphi_k \circ (\coprod_{i \in J_k} i)) && \text{(universality)}\\
&\cong \mathrm{col}_{\mathtt{S}_k}(\partial\varphi_k \circ \tau_k) && \text{(definition)}
\end{aligned}$$

It follows from Proposition 1.33 that, since f has the rlp with respect to $\mathrm{col}_{\mathtt{S}_k}(\partial\varphi_k \circ \tau_k)$, it has the rlp with respect to any of its pushouts, and in particular χ_k for any $k \in \kappa$. It finally follows from Proposition 5.9 and the fact that the context functor $G : \mathbf{O}(\kappa + 1) \to \mathcal{C}$ is sequential that f has the rlp with respect to the arrow $\chi_0^\kappa : G_0 \to G_\kappa$ in $\mathcal{C}$. □

5.4.2.4. *Morphisms of oeuvres and narratives.* Let $(\mathcal{C}, \kappa)$ be a numbered category. For every pair of oeuvres $\mathfrak{O} : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$ and $\mathfrak{O}' : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$, of respective themes Q and Q', a *morphism of oeuvres* from $\mathfrak{O}$ to $\mathfrak{O}'$ consists, for every ordinal $k \in \kappa$, of a regular morphism of tomes

$$(x_k, y_k, \sigma_k) : \mathfrak{O}_k \Rightarrow \mathfrak{O}'_k \qquad (\text{with } y_k : Q \to Q')$$

such that the underlying loose morphisms $(x_k, y_k) : \mathfrak{O}_k \overset{\star}{\Rightarrow} \mathfrak{O}'_k$ induce a morphism $\mathfrak{O} \Rightarrow \mathfrak{O}$ in the functor category $\mathbf{Ltom}(\mathcal{C})^{\mathbf{O}(\kappa+1)}$. Such a definition implies that all the arrows y_k are equal to the same morphism $y : Q \to Q'$ for every $k \in \kappa + 1$. In addition, it forces the following diagram to commute $\mathcal{C}$ for every $k \in \kappa$.

(5.26)
$$\begin{tikzcd} G_k \arrow[r, "x_k"] \arrow[d, "\chi_k"'] & G'_k \arrow[d, "\chi'_k"] \\ G_{k+1} \arrow[r, "x_{k+1}"'] & G'_{k+1} \end{tikzcd}$$

The category whose objects are oeuvres for the numbered category $(\mathcal{C}, \kappa)$ and whose arrows are morphisms of oeuvres will be denoted by $\mathbf{Oeuv}(\mathcal{C}, \kappa)$.

Now, if the oeuvres $\mathfrak{O}$ and $\mathfrak{O}'$ are equipped with structures of narratives whose respective sets of events are given by J_k and J'_k for every $k \in \kappa$, a *morphism of narratives* from $\mathfrak{O}$ to $\mathfrak{O}'$ is a morphism of oeuvres $\mathfrak{O} \Rightarrow \mathfrak{O}'$, say encoded, at rank $k \in \kappa$, by the following components

(5.27)
$$(x_k, y, \sigma_k) : (h_k, \mathtt{S}_k, \varphi_k) \Rightarrow (h'_k, \mathtt{S}'_k, \varphi'_k)$$

such that for every functor $i : \mathtt{T} \to \mathtt{S}_k$ in J_k and $k \in \kappa$, the composite functor $\sigma_k \circ i : \mathtt{T} \to \mathtt{S}'_k$ belongs to the set J'_k. In other words, the post-composition by σ_k induces a function $\sigma_k : J_k \to J'_k$. The category whose objects are narratives for the numbered category $(\mathcal{C}, \kappa)$ and whose arrows are morphisms of narratives will be denoted by $\mathbf{Narr}(\mathcal{C}, \kappa)$.

Remark 5.46. Let k be some ordinal in κ and $i : \mathtt{T} \to \mathtt{S}_k$ be a functor in J_k. For every morphism of narratives $\mathfrak{O} \Rightarrow \mathfrak{O}'$ as defined above, the function $\sigma_k : J_k \to J'_k$ induces an inclusion functor $\jmath_k : \coprod_{j \in J_k} \mathrm{dom}(j) \to \coprod_{j' \in J'_k} \mathrm{dom}(j')$ making the following right diagram commute, where the notations κ and τ are defined in section 5.4.2.3.

$$\begin{cases} \mathtt{R}_k := \coprod_{j \in J_k} \mathrm{dom}(j) \\ \\ \mathtt{R}'_k := \coprod_{j' \in J'_k} \mathrm{dom}(j') \end{cases} \qquad \begin{tikzcd} \mathtt{T} \arrow[r, "\kappa_i"] \arrow[rr, bend left, "i"] \arrow[d, equal] & \mathtt{R}_k \arrow[r, "\tau_k"] \arrow[d, "\mathtt{J}_k"'] & \mathtt{S}_k \arrow[d, "\sigma_k"] \\ \mathtt{T} \arrow[r, "\kappa_{\sigma_k \circ i}"] \arrow[rr, bend right, "\sigma_k \circ i"'] & \mathtt{R}'_k \arrow[r, "\tau'_k"] & \mathtt{S}'_k \end{tikzcd}$$

The next proposition will only be used for expository purposes.

Proposition 5.47. *Consider a morphism of narratives $\mathfrak{D} \Rightarrow \mathfrak{D}'$ as given by equation (5.27) where the narratives $\mathfrak{D}$ to $\mathfrak{D}'$ are strict. For every functor $i : \mathtt{T} \to \mathtt{S}_k$ in J_k and $k \in \kappa$, pasting the (pushout) square associated with the point of view of $\mathfrak{D}$ at rank k along i with diagram (5.26) gives the (pushout) square associated with the point of view of $\mathfrak{D}'$ at rank k along $i' := \sigma_k \circ i$.*

$$\begin{tikzcd} \mathrm{col}_{\mathtt{T}}(\mathbf{A}_k \circ i) \arrow[r, "\upsilon(i)"] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\partial\varphi_k \circ i)"'] & G_k \arrow[r, "x_k"] \arrow[d, "\chi_k"'] & G'_k \arrow[d, "\chi'_k"] \\ \mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \arrow[r, "\pi_k^i"'] & N_k \arrow[r, "x_{k+1}"'] & N'_k \end{tikzcd} \quad = \quad \begin{tikzcd} \mathrm{col}_{\mathtt{T}}(\mathbf{A}'_k \circ i') \arrow[r, "\upsilon'(i')"] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\partial\varphi'_k \circ i')"'] & G'_k \arrow[d, "\chi'_k"] \\ \mathrm{col}_{\mathtt{T}}(\mathbf{B}'_k \circ i') \arrow[r, "\pi_k^{i'}"'] & N'_k \end{tikzcd}$$

Proof. By definition of a morphism of oeuvres $\mathfrak{D} \Rightarrow \mathfrak{D}'$, the following left diagram commutes. By pre-composing this diagram with the total functor $\tau_k : \mathtt{R}_k \to \mathtt{S}_k$ and using Remark 5.46, we obtain the commutative diagram given below on the right.

$$\begin{tikzcd} \mathtt{S}_k \arrow[r, "\sigma_k"] \arrow[d, "\varphi_k"'] & \mathtt{S}'_k \arrow[d, "\varphi'_k"] \\ \mathcal{C}^2/h_k \arrow[r, "\mathcal{C}^2/(x_k, y)"'] & \mathcal{C}^2/h'_k \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathtt{R}_k \arrow[r, "\mathtt{J}_k"] \arrow[d, "\varphi_k \circ \tau_k"'] & \mathtt{R}'_k \arrow[d, "\varphi'_k \circ \tau'_k"] \\ \mathcal{C}^2/h_k \arrow[r, "\mathcal{C}^2/(x_k, y)"'] & \mathcal{C}^2/h'_k \end{tikzcd}$$

The preceding right diagram then induces a morphism of diagram between the content of $\mathfrak{D}_k$ along τ_k and the content of $\mathfrak{D}'_k$ along τ'_k. More specifically, this means that the following two commutative diagrams are equal.

$$\begin{tikzcd} \mathrm{col}_{\mathtt{R}_k}(\mathbf{A}_k \circ \tau_k) \arrow[rr, "\upsilon(\tau_k)"] \arrow[dd, "\mathrm{col}_{\mathtt{R}_k}(\partial\varphi_k \circ \tau_k)"'] & & G_k \arrow[r, "x_k"] \arrow[dl, "\chi_k"'] \arrow[dd, "h_k"] & G'_k \arrow[dd, "h'_k"] \\ & N_k \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\alpha_k"] & & \\ \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}_k \circ \tau_k) \arrow[ur, "\pi_k"] \arrow[rr, "\overline{\upsilon}(\tau_k)"'] & & Q \arrow[r, "y"'] & Q' \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{col}_{\mathtt{R}_k}(\mathbf{A}'_k \circ \tau'_k \circ \mathtt{J}_k) \arrow[r, "\xi_{\mathtt{J}_k}(\mathbf{A}'_k \circ \tau'_k)"] \arrow[dd, "\mathrm{col}_{\mathtt{R}_k}(\partial\varphi'_k \circ \tau'_k \circ \mathtt{J}_k)"'] & \mathrm{col}_{\mathtt{R}'_k}(\mathbf{A}'_k \circ \tau'_k) \arrow[rr, "\upsilon'(\tau'_k)"] \arrow[dd, "\mathrm{col}_{\mathtt{R}'_k}(\partial\varphi'_k \circ \tau'_k)"] & & G'_k \arrow[dl, "\chi'_k"'] \arrow[dd, "h'_k"] \\ & & N'_k \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\alpha_k"] & \\ \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}'_k \circ \tau'_k \circ \mathtt{J}_k) \arrow[r, "\xi_{\mathtt{J}_k}(\mathbf{B}'_k \circ \tau'_k)"'] & \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}'_k \circ \tau'_k) \arrow[ur, "\pi'_k"] \arrow[rr, "\overline{\upsilon}'(\tau'_k)"'] & & Q' \end{tikzcd}$$

The leftmost vertical arrows of the previous two diagrams are indeed equal since post-composing the domain functor $\partial : \mathcal{C}^2/h'_k \to \mathcal{C}^2$ with the functor $\mathcal{C}^2/(x_k, y) : \mathcal{C}^2/h'_k \to \mathcal{C}^2/h'_k$ gives the domain functor $\partial : \mathcal{C}^2/h_k \to \mathcal{C}^2$. Now, it follows from the universality of N_k that the preceding equality of diagrams induces a canonical arrow from N_k to N'_k, which must necessarily be equal to the arrow $x_{k+1} : N_k \to N'_k$ by strictness of $\mathfrak{D}$ and $\mathfrak{D}'$. In other words, we obtain the following equality.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(\mathbf{A}_k \circ \tau_k) \arrow[r, "\upsilon(\tau_k)"] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\partial\varphi_k \circ \tau_k)"'] & G_k \arrow[r, "x_k"] \arrow[d, "\chi_k"'] & G'_k \arrow[d, "\chi'_k"] & = & \mathrm{col}_{\mathtt{T}}(\mathbf{A}'_k \circ \tau'_k \circ \mathtt{J}_k) \arrow[r, "\upsilon'(\tau'_k \circ \mathtt{J}_k)"] \arrow[d, "\mathrm{col}_{\mathtt{T}}(\partial\varphi'_k \circ \tau'_k)"'] & G'_k \arrow[d, "\chi'_k"] \\
\mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ \tau_k) \arrow[r, "\pi_k"'] & N_k \arrow[r, "x_{k+1}"'] & N'_k & & \mathrm{col}_{\mathtt{T}}(\mathbf{B}'_k \circ \tau'_k \circ \mathtt{J}_k) \arrow[r, "\pi_k \circ \xi_{\mathtt{J}_k}(\mathbf{B}'_k \circ \tau'_k)"'] & N'_k
\end{tikzcd}$$

Pre-composing the above two diagrams with the following commutative square (universal shifting along $\kappa_i : \mathtt{T} \to \mathtt{R}_k$) finally leads to the statement since both relations $\tau_k \circ \kappa_i = i$ and $\tau'_k \circ \mathtt{J}_k \circ \kappa_i = i'$ hold by Remark 5.46.

$$\begin{tikzcd}
\mathrm{col}_{\mathtt{T}}(\mathbf{A}_k \circ i) \arrow[r, "\xi_{\kappa_i}(\mathbf{A}_k \circ \tau_k)"] \arrow[d, "\mathrm{col}_{\mathtt{T}}\partial\varphi_k \circ i"'] & \mathrm{col}_{\mathtt{R}_k}(\mathbf{A}_k \circ \tau_k) \arrow[d, "\mathrm{col}_{\mathtt{R}_k}\partial\varphi_k \circ \tau_k"] \\
\mathrm{col}_{\mathtt{T}}(\mathbf{B}_k \circ i) \arrow[r, "\xi_{\kappa_i}(\mathbf{B}_k \circ \tau_k)"] & \mathrm{col}_{\mathtt{R}_k}(\mathbf{B}_k \circ \tau_k)
\end{tikzcd}$$

□

5.4.3. Constructors and their tomes.

5.4.3.1. *Notations.* Let $\mathcal{A}$ and $\mathcal{C}$ be two categories and $F : \mathcal{A} \to \mathcal{C}^2$ be a functor. In order to make our reasonings less cumbersome, the image $F(X)$ of an object X of $\mathcal{A}$ in the arrow category $\mathcal{C}^2$ will be denoted as $F(X) : F^\circ(X) \to F^\bullet(X)$. This implies that every morphism $f : X \to Y$ in $\mathcal{A}$ implies a commutative diagram as follows.

$$\begin{tikzcd}
F^\circ(X) \arrow[r, "F(X)"] \arrow[d, "F^\circ(f)"'] & F^\bullet(X) \arrow[d, "F^\bullet(f)"] \\
F^\circ(Y) \arrow[r, "F(Y)"'] & F^\bullet(Y)
\end{tikzcd}$$

Remark 5.48. For every functor $G : \mathcal{A} \to \mathcal{C}$, the identity natural transformation $\mathrm{id}_G : G \Rightarrow G$ may be seen as a functor $\mathcal{A} \to \mathcal{C}^2$ whose images are given by identities in $\mathcal{C}$. In this case, the equation $\mathrm{id}_G^\circ = G$ holds.

Let now $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three categories. The image of any functor of the form $G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ will later be denoted as $F_a(b)$ for any pair of objects (a, b) in $\mathcal{A} \times \mathcal{B}$ – instead of the usual notation $F(a, b)$.

5.4.3.2. *Constructors.* Let $\mathcal{C}$, $\mathcal{B}$ be two categories and K, D be two small categories. A *constructor of type* $[K \downarrow D] \times \mathcal{B}$ *in* $\mathcal{C}$ consists of a portfolio $\mathtt{V}$ of vertebrae in $\mathcal{C}$ over K, a functor $H : K \to D$ and two functors $\mathfrak{I} : D \times \mathcal{B} \to \mathcal{C}$ and $\mathfrak{L} : K \times \mathcal{B} \to \mathcal{C}^2$ making the following diagram commute.

$$\begin{tikzcd}
K \times \mathcal{B} \arrow[r, "\mathfrak{L}^\circ"] \arrow[d, "H \times \mathrm{id}_{\mathcal{B}}"'] & \mathcal{C} \\
D \times \mathcal{B} \arrow[ur, "\mathfrak{I}"'] &
\end{tikzcd}$$

Such a structure will later be denoted as a 4-tuple $(\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$.

Definition 5.49 (Circle operation)**.** In this definition, the superfix $\circ$ does not strictly denote the operator defined in section 5.4.3.1, but refers to its action as will be seen later. Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor as defined above. Denote by $\mathtt{V}^\circ$ the portfolio over D whose component $\mathtt{V}^\circ_d$ at an object d in D contains all the 'degenerate' vertebrae in $\mathcal{C}$ of the form given below for every vertebra $p \cdot \beta$ in $\mathtt{V}_\theta$ and object θ in K satisfying the equation $H(\theta) = d$.

$$\begin{tikzcd} \mathbb{S}' \arrow[r, equal] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{S}' \arrow[d, "\beta"] & \\ \mathbb{D}' \arrow[r, equal] & \mathbb{D}' \arrow[r, equal] & \mathbb{D}' \end{tikzcd}$$

We shall let Γ° denote the constructor of type $[D \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ made of the quadruple $(\mathtt{V}^\circ, \mathrm{id}_D, \mathfrak{I}, \mathrm{id}_{\mathfrak{I}(\_)})$ where $\mathrm{id}_{\mathfrak{I}(\_)}$ denotes the identity natural transformation on $\mathfrak{I}$. The fact that such a quadruple is a constructor follows from Remark 5.48.

Definition 5.50 (Star operation)**.** Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor as defined above. Denote by $\mathtt{V}^\star$ the portfolio over K whose component $\mathtt{V}^\star_\theta$ at an object θ in K contains all the vertebrae of the following form in $\mathcal{C}^2$ for every vertebra $v = \|\gamma, \gamma'\| \cdot \beta$ in $\mathtt{V}_\theta$.

$$\begin{tikzcd} \mathrm{id}_{\mathbb{S}} \arrow[r, Rightarrow, "\gamma'"] \arrow[d, Rightarrow, "\gamma"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{id}_{\mathbb{D}_1} \arrow[d, Rightarrow, "\delta_1"] & \\ \mathrm{id}_{\mathbb{D}_2} \arrow[r, Rightarrow, "\delta_2"'] & \mathrm{id}_{\mathbb{S}'} \arrow[r, Rightarrow, "\beta"] & \mathrm{id}_{\mathbb{D}'} \end{tikzcd}$$

Such a vertebra will later be denoted as id_v as it may be seen as the identity morphism on v in $\mathbf{Mod}(\mathtt{Vert})$. We shall let $\Gamma^\star$ denote the constructor of type $[K \downarrow K] \times \mathcal{B}$ in $\mathcal{C}^2$ made of the quadruple $(\mathtt{V}^\star, \mathrm{id}_K, \mathfrak{L}, \mathrm{id}_{\mathfrak{L}(\_)})$. Again, the fact that such a quadruple is a constructor follows from Remark 5.48.

Example 5.51 (Categories of premodels)**.** Let $\mathcal{C}$ be a complete category, D be a small category and (K, T) be a conical croquis in D. Suppose given an adjunction $L \dashv R$ where R and L are endofunctors of $\mathcal{C}$. It follows that the functors R and L commute with limits and colimits, respectively. This means that the image of any vertebra v in $\mathcal{C}$ via L defines a vertebra $L(v)$ in $\mathcal{C}$. Let us consider a portfolio $\mathtt{V}$ of vertebrae in $\mathcal{C}$ over K such that the mapping $v \mapsto L(v)$ defines a function $\mathtt{V}_c \to \mathtt{V}_{T \cdot c}$ for every cylinder c in K. Any portfolio may be completed in such a way[8]. Let $\mathcal{P} \subseteq \mathbf{Np}_{\mathcal{C}}(D, R, T)$ be a category of R-premodels for K. Since the equation

$$\mathrm{dom}_{\mathcal{C}}(\mathcal{G}\mathrm{l}^K(X, e)(c)) = X(\mathtt{peak}_K(c))$$

holds (see section 5.3.1.3 for $\mathtt{peak}_K$), the category $\mathcal{P}$ is equipped with a natural constructor $(\mathtt{V}, \mathtt{peak}_K, \mathfrak{I}, \mathfrak{L})$ of type $[K \downarrow D] \times \mathcal{P}$ in $\mathcal{C}$ where $\mathfrak{I}$ and $\mathfrak{L}$ are the obvious functors defined as follows on the objects.

$$\mathfrak{I} : \left(\begin{array}{lcl} D \times \mathcal{P} & \to & \mathcal{C} \\ (d, (X, e)) & \mapsto & X(d) \end{array} \right) \qquad \mathfrak{L} : \left(\begin{array}{lcl} K \times \mathcal{P} & \to & \mathcal{C}^2 \\ (c, (X, e)) & \mapsto & \mathcal{G}\mathrm{l}^K(X, e)(c) \end{array} \right)$$

This constructor will later be referred to as Γ_K.

5.4.3.3. *Playgrounds.* The term 'playground' refers to the fact that most of our (combinatorial) activities will take place there. Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. For every object d in D and morphism $f : X \to Y$ in $\mathcal{B}$, the *playground of* Γ *at* (f, d) is the set of 4-tuples $(\theta, v, t, \mathbf{c})$ consisting of an object θ in K, a vertebra v in $\mathtt{V}_\theta$, an arrow $t : H(\theta) \to d$ in D and a commutative cube $\mathbf{c}$ in $\mathcal{C}$ encoding an element of the hom-set $\mathcal{C}^{\mathbf{sq}}(\mathbf{disk}(v), \mathfrak{L}_\theta(f))$. Such a set will be denoted by $\mathcal{S}_\Gamma(f, d)$.

---

[8] Any portfolio $\mathtt{V}'$ of vertebrae in $\mathcal{C}$ over K generates a portfolio of the previous form by considering the collection made of the set $\mathtt{V}_c := \bigcup_{c = T^n \cdot c_\flat} L^n(\mathtt{V}'_{c_\flat})$ for every object c in K.

Remark 5.52. In the case where v is of the form $\|\gamma,\gamma'\|\cdot\beta$, the commutative cube $\mathbf{c}$ may be seen as a commutative square in $\mathcal{C}^2$ of the form given below on the left, which represents the right commutative cube when viewed from above.

(5.28)
$$\begin{tikzcd} \gamma \arrow[r, Rightarrow, "x"] \arrow[d, Rightarrow, "\mathbf{disk}(v)"'] & \mathfrak{I}_{H(\theta)}(f) \arrow[d, Rightarrow, "\mathfrak{L}_\theta(f)"] \\ \beta\circ\delta_1 \arrow[r, Rightarrow, "y"'] & \mathfrak{L}^\bullet_\theta(f) \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[rr, "\mathfrak{L}_\theta(X)"] \arrow[dd] & & \cdot \arrow[dd, "\mathfrak{L}_\theta(f)"] \\ \cdot \arrow[ur] \arrow[rr, "\gamma'"] \arrow[dd, "\gamma"] & & \cdot \arrow[ur] \arrow[dd, "\beta\circ\delta_1"] & \\ & \cdot \arrow[rr] & & \cdot \\ \cdot \arrow[ur, "\mathbf{disk}(v)"] \arrow[rr, "\beta\circ\delta_2"'] & & \cdot \arrow[ur] & \end{tikzcd}$$

Remark 5.53. The function $(f,d)\mapsto\mathcal{S}_\Gamma(f,d)$ extends to an obvious functor $\mathcal{B}^2\times D\to\mathbf{Set}$ by mapping any arrow $s:d\to d'$ in D and morphism $\eta:f\Rightarrow f'$ in $\mathcal{B}^2$ to the function $\mathcal{S}_\Gamma(f,d)\to\mathcal{S}_\Gamma(f,d')$ satisfying the mapping rule $(\theta,v,t,\mathbf{c})\mapsto(\theta,v,s\circ t,\mathfrak{L}_\theta(\eta)\circ\mathbf{c})$ where the composite $\mathfrak{L}_\theta(\eta)\circ\mathbf{c}$ is more explicitly as follows.

$$\mathbf{disk}(v) \xRightarrow{\mathbf{c}} \mathfrak{L}_\theta(f) \xRightarrow{\mathfrak{L}_\theta(\eta)} \mathfrak{L}_\theta(f')$$

The functor $\mathcal{S}_\Gamma(f,\_):D\to\mathbf{Set}$ will later be referred to as the *playground of the constructor* Γ *at* f. In the sequel, this functor will be seen as a functor valued in $\mathbf{Cat}(1)$ by identifying its images with discrete categories.

Example 5.54. Let $\Gamma=(\mathtt{V},H,\mathfrak{I},\mathfrak{L})$ be a constructor of type $[K\downarrow D]\times\mathcal{B}$ in $\mathcal{C}$. For every object d in D and morphism f in $\mathcal{B}$, the playground of Γ° at (f,d) is the set of 4-tuples $(d',v,t,\mathbf{c})$ consisting of an object d' in D, a degenerate vertebra v in $\mathtt{V}^\circ_{d'}$, a morphism $t:d'\to d$ in D and commutative cube $\mathbf{c}$ in $\mathcal{C}$ encoding an arrow of the following form in $\mathcal{C}^{\mathbf{sq}}$.

$$\mathbf{c}:\mathbf{disk}(v)\Rightarrow\mathrm{id}_{\mathfrak{I}_{d'}(f)}$$

Because of the particular form of v, the cube $\mathbf{c}$ may instead be seen as an arrow

$$\mathbf{s}:\beta\Rightarrow\mathfrak{I}_{d'}(f)$$

in $\mathcal{C}^2$ when v is equal to $\|\beta,\mathrm{id}\|\cdot\mathrm{id}$. This is how we will later regard the cubes associated with the elements of $\mathcal{S}_{\Gamma^\circ}(f,d)$.

Example 5.55. Let $\Gamma=(\mathtt{V},H,\mathfrak{I},\mathfrak{L})$ be a constructor of type $[K\downarrow D]\times\mathcal{B}$ in $\mathcal{C}$. For every object θ in K and morphism f in $\mathcal{B}$, the playground of $\Gamma^\star$ at (f,θ) is the set of 4-tuples $(\theta',\mathrm{id}_v,t,\mathbf{h})$ consisting of an object θ in K, a vertebra id_v in $\mathtt{V}^\star_{\theta'}$, a morphism $t:\theta'\to\theta$ in K and a commutative hypercube $\mathbf{h}$ in $\mathcal{C}$ encoding an arrow of the following form in $(\mathcal{C}^{\mathbf{sq}})^2$.

$$\mathbf{h}:\mathbf{disk}(\mathrm{id}_v)\Rightarrow\mathrm{id}_{\mathfrak{L}_\theta(f)}$$

It follows from the particular form of the vertebrae in $\mathtt{V}^\star$ that the hypercube $\mathbf{c}$ may be seen as an obvious commutative cube in $\mathcal{C}$ or an arrow as follows in $\mathcal{C}^{\mathbf{sq}}$.

$$\mathbf{c}:\mathbf{disk}(v)\Rightarrow\mathfrak{L}_\theta(f)$$

This is how we will later regard the hypercubes of the elements of $\mathcal{S}_{\Gamma^\star}(f,\theta)$.

Example 5.56 (Categories of premodels)**.** Consider the constructor Γ_K defined in Example 5.51 for some category of premodels $\mathcal{P}\subseteq\mathbf{Np}_{\mathcal{C}}(D,R,T)$ over a conical croquis (K,T). For every morphism $f:(X,e)\Rightarrow(Y,e')$ in $\mathcal{P}$, we are going to show that there exists a natural transformation as follows.

$$\mathcal{S}_{\Gamma_K}(f,\_)\Rightarrow\mathcal{S}_{\Gamma_K}(f,T(\_))$$

For every object d in D, consider an object $(c, v, t, \mathbf{c})$ in $\mathcal{S}_{\ulcorner K}(f, d)$ where c is a cylinder in K, $t : \mathtt{peak}_K(c) \to d$ is an arrow in D and $\mathbf{c}$ is as shown in diagram (5.28). The cylinder c will be encoded by the following left diagram so that the object $\mathtt{peak}_K(c)$ in D is given by $d_\flat$. It follows that $\mathbf{c}$ has the form of the right commutative square.

$$\begin{tikzcd} \mathbf{1} \arrow[d, "d_\flat"'] & A_\dagger \arrow[l, "!"'] \arrow[d, "d_\dagger"] \arrow[dl, Rightarrow, "t_\flat"'] \\ D \arrow[r, equal] & D \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \gamma \arrow[r, Rightarrow, "x"] \arrow[d, Rightarrow, "\mathbf{disk}(v)"'] & \lim_{d_\flat} f \arrow[d, Rightarrow, "\mathfrak{L}_c(f)"] \\ \beta \circ \delta_1 \arrow[r, Rightarrow, "y"'] & \lim_{d_\dagger} RfT \end{tikzcd} \tag{5.29}$$

Now, it comes from the naturality of $e : X \Rightarrow RXT$ and $e' : Y \Rightarrow RYT$; the functoriality of $\mathcal{G}\mathrm{l}^K(\_)(c)$ and $\mathcal{G}\mathrm{l}^K(\_)(T \cdot c)$ and the definition of the morphism $f : (X, e) \Rightarrow (Y, e')$ in $\mathcal{P}$ that the following cuboid commutes in $\mathcal{C}$. Note that the commutativity of R with the limits of $\mathcal{C}$ is used implicitly on the left face.

$$\begin{tikzcd} \lim_{d_\flat} X \arrow[rr, "\lim_{d_\flat} e"] \arrow[dr, "\mathfrak{L}_c(X)"'] \arrow[dd, "\lim_{d_\flat} f"'] & & \lim_{d_\flat} RXT \arrow[dr, "R\mathfrak{L}_{T\cdot c}(X)"] \arrow[dd, "\lim_{d_\flat} RfT"' near start] & \\ & \lim_{d_\dagger} RXT \arrow[rr, "\lim_{d_\dagger} ReT"' near start] \arrow[dd, "\lim_{d_\dagger} RfT" near start] & & \lim_{d_\dagger} R^2XT^2 \arrow[dd, "\lim_{d_\dagger} R^2fT^2"] \\ \lim_{d_\flat} Y \arrow[rr, "\lim_{d_\flat} e'"' near end] \arrow[dr, "\mathfrak{L}_c(Y)"'] & & \lim_{d_\flat} RYT \arrow[dr, "R\mathfrak{L}_{T\cdot c}(Y)"'] & \\ & \lim_{d_\dagger} RYT \arrow[rr, "\lim_{d_\dagger} Re'T"'] & & \lim_{d_\dagger} R^2YT^2 \end{tikzcd}$$

The above cuboid may be expressed in terms of the left commutative diagram, below, in $\mathcal{C}^2$ when seen from above. Pasting this commutative diagram with the right commutative square of (5.29) then leads to the commutative square given below on the right.

$$\begin{tikzcd} \lim_{d_\flat} f \arrow[r, Rightarrow, "\lim_d e_f"] \arrow[d, Rightarrow, "\mathfrak{L}_c(f)"'] & \lim_{d_\flat} RfT \arrow[d, Rightarrow, "R\mathfrak{L}_{T\cdot c}(f)"] \\ \lim_{d_\dagger} RfT \arrow[r, Rightarrow, "\lim_{d_\dagger} Re_fT"'] & \lim_{d_\dagger} R^2fT^2 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \gamma \arrow[r, Rightarrow, "\lim_d e_f \circ x"] \arrow[d, Rightarrow, "\mathbf{disk}(v)"'] & \lim_{d_\flat} RfT \arrow[d, Rightarrow, "R\mathfrak{L}_{T\cdot c}(f)"] \\ \beta \circ \delta_1 \arrow[r, Rightarrow, "\lim_{d_\dagger} Re_fT \circ y"'] & \lim_{d_\dagger} R^2fT^2 \end{tikzcd}$$

As previously assumed, the preceding right commutative diagram may be considered up to canonical isomorphisms of the form $R\mathrm{lim} \cong \mathrm{lim}R$. In this case, applying the functor $L : \mathcal{C} \to \mathcal{C}$ on the resulting diagram provides the following left-hand commutative square in $\mathcal{C}^2$. Post-composing this diagram with the naturality square of the counit $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathcal{C}}$ of the adjunction $L \vdash R$ leads to the succeeding right commutative diagram.

$$\begin{tikzcd} L(\gamma) \arrow[r, Rightarrow, "L(\lim_d e_f \circ x)"] \arrow[d, Rightarrow, "L(\mathbf{disk}(v))"'] & LR\lim_{d_\flat} fT \arrow[d, Rightarrow, "LR\mathfrak{L}_{T\cdot c}(f)"] \\ L(\beta \circ \delta_1) \arrow[r, Rightarrow, "LR(\lim_{d_\dagger} e_fT \circ y)"'] & LR\lim_{d_\dagger} RfT^2 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} L(\gamma) \arrow[r, Rightarrow, "\varepsilon_* \circ L(\lim_d e_f \circ x)"] \arrow[d, Rightarrow, "L(\mathbf{disk}(v))"'] & \lim_{d_\flat} fT \arrow[d, Rightarrow, "\mathfrak{L}_{T\cdot c}(f)"] \\ L(\beta \circ \delta_1) \arrow[r, Rightarrow, "\varepsilon_* \circ L(\lim_{d_\dagger} Re_fT \circ y)"'] & \lim_{d_\dagger} RfT^2 \end{tikzcd}$$

The notation ε_* is used to make the previous diagram less cumbersome, but the star should rigorously be thought of as the appropriate object. In the end, by putting the functor T in the indices of the limits, the preceding right diagram provides a commutative cube $\mathbf{c}'$ as follows.

$$\begin{tikzcd} L(\gamma) \arrow[rr, Rightarrow, "x'"] \arrow[d, Rightarrow, "\mathbf{disk}(L(v))"'] & & \lim_{T\circ d_\flat} f \arrow[d, Rightarrow, "\mathfrak{L}_{T\cdot c}(f)"] \\ L(\beta\circ\delta_1) \arrow[rr, Rightarrow, "y'"'] & & \lim_{T\circ d_\dagger} Rf \end{tikzcd}$$

Since the vertebra $L(v)$ belongs to $\mathtt{V}_{T\cdot c}$, the preceding cube is associated with the quadruple $(T\cdot c, L(v), T(t), \mathbf{c}')$ in the discrete category $\mathcal{S}_{\Gamma_K}(f, T(d))$. In fact, the mapping $(c, v, t, \mathbf{c}) \mapsto (T\cdot c, L(v), T(t), \mathbf{c}')$ defines a functor from $\mathcal{S}_{\Gamma_K}(f, d)$ to $\mathcal{S}_{\Gamma_K}(f, T(d))$, which is obviously natural in the variable d by functoriality of T. The induced natural transformation will later be denoted as follows.

$$\zeta_{\Gamma_K} : \mathcal{S}_{\Gamma_K}(f, \_) \Rightarrow \mathcal{S}_{\Gamma_K}(f, T(\_))$$

Example 5.57 (Categories of premodels)**.** The construction of Example 5.56 also holds for the constructor Γ_K° as its underlying portfolio $\mathtt{V}^\circ$ of vertebrae is also equipped with mappings $v \mapsto L(v)$ from $\mathtt{V}_d^\circ$ to $\mathtt{V}_{T(d)}^\circ$ when $\mathtt{V}$ is. Specifically, for every morphism $f : (X, e) \Rightarrow (Y, e')$ in the category of premodels $\mathcal{P}$, the natural transformation

$$\zeta_{\Gamma_K^\circ} : \mathcal{S}_{\Gamma_K^\circ}(f, \_) \Rightarrow \mathcal{S}_{\Gamma_K^\circ}(f, T(\_))$$

maps a 4-tuple $(d', v, t, \mathbf{s})$, where d' is an object of D and $\mathbf{s}$ is an arrow $\beta \Rightarrow f(d')$ in $\mathcal{C}^2$ (see Example 5.54), to the 4-tuple $(T(d'), L(v), T(t), \mathbf{s}')$ where $\mathbf{s}'$ is the arrow $\varepsilon_* \circ L(e(d') \circ \mathbf{s}) : L(\beta) \Rightarrow fT(d')$ in $\mathcal{C}^2$.

5.4.3.4. *Tomes of a constructor.* Let $\mathcal{C}$ be a category that admits all coproducts and $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. For every object d in D and morphism f in $\mathcal{B}$, the *tome of* Γ *at* (f, d) is the functor $\varphi_{\Gamma,d} : \mathcal{S}_\Gamma(f, d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$ that maps a 4-tuple $(\theta, v, t, \mathbf{c})$ to the composite arrow

$$\gamma \xRightarrow{\mathrm{top}(\mathbf{c})} \mathfrak{I}_{H(\theta)}(f) \xRightarrow{\mathfrak{I}_t(f)} \mathfrak{I}_d(f)$$

where $\mathrm{top} : \mathcal{C}^{\mathbf{sq}} \to \mathcal{C}^2$ is the domain functor of $\mathcal{C}^2$, which sends any commutative cube living in $\mathcal{C}^{\mathbf{sq}}(\mathbf{disk}(v), \mathfrak{L}_\theta(f))$ to the underlying commutative square in $\mathcal{C}^2(\gamma, \mathfrak{I}_{H(\theta)}(f))$ (diagram (5.28) might turn out to be useful). We will later refer to this tome as a triple of the following form.

$$\mathbb{T}^\Gamma(f, d) := (\mathfrak{I}_d(f), \mathcal{S}_\Gamma(f, d), \varphi_{\Gamma,d})$$

Example 5.58. Let $\mathcal{C}$ be a category admitting all coproducts and $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. The tome associated with the constructor Γ° provides for every object d in D and morphism f in $\mathcal{B}$, a functor $\varphi_{\Gamma^\circ,d} : \mathcal{S}_{\Gamma^\circ}(f, d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$ mapping a 4-tuple $(d', v, t, \mathbf{s})$ to the following composite arrow in $\mathcal{C}^2$ (see convention in Example 5.54).

$$\beta \xRightarrow{\mathbf{s}} \mathfrak{I}_{d'}(f) \xRightarrow{\mathfrak{I}_t(f)} \mathfrak{I}_d(f)$$

Example 5.59. Let $\mathcal{C}$ be a category admitting coproducts and $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. The tome of the constructor $\Gamma^\star$ provides for every object θ in K and morphism f in $\mathcal{B}$, a functor $\varphi_{\Gamma^\star,\theta} : \mathcal{S}_{\Gamma^\star}(f, \theta) \to \mathcal{C}^{\mathbf{sq}}/\mathfrak{L}_\theta(f)$ mapping a 4-tuple $(\theta', \mathrm{id}_v, t, \mathbf{c})$ to the following composite arrow in $\mathcal{C}^2$ (see convention in Example 5.55).

$$\mathbf{disk}(v) \xRightarrow{\mathbf{c}} \mathfrak{L}_{\theta'}(f) \xRightarrow{\mathfrak{L}_t(f)} \mathfrak{L}_\theta(f)$$

Proposition 5.60. *The functor $\varphi_{\Gamma,d} : \mathcal{S}_\Gamma(f, d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$ is natural in d over D and f over $\mathcal{B}^2$. This is equivalent to saying that the mapping $(f, d) \mapsto \mathbb{T}^\Gamma(f, d)$ induces a functor $\mathcal{B}^2 \times D \to \mathbf{Tome}(\mathcal{C})$.*

Proof. Let $f : x \to Y$ and $f' : X' \to Y'$ be two arrows in $\mathcal{B}$. For every pair of morphisms $\eta : f \Rightarrow f'$ in $\mathcal{B}^2$ and $s : d \to d'$ in D, the following diagram commutes, which turns the mapping $(f,d) \mapsto \mathbb{T}^\Gamma(f,d)$ into a first functor $\mathcal{B}^2 \times D \to \mathbf{LTom}(\mathcal{C})$ by functoriality of $\mathfrak{I} : D \times \mathcal{B} \to \mathcal{C}$.

$$\begin{tikzcd}
\mathfrak{I}_d(X) \arrow[r, "\mathfrak{I}_d(\eta)"] \arrow[d, "\mathfrak{I}_d(f)"'] & \mathfrak{I}_d(X') \arrow[r, "\mathfrak{I}_t(X')"] \arrow[d, "\mathfrak{I}_d(f')"] & \mathfrak{I}_{d'}(X') \arrow[d, "\mathfrak{I}_{d'}(f')"] \\
\mathfrak{I}_d(Y) \arrow[r, "\mathfrak{I}_d(\eta)"'] & \mathfrak{I}_d(Y') \arrow[r, "\mathfrak{I}_s(Y')"'] & \mathfrak{I}_{d'}(Y')
\end{tikzcd}$$

We are now going to show that this functor lift to $\mathbf{Tome}(\mathcal{C})$. For this, we need to show that the action of this commutative square as a functor $\mathcal{C}^2/\mathfrak{I}_d(f) \to \mathcal{C}^2/\mathfrak{I}_{d'}(f')$ is compatible with the tomes $\varphi_{\Gamma,d} : \mathcal{S}_\Gamma(f,d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$ and $\varphi_{\Gamma,d'} : \mathcal{S}_\Gamma(f',d') \to \mathcal{C}^2/\mathfrak{I}_{d'}(f')$ along the playground functor $\mathcal{S}_\Gamma(\eta,s) : \mathcal{S}_\Gamma(f,d) \to \mathcal{S}_\Gamma(f',d')$. In this respect, consider the above commutative diagram in $\mathcal{C}^2$ and paste the image $\varphi_{\Gamma,d}(\theta,v,t,\mathbf{c})$ next to it as follows.

$$\gamma \xRightarrow{\mathrm{top}(\mathbf{c})} \mathfrak{I}_{H(\theta)}(f) \xRightarrow{\mathfrak{I}_t(f)} \mathfrak{I}_d(f) \xRightarrow{\mathfrak{I}_d(\eta)} \mathfrak{I}_d(f') \xRightarrow{\mathfrak{I}_s(f')} \mathfrak{I}_{d'}(f')$$

After using the functoriality of $\mathfrak{I} : D \times \mathcal{B} \to \mathcal{C}$, the above sequence of arrows may be turned into the following one.

$$\gamma \xRightarrow{\mathrm{top}(\mathbf{c})} \mathfrak{I}_{H(\theta)}(f) \xRightarrow{\mathfrak{I}_{H(\theta)}(\eta)} \mathfrak{I}_{H(\theta)}(f') \xRightarrow{\mathfrak{I}_{sot}(f')} \mathfrak{I}_{d'}(f') \tag{5.30}$$

Because the equation

$$\mathfrak{I}_{H(\theta)}(\eta) \circ \mathrm{top}(\mathbf{c}) = \mathrm{top}(\mathfrak{L}_\theta(\eta) \circ \mathbf{c})$$

holds (by dedinition of a constructor), the composite of the sequence of arrows (5.30) belongs to $\varphi_{\Gamma,d'} \circ \mathcal{S}_\Gamma(\eta,s)(\theta,v,t,\mathbf{c})$ by Remark 5.53. This shows that the mapping $(f,d) \mapsto \mathbb{T}^\Gamma(f,d)$ defines a functor to $\mathbf{Tome}(\mathcal{C})$. □

5.4.4. Modifiers and obstruction squares.

5.4.4.1. *Local modifiers.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. For every morphism f in $\mathcal{B}$, a *local modifier of* Γ *at* f is a subfunctor of the playground of Γ° at f. In other words, a local modifier of Γ at f is a functor $\mathfrak{m}_f : D \to \mathbf{Set}$ such that

1) for every object d in D, the inclusion $\mathfrak{m}_f(d) \subseteq \mathcal{S}_{\Gamma^\circ}(f,d)$ holds;
2) for every morphism $t : d \to d'$ in D, the function $\mathfrak{m}_f(t) : \mathfrak{m}_f(d) \to \mathfrak{m}_f(d')$ is the restriction of $\mathcal{S}_{\Gamma^\circ}(f,t)$ along the respective inclusions of the domains and codomains.

5.4.4.2. *Modifiers.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. A *modifier* of Γ is a collection of local modifiers $\mathfrak{m}_f$ at every morphism f in $\mathcal{B}$.

Remark 5.61. According to Example 5.54, for every object d in D and morphism $f : X \to Y$ in $\mathcal{B}$, the image $\mathfrak{m}_f(d)$ of some modifier $\mathfrak{m}$ of Γ contains data consisting of an object d' in D, a stem $\beta : \mathbb{S}' \to \mathbb{D}'$ of some vertebra in $\mathtt{V}_{d'}$, a morphism $t : d' \to d$ in D and a commutative square in $\mathcal{C}$ as follows.

$$\begin{tikzcd}
\mathbb{S}' \arrow[r, "x"] \arrow[d, "\beta"'] & \mathfrak{I}_{d'}(X) \arrow[d, "\mathfrak{I}_{d'}(f)"] \\
\mathbb{D}' \arrow[r, "y"'] & \mathfrak{I}_{d'}(Y)
\end{tikzcd}$$

For every modifier $\mathfrak{m}$ as defined above, we will denote by $\tilde{\mathfrak{m}}_f$ the functor $D \to \mathbf{Set}$ obtained by the image factorisation (i.e. epi-mono factorisation) of the natural transformation resulting from the composition of the inclusion $\mathfrak{m}_f \subseteq \mathcal{S}_{\Gamma^\circ}(f,\_)$ together with the tome $\varphi_{\Gamma^\circ} : \mathcal{S}_{\Gamma^\circ}(f,\_) \Rightarrow \mathcal{C}^2 \downarrow_\circ \mathfrak{I}(f)$. In other words, we obtain a canonical factorisation as follows.

$$\mathfrak{m}_f \Longrightarrow\!\!\!\!\!\Rightarrow \tilde{\mathfrak{m}}_f \xRightarrow{\tilde{\varphi}_{\Gamma^\circ}} \mathcal{C}^{\mathbf{2}} \downarrow_\circ \Upsilon(f)$$

For every object $d \in D$, the image $\tilde{\mathfrak{q}}(d)$ will be thought of as a quotiented set of $\mathfrak{q}(d)$

5.4.4.3. *Modified playground.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ and $\mathfrak{m}$ be a modifier of Γ. The *modified playground* Γ *along* $\mathfrak{m}$ is the functor $\mathtt{S}_{\mathfrak{m}}(f,\_) : D \to \mathbf{Set}$ defined by the following formal sum in $\mathbf{Set}$ for every object d in D.

$$\mathtt{S}_{\mathfrak{m}}(f,d) = \tilde{\mathfrak{m}}_f(d) + \mathcal{S}_\Gamma(f,d)$$

In the sequel, any modified playground and modifier will be seen as a functor valued in $\mathbf{Cat}(1)$ by identifying their images with discrete categories.

Example 5.62 (Categories of premodels)**.** In the sequel, a modifier $\mathfrak{m}$ will be said to be *admissible* for the constructor Γ_K if the transformation $\zeta_{\Gamma_K^\circ}(f,\_) : \mathcal{S}_{\Gamma_K}(f,\_) \Rightarrow \mathcal{S}_{\Gamma_K}(f,T(\_))$ restricts on the domain and lifts on the codomain to a natural transformation $\zeta_{\Gamma_K^\circ}(f,\_) : \mathfrak{m}_f(\_) \Rightarrow \mathfrak{m}_f \circ T(\_)$ for every morphism f in $\mathcal{P}$. The mapping rules of $\zeta_\Gamma(f,\_)$ and $\zeta_{\Gamma_K^\circ}(f,\_)$ then extend to the modified playground along $\mathfrak{m}$ in terms of a natural transformation as follows.

$$\zeta_{\mathfrak{m}}(f,\_) : \mathtt{S}_{\mathfrak{m}}(f,\_) \Rightarrow \mathtt{S}_{\mathfrak{m}}(f,T(\_))$$

5.4.4.4. *Modified tomes of a constructor.* Let $\mathcal{C}$ be a category that admits all coproducts, $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ and $\mathfrak{m}$ be a modifier of Γ. The *modified tome of* Γ *along* $\mathfrak{m}$ is the functor $\varphi_d^{\mathfrak{m}} : \mathtt{S}_{\mathfrak{m}}(f,d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$ defined as the coproduct in $\mathbf{Cat}(1)$ of the cocone induced by the following two functors.

$$\varphi_{\Gamma,d} : \mathcal{S}_\Gamma(f,d) \to \mathcal{C}^2/\mathfrak{I}_d(f) \qquad\qquad \tilde{\varphi}_{\Gamma^\circ,d} : \tilde{\mathfrak{m}}_f(d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$$

This functor is therefore equipped with the following mapping rules.

$$\begin{cases} (\theta, v, t, \mathbf{c}) & \mapsto & \mathfrak{I}_t(f) \circ \mathrm{top}(\mathbf{c}) & : \gamma \Rightarrow \mathfrak{I}_d(f) & \text{over } \mathcal{S}_\Gamma(f,d) \\ (d', v, t, \mathbf{s}) & \mapsto & \mathfrak{I}_t(f) \circ \mathbf{s} & : \beta \Rightarrow \mathfrak{I}_d(f) & \text{over } \mathfrak{m}_f(d) \end{cases}$$

The above tome will later be referred to by the following triple.

$$\mathbb{T}_{\mathfrak{m}}^{\Gamma}(f,d) := (\mathfrak{I}_d(f), \mathtt{S}_{\mathfrak{m}}(f,d), \varphi_d^{\mathfrak{m}})$$

Proposition 5.63. *The functor $\varphi_d^{\mathfrak{m}} : \mathtt{S}_{\mathfrak{m}}(f,d) \to \mathcal{C}^2/\mathfrak{I}_d(f)$ is natural in d over D. This is equivalent to saying that the mapping $d \mapsto \mathbb{T}_{\mathfrak{m}}^{\Gamma}(f,d)$ induces a functor $D \to \mathbf{Tome}(\mathcal{C})$ defined by post-composition.*

Proof. Follows from Proposition 5.60, the definition of $\varphi_d^{\mathfrak{m}}$ as a coproduct functor and the fact that $\mathfrak{m}_f$ is a subfunctor of the playground $\mathcal{S}_{\Gamma^\circ}(f,\_)$. □

Remark 5.64. The naturality of $\varphi^{\mathfrak{m}} : \mathtt{S}_{\mathfrak{m}}(f,\_) \Rightarrow \mathcal{C}^2 \downarrow \mathfrak{I}_{\_}(f)$ over D extends to the content of the tome $\mathbb{T}_{\mathfrak{m}}^{\Gamma}(f,d)$. Specifically, recall that the content of the tome $\mathbb{T}_{\mathfrak{m}}^{\Gamma}(f,d)$ is obtain by applying the colimit functor on $\varphi_d^{\mathfrak{m}}$ (see section 5.4.1.5). If we denote by $\mathtt{S}_d$ the category $\mathtt{S}_{\mathfrak{m}}(f,d)$, the functoriality of the colimit functor implies that the content given below on the left-hand side in $\mathcal{C}$ extends to the right commutative square in $\mathcal{C}^D$.

$$\begin{tikzcd} \mathrm{col}_{\mathtt{S}_d}\mathbf{A}_d \arrow[r, "\mathrm{col}_{\mathtt{S}_d} u"] \arrow[d, "\mathrm{col}_{\mathtt{S}_d}\partial\varphi_d^{\mathfrak{m}}"'] & \mathfrak{I}_d(X) \arrow[d, "\mathfrak{I}_d(f)"] \\ \mathrm{col}_{\mathtt{S}_d}\mathbf{B}_d \arrow[r, "\mathrm{col}_{\mathtt{S}_d} v"'] & \mathfrak{I}_d(Y) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{col}_{\mathtt{S}}\mathbf{A} \arrow[r, Rightarrow, "\mathrm{col}_{\mathtt{S}} u"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathtt{S}}\partial\varphi^{\mathfrak{m}}"'] & \mathfrak{I}(X) \arrow[d, Rightarrow, "\mathfrak{I}(f)"] \\ \mathrm{col}_{\mathtt{S}}\mathbf{B} \arrow[r, Rightarrow, "\mathrm{col}_{\mathtt{S}} v"'] & \mathfrak{I}(Y) \end{tikzcd}$$

The above right diagram will be referred to as the *functorial content of* $\mathbb{T}_{\mathfrak{m}}^{\Gamma}(f,\_)$.

Proposition 5.65. *For every object d in D, the mapping $f \mapsto \mathbb{T}^{\Gamma}_{\mathfrak{m}}(f,d)$ induces an obvious functor $\mathcal{B}^2 \to \mathbf{Ltom}(\mathcal{C})$.*

Proof. It suffices to notice that the following diagram commutes for every morphism $\eta : f \Rightarrow f'$ in $\mathcal{B}^2$ where $f : X \to Y$ and $f' : X \to Y'$.

$$\begin{tikzcd} \mathfrak{I}_d(X) \arrow[r, "\mathfrak{I}_d(\eta)"] \arrow[d, "\mathfrak{I}_d(f)"'] & \mathfrak{I}_d(X') \arrow[d, "\mathfrak{I}_d(f')"] \\ \mathfrak{I}_d(Y) \arrow[r, "\mathfrak{I}_d(\eta)"'] & \mathfrak{I}_d(Y') \end{tikzcd}$$

□

5.4.4.5. *Combinatorial constructors.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ and $\mathfrak{m}$ be a modifier of Γ. The functorial content of $\mathbb{T}^{\Gamma}_{\mathfrak{m}}(f,\_)$ will be said to *admit a pushout in $\mathcal{C}^D$* if one may form a pushout square inside the functorial content of $\mathbb{T}^{\Gamma}_{\mathfrak{m}}(f,\_)$ as shown below.

$$\begin{tikzcd} \mathrm{col}_{\mathbb{S}}\mathbf{A} \arrow[r, Rightarrow, "\mathrm{col}_{\mathbb{S}}u"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathbb{S}}\partial\varphi^{\mathfrak{m}}"'] & \mathfrak{I}(X) \arrow[d, Rightarrow, "\mathfrak{q}^{\mathfrak{m}}_f"] \arrow[dr, Rightarrow, bend left, "\mathfrak{I}(f)"] & \\ \mathrm{col}_{\mathbb{S}}\mathbf{B} \arrow[r, Rightarrow, "\pi^{\mathfrak{m}}_f"'] \arrow[rr, Rightarrow, bend right, "\mathrm{col}_{\mathbb{S}}v"'] \arrow[ur, phantom, "\ulcorner"] & \mathfrak{N}^{\mathfrak{m}}_f \arrow[r, Rightarrow, "\mathfrak{a}^{\mathfrak{m}}_f"] & \mathfrak{I}(Y) \end{tikzcd} \tag{5.31}$$

The functor $d \mapsto \mathfrak{N}^{\mathfrak{m}}_f(d)$ will then be called the *pushout construction under* $\mathfrak{m}$ *at* f.

Definition 5.66 (Combinatorial constructors)**.** A constructor $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ will be said to be *combinatorial along a modifier* $\mathfrak{m}$ *at a morphism* $f : X \to Y$ *in* $\mathcal{B}$ if

1) the functorial content of $\mathbb{T}^{\Gamma}_{\mathfrak{m}}(f,\_)$ admits a pushout in $\mathcal{C}^D$ as given in (5.31);
2) there exists a factorisation of $f : X \to Y$ in $\mathcal{B}$ of the form given below, on the left, that lifts the corresponding right-hand factorisation of $\mathfrak{I}(f)$ along $\mathfrak{I} : \mathcal{B} \to \mathcal{C}^D$.

$$\begin{tikzcd} X \arrow[r, "\{f\}_{\mathfrak{m}}"'] \arrow[rr, bend left, "f"] & N^{\mathfrak{m}}_f \arrow[r, "\lfloor f \rfloor_{\mathfrak{m}}"'] & Y & \overset{\mathfrak{I}}{\longmapsto} & \mathfrak{I}(X) \arrow[r, Rightarrow, "\mathfrak{q}^{\mathfrak{m}}_f"'] \arrow[rr, Rightarrow, bend left, "\mathfrak{I}(f)"] & \mathfrak{N}^{\mathfrak{m}}_f \arrow[r, Rightarrow, "\mathfrak{a}^{\mathfrak{m}}_f"'] & \mathfrak{I}(Y) \end{tikzcd} \tag{5.32}$$

A constructor Γ of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ that is combinatorial[9] along a modifier $\mathfrak{m}$ at every morphism of $\mathcal{B}$ will be said to be *combinatorial along* $\mathfrak{m}$ and will sometimes be referred to as a pair $(\Gamma, \mathfrak{m})$.

Example 5.67 (Categories of premodels)**.** Suppose that the category $\mathcal{C}$ of Example 5.56 admits pushouts. By Proposition 1.28, the functor category $\mathcal{C}^D$ has componentwise pushouts. We are going to show that the constructor Γ_K is combinatorial along any admissible modifier $\mathfrak{m}$ when the category of R-premodels $\mathcal{P}$ is equal to $\mathbf{Np}_{\mathcal{C}}(D,R,T)$. Let $f : (X,e) \Rightarrow (Y,e')$ be a morphism in $\mathbf{Np}_{\mathcal{C}}(D,R,T)$. To start with, define the obvious natural transformation over D of the form

$$\zeta'_{\Gamma_K} : \mathcal{C}^2 \mathbin{\downarrow_{\!\circ}} f \Rightarrow \mathcal{C}^2 \mathbin{\downarrow_{\!\circ}} fT$$

[9]Here, the term 'combinatorial' refers to the common practice, in homotopy theory, of calling a structure 'combinatorial' to mean that it allows the application of the small object argument.

that maps an object $z : \delta \Rightarrow f(d)$ in $\mathcal{C}^2/f(d)$ to the composite arrow displayed below, on the left, where $e_f(d)$ denotes the arrow $f(d) \Rightarrow RfT(d)$ in $\mathcal{C}^2$ as displayed in brackets on the right.

$$\varepsilon_* \circ L(e_f(d) \circ z) : L(\delta) \Rightarrow fT(d) \qquad \left(\begin{tikzcd} X(d) \arrow[r, "e(d)"] \arrow[d, "f(d)"'] & RXT(d) \arrow[d, "RfT(d)"] \\ Y(d) \arrow[r, "e'(d)"'] & RYT(d) \end{tikzcd}\right)$$

Let us show that this natural transformation makes the following left diagram commute, the underlying mapping rules being given on the right.

$$\begin{tikzcd} \mathtt{S}_{\mathfrak{m}}(f, \_) \arrow[r, Rightarrow, "\zeta_{\mathfrak{m}}"] \arrow[d, Rightarrow, "\varphi^{\mathfrak{m}}"'] & \mathtt{S}_{\mathfrak{m}}(f, T(\_)) \arrow[d, Rightarrow, "\varphi^{\mathfrak{m}}_T"'] \\ \mathcal{C}^2 \downarrow f \arrow[r, Rightarrow, "\zeta'_{\ulcorner_K}"'] & \mathcal{C}^2 \downarrow fT \end{tikzcd} \qquad \left(\begin{tikzcd} (c, v, t, \mathbf{c}) \arrow[r, mapsto, "\zeta_{\mathfrak{m}}"] \arrow[d, mapsto, "\varphi^{\mathfrak{m}}"'] & (T \cdot c, L(v), T(t), \mathbf{c}') \arrow[d, mapsto, "\varphi^{\mathfrak{m}}_T"'] \\ \mathfrak{I}_t(f) \circ \mathrm{top}(\mathbf{c}) \arrow[r, mapsto, "\zeta'_{\ulcorner_K}"'] & \mathfrak{I}_{T(t)}(f) \circ \mathrm{top}(\mathbf{c}') \end{tikzcd}\right)$$

Specifically, the commutativity of the preceding diagram at an object $(c, v, t, \mathbf{c})$ (with the same notation as in Example 5.56) follows from the next few equations, where the limit $\lim_d e_f$ over a functor $d : \mathbf{1} \to D$ has been replaced with the evaluation $e_f(d)$.

$$\begin{aligned} \zeta'_{\ulcorner_K, d}(\varphi^{\mathfrak{m}}_d(c, v, t, \mathbf{c})) &= \varepsilon_* \circ L(e_f(d) \circ \mathfrak{I}_t(f) \circ x) && \text{(definition)} \\ &= \varepsilon_* \circ L(R\mathfrak{I}_{T(t)}(f) \circ e_f(d_\flat) \circ x) && \text{(naturality of } e_f) \\ &= \mathfrak{I}_{T(t)}(f) \circ \varepsilon_* \circ L(e_f(d_\flat) \circ x) && \text{(naturality of } \varepsilon) \\ &= \varphi^{\mathfrak{m}}_{T(d)}(\zeta_{\mathfrak{m},d}(c, v, t, \mathbf{c})) && \text{(definition)} \end{aligned}$$

Interestingly, the preceding equation between the first and last terms allows us to express the content of the tome $\mathbb{T}^{\ulcorner_K}_{\mathfrak{m}}(f, T(d))$ along $\zeta_{\mathfrak{m},d} : \mathtt{S}_{\mathfrak{m}}(f, d) \to \mathtt{S}_{\mathfrak{m}}(f, T(d))$ in terms of the content of the tome $\mathbb{T}^{\ulcorner_K}_{\mathfrak{m}}(f, d)$. To show this, let us regard (as usual) the functor $\partial\varphi^{\mathfrak{m}}_d : \mathtt{S}_{\mathfrak{m}}(f, d) \to \mathcal{C}^2$ as a natural transformation $\partial\varphi^{\mathfrak{m}}_d : \mathbf{A}_d \Rightarrow \mathbf{B}_d$ in $\mathcal{C}$ over $\mathtt{S}_{\mathfrak{m}}(f, d)$. Under the correspondence established in section 5.4.1.5 between tomes and their contents, the equation

$$\varphi^{\mathfrak{m}}_T \circ \zeta_{\mathfrak{m}} = \zeta'_{\ulcorner_K} \circ \varphi^{\mathfrak{m}} \tag{5.33}$$

says that the functorial content of $\varphi^{\mathfrak{m}}_T$ shifted along $\zeta_{\mathfrak{m}} : \mathtt{S}_{\mathfrak{m}}(f, \_) \Rightarrow \mathtt{S}_{\mathfrak{m}}(f, T(\_))$, namely

$$\begin{tikzcd} \mathrm{col}_{\mathtt{S}}\mathbf{A}_T \circ \zeta_{\mathfrak{m}} \arrow[r, Rightarrow, "\xi_{\zeta_{\mathfrak{m}}}(\mathbf{A}_T)"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathtt{S}}\partial\varphi^{\mathfrak{m}}_T \circ \zeta_{\mathfrak{m}}"'] & \mathrm{col}_{\mathtt{S}}\mathbf{A}_T \arrow[r, Rightarrow, "\mathrm{col}_{\mathtt{S}} u_T"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathtt{S}}\partial\varphi^{\mathfrak{m}}_T"'] & XT \arrow[d, Rightarrow, "fT"] \\ \mathrm{col}_{\mathtt{S}}\mathbf{B}_T \circ \zeta_{\mathfrak{m}} \arrow[r, Rightarrow, "\xi_{\zeta_{\mathfrak{m}}}(\mathbf{B}_T)"'] & \mathrm{col}_{\mathtt{S}}\mathbf{B}_T \arrow[r, Rightarrow, "\mathrm{col}_{\mathtt{S}} v_T"'] & YT, \end{tikzcd} \tag{5.34}$$

is equal to the image of the contents of $\mathbb{T}_{\mathfrak{m}}(f, \_)$ via the components of the transformation $\zeta'_{\ulcorner_K} : \mathcal{C}^2 \downarrow f \Rightarrow \mathcal{C}^2 \downarrow fT$ (up to canonical isomorphism $\mathrm{col}_{\mathtt{S}} L \cong L\mathrm{col}_{\mathtt{S}}$), which is given below.

$$\begin{tikzcd} \mathrm{col}_{\mathtt{S}} L\mathbf{A} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathtt{S}} L\partial\varphi^{\mathfrak{m}}"'] & L\mathrm{col}_{\mathtt{S}}\mathbf{A} \arrow[r, Rightarrow, "L(\mathrm{col}_{\mathtt{S}} u)"] \arrow[d, Rightarrow, "L\mathrm{col}_{\mathtt{S}}\partial\varphi^{\mathfrak{m}}"'] & LX \arrow[r, Rightarrow, "Le"] \arrow[d, Rightarrow, "Lf"] & LRXT \arrow[r, Rightarrow, "\varepsilon_*"] \arrow[d, Rightarrow, "LRfT"] & XT \arrow[d, Rightarrow, "fT"] \\ \mathrm{col}_{\mathtt{S}} L\mathbf{B} \arrow[r, Rightarrow, "\cong"'] & L\mathrm{col}_{\mathtt{S}}\mathbf{B} \arrow[r, Rightarrow, "L(\mathrm{col}_{\mathtt{S}} v)"'] & LY \arrow[r, Rightarrow, "Le'"'] & LRYT \arrow[r, Rightarrow, "\varepsilon_*"'] & YT \end{tikzcd} \tag{5.35}$$

Remark 5.68. Because the equation $\partial \circ \zeta'_{\ulcorner_K} = L \circ \partial$ holds, equation (5.33) implies the equality $\partial \circ \varphi^{\mathfrak{m}}_T \circ \zeta_{\mathfrak{m}} = L \circ \partial\varphi^{\mathfrak{m}}$, which corroborates the fact that the leftmost vertical arrows of the previous two diagrams are the same.

After forming the pushout construction under $\mathfrak{m}$ at f in the functorial content given by diagram (5.34), the equality between (5.34) and (5.35) allows us to show that the following diagram commutes where the bottom and top commutative parts come from diagram (5.35).

$$\begin{tikzcd}
L\mathrm{col}_{\mathbf{S}}\mathbf{A} \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "L\mathrm{col}_{\mathbf{S}}\partial\varphi^{\mathfrak{m}}"'] \arrow[rrr, Rightarrow, bend left, "\varepsilon_*\circ L(e\circ\mathrm{col}_{\mathbf{S}}u)"] & \mathrm{col}_{\mathbf{S}}L\mathbf{A} \arrow[r, Rightarrow, "\xi_{\zeta_{\mathfrak{m}}}(\mathbf{A}_T)"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathbf{S}}L\partial\varphi^{\mathfrak{m}}"'] & \mathrm{col}_{\mathbf{S}}\mathbf{A}_T \arrow[r, Rightarrow, "\mathrm{col}_{\mathbf{S}}u_T"] \arrow[d, Rightarrow, "\mathrm{col}_{\mathbf{S}}\partial\varphi^{\mathfrak{m}}_T"'] & XT \arrow[d, Rightarrow, "\mathfrak{q}^{\mathfrak{m}}_f T"] \arrow[dr, Rightarrow, "fT"] & \\
L\mathrm{col}_{\mathbf{S}}\mathbf{B} \arrow[r, Rightarrow, "\cong"'] \arrow[rrrr, Rightarrow, bend right, "\varepsilon_*\circ L(e'\circ\mathrm{col}_{\mathbf{S}}v)"'] & \mathrm{col}_{\mathbf{S}}L\mathbf{B} \arrow[r, Rightarrow, "\xi_{\zeta_{\mathfrak{m}}}(\mathbf{B}_T)"'] & \mathrm{col}_{\mathbf{S}}\mathbf{B}_T \arrow[r, Rightarrow, "\pi^{\mathfrak{m}}T"] \arrow[rr, Rightarrow, bend right, "\mathrm{col}_{\mathbf{S}}v_T"'] & \mathfrak{N}^{\mathfrak{m}}_f T \arrow[r, Rightarrow, "\mathfrak{a}^{\mathfrak{m}}_f T"] \arrow[ul, phantom, "\ulcorner"] & YT
\end{tikzcd}$$

If $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow RL$ denotes the unit of the adjunction $L \dashv R$, it follows from the definition of an adjunction that the function $R(\_) \circ \eta_*$ is inverse of $\varepsilon_* \circ L(\_)$. Applying the function $R(\_) \circ \eta_*$ on the earlier diagram therefore provides the following commutative diagram.

$$\begin{tikzcd}
\mathrm{col}_{\mathbf{S}}\mathbf{A} \arrow[r, Rightarrow, "\mathrm{col}_{\mathbf{S}}u"] \arrow[dd, Rightarrow, "\mathrm{col}_{\mathbf{S}}\partial\varphi^{\mathfrak{m}}"'] & X \arrow[r, Rightarrow, "e"] & RXT \arrow[dl, Rightarrow, "R\mathfrak{q}^{\mathfrak{m}}_f T"'] \arrow[dd, Rightarrow, "RfT"] \\
 & R\mathfrak{N}^{\mathfrak{m}}_f T \arrow[dr, Rightarrow, "R\mathfrak{a}^{\mathfrak{m}}_f T"] & \\
\mathrm{col}_{\mathbf{S}}\mathbf{B} \arrow[r, Rightarrow, "\mathrm{col}_{\mathbf{S}}v"'] \arrow[ur, Rightarrow, "R\pi T"] & Y \arrow[r, Rightarrow, "e'"'] & RYT
\end{tikzcd}$$

Now, because the top left corner of the previous diagram corresponds to the top left corner of the commutative square defining the pushout construction $\mathfrak{N}^{\mathfrak{m}}_f$, it follows that there exists of a natural transformation $e_{\mathfrak{m}} : \mathfrak{N}^{\mathfrak{m}}_f \Rightarrow R\mathfrak{N}^{\mathfrak{m}}_f T$ making the following diagram commute.

$$\begin{tikzcd}
\mathrm{col}_{\mathbf{S}}\mathbf{A} \arrow[rr, Rightarrow, "\mathrm{col}_{\mathbf{S}}u"] \arrow[dd, Rightarrow, "\mathrm{col}_{\mathbf{S}}\partial\varphi^{\mathfrak{m}}"'] & & X \arrow[r, Rightarrow, "e"] \arrow[dl, Rightarrow, "\mathfrak{q}^{\mathfrak{m}}_f"'] & RXT \arrow[dl, Rightarrow, "R\mathfrak{q}^{\mathfrak{m}}_f T"'] \arrow[dd, Rightarrow] \\
 & \mathfrak{N}^{\mathfrak{m}}_f \arrow[r, Rightarrow, "e_{\mathfrak{m}}"] \arrow[ul, phantom, "\ulcorner"] & R\mathfrak{N}^{\mathfrak{m}}_f T \arrow[dr, Rightarrow, "R\mathfrak{a}^{\mathfrak{m}}_f T"] & \\
\mathrm{col}_{\mathbf{S}}\mathbf{B} \arrow[rr, Rightarrow, "\mathrm{col}_{\mathbf{S}}v"'] \arrow[ur, Rightarrow, "\pi"] \arrow[urr, Rightarrow, "R\pi T"'] & & Y \arrow[r, Rightarrow, "e'"'] & RYT
\end{tikzcd}$$

The previous diagram provides a morphism $\mathfrak{q}^{\mathfrak{m}}_f : (X, e) \Rightarrow (\mathfrak{N}^{\mathfrak{m}}_f, e_{\mathfrak{m}})$ in the category of R-premodels $\mathbf{Np}_{\mathcal{C}}(D, R, T)$. The universality of $\mathfrak{N}^{\mathfrak{m}}_f$ also provides a morphism $\mathfrak{a}^{\mathfrak{m}}_f : (\mathfrak{N}^{\mathfrak{m}}_f, e_{\mathfrak{m}}) \Rightarrow (Y, e')$ in $\mathbf{Np}_{\mathcal{C}}(D, R, T)$. These two morphisms obviously define a factorisation of the morphism $f : (X, e) \Rightarrow (Y, e')$ in $\mathbf{Np}_{\mathcal{C}}(D, R, T)$.

In other words, the factorisation of $f : X \Rightarrow Y$ in $\mathcal{C}^D$ induced by the pushout construction under $\mathfrak{m}$ at f lifts to the category of premodels $\mathbf{Np}_{\mathcal{C}}(D, R, T)$. This shows that when $\mathcal{C}$ admits all pushouts, the constructor Γ_K is combinatorial along any admissible modifier $\mathfrak{m}$ at any morphism f in $\mathbf{Np}_{\mathcal{C}}(D, R, T)$.

Definition 5.69 (Fibered category of premodels)**.** Let $\mathcal{C}$ be a category admitting all pushouts, $\mathcal{P}$ be a category of R-premodels in $\mathbf{Np}_{\mathcal{C}}(D, R, T)$ over some croquis (K, T), Γ_K be the associated constructor and $\mathfrak{m}$ be an admissible modifier of Γ_K. The category $\mathcal{P}$ will be said to be *fibered along* $\mathfrak{m}$ if for every morphism $f : (X, e) \Rightarrow (Y, e')$ in $\mathcal{P}$, the factorisation

$$(X, e) \xRightarrow{\mathfrak{q}_f^{\mathfrak{m}}} (\mathfrak{N}_f^{\mathfrak{m}}, e_{\mathfrak{m}}) \xRightarrow{\mathfrak{a}_f^{\mathfrak{m}}} (Y, e')$$

belongs to the subcategory $\mathcal{P} \subseteq \mathbf{Np}_{\mathcal{C}}(D, R, T)$.

- ▷ If e and e' are identity natural transformations, then so is $e_{\mathfrak{m}}$. This means that a functor category $\mathcal{C}^D \subseteq \mathbf{Np}_{\mathcal{C}}(D, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_D)$ is fibered along any admissible modifier $\mathfrak{m}$.
- ▷ If the equation $\mathcal{P} = \mathbf{Np}_{\mathcal{C}}(D, R, T)$ holds, then $\mathcal{P}$ is fibered along any admissible modifier $\mathfrak{m}$.

5.4.4.6. *Trigger functors.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ equipped with one of its modifiers $\mathfrak{m}$. For every object d in D and morphism $f : X \to Y$ in $\mathcal{B}$, a *trigger functor at* (f, d) *along* $\mathfrak{m}$ is a functor $i : \mathbf{1} \to \mathtt{S}_{\mathfrak{m}}(f, d)$ that picks out a 4-tuple $(\theta, v, t, \mathbf{c})$ in the subcategory $\mathcal{S}_\Gamma(f, d) \subseteq \mathtt{S}_{\mathfrak{m}}(f, d)$ such that the arrow $t : H(\theta) \to d$ is an identity in $\mathcal{C}$.

$$\begin{tikzcd} & \mathcal{S}_\Gamma(f,d) \arrow[d, "\subseteq"] \\ \mathbf{1} \arrow[ur, dashed] \arrow[r, "i"'] & \mathtt{S}_{\mathfrak{m}}(f,d) \end{tikzcd} \qquad i(*) = (\theta, v, \mathrm{id}_{H(\theta)}, \mathbf{c})$$

In the sequel, the term Γ-*collection of triggers along* $\mathfrak{m}$ will be used to refer to any collection of sets $J(f, d)$, where the indices run over every object d in D and morphism f in $\mathcal{B}$, whose elements are trigger functors at (f, d) along $\mathfrak{m}$.

Definition 5.70 (Upper star operation)**.** For every trigger functor i picking out the 4-tuple $(\theta, v, \mathrm{id}_{H(\theta)}, \mathbf{c})$ in the modifier playground $\mathtt{S}_{\mathfrak{m}}(f, d)$ associated with Γ, we shall denote by $i^\star$ the trigger functor picking out the 4-tuple $(\theta, \mathrm{id}_v, \mathrm{id}_\theta, \mathbf{c})$ in the playground $\mathcal{S}_{\Gamma^\star}(f, \theta)$ associated with the constructor $\Gamma^\star$.

Definition 5.71 (Lower star operation)**.** For every trigger functor i picking out the 4-tuple $(\theta, \mathrm{id}_v, \mathrm{id}_\theta, \mathbf{c})$ in $\mathcal{S}_{\Gamma^\star}(f, \theta)$ associated with the constructor $\Gamma^\star$, we shall denote by $i_\star$ the trigger functor picking out the 4-tuple $(\theta, v, \mathrm{id}_{H(\theta)}, \mathbf{c})$ in the modifier playground $\mathtt{S}_{\mathfrak{m}}(f, d)$ associated with Γ.

Remark 5.72. It follows from the preceding definitions that when the lower star operation is seen as a function (or functor) of the form $\mathcal{S}_{\Gamma^\star}(f, \theta) \hookrightarrow \mathtt{S}_{\mathfrak{m}}(f, H(\theta))$, then the upper star operation turns it into a one-to-one correspondence.

5.4.4.7. *Obstruction squares.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ combinatorial along a modifier $\mathfrak{m}$ at some morphism $f : X \to Y$ in $\mathcal{B}$ and d be an object in D. For every functor $i : \mathbf{1} \to \mathtt{S}_{\mathfrak{m}}(f, d)$ picking out a quadruple $(\theta, v, t, \mathbf{c})$ in the modified playground $\mathtt{S}_{\mathfrak{m}}(f, d)$, the content of the tome

$$\mathbb{T}_{\mathfrak{m}}^{\Gamma}(f, d) = (\mathfrak{I}_d(f), \mathtt{S}_{\mathfrak{m}}(f, d), \varphi_d^{\mathfrak{m}})$$

along the functor i is given by the composite $\varphi_d^{\mathfrak{m}} \circ i$. By definition, this commutative square is the composite $\mathfrak{I}_t(f) \circ \mathrm{top}(\mathbf{c})$. Wherever the image of the functor i lands in $\mathtt{S}_{\mathfrak{m}}(f, d)$, this square is of the form given below on the left, where $\delta : \mathbb{S} \to \mathbb{D}$ stands for either a seed or a stem of a vertebra in $\mathtt{V}_\theta$. Extracting this square in the pushout construction under $\mathfrak{m}$ at f (see diagram (5.31)) via the universal shift along i leads to the following right commutative diagram.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\mathfrak{I}_t(X)\circ x"] \arrow[d, "\delta"'] & \mathfrak{I}_d(Y) \arrow[d, "\mathfrak{I}_d(f)"] \\
\mathbb{D} \arrow[r, "\mathfrak{I}_t(Y)\circ x'"'] & \mathfrak{I}_d(Y)
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathrm{col}_{\mathsf{S}}\mathbf{A}_d \circ i \arrow[r, "\xi_i(\mathbf{A})"] \arrow[d, "\delta"'] \arrow[rr, bend left, "\mathfrak{I}_t(X)\circ x"] & \mathrm{col}_{\mathsf{S}_d}\mathbf{A}_d \arrow[r, "\mathrm{col}_{\mathsf{S}_d} u"] \arrow[d, "\mathrm{col}_{\mathsf{S}_d}\partial\varphi_d^{\mathfrak{m}}"] & \mathfrak{I}_d(X) \arrow[d, "\mathfrak{q}_f^{\mathfrak{m}}(d)"] \arrow[dr, bend left, "\mathfrak{I}_d(f)"] & \\
\mathrm{col}_{\mathsf{S}_d}\mathbf{B}_d \circ i \arrow[r, "\xi_i(\mathbf{B}_d)"'] \arrow[rrr, bend right, "\mathfrak{I}_t(Y)\circ x'"'] & \mathrm{col}_{\mathsf{S}_d}\mathbf{B}_d \arrow[r, "\pi_f^{\mathfrak{m}}(d)"'] \arrow[rr, bend right, "\mathrm{col}_{\mathsf{S}_d} v"] & \mathfrak{N}_f^{\mathfrak{m}}(d) \arrow[r, "\mathfrak{a}_f^{\mathfrak{m}}(d)"] & \mathfrak{I}_d(Y)
\end{tikzcd}
$$

If we denote the composite $\pi_f^{\mathfrak{m}}(d) \circ \xi_i(\mathbf{B}_d)$ by the arrow $\pi_f^i(d) : \mathbb{D} \to \mathfrak{N}_f^{\mathfrak{m}}(d)$, the earlier right diagram may be simplified into the following left one.

(5.36)
$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "x"] \arrow[dd, "\delta"'] & \mathfrak{I}_d(X) \arrow[d, "\mathfrak{q}_f^{\mathfrak{m}}(d)"] \\
& \mathfrak{N}_f^{\mathfrak{m}}(d) \arrow[d, "\mathfrak{a}_f^{\mathfrak{m}}(d)"] \\
\mathbb{D} \arrow[r, "x'"'] \arrow[ur, "\pi_f^i(d)"] & \mathfrak{I}_d(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{I}_d(X) \arrow[r, "\mathfrak{L}_\theta(X)"] \arrow[d, "\mathfrak{q}_f^{\mathfrak{m}}(d)"'] & \mathfrak{L}_\theta^\bullet(X) \arrow[d, "\mathfrak{L}_\theta^\bullet(\{f\}_{\mathfrak{m}})"] \\
\mathfrak{N}_f^{\mathfrak{m}}(d) \arrow[r, "\mathfrak{L}_\theta(N_f^{\mathfrak{m}})"] \arrow[d, "\mathfrak{a}_f^{\mathfrak{m}}(d)"'] & \mathfrak{L}_\theta^\bullet(N_f^{\mathfrak{m}}) \arrow[d, "\mathfrak{L}_\theta^\bullet(\lfloor f\rfloor_{\mathfrak{m}})"] \\
\mathfrak{I}_d(X) \arrow[r, "\mathfrak{L}_\theta(Y)"] & \mathfrak{L}_\theta^\bullet(Y)
\end{tikzcd}
$$

From now on, suppose that i is a trigger functor that picks out a quadruple of the form $(\theta, v, \mathrm{id}_d, \mathbf{c})$. In this case, the morphism $\delta : \mathbb{S} \to \mathbb{D}$ corresponds to the seed of the vertebra v, which will be denoted by γ. Because the identity $H(\theta) = d$ holds, the image of the factorisation $f = \lfloor f\rfloor_{\mathfrak{m}} \circ \{f\}_{\mathfrak{m}}$ via the functor $\mathfrak{L}_\theta : \mathcal{B} \to \mathcal{C}^2$ is of the form given on the above right. Note that the two diagrams of (5.36) may be pasted in an obvious way, so that pasting these and merging the resulting diagram with the commutative cube defined by $\mathbf{c} : \mathbf{disk}(v) \Rightarrow \mathfrak{L}_\theta(f)$ leads to the following factorisation of the cube $\mathbf{c}$ itself for the notations $v = \|\gamma, \gamma'\| \cdot \beta$.

(5.37)
$$
\begin{tikzcd}
\mathbb{S} \arrow[rr, "x"] \arrow[dr, "\gamma'"'] \arrow[ddd, "\gamma"'] & & \mathfrak{I}_d(X) \arrow[dr, "\mathfrak{L}_\theta(X)"] \arrow[dd] & \\
& \mathbb{D}_1 \arrow[rr, "y"] \arrow[ddd, "\beta\circ\delta_1"] & & \mathfrak{L}_\theta^\bullet(X) \arrow[dd, "\mathfrak{L}_\theta^\bullet(\{f\}_{\mathfrak{m}})"] \\
& & \mathfrak{N}_f^{\mathfrak{m}}(d) \arrow[d] \arrow[dr, "\mathfrak{L}_\theta(N_f^{\mathfrak{m}})"] & \\
\mathbb{D}_2 \arrow[rr, "x'"] \arrow[urr, dashed, "\pi_f^i(d)"] \arrow[dr, "\beta\circ\delta_2"'] & & \mathfrak{I}_d(Y) \arrow[dr, "\mathfrak{L}_\theta(Y)"] & \mathfrak{L}_\theta^\bullet(N_f^{\mathfrak{m}}) \arrow[d, "\mathfrak{L}_\theta^\bullet(\lfloor f\rfloor_{\mathfrak{m}})"] \\
& \mathbb{D}' \arrow[rr, "y'"] & & \mathfrak{L}_\theta^\bullet(Y)
\end{tikzcd}
$$

Now, notice that the above commutative cube provides the following left commutative square. By definition of the vertebra $\|\gamma, \gamma'\| \cdot \beta$, we may form a pushout in it so that we obtain a canonical arrow $w : \mathbb{S}' \to \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{m}}_f)$ making the next right diagram commute.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\mathfrak{L}^{\bullet}_{\theta}(\{f\}_{\mathfrak{m}})\circ y"] \\ \mathbb{D}_2 \arrow[r, "\mathfrak{L}_{\theta}(N^{\mathfrak{m}}_f)\circ\pi^i_f(d)"'] & \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{m}}_f) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] \arrow[dr, bend left, "\mathfrak{L}^{\bullet}_{\theta}(\{f\}_{\mathfrak{m}})\circ y"] & \\ \mathbb{D}_2 \arrow[r, "\delta_2"] \arrow[rr, bend right, "\mathfrak{L}_{\theta}(N^{\mathfrak{m}}_f)\circ\pi^i_f(d)"'] & \mathbb{S}' \arrow[r, dashed, "w"] & \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{m}}_f) \end{tikzcd}$$

It is then not hard to deduce from the universality of this pushout that the two arrows

$$\mathfrak{L}^{\bullet}_{\theta}(\lfloor f\rfloor_{\mathfrak{m}}) \circ w : \mathbb{S}' \to \mathfrak{L}^{\bullet}_{\theta}(Y) \quad \text{and} \quad y' \circ \beta : \mathbb{S}' \to \mathfrak{L}^{\bullet}_{\theta}(Y)$$

are solutions for the same universal problem over $\mathbb{S}'$ (diagram (5.37) might come in handy to visualise this fact). By uniqueness of a universal solution, the following diagram must then commute.

$$\begin{tikzcd} \mathbb{S}' \arrow[r, "w"] \arrow[d, "\beta"'] & \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{m}}_f) \arrow[d, "\mathfrak{L}^{\bullet}_{\theta}(\lfloor f\rfloor_{\mathfrak{m}})"] \\ \mathbb{D}' \arrow[r, "y'"'] & \mathfrak{L}^{\bullet}_{\theta}(Y) \end{tikzcd}$$

For any trigger functor $i : \mathbf{1} \to \mathtt{S}_{\mathfrak{m}}(f,d)$, the previous commutative square will be referred to as the *obstruction square of* $(\Gamma, \mathfrak{m})$ *triggered by* i. This terminology has a certain topological connotation (e.g. obstruction theory) and mainly refers to the fact that the previous commutative square is representative of the obstruction generated by $\mathfrak{L}^{\bullet}_{\theta}(\lfloor f\rfloor_{\mathfrak{m}})$ to complete the object $\mathbb{S}'$, which one would like to think of as an abstract sphere, into the object $\mathbb{D}'$, which one would like to think of as an abstract disc.

5.4.4.8. *Rectifying modifiers.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$ be a constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ combinatorial along a modifier $\mathfrak{m}$ at some morphism $f : X \to Y$ in $\mathcal{B}$ and d be an object in D. For every Γ-collection of triggers J along $\mathfrak{m}$, a modifier $\mathfrak{u}$ of Γ will be said to *rectify the modifier* $\mathfrak{m}$ *over* J *at* f, which will be denoted by the relation $\mathfrak{m} \prec^J_f \mathfrak{u}$, if

1) the constructor Γ is combinatorial along $\mathfrak{u}$ at $\lfloor f\rfloor_{\mathfrak{m}} : N^{\mathfrak{m}}_f \to Y$;
2) for every object d in D and functor i in $J(f,d)$ picking out a quadruple $(\theta, v, t, \mathbf{c})$ in $\mathtt{S}_{\mathfrak{m}}(f,d)$, it is equipped with an arrow

$$\varpi^i_{\mathfrak{u}}(f,d) : \mathbb{D}' \to \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{u}}_{\lfloor f\rfloor_{\mathfrak{m}}})$$

factorising the obstruction square of $(\Gamma, \mathfrak{m})$ triggered by i as follows, where the factorisation $\lfloor f\rfloor_{\mathfrak{m}} = \lfloor\lfloor f\rfloor_{\mathfrak{m}}\rfloor_{\mathfrak{u}} \circ \{\lfloor f\rfloor_{\mathfrak{m}}\}_{\mathfrak{u}}$ holds in $\mathcal{B}$ by item 1).

(5.38)
$$\begin{tikzcd} \mathbb{S}' \arrow[rr, "w"] \arrow[dd, "\beta"'] & & \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{m}}_f) \arrow[dd, "\mathfrak{L}^{\bullet}_{\theta}(\lfloor f\rfloor_{\mathfrak{m}})"] \arrow[dl, "\mathfrak{L}^{\bullet}_{\theta}(\{\lfloor f\rfloor_{\mathfrak{m}}\}_{\mathfrak{u}})"'] \\ & \mathfrak{L}^{\bullet}_{\theta}(N^{\mathfrak{u}}_{\lfloor f\rfloor_{\mathfrak{m}}}) \arrow[dr, "\mathfrak{L}^{\bullet}_{\theta}(\lfloor\lfloor f\rfloor_{\mathfrak{m}}\rfloor_{\mathfrak{u}})"] & \\ \mathbb{D}' \arrow[ur, dashed, "\varpi^i_{\mathfrak{u}}(f,d)"] \arrow[rr, "y'"'] & & \mathfrak{L}^{\bullet}_{\theta}(Y) \end{tikzcd}$$

Remark 5.73 (Points of view)**.** Let $\mathfrak{u}$ be a modifier of Γ that rectifies the modifier $\mathfrak{m}$ over a collection of triggers J at a morphism f in $\mathcal{B}$. For every trigger functor i in $J(f,d)$ picking out a quadruple $(\theta, v, \mathrm{id}_d, \mathbf{c})$ in $\mathtt{S}_{\mathfrak{m}}(f,d)$, diagram (5.38) may be used to complete diagram (5.37) into diagram (5.39), where, for the sake of convenience, the compositions

$$\mathfrak{L}^{\bullet}_{\theta}(\{\lfloor f\rfloor_{\mathfrak{m}}\}_{\mathfrak{u}}) \circ \mathfrak{L}_{\theta}(N^{\mathfrak{m}}_f) \quad \text{and} \quad \mathfrak{L}^{\bullet}_{\theta}(\{\lfloor f\rfloor_{\mathfrak{m}}\}_{\mathfrak{u}}) \circ \mathfrak{L}^{\bullet}_{\theta}(\{f\}_{\mathfrak{m}})$$

have been shortened to the symbols $h_\theta(f|\mathfrak{m},\mathfrak{u})$ and $h'_\theta(f|\mathfrak{m},\mathfrak{u})$, respectively.

(5.39)
$$\begin{tikzcd}
\mathbb{S} \arrow[rr, "x"] \arrow[dr, "\gamma'"'] \arrow[ddd, "\gamma"'] & & \mathfrak{I}_d(X) \arrow[dr, "\mathfrak{L}_\theta(X)"] \arrow[dd] & \\
 & \mathbb{D}_1 \arrow[rr, "y"] \arrow[ddd, "\beta\circ\delta_1"] & & \mathfrak{L}^\bullet_\theta(X) \arrow[dd, "h'_\theta(f|\mathfrak{m},\mathfrak{u})"] \\
 & & \mathfrak{N}^{\mathfrak{m}}_f(d) \arrow[d] \arrow[r, "h_\theta(f|\mathfrak{m},\mathfrak{u})"] & \mathfrak{L}^\bullet_\theta(N^{\mathfrak{u}}_{\lfloor f\rfloor_{\mathfrak{m}}}) \arrow[d, "\mathfrak{L}^\bullet_\theta(\lfloor\lfloor f\rfloor_{\mathfrak{m}}\rfloor_{\mathfrak{u}})"] \\
\mathbb{D}_2 \arrow[urr, dashed, "\pi^i_f(d)"] \arrow[rr, "x'"] \arrow[dr, "\beta\circ\delta_2"'] & & \mathfrak{I}_d(Y) \arrow[dr] & \mathfrak{L}^\bullet_\theta(Y) \\
 & \mathbb{D}' \arrow[urr, dashed, "\varpi^i_{\mathfrak{u}}(f,d)"] \arrow[rr, "y'"] & & \mathfrak{L}^\bullet_\theta(Y)
\end{tikzcd}$$

Interestingly, the outer commutative cube of diagram (5.39) exactly corresponds to the content of $\mathbb{T}^{\Gamma^\star}(f,\theta)$ along the trigger functor $i^\star : \mathbf{1} \to \mathcal{S}_{\Gamma^\star}(f,\theta)$ picking out the 4-tuple $(\theta, \mathrm{id}_v, \mathrm{id}_\theta, \mathbf{c})$.

Example 5.74 (Categories of premodels)**.** This part continues the discussion implicitly suggested by Definition 5.69 by considering a category of R-premodels $\mathcal{P} \subseteq \mathbf{Np}_{\mathcal{C}}(D,R,T)$ over some conical croquis (K,T) that is fibered along some admissible modifier $\mathfrak{m}$. The goal is to define a modifier $\mathfrak{u}$ rectifying the modifier $\mathfrak{m}$ over any Γ-collection of triggers J.

Let us consider some Γ-collection of triggers J and a morphism $g : (X,e) \Rightarrow (Y,e')$ in $\mathcal{P}$. Recall that, for every object $d_\flat$ in D, the obstruction square of $(\Gamma_K, \mathfrak{m})$ along a functor $i : \mathbf{1} \to \mathtt{S}_{\mathfrak{m}}(g,d_\flat)$ in $J(g,d_\flat)$ that picks out a 4-tuple $(c, v, \mathrm{id}_{d_\flat}, \mathbf{c})$ is of the following form when the cylinder c is of the form $(!,t) : d_\flat \Rightarrow d_\dagger$.

$$\begin{tikzcd}
\mathbb{S}' \arrow[r, "w"] \arrow[d, "\beta"'] & \mathfrak{L}^\bullet_c(N^{\mathfrak{m}}_g) \arrow[d, "\mathfrak{L}^\bullet_c(\lfloor g\rfloor_{\mathfrak{m}})"] \\
\mathbb{D}' \arrow[r, "y'"'] & \mathfrak{L}^\bullet_c(Y)
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S}' \arrow[r, "w"] \arrow[d, "\beta"'] & \lim_{d_\dagger} R\mathfrak{N}^{\mathfrak{m}}_g T \arrow[d, "\lim_{d_\dagger} R\mathfrak{a}^{\mathfrak{m}}_g T"] \\
\mathbb{D}' \arrow[r, "y'"'] & \lim_{d_\dagger} RYT
\end{tikzcd}$$

Applying the functor $L : \mathcal{C} \to \mathcal{C}$ on the previous right commutative square and post-composing with the counit $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathcal{C}}$ (up to canonical isomorphism $R\mathrm{lim} \cong \mathrm{lim}R$) leads to the following one.

$$\begin{tikzcd}
L(\mathbb{S}') \arrow[r, "L(w)"] \arrow[d, "L(\beta)"'] & L(\lim_{d_\dagger} R\mathfrak{N}^{\mathfrak{m}}_g T) \arrow[r, "\cong", "r"'] \arrow[d, "L(\lim_{d_\dagger} R\mathfrak{a}^{\mathfrak{m}}_g T)"] & LR\lim_{d_\dagger}\mathfrak{N}^{\mathfrak{m}}_g T \arrow[r, "\varepsilon_*"] \arrow[d, "LR\lim_{d_\dagger}\mathfrak{a}^{\mathfrak{m}}_g T"] & \lim_{d_\dagger}\mathfrak{N}^{\mathfrak{m}}_g T \arrow[d, "\lim_{d_\dagger}\mathfrak{a}^{\mathfrak{m}}_g T"] \\
L(\mathbb{D}') \arrow[r, "L(y')"'] & L(\lim_{d_\dagger} RYT) \arrow[r, "r", "\cong"'] & LR\lim_{d_\dagger} YT \arrow[r, "\varepsilon_*"'] & \lim_{d_\dagger} YT
\end{tikzcd}$$

The symbols r refers to the canonical isomorphisms making the functor R commute with the limits of $\mathcal{C}$. Then, using the universal cone associated with the limits involved in the rightmost vertical arrow of the above diagram, say $\varsigma : \Delta_{A_\dagger}\lim_{A_\dagger} \Rightarrow \mathrm{id}_{\mathcal{C}}$, provides the following commutative square for every object z in the domain $A_\dagger$ of $d_\dagger$.

$$\begin{tikzcd}
L(\mathbb{S}') \arrow[r, "\varepsilon_* \circ r \circ L(w)"] \arrow[d, "L(\beta)"'] & \lim_{d_\dagger} \mathfrak{N}^{\mathfrak{m}}_g T \arrow[r, dashed, "\varsigma_z"] \arrow[d, dashed, "\lim_{d_\dagger} \mathfrak{a}^{\mathfrak{m}}_g T"] & \mathfrak{N}^{\mathfrak{m}}_g T d_\dagger(z) \arrow[d, "\mathfrak{a}^{\mathfrak{m}}_g T d_\dagger(z)"] \\
L(\mathbb{D}') \arrow[r, "\varepsilon_* \circ r \circ L(y')"'] & \lim_{d_\dagger} YT \arrow[r, dashed, "\varsigma_z"'] & YTd_\dagger(z)
\end{tikzcd}$$

The outer square of the earlier commutative diagram, which will be referred to as a morphism $\mathbf{s}'_i(z) : L(\beta) \Rightarrow \mathfrak{a}^{\mathfrak{m}}_g T d_\dagger(z)$ in $\mathcal{C}^2$, gives rise to a quadruple

$$\mathrm{mod}_i(z,t) := (Td_\dagger(z), L(\|\beta, \mathrm{id}\| \cdot \mathrm{id}), t, \mathbf{s}'_i(z))$$

in the set $\mathcal{S}_{\Gamma^\circ_K}(\mathfrak{a}^{\mathfrak{m}}_g, d)$ for any arrow $t : T(d_\dagger(z)) \to d$ in D. These quadruples are going to 'generate' the rectifying modifier we are looking for. Before defining such a modifier, let us first define for every object d in D the set $\mathfrak{v}_g(d)$ of all quadruples of the form $\mathrm{mod}_i(z,t)$ for every

- object $d_\flat$ in D and trigger functor $i(*) = (c, v, \mathrm{id}_{d_\flat}, \mathbf{c})$ in $\mathtt{S}_{\mathfrak{m}}(g, d_\flat)$;
- object z in the small category $A_\dagger$ associated with the cylinder c;
- arrow $t : Td_\dagger(z) \to d$ in D.

The sets $\mathfrak{v}_g(d)$, which are subsets of $\mathcal{S}_{\Gamma^\circ_K}(\mathfrak{a}^{\mathfrak{m}}_g, d)$, inherit the functoriality of the functor $\mathcal{S}_{\Gamma^\circ_K}(\mathfrak{a}^{\mathfrak{m}}_g, \_)$ and thus define a local modifier of Γ_K at the morphism $\mathfrak{a}^{\mathfrak{m}}_g$. This local modifier may be extended to an obvious modifier $\mathfrak{u}$ of Γ_K by considering the equations given below.

(5.40)
$$\begin{cases} \mathfrak{u}_f(d) := \mathfrak{v}_g(d) & \text{if } f = \mathfrak{a}^{\mathfrak{m}}_g \\ \mathfrak{u}_f(d) := \emptyset & \text{otherwise} \end{cases}$$

However, this modifier is not admissible in the sense of Example 5.56. This may be corrected by considering the smallest subfunctor $\mathfrak{u}'_f$ containing the images of $\mathfrak{u}_f$ that is stable under application of the following function.

$$\zeta_{\Gamma_K,d} : \mathcal{S}_{\Gamma^\circ_K}(\mathfrak{a}^{\mathfrak{m}}_f, d) \to \mathcal{S}_{\Gamma^\circ_K}(\mathfrak{a}^{\mathfrak{m}}_f, T(d))$$

Specifically, since the category $\mathbf{Set}^D$ is complete, the subfunctor $\mathfrak{u}'_f$ is given by the pullback of all the inclusions $m_f(\_) \hookrightarrow \mathcal{S}_{\Gamma^\circ_K}(\mathfrak{a}^{\mathfrak{m}}_f, \_)$ where $m_f(\_)$ is stable under ζ_{Γ_K} and $\mathfrak{u}_f$ is a subfunctor of $m_f(\_)$. This leads to the following formula.

$$\mathfrak{u}'_f(d) := \{(d_\flat, v, t, \mathbf{s}) \mid t : d_\flat \to d \text{ in } D \text{ and } \exists n : \ (d_\flat, v, *, \mathbf{s}) \in \zeta^n_{\Gamma_K, d_\flat}(\mathfrak{u}_f(d_\flat))\}$$

The collection $\mathfrak{u}'$ consisting of the functor $\mathfrak{u}'_f$ for every morphism $f : (X,e) \Rightarrow (Y,e')$ in $\mathcal{P}$ finally induces an admissible modifier for Γ_K. It directly follows from Definition 5.69 that Γ_K is combinatorial along $\mathfrak{u}'$ at $\mathfrak{a}^{\mathfrak{m}}_g$ when, for instance, one of the following equations hold.

$$\mathcal{P} = \mathcal{C}^D \qquad\qquad \mathcal{P} = \mathbf{Np}_{\mathcal{C}}(D, R, T)$$

This therefore proves the first item of the definition of a rectifying modifier for such categories of R-premodels. From now on, we shall assume that $\mathcal{P}$ is such that Γ_K is combinatorial along $\mathfrak{u}'$ at $\mathfrak{a}^{\mathfrak{m}}_g$.

The rest of the section shows that $\mathfrak{u}'$ actually rectifies the modifier $\mathfrak{m}$ over J at g by showing that item 2) of section 5.4.4.8 is satisfied. In this respect, consider an object d in D and a trigger functor $i(*) = (c, v, \mathrm{id}_{d_\flat}, \mathbf{c})$ in $\mathtt{S}_{\mathfrak{m}}(g, d_\flat)$ in $J(g,d)$. For every object z in the category $A_\dagger$ associated with c, denote by $j_z : \mathbf{1} \to \mathfrak{u}'(\mathfrak{a}^{\mathfrak{m}}_g, Td_\dagger(z))$ the functor picking out the quadruple $\mathrm{mod}_i(z, \mathrm{id}_{Td_\dagger(z)})$ defined above. In particular, the following equation hold by definition of $\varphi^{\mathfrak{u}'}$.

$$\varphi^{\mathfrak{u}'}_{Td_\dagger(z)} \circ j_z = \mathbf{s}'_i(z)$$

It follows from the preceding equation that the content of the tome $\mathbb{T}_{\mathfrak{u}'}(\mathfrak{a}^{\mathfrak{m}}_{g}, Td_{\dagger}(z))$ along the functor j_z is equal to the outer square of the next commutative diagram, wherein the pushout construction under $\mathfrak{u}'$ at the morphism $\mathfrak{a}^{\mathfrak{m}}_{g}$ has been added (since Γ_K is combinatorial along $\mathfrak{u}'$ at $\mathfrak{a}^{\mathfrak{m}}_{g}$).

$$\begin{tikzcd}
L(\mathbb{S}') \arrow[r, "\varepsilon_*\circ r\circ L(w)"] \arrow[dd, "L(\beta)"'] & \lim_{d_\dagger}\mathfrak{N}^{\mathfrak{m}}_{f}T \arrow[r, "\varsigma_z"] & \mathfrak{N}^{\mathfrak{m}}_{f}Td_\dagger(z) \arrow[dl, "\mathfrak{q}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}Td_\dagger(z)"] \arrow[dd, "\mathfrak{a}^{\mathfrak{m}}_{g}Td_\dagger(z)"] \\
 & \mathfrak{N}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}Td_\dagger(z) \arrow[dr, "\mathfrak{a}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}Td_\dagger(z)"] & \\
L(\mathbb{D}') \arrow[ur, dashed, "\pi^{j_z}_{\mathfrak{a}^{\mathfrak{m}}_{g}}Td_\dagger(z)"] \arrow[r, "\varepsilon_*\circ r\circ L(y')"'] & \lim_{d_\dagger}YT \arrow[r, "\varsigma_z"'] & YTd_\dagger(z)
\end{tikzcd}$$

By definition of the quotient $\tilde{\mathfrak{u}}'$, the functor $j_z : \mathbf{1} \to \tilde{\mathfrak{u}}'(\mathfrak{a}^{\mathfrak{m}}_{g}, Td_{\dagger}(z))$ must be functorial in z (this functoriality was the main reason for defining such a quotient), which implies that the above commutative diagram is functorial in the variable z. This means that we may form the limit of the underlying cone above $A_\dagger$ as follows.

$$\begin{tikzcd}
L(\mathbb{S}') \arrow[r, "\varepsilon_*\circ r\circ L(w)"] \arrow[dd, "L(\beta)"'] & \lim_{d_\dagger}\mathfrak{N}^{\mathfrak{m}}_{f}T \arrow[r, equal] & \lim_{d_\dagger}\mathfrak{N}^{\mathfrak{m}}_{f}T \arrow[dl, "\lim_{d_\dagger}\mathfrak{q}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T"] \arrow[dd, "\lim_{d_\dagger}\mathfrak{a}^{\mathfrak{m}}_{g}T"] \\
 & \lim_{d_\dagger}\mathfrak{N}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T \arrow[dr, "\lim_{d_\dagger}\mathfrak{a}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T"] & \\
L(\mathbb{D}') \arrow[ur, dashed, "\lim_{d_\dagger}\pi^{j_z}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T"] \arrow[r, "\varepsilon_*\circ r\circ L(y')"'] & \lim_{d_\dagger}YT \arrow[r, equal] & \lim_{d_\dagger}YT
\end{tikzcd}$$

Now, using the universal property of the adjunction $L \dashv R$ and the fact that the isomorphisms $r : \lim_{A_\dagger}R \cong R\lim_{A_\dagger}$ are universal, the above diagram may be transformed into the following one, which proves that $\mathfrak{u}'$ rectifies $\mathfrak{m}$ over J at f.

$$\begin{tikzcd}
\mathbb{S}' \arrow[rr, "w"] \arrow[dd, "\beta"'] & & \lim_{d_\dagger}R\mathfrak{N}^{\mathfrak{m}}_{f}T \arrow[dl, "\lim_{d_\dagger}R\mathfrak{q}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T"] \arrow[dd, "\lim_{d_\dagger}R\mathfrak{a}^{\mathfrak{m}}_{g}T"] \\
 & \lim_{d_\dagger}R\mathfrak{N}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T \arrow[dr, "\lim_{d_\dagger}R\mathfrak{a}^{\mathfrak{u}'}_{\mathfrak{a}^{\mathfrak{m}}_{g}}T"] & \\
\mathbb{D}' \arrow[ur, dashed, "\varpi^{i}_{\mathfrak{u}'}(g,d)"] \arrow[rr, "y'"'] & & \lim_{d_\dagger}RYT
\end{tikzcd}$$

For convenience, the rectifying modifier $\mathfrak{u}'$ will later be denoted as $\mathrm{Rec}[\mathfrak{m}|f]$.

5.4.5. Combinatorial categories.

5.4.5.1. *Numbered constructors.* Let $\mathcal{B}$, $\mathcal{C}$ be two categories and K, D be two small categories. A *numbered constructor of type* $[K \downarrow D] \times \mathcal{B}$ *in* $\mathcal{C}$ consists of a constructor Γ of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$ together with a limit ordinal κ such that the category $\mathcal{B}$ admits colimits over the category $\mathbf{O}(\lambda)$ for every limit ordinal λ in $\mathbf{O}(\kappa + 1)$. Such a structure will be denoted as a pair (Γ, κ) and usually defined via an equation of the form $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})_\kappa$ when the constructor Γ is of the form $(\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})$.

5.4.5.2. *Factorisable morphisms of a numbered constructor.* Let $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})_\kappa$ be a numbered constructor of type $[K \downarrow D] \times \mathcal{B}$ in $\mathcal{C}$. A morphism $f : X \to Y$ in $\mathcal{B}$ will be said to be (Γ, κ)-*factorisable* if it is equipped with

- a sequence $(\mathfrak{u}_n)_{n\in\kappa}$ of modifiers of Γ;
- a sequence of Γ-collections of triggers $(J_n)_{n\in\kappa}$ relative to the respective modifiers of $(\mathfrak{u}_n)_{n\in\kappa}$ for the corresponding indices;

satisfying the following inductive conditions:

▷ **initial case:** The constructor Γ is combinatorial along $\mathfrak{u}_0$ at f_0 where we symbolically set the notation $f_0 := f$;

▷ **successor cases:** The modifier $\mathfrak{u}_{n+1}$ rectifies the modifier $\mathfrak{u}_n$ at f_n over J_n where we inductively define $f_{n+1} := \lfloor f_n \rfloor_{\mathfrak{u}_n}$;

▷ **limit cases:** for any (infinite) limit ordinal λ, the constructor Γ is combinatorial along $\mathfrak{u}_\lambda$ at f_λ where f_λ is the colimit $\mathrm{col}_{n\in\mathbf{O}(\lambda)} f_n$ in $\mathcal{B}$ of the diagram, below.

(5.41)
$$\begin{tikzcd}
X \arrow[r, "\{f_0\}_{\mathfrak{u}_0}"] \arrow[d, "f_0"'] & N^{\mathfrak{u}_0}_{f_0} \arrow[r, "\{f_1\}_{\mathfrak{u}_1}"] \arrow[d, "f_1"'] & N^{\mathfrak{u}_1}_{f_1} \arrow[r, "\{f_2\}_{\mathfrak{u}_2}"] \arrow[d, "f_2"'] & \dots \arrow[r] & N^{\mathfrak{u}_{n-1}}_{f_{n-1}} \arrow[r, "\{f_n\}_{\mathfrak{u}_n}"] \arrow[d, "f_n"'] & N^{\mathfrak{u}_n}_{f_n} \arrow[r] \arrow[d, "f_{n+1}"'] & \dots \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & \dots \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & \dots
\end{tikzcd}$$

Specifically, this is the colimit of the cocone formed by the morphisms $f_{n+1} : N^{\mathfrak{u}_n}_{f_n} \to Y$ over the sequence of composable morphisms $\{f_n\}_{\mathfrak{u}_n}$ for every $n \in \lambda$. It follows that the domain N^{λ}_{f} of the colimit

$$f_\lambda : N^{\lambda}_{f} \to Y,$$

thus obtained, is also a colimit, namely the colimit of the sequence of arrows $\{f_n\}_{\mathfrak{u}_n}$ where n runs over λ (see diagram (5.41)). We will denote by $\chi^{\lambda}_{n}(f)$ the canonical arrow

$$N^{\mathfrak{u}_n}_{f_n} \to N^{\lambda}_{f}$$

induced by the universal cocone of this colimit in $\mathcal{B}$. By induction, these arrows give rise to a sequential functor $G(f) : \mathbf{O}(\kappa+1) \to \mathcal{B}$ with the following mapping rules.

$n+1$	$\mapsto$	$N^{\mathfrak{u}_n}_{f_n}$	(succ. objects)
λ	$\mapsto$	X if $\lambda = 0$ and N^{λ}_{f} otherwise.	(lim. objects)
$n+1 < n+2$	$\mapsto$	$\{f_n\}_{\mathfrak{u}_n}$	(succ. arrows)
$n+1 < \lambda$	$\mapsto$	$\chi^{\lambda}_{n}(f)$	(lim. arrows)
$\lambda < \lambda+1$	$\mapsto$	$\{f_\lambda\}_{\mathfrak{u}_\lambda}$	(lim. arrows)

Remark 5.75. The functor $G(f) : \mathbf{O}(\kappa+1) \to \mathcal{B}$ turns the mapping $n \mapsto f_n$ into an obvious functor $\mathrm{spc}(f) : \mathbf{O}(\kappa+1) \to \mathcal{B}^2$ (see diagram (5.41)).

In terms of pictural representation, a (Γ,κ)-factorisable morphism is associated with a sequence of modifiers satisfying the following relations where n runs over κ.

$$\mathfrak{u}_0 \prec^{f}_{J_0} \mathfrak{u}_1 \prec^{f_1}_{J_1} \mathfrak{u}_2 \prec^{f_2}_{J_2} \dots \prec^{f_{n-1}}_{J_{n-1}} \mathfrak{u}_n \prec^{f_n}_{J_n} \mathfrak{u}_{n+1} \prec^{f_{n1+1}}_{J_{n+1}} \dots$$

Example 5.76 (Categories of premodels)**.** This part continues the discussion of Example 5.67 for the category of R-premodels $\mathbf{Np}_{\mathcal{C}}(D,R,T)$. We shall assume that $\mathcal{C}$ is cocomplete. Let κ denote a limit ordinal. We are going to show that every morphism $f : (X,e) \Rightarrow (Y,e')$ of $\mathbf{Np}_{\mathcal{C}}(D,R,T)$ may be equipped with a structure of a (Γ_K,κ)-factorisable morphism.

In this respect, because the empty subfunctor $\emptyset : D \to \mathbf{Set}$ of $\mathcal{S}_{\Gamma^{\circ}_{K}}(f, \_)$ is admissible, let us define our first modifier $\mathfrak{u}_0$ as the empty subfunctor. Because a Γ_K-collection of triggers never depends on the modifier with which Γ_K is associated, we may consider any sequence of Γ_K-collection of triggers $(J_n)_{n\in\kappa}$ for the structure characterising the factorisability of f. Now, let us define the sequence of modifiers $(\mathfrak{u}_n)_{n\in\kappa}$ as follows:

▷ **initial case:** Denote $f_0 := f$;

▷ **successor cases:** Define $\mathfrak{u}_{n+1} := \mathrm{Rec}[\mathfrak{u}_n|f_n]$, which rectifies the modifier $\mathfrak{u}_n$ at f_n over J_n according to Example 5.74 and denote $f_{n+1} := \lfloor f_n \rfloor_{\mathfrak{u}_n}$;

▷ **limit cases:** for any (infinite) limit ordinal λ, define $\mathfrak{u}_\lambda := \emptyset$ and take f_λ to be the colimit $\mathrm{col}_{n\in\mathbf{O}(\lambda)} f_n$ in $\mathcal{B}$ of the following diagram.

$$\begin{tikzcd} X \arrow[r, "\{f_0\}_{\mathfrak{u}_0}"] \arrow[d, "f_0"'] & N^{\mathfrak{u}_0}_{f_0} \arrow[r, "\{f_1\}_{\mathfrak{u}_1}"] \arrow[d, "f_1"'] & N^{\mathfrak{u}_1}_{f_1} \arrow[r, "\{f_2\}_{\mathfrak{u}_2}"] \arrow[d, "f_2"'] & \dots \arrow[r] & N^{\mathfrak{u}_{n-1}}_{f_{n-1}} \arrow[r, "\{f_n\}_{\mathfrak{u}_n}"] \arrow[d, "f_n"'] & N^{\mathfrak{u}_n}_{f_n} \arrow[r] \arrow[d, "f_{n+1}"'] & \dots \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & \dots \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & \dots \end{tikzcd}$$

It is easy to check that the preceding construction gives the structure of a (Γ_K, κ)-factorisable morphism to f.

Example 5.77 (Categories of premodels)**.** Let $\mathcal{C}$ be a cocomplete category, $\mathcal{P}$ be a category of R-premodels in $\mathbf{Np}_{\mathcal{C}}(D, R, T)$ over some conical croquis (K, T) and Γ_K be the associated constructor. If the category $\mathcal{P}$ is fibered along any admissible modifier of Γ_K, then the discussion of Example 5.76 implies that every morphism $f : (X, e) \Rightarrow (Y, e')$ of $\mathcal{P}$ is (Γ_K, κ)-factorisable for every limit ordinal κ. According to Definition 5.69, the categories of R-premodels $\mathcal{C}^D$ and $\mathbf{Np}_{\mathcal{C}}(D, R, T)$ are examples.

Proposition 5.78. *Let $f : X \to Y$ be a (Γ, κ)-factorisable morphism. For every object d in D, the mapping $n \mapsto \mathbb{T}^{\Gamma}_{\mathfrak{u}_n}(f_n, d)$ induces an oeuvre $\mathfrak{O}_f(d) : \mathbf{O}(\kappa + 1) \to \mathbf{Ltom}(\mathcal{C})$ of theme $\mathfrak{I}_d(Y)$. The mapping $\mathfrak{O}_f : d \mapsto \mathfrak{O}_f(d)$ may then be equipped with a structure of functor $D \to \mathbf{Narr}(\mathcal{C}, \kappa)$ whose images are strict narratives.*

Proof. The fact that the mapping $n \mapsto \mathbb{T}^{\Gamma}_{\mathfrak{u}_n}(f_n, d)$ induces an oeuvre directly follows from Proposition 5.65 and Remark 5.75. One thus obtains an oeuvre $\mathfrak{O}_f(d) : \mathbf{O}(\kappa+1) \to \mathbf{Ltom}(\mathcal{C})$ of theme $\mathfrak{I}_d(Y)$. Proposition 5.63 shows that the mapping $d \mapsto \mathfrak{O}_f(d)$ defines a functor $D \to \mathbf{Oeuv}(\mathcal{C}, \kappa)$. The narrative structure is defined as follows:

1) for every $n \in \kappa$, the set of events J^d_n contains all the functors $\mathbf{1} \to \mathtt{S}_{\mathfrak{u}_n}(f_n, d)$;
2) for every $n \in \kappa$, the transition factorisation is given by the commutative diagram

$$\begin{tikzcd} \mathfrak{N}^{\mathfrak{u}_n}_{f_n}(d) \arrow[r, equal] \arrow[d, "\mathfrak{I}_d(f_{n+1})"'] & \mathfrak{N}^{\mathfrak{u}_n}_{f_n}(d) \arrow[r, "\mathfrak{q}^{\mathfrak{u}_{n+1}}_{f_{n+1}}(d)"] \arrow[d, "\mathfrak{I}_d(f_{n+1})"'] & \mathfrak{N}^{\mathfrak{u}_{n+1}}_{f_{n+1}}(d) \arrow[d, "\mathfrak{I}_d(f_{n+2})"] \\ \mathfrak{I}_d(Y) \arrow[r, equal] & \mathfrak{I}_d(Y) \arrow[r, equal] & \mathfrak{I}_d(Y) \end{tikzcd}$$

which is deduced from the factorisation of f_{n+1} in terms of f_{n+2};
3) for every $n \in \kappa$ and functor $i : \mathbf{1} \to \mathtt{S}_{\mathfrak{u}_n}(f_n, d)$ in J^d_n, the point of view is given by the morphism $\pi^i_{f_n}(d)$ defined in section 5.4.4.7 for the rectifying modifier $\mathfrak{u}_n$, which makes the next diagram commute according to the left diagram of (5.36).

$$\begin{tikzcd} \mathrm{col}_{\mathbf{1}}(\mathbf{A}_k \circ i) \arrow[rr, "\upsilon_k(i)"] \arrow[dd, "\mathrm{col}_{\mathbf{1}}(\partial\varphi^{\mathfrak{u}_n} \circ i)"'] & & \mathfrak{I}_d(N^{\mathfrak{u}_{n-1}}_{f_{n-1}}) \arrow[dl, "\mathfrak{q}^{\mathfrak{u}_n}_{f_n}(d)"'] \arrow[dd, "\mathfrak{I}_d(f_n)"] \\ & \mathfrak{N}^{\mathfrak{u}_n}_{f_n}(d) \arrow[dr, "\mathfrak{I}_d(f_{n+1})"] & \\ \mathrm{col}_{\mathbf{1}}(\mathbf{B}_k \circ i) \arrow[rr, "\overline{\upsilon}_k(i)"'] \arrow[ur, "\pi^i_{f_n}(d)"] & & \mathfrak{I}_d(Y) \end{tikzcd}$$

The object $N^{\mathfrak{u}_{n-1}}_{f_{n-1}}$ stands for X when $n = 0$.

It follows from the definition of section 5.4.2.3 that such a narrative is strict. Since, for every morphism $t : d \to d'$, the functor $\mathtt{S}_{\mathfrak{u}_n}(f_n, t) : \mathtt{S}_{\mathfrak{u}_n}(f_n, d) \to \mathtt{S}_{\mathfrak{u}_n}(f_n, d')$ induces a functor $J_n^d \to J_n^{d'}$ by post-composition, the mapping $d \mapsto \mathfrak{O}_f(d)$ is functorial in $\mathbf{Narr}(\mathcal{C})$. $\square$

Proposition 5.79. *Let $f : X \to Y$ be a (Γ, κ)-factorisable morphism. For every object θ in K, the mapping $n \mapsto \mathbb{T}^{\Gamma^\star}(f_n, \theta)$ induces an oeuvre $\mathfrak{O}_f^\star(\theta) : \mathbf{O}(\kappa + 1) \to \mathbf{Ltom}(\mathcal{C}^2)$ of theme $\mathfrak{L}_\theta(Y)$. The mapping $\mathfrak{O}_f^\star : \theta \mapsto \mathfrak{O}_f^\star(\theta)$ induces a functor $K \to \mathbf{Oeuv}(\mathcal{C}^2, \kappa)$ whose images are equipped with narrative structures.*

Proof. The fact that the mapping $n \mapsto \mathbb{T}^{\Gamma^\star}(f_n, \theta)$ induces an oeuvre directly follows from Proposition 5.60 applied to the constructor $\Gamma^\star$ and Remark 5.75. One thus obtains an oeuvre $\mathfrak{O}_f^\star(\theta) : \mathbf{O}(\kappa + 1) \to \mathbf{Ltom}(\mathcal{C}^2)$ of theme $\mathfrak{L}_\theta(Y)$. Proposition 5.60 applied to the constructor $\Gamma^\star$ also shows that the mapping $\theta \mapsto \mathfrak{O}_f^\star(\theta)$ defines a functor $K \to \mathbf{Oeuv}(\mathcal{C}, \kappa)$. The narrative structure is defined as follows:

1) for every $n \in \kappa$, the set of events $J_n^{\star\theta}$ contains all the trigger functors of the form $\mathbf{1} \to \mathcal{S}_{\Gamma^\star}(f_n, \theta)$ along the empty modifier;

2) for every $n \in \kappa$, the transition factorisation is given by the following commutative diagram where $h_\theta(f_n|\mathfrak{u}_n, \mathfrak{u}_{n+1})$ denotes the arrow defined in Remark 5.73 when applied to the morphism f_n and the relation $\mathfrak{u}_n \prec_{J_n}^{f_n} \mathfrak{u}_{n+1}$.

$$\begin{tikzcd}
 & \mathfrak{N}_{f_n}^{\mathfrak{u}_n}(d) \arrow[rr, equal] \arrow[d, "\mathfrak{I}_d(f_{n+1})"] \arrow[ddl, "\mathfrak{L}_\theta(N_{f_n}^{\mathfrak{u}_n})"'] & & \mathfrak{N}_{f_n}^{\mathfrak{u}_n}(d) \arrow[rr, "\mathfrak{q}_{f_{n+1}}^{\mathfrak{u}_{n+1}}(d)"] \arrow[d, "\mathfrak{I}_d(f_{n+1})"] \arrow[ddl, "h_\theta(f_n|\mathfrak{u}_n{,}\mathfrak{u}_{n+1})"] & & \mathfrak{N}_{f_{n+1}}^{\mathfrak{u}_{n+1}}(d) \arrow[d, "\mathfrak{I}_d(f_{n+2})"] \arrow[ddl, "\mathfrak{L}_\theta(N_{f_{n+1}}^{\mathfrak{u}_{n+1}})"'] \\
 & \mathfrak{I}_d(Y) \arrow[rr, dotted, equal] \arrow[ddl, dotted] & & \mathfrak{I}_d(Y) \arrow[rr, dotted, equal] \arrow[ddl, dotted] & & \mathfrak{I}_d(Y) \arrow[ddl, "\mathfrak{L}_\theta(Y)"] \\
\mathfrak{L}_\theta^\bullet(N_{f_n}^{\mathfrak{u}_n}) \arrow[rr, "\mathfrak{L}_\theta^\bullet(\{f_{n+1}\}_{\mathfrak{u}_{n+1}})"] \arrow[d, "\mathfrak{L}_\theta^\bullet(f_{n+1})"'] & & \mathfrak{L}_\theta^\bullet(N_{f_{n+1}}^{\mathfrak{u}_{n+1}}) \arrow[rr, equal] \arrow[d, "\mathfrak{L}_\theta^\bullet(f_{n+2})"'] & & \mathfrak{L}_\theta^\bullet(N_{f_{n+1}}^{\mathfrak{u}_{n+1}}) \arrow[d, "\mathfrak{L}_\theta^\bullet(f_{n+2})"'] & \\
\mathfrak{L}_\theta^\bullet(Y) \arrow[rr, equal] & & \mathfrak{L}_\theta^\bullet(Y) \arrow[rr, equal] & & \mathfrak{L}_\theta^\bullet(Y) &
\end{tikzcd}$$

3) for every $n \in \kappa$ and functor $i : \mathbf{1} \to \mathcal{S}_{\Gamma^\star}(f_n, \theta)$ in $J_n^{\star\theta}$, the point of view is given by the pair of morphisms $\pi_{f_n}^{i_\star}(d)$ and $\varpi_{\mathfrak{u}_{n+1}}^{i_\star}(f_n, d)$ defined in section 5.4.4.7 for the trigger functor $i_\star$ (see Definition 5.71). The version of diagram (5.39) for the morphism f_n and the rectification of $\mathfrak{u}_n$ by $\mathfrak{u}_{n+1}$ then provides a factorisation of the content of

$\mathbb{T}^{\Gamma^\star}(f_n,\theta)$ along the trigger i as follows (see end of Remark 5.73 and Remark 5.72).

$$
\begin{tikzcd}
\mathbb{S} \arrow[rr] \arrow[dr, "\gamma'"] \arrow[ddd, "\gamma"'] & & \mathfrak{N}^{\mathfrak{u}_{n-1}}_{f_{n-1}}(d) \arrow[dr, "\mathfrak{L}_\theta(N^{\mathfrak{u}_{n-1}}_{f_{n-1}})"] \arrow[dd, "\mathfrak{q}^{\mathfrak{u}_n}_{f_n}"] & \\
& \mathbb{D}_1 \arrow[rr] \arrow[ddd, "\beta\circ\delta_1"] & & \mathfrak{L}^\bullet_\theta(N^{\mathfrak{u}_{n-1}}_{f_{n-1}}) \arrow[dd, "h'_\theta(f_n|\mathfrak{u}_n,\mathfrak{u}_{n+1})"] \\
& & \mathfrak{N}^{\mathfrak{u}_n}_{f_n}(d) \arrow[d] \arrow[dr, "h_\theta(f_n|\mathfrak{u}_n,\mathfrak{u}_{n+1})"] & \\
\mathbb{D}_2 \arrow[urr, dashed, "\pi^{i_\star}_{f_n}(d)"] \arrow[rr] \arrow[dr, "\beta\circ\delta_2"'] & & \mathfrak{I}_d(Y) \arrow[dr] & \mathfrak{L}^\bullet_\theta(N^{\mathfrak{u}_{n+1}}_{f_{n+1}}) \arrow[d, "\mathfrak{L}^\bullet_\theta(f_{n+2})"] \\
& \mathbb{D}' \arrow[urr, dashed, "\varpi^{i_\star}_{\mathfrak{u}_{n+1}}(f_n,d)"] \arrow[rr] & & \mathfrak{L}^\bullet_\theta(Y)
\end{tikzcd}
$$

The object $N^{\mathfrak{u}_{n-1}}_{f_{n-1}}$ stands for X when $n=0$.

□

5.4.5.3. *Notations.* Let D be a small category, $\mathcal{C}$ be a category and $\mathtt{V}$ be a portfolio of vertebrae in $\mathcal{C}$ over K. We shall let $\mathcal{G}\mathbf{en}(\mathtt{V})$ denote the set of the domains and codomains of every coseed of any vertebra in $\mathtt{V}_\theta$ for every object θ in K. Similarly, we shall let $\mathcal{D}\mathbf{isk}(\mathtt{V})$ denote the set of the coseeds of every vertebra in $\mathtt{V}_\theta$ for every object θ in K, which may alternatively be seen as the set of the domains of every codiskad $\mathbf{disk}(v^{\mathrm{rv}}) : \gamma' \Rightarrow \beta\circ\delta_2$ (seen as arrows in $\mathcal{C}^2$) of any vertebra v in $\mathtt{V}_\theta$ for every object θ in K.

5.4.5.4. *Combinatorial categories.* Let $\mathcal{C}$ be a category. A category $\mathcal{B}$ will be said to be *combinatorial* in $\mathcal{C}$ if it is equipped with a numbered constructor $\Gamma = (\mathtt{V}, H, \mathfrak{I}, \mathfrak{L})_\kappa$ of type $[K\downarrow D]\times\mathcal{B}$ in $\mathcal{C}$ such that

1) every morphism in $\mathcal{B}$ is (Γ,κ)-factorisable;
2) for every morphism f in $\mathcal{B}$ and object θ in K, the context functor

$$\mathfrak{L}_\theta\circ G(f) : \mathbf{O}(\kappa+1)\to\mathcal{C}^2$$

of the oeuvre $\mathfrak{O}^\star_f(\theta)$ is $\mathcal{D}\mathbf{isk}(\mathtt{V})$-convergent.

Remark 5.80. In practice, it is easy to prove that for every morphism f in $\mathcal{B}$ and object d in D, the context functor $\mathfrak{I}_d\circ G(f) : \mathbf{O}(\kappa+1)\to\mathcal{C}$ of the oeuvre $\mathfrak{O}_f(d)$ is $\mathcal{G}\mathbf{en}(\mathtt{V})$-convergent. This is generally due to the fact that the context functor is the result of colimit constructions using the elements in the image of the functor $\partial\varphi^{\mathfrak{u}^n}_d : \mathtt{S}_{\mathfrak{u}_n}(f_n,d)\to\mathcal{C}^{\mathbf{2}}$, which consists of the seeds and stems of $\mathtt{V}$.

Example 5.81 (Categories of premodels)**.** Let $\mathcal{C}$ be a cocomplete category, $\mathcal{P}$ be a category of R-premodels in $\mathbf{Np}_{\mathcal{C}}(D,R,T)$ over some conical croquis (K,T) and Γ_K be the associated constructor for a given portfolio $\mathtt{V}$ of vertebrae in $\mathcal{C}$ over K. Suppose that the category $\mathcal{P}$ is fibered along any admissible modifier of Γ_K. In this case, Example 5.77 shows that every morphism in $\mathcal{P}$ is (Γ_K,κ)-factorisable for any limit ordinal κ. Let us prove that the category $\mathcal{P}$ becomes combinatorial if κ is a well-chosen ordinal and the statement of Remark 5.80 holds.

As specified by Remark 5.80, for every morphism $f : (X,e)\Rightarrow(Y,e')$ in $\mathcal{P}$ and object d in D, the context functor $\mathfrak{N}^d_f : \mathbf{O}(\kappa{+}1)\to\mathcal{C}$ of the oeuvre $\mathfrak{O}_f(d)$ is generally $\mathcal{G}\mathbf{en}(\mathtt{V})$-convergent.

Recall that this functor lifts to the category $\mathcal{P}$ in the form of a functor $\mathfrak{N}_f : \mathbf{O}(\kappa+1) \to \mathcal{P}$ satisfying the following mapping rule.

$$n \mapsto (\mathfrak{N}^{\mathtt{u}_{n-1}}_{f_{n-1}}, e_{\mathtt{u}_{n-1}}) \quad \text{where} \quad (\mathfrak{N}^{\mathtt{u}_{-1}}_{f_{-1}}, e_{\mathtt{u}_{-1}}) = (X, e)$$

Let c denote a cylinder of the form $(!, t) : d \Rightarrow d_\dagger$ in K where $d_\dagger$ is a functor $A_\dagger \to D$. Let also g denote the functor $(\mathcal{C}^D)^2 \to \mathcal{C}^2$ defined in Remark 5.7 where the cone 'r' thereof is replaced with the natural transformation $t : \Delta_{A_\dagger}(d) \Rightarrow d_\dagger$. By definition, the following equation holds.

$$\mathcal{G}\mathrm{l}^K(\mathfrak{N}^{\mathtt{u}_{n-1}}_{f_{n-1}}, e_{\mathtt{u}_{n-1}})(c) = g(e_{\mathtt{u}_{n-1}})$$

In the case where the inequality $|A_\dagger| \leq \kappa$ holds, Remark 5.7 then says that the following isomorphism holds for every coseed γ' of a vertebra in $\mathtt{V}$

$$\mathcal{C}^2(\gamma', \mathcal{G}\mathrm{l}^K(\mathfrak{N}_f(\kappa), \mathrm{id})(c)) \cong \mathrm{col}_{\mathbf{O}(\kappa)} \mathcal{C}^2(\gamma', \mathcal{G}\mathrm{l}^K(\mathfrak{N}_f(\iota_\kappa(\_)), \mathrm{id})(c))$$

In other words, this shows that if κ is equal to the cardinality $|(K,T)|$ defined in section 5.3.1.4, then the composite of the functor $\mathfrak{N}_f : \mathbf{O}(\kappa+1) \to \mathcal{P}$ with the functor $\mathfrak{L}_c : \mathcal{P} \to \mathcal{C}^2$ is $\mathcal{D}\mathbf{isk}(\mathtt{V})$-convergent. To put it differently, this shows that the context functor of the oeuvre $\mathfrak{O}^\star_f(\theta)$ is $\mathcal{D}\mathbf{isk}(\mathtt{V})$-convergent.

Definition 5.82 (Lifting system)**.** Let $\mathcal{B}$ be a combinatorial category as defined above and θ be an object in K. Denote by $\mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V})$ the discrete subcategory of $(\mathcal{C}^2)^2$ containing the codiskad $\mathbf{disk}(v^{\mathrm{rv}}) : \gamma' \Rightarrow \beta \circ \delta_2$ of every vertebra v in $\mathtt{V}_\theta$. By definition, this provides an inclusion of categories $\varphi^{\mathrm{soa}}_\theta : \mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V}) \hookrightarrow (\mathcal{C}^2)^2$. If κ is not zero, the functor $\varphi^{\mathrm{soa}}_\theta$ may be equipped with an obvious structure of a lifting system by taking the associated set of functors J^{soa}_θ to be the set containing all the functors of the form $\mathbf{1} \to \mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V})$ picking out a codiskad of a vertebra in $\mathtt{V}_\theta$.

Let us now show that the lifting system $(J^{\mathrm{soa}}_\theta, \varphi^{\mathrm{soa}}_\theta) : \mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V}) \hookrightarrow (\mathcal{C}^2)^2$ agrees with the narrative $\mathfrak{O}^\star_f(\theta)$ for every object θ in K and morphism f in $\mathcal{B}$. In this respect, consider an ordinal $n \in \kappa$, functor $i : \mathbf{1} \to \mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V})$ in J^{soa}_θ and a functor $\psi : \mathbf{1} \to (\mathcal{C}^2)^2/\mathfrak{L}_\theta(f_n)$ making the following left diagram commute. When $i : \mathbf{1} \to \mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V})$ picks out the codiskad of a vertebra $v = \|\gamma, \gamma'\| \cdot \beta$, this exactly means that the functor ψ picks out a commutative cube of the form given on the right.

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\varphi\circ i"] \arrow[dr, "\psi"'] & (\mathcal{C}^2)^2 \\
& (\mathcal{C}^2)^2/\mathfrak{L}_\theta(f_n) \arrow[u, "\partial"']
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
& & \cdot \arrow[rr, "\mathfrak{L}_\theta(N^{\mathtt{u}_n}_{f_n})"] \arrow[dd] & & \cdot \arrow[dd, "\mathfrak{L}_\theta(f_n)"] \\
& \cdot \arrow[ur] \arrow[rr, "\gamma'"] \arrow[dd, "\gamma"] & & \cdot \arrow[ur] \arrow[dd, "\beta\circ\delta_1"'] & \\
\mathbf{disk}(v) \arrow[r] & {} & \cdot \arrow[rr] & & \cdot \\
& \cdot \arrow[ur] \arrow[rr, "\beta\circ\delta_2"] & & \cdot \arrow[ur] &
\end{tikzcd}$$

Now, observe that this commutative cube is associated with the trigger functor $i' : \mathbf{1} \to \mathcal{S}_{\Gamma^\star}(f_n, \theta)$ picking out the 4-tuple $(\theta, \|\gamma, \gamma'\| \cdot \beta, \mathrm{id}_\theta, \psi(*))$. In particular, this trigger functor makes the following diagram commute.

$$\begin{tikzcd}
\mathtt{T} \arrow[r, "\varphi\circ i"] \arrow[d, "i'"'] \arrow[dr, "\psi"] & (\mathcal{C}^2)^2 \\
\mathcal{S}_{\Gamma^\star}(f_n, \theta) \arrow[r, "\varphi_{\Gamma^\star,\theta}"'] & (\mathcal{C}^2)^2/\mathfrak{L}_\theta(f_n) \arrow[u, "\partial"']
\end{tikzcd}$$

This thus proves that the lifting system $(J^{\mathrm{soa}}_\theta, \varphi^{\mathrm{soa}}_\theta) : \mathcal{S}^{\mathrm{soa}}_\theta(\mathtt{V}) \hookrightarrow (\mathcal{C}^2)^2$ agrees with the narrative $\mathfrak{O}^\star_f(\theta)$ for every object θ in K and morphism f in $\mathcal{B}$.

Theorem 5.83. *Let $\mathcal{B}$ be a combinatorial category as defined above where κ is non-zero. Every morphism $f : X \to Y$ may be factorised into two arrows*

$$X \xrightarrow{\chi_0^\kappa(f)} G(f)(\kappa) \xrightarrow{f_\kappa} Y$$

such that for every object θ in K, the arrow $\mathfrak{L}_\theta(f_\kappa) : \mathfrak{L}_\theta(G(f)(\kappa)) \to \mathfrak{L}_\theta(Y)$ in $\mathcal{C}^2$ has the rlp with respect to the codiskad of every vertebra in $\mathtt{V}_\theta$ and for every object d in D, the arrow $\mathfrak{I}_d(\chi_0^\kappa(f)) : \mathfrak{I}_d(X) \to \mathfrak{I}_d(G(f)(\kappa))$ has the llp with respect to every morphism in $\mathbf{rlp}_\kappa(J_n^d, \partial\varphi^{\mathrm{u}_n})$ for every $n \in \kappa$.

Proof. The factorisation is given by the image of the arrow $0 \to \kappa$ in $\mathbf{O}(\kappa+1)$ via the functor $\mathrm{spc}(f) : \mathbf{O}(\kappa+1) \to \mathcal{B}^2$ defined in Remark 5.75. The statement on the arrow $\mathfrak{L}_\theta(f_\kappa) : \mathfrak{L}_\theta(G(f)(\kappa)) \to \mathfrak{L}_\theta(Y)$ follows from Proposition 5.44 applied to $(J_\theta^{\mathrm{soa}}, \varphi_\theta^{\mathrm{soa}}) : \mathcal{S}_\theta^{\mathrm{soa}}(\mathtt{V}) \hookrightarrow (\mathcal{C}^2)^2$ since the context functor

$$\mathfrak{L}_\theta \circ G(f) : \mathbf{O}(\kappa+1) \to \mathcal{C}^2$$

of the oeuvre $\mathfrak{O}_f^\star(\theta)$ is $\mathcal{D}\mathbf{isk}(\mathtt{V})$-convergent (and hence $(\mathrm{dom} \circ \varphi_\theta^{\mathrm{soa}})$-convergent). The statement on the arrow $\mathfrak{I}_d(\chi_0^\kappa(f)) : \mathfrak{I}_d(X) \to \mathfrak{I}_d(G(f)(\kappa))$ follows from Proposition 5.78, which ensures that $\mathfrak{O}_f(d)$ is a strict narrative for every object d in D and Proposition 5.45. $\square$

Example 5.84. For a system of models $(\mathcal{P}, K, T, \mathtt{V})$ where $\mathcal{P} \subseteq \mathbf{Np}_\mathcal{C}(D, R, T)$ is equipped with the structure of a combinatorial category as discussed in Example 5.81, Theorem 5.83 provides any arrow $(X, e) \Rightarrow \mathbf{1}$ in $\mathcal{P}$ with a factorisation of the form

$$X \xRightarrow{\chi(f)} G(X) \Longrightarrow \mathbf{1}$$

where $G(X)$ is an R-model and the arrow $\chi(f)$ satisfies good lifting properties.

Remark 5.85. In fact, the last assertion of Theorem 5.83 may be used to prove that the given factorisation gives rise to a reflection from the premodels to the models of a system of models. The proof is however not straightforward and uses results taking advantage of the strictness of $\mathfrak{O}_f$, such as Proposition 5.47, to transform the lifting properties of the arrows $\mathfrak{I}_d(\chi_0^\kappa(f))$ into a lifting property for $\mathfrak{I}(\chi_0^\kappa(f))$.

Let κ be a non-zero limit ordinal. A category $\mathcal{C}$ will be said to be *trivially κ-combinatorial* over a set $\mathtt{G}$ of arrows in $\mathcal{C}$ if it is combinatorial as a category of $\mathrm{id}_\mathcal{C}$-premodels in $\mathbf{Np}(\mathbf{1}, \mathrm{id}, \mathrm{id})$ for the numbered constructor $(\mathtt{G}, \mathrm{id}_\mathbf{1}, \mathrm{id}_\mathcal{C}, \mathrm{id}_{\mathrm{id}_\mathcal{C}})_\kappa$ of type $[\mathbf{1} \downarrow \mathbf{1}] \times \mathcal{C}$ in $\mathcal{C}$ where the set $\mathtt{G}$ is regarded as a portfolio of one set consisting of degenerate vertebrae of the following form for every $\delta \in \mathtt{G}$.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, equal] \arrow[d, "\delta"'] & \mathbb{S} \arrow[d, "\delta"] & \\
\mathbb{D} \arrow[r, equal] \arrow[ur, phantom, "\ulcorner"] & \mathbb{D} \arrow[r, equal] & \mathbb{D}
\end{tikzcd}$$

Corollary 5.86. *Let $\mathcal{C}$ be a cocomplete (and optionally complete) category that is trivially combinatorial over a set of arrows $\mathtt{G}$ in $\mathcal{C}$. Every morphism $f : X \to Y$ in $\mathcal{C}$ may be factorised into two arrows $\chi_0^\kappa(f) : X \to G(f)(\kappa)$ and $f_\kappa : G(f)(\kappa) \to Y$ where the arrow f_κ is in the class $\mathbf{rlp}(\mathtt{G})$ and the arrow $\chi_0^\kappa(f)$ is in the class $\mathbf{llp}(\mathbf{rlp}(\mathtt{G}))$.*

Proof. Theorem 5.83 and Example 5.81 imply that every morphism $f : X \to Y$ in $\mathcal{C}$ may be factorised into two arrows $\chi_0^\kappa(f) : X \to G(f)(\kappa)$ and $f_\kappa : G(f)(\kappa) \to Y$ where the arrow f_κ is in the class $\mathbf{rlp}(\mathtt{G})$ and the arrow $\chi_0^\kappa(f)$ has the llp with respect to every morphism in $\mathbf{rlp}_\kappa(J_n^d, \partial\varphi^{\mathrm{u}_n})$ for every $n \in \kappa$. But because of the triviality of all our data, it follows from formula (5.18) that the equality $\mathbf{rlp}_\kappa(J_n^d, \partial\varphi^{\mathrm{u}_n}) = \mathbf{rlp}(\mathtt{G})$ holds for every $n \in \kappa$. $\square$

5.5. From spinal categories to homotopy theories

The aim of this section is to explain how to extend the notion of spinal category to that of spinal theory in order to be able to recover the type of homotopy theory usually defined for sheaves and other similar structures. The section is deliberately concise regarding sheaf-like structures and rather focuses on the construction of model categories (see section 5.5.2).

5.5.1. Spinal and vertebral theories.

5.5.1.1. *Recapitulation on transfers and some notations.* Let $\mathcal{E}$ be a category equipped with the structure of a discrete system of vertebrae $\hat{\mathcal{E}}$ and K be some small category. Chapter 4 shows that if the left Kan extension $\mathrm{Lan}_c : \mathcal{E} \to \mathcal{E}^K$ exists for some object c in K, then it gives rise to a (discrete) system of vertebrae $\mathrm{Lan}_c(\hat{\mathcal{E}})$ in $\mathcal{E}^K$ (see Example 4.81). In this case, the underlying transfer of structure $\mathrm{Lan}_c : \hat{\mathcal{E}} \to \mathrm{Lan}_c(\hat{\mathcal{E}})$ is 0-regular and pseudo-1-regular. As seen in Example 4.94, this implies that the right adjoint $\nabla_c : \mathcal{E}^K \to \mathcal{E}$ of the functor Lan_c is a pseudo-opcovertebral and covertebral for every object c in K.

Remark 5.87 (Notations)**.** Later on, it will come in handy to denote any composition of functors $U : \mathcal{C} \to \mathcal{E}^K$ and $\nabla_c : \mathcal{E}^K \to \mathcal{E}$ as a functor of the form $U_c : \mathcal{C} \to \mathcal{E}$.

5.5.1.2. *Local configurations.* Let K and D be two small categories. A *local configuration* is a functor $H : K \to D$ such that for every object d in D, the fiber $H^{-1}(d)$ above d may be equipped with the structure of a partially ordered set such that for every object c and c' in $H^{-1}(d)$, there exists an object c'' in $H^{-1}(d)$ satisfying the inequalities $c, c' \leq c''$. A fiber of the form $H^{-1}(d)$ will later be denoted by H_d.

Example 5.88. The identity functor on a small category is a local configuration.

Example 5.89. Any functor $H : K \to D$ such that, for every object d in D, the fiber H_d above d admits coproducts, is associated with the structure of an obvious local configuration in which the partial order $c \leq c'$ is induced by the existence of an arrow of the form $c \to c'$.

5.5.1.3. *Spinal and vertebral theories.* A *spinal theory* (resp. *vertebral theory*) consists of a category $\mathcal{C}$, a local configuration $H : K \to D$ whose domain will be called the *sketch* of the theory, a category $\mathcal{E}$ equipped, for every object c in K, with the structure of a discrete spinal (resp. vertebral) category symbolically denoted as $\hat{\mathcal{E}}_c$ and a functor $U : \mathcal{C} \to \mathcal{E}^K$ such that

1) for every morphism f in $\mathcal{C}$, object d in D and pair of objects $c \leq c'$ in H_d, if $U_c(f)$ is a weak equivalence in $\hat{\mathcal{E}}_c$, then so is $U_{c'}(f)$ in $\hat{\mathcal{E}}_{c'}$;
2) for every object c in K, the transfer of structure $\mathrm{Lan}_c : \hat{\mathcal{E}}_c \to \mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ exists;

Such a structure will be denoted as an arrow $U : \mathcal{C} \to \hat{\mathcal{E}}^H$. A spinal (resp. vertebral) theory $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ will be said to be

- *fully faithful* (abbrev. ff) if its underlying functor $U : \mathcal{C} \to \mathcal{E}^K$ is so;
- *cartesian* if its underlying functor $U : \mathcal{C} \to \mathcal{E}^K$ preserves pullbacks;
- *standard* if its local configuration is an identity functor.

Example 5.90. Any system of vertebrae $\hat{\mathcal{C}}$ equipped with a discrete vertebral (resp. spinal) category defines a standard fully faithful vertebral (resp. spinal) theory whose functor U is given by the identity functor on the ambient category $\mathcal{C}$ and whose underlying sketch is given by any terminal category.

Example 5.91. Let $\mathcal{C}$ be a cocomplete category equipped with a structure of discrete vertebral (resp. spinal) category whose system of vertebrae is given by $\hat{\mathcal{C}}$ and D be a small category. Because $\mathcal{C}$ is cocomplete, Remark 1.29 implies that $\mathrm{Lan}_d : \mathcal{C} \to \mathcal{C}^D$ exists for every object d in D. The functor category $\mathcal{C}^D$ is then associated with a standard fully faithful vertebral (resp. spinal) theory $U : \mathcal{C}^D \to \mathcal{C}^D$ given by the identity functor on $\mathcal{C}^D$. The underlying sketch is given by the category D.

Example 5.92 (Sheaves and Stacks)**.** Let (D^{op}, J) be a site. It follows from the locality and stability axioms for Grothendieck's pretopologies that $\mathtt{peak}_{K_J} : K_J \to D$ defines a local configuration in the sense of Example 5.89. It is a consequence of Proposition 5.31 and Proposition 5.24 that the system of $\mathrm{id}_{\mathbf{Set}}$-models for sheaves (see Example 5.39) induces a fully faithful vertebral theory along the local configuration peak_K given by the following functor.

$$\mathcal{G}\mathrm{l}_0^K : \mathcal{P} \hookrightarrow (\mathbf{Set}^\omega)^K$$

The vertebral structure on $\mathbf{Set}^\omega$ is provided by the cohesive set of vertebrae defined in Example 5.39. The axioms of a vertebral theory follows from the combinatorial properties of $\mathbf{Set}$ (for item 2)) and the functoriality of sheaves (for item 1)). The astute reader might notice that the above reasoning may be generalised to any category of sheaves valued in any nice category (e.g. $\mathbf{Cat}(1)$).

Example 5.93 (System of models)**.** Let D be a small category, (K, T) be a croquis in D, $\mathcal{C}$ be a cocomplete and complete category and $R : \mathcal{C} \to \mathcal{C}$ be an endofunctor commuting with the limits of $\mathcal{C}$. Proposition 5.31 implies that any system of R-models $(\mathtt{V}, \mathcal{P}, K, T)$ in $\mathcal{C}$ induces a standard vertebral theory along the identity local configuration on K given by the following functor.

$$\mathcal{G}\mathrm{l}_0^K : \mathcal{P} \hookrightarrow (\mathcal{C}^\omega)^K$$

The vertebral structure on $\mathcal{C}^\omega$ is induced by the cohesive sets $\mathtt{V}_c$ for every cylinder c in K. Because the functor $\mathcal{G}\mathrm{l}_0^K$ consists exclusively out of right adjoint functors, it commutes with pullbacks so that the vertebral theory is cartesian.

5.5.1.4. *Refined spinal and vertebral theories.* A spinal (resp. vertebral) theory $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ along a configuration $H : K \to D$ will be said to be *refined* if the underlying system of vertebrae $\hat{\mathcal{E}}_c$ is refined for every object c in K. A spinal (resp. vertebral) theory that is fully faithful and refined will be said to be *ffr*.

5.5.1.5. *Quasi-small spinal and vertebral theories.* A spinal (resp. vertebral) theory $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ will be said to be *quasi-small* if for every object c in K where $\hat{\mathcal{E}}_c = (\mathcal{E}, A_c, A'_c, E_c)$ the object-class of the magmoid A'_c is a set.

Remark 5.94. In the case of a quasi-small theory as defined above, the transferred graph $\mathrm{Lan}_c(A'_c)$ of $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ is a set of vertebrae in $\mathcal{E}^K$. This set will later be denoted as $\mathtt{GCof}_c(U)$. The union of all the sets $\mathtt{GCof}_c(U)$ where c runs over the set of objects K will be denoted as $\mathtt{GCof}(U)$.

An object X in $\mathcal{C}$ will be said to be *fibrant* if for every trivial stem $\beta \circ \delta_1 : \mathbb{D}_1 \to \mathbb{D}'$ of a vertebra in $\mathtt{GCof}(U)$ and morphism $f : \mathbb{D}_1 \to U(X)$, the arrow f factorises through the trivial stem $\beta \circ \delta_1$. When the functor category $\mathcal{E}^K$ has a terminal object, say $\mathbf{1}$, this is equivalent to saying that $U(X) \to \mathbf{1}$ is a fibration for the system of vertebrae $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ for every object c in K.

Example 5.95 (System of models)**.** Any system of R-models $\mathcal{P} \subseteq \mathbf{Pm}_{\mathcal{C}}(K, R, T)$ as defined in Example 5.93 defines a quasi-small vertebral category. Since the right adjoint ∇_c of Lan_c is opcovertebral, pseudo-covertebral (see Example 4.94) and preserves terminal objects, it follows from Proposition 4.83 and Proposition 4.86 that an object X in $\mathcal{P}$ defines an R-model in $\mathcal{P}$ if and only if it is a fibrant object for the vertebral theory $\mathcal{G}\mathrm{l}^K : \mathcal{P} \hookrightarrow (\mathcal{C}^\omega)^K$.

5.5.1.6. *Zoo of a spinal or vertebral theory.* Vertebral and spinal theories inherit a 'natural' notion of zoo from their underlying system of vertebrae. Let $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ be a vertebral theory. A morphism $f : X \to Y$ in $\mathcal{C}$ will be said to be a *i) fibration; ii) weak equivalence* for U if, for every object d of D, the image $U_c(f)$ is

i) a fibration in $\hat{\mathcal{E}}_c$ for every object c in H_d;

ii) a weak equivalence in $\hat{\mathcal{E}}_c$ for some object c in H_d;

A morphism $f : X \to Y$ in $\mathcal{C}$ will be said to be a *iii) cofibration; iv) trivial cofibration* for U if, for every object d in D, the image $U(f)$ has the llp with respect to the morphisms of $\mathcal{E}^D$ that are

iii) trivial fibrations in $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ for some object c in H_d;

iv) fibrations in $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ for every object c in H_d.

Any fibration (resp. cofibration) for U that is also a weak equivalence for U will later be called an *acyclic fibration* (resp. *acyclic cofibration*) for U. The zoo of a spinal theory is the same as its underlying vertebral theory.

Example 5.96. In the case of Example 5.92, the local configuration $\mathtt{peak}_K$ forces the weak equivalences to be what are usually called *local weak equivalences* in the literature. In the case where the configuration is an identity (see Example 5.90, Example 5.91 or Example 5.93), the weak equivalences are *component-wise weak equivalences.*

Lemma 5.97. *Let $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ be a vertebral theory as above. A morphism $f : X \to Y$ in $\mathcal{C}$ is a fibration for U if and only if its image $U(f)$ is a fibration in* $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ *for every object c of K.*

Proof. Suppose that f is a fibration for U and choose an object c in K. By definition, the image $\nabla_c \circ U(f)$ is a fibration in the vertebral category $\hat{\mathcal{E}}_c$. Because the functor ∇_c is covertebral, it follows from Proposition 4.83 that the image $U(f)$ is a fibration in the system of vertebrae $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$.

Suppose that $U(f)$ is a fibration in $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$ for every object c of K. Because the functor ∇_c is pseudo-opcovertebral, it follows from Proposition 4.86 that the image $\nabla_c \circ U(f)$ is a fibration in the vertebral category $\hat{\mathcal{E}}_c$. By definition, this means that f is a fibration for U. □

Lemma 5.98. *Let $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ be a refined vertebral theory as above. A morphism $f : X \to Y$ in $\mathcal{C}$ is an acyclic fibration for U if and only if for every object d of D, there exists an object c in H_d such that $U(f)$ is a trivial fibration in* $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$.

Proof. Suppose that f is an acyclic fibration for U and choose an object d in D. By definition, there exists an object c in $H^{-1}(d)$ such that $\nabla_c \circ U(f)$ is an acyclic fibration in the refined vertebral category $\hat{\mathcal{E}}_c$. By Theorem 4.54 (refinement), the image $\nabla_c \circ U(f)$ is a trivial fibration. Because the functor ∇_c is covertebral, it follows from Proposition 4.83 that the image $U(f)$ is a trivial fibration in the system of vertebrae $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$. Conversely, suppose that for every object d of D, there exists an object c in $H^{-1}(d)$ such that $U(f)$ is a trivial fibration in $\mathrm{Lan}_c(\hat{\mathcal{E}}_c)$. Because the functor ∇_c is pseudo-opcovertebral, it follows from Proposition 4.86 that the image $\nabla_c \circ U(f)$ is a trivial fibration in the refined vertebral category $\hat{\mathcal{E}}_c$. By Theorem 4.54 (refinement), the image $\nabla_c \circ U(f)$ is an acyclic fibration in $\hat{\mathcal{E}}_c$. □

Proposition 5.99. *Any vertebral theory $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ satisfies the properties from S0 to S6. Any spinal theory $U : \mathcal{C} \to \hat{\mathcal{E}}^H$ satisfies the properties from S0 to S7.*

S0 *Weak equivalences, fibrations and cofibrations form coherent $\mathcal{C}$-classes;*

S1 *If U is cartesian, then every fibration is stable under pullbacks. If U is refined and cartesian, then every acyclic fibration is stable under pullbacks;*

S2 *The classes of weak equivalences and fibrations are stable under retracts;*

S3 *The classes of cofibrations and trivial cofibrations are stable under retracts;*

S4 *If U is ffr, then cofibrations have the llp with respect acyclic fibrations;*

S5 *If U is ff, then trivial cofibrations have the llp with respect to fibrations;*

S6 *If $f \circ g$ and $g \circ h$ are weak equivalences, then so are h and $f \circ g \circ h$;*

S7 *If $f \circ g$ and $g \circ h$ are weak equivalences, then so are f, g, h and $f \circ g \circ h$;*

Proof. Let us first deal with the case of a vertebral theory.

Let us prove Axiom S0 for weak equivalences. Let $f : Y \to Z$ and $g : X \to Y$ be two weak equivalences in $\mathcal{C}$. By assumption, for every object d in D, there exist objects c and c' in H_d such that $U_c(f)$ and $U_{c'}(g)$ are weak equivalences in $\hat{\mathcal{E}}_c$ and $\hat{\mathcal{E}}_{c'}$, respectively. By definition of a vertebral theory, there exists c'' in H_d such that $U_{c''}(f)$ and $U_{c''}(g)$ are weak equivalences in $\hat{\mathcal{E}}_{c''}$. By Theorem 4.53, the composite $U_{c''}(f \circ g)$ is a weak equivalences in $\hat{\mathcal{E}}_{c''}$, which proves that $f \circ g$ is a weak equivalence.

More generally, axioms S0, S2 and S6 for weak equivalences and fibrations follow from the axioms of section 5.5.1.3, the functoriality of $U_c : \mathcal{C} \to \mathcal{E}$ for every object c in K and the properties of section 4.4.1.2. Axiom S0 for cofibrations follows from Proposition 1.34, Remark 1.36 and the functoriality of $U : \mathcal{C} \to \mathcal{E}^D$. Axiom S3 for cofibrations and trivial cofibrations follows from the properties listed in section 1.2.2.2 about retracts and the functoriality of U.

Let us prove Axiom S1 for acyclic fibrations, the case of fibrations being similar and even simpler. Let $f : X \to Y$ be an acyclic fibration for U and $f^* : X^* \to Z$ be a pullback of it along an arrow $y : Z \to Y$. For every object c in K, the arrow $\nabla_c(U(f))$ is the pullback of $\nabla_c(U(f^*))$ along $\nabla_c(U(y))$ as the right adjoint $\nabla_c : \mathcal{E}^K \to \mathcal{E}$ of Lan_c preserves limits. By assumption, for every object d in D, there exists an object c in H_d such that the arrow $\nabla_c(U(f))$ is an acyclic fibration in $\hat{\mathcal{E}}_c$ and hence a trivial fibration by Theorem 4.54 (refinement). By Proposition 4.38, it follows that the pullback $\nabla_c(U(f^*))$ is a trivial fibration and hence an acyclic fibration in $\hat{\mathcal{E}}_c$ (by refinement).

To prove axiom S4, consider a commutative square of the form given below on the left, where i is a cofibration and f is an acyclic fibration. It follows from Lemma 5.98 that for every object d in D, there exists c in H_d such that morphism $U(f)$ is a trivial fibration in $\mathrm{Lan}_c(\mathcal{E}_c)$. As a result, the middle diagram, below, admits a lift, which is lifted to the category $\mathcal{C}$ by fully faithfulness of U.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} U(A) \arrow[r, "U(u)"] \arrow[d, "U(i)"'] & U(X) \arrow[d, "U(f)"] \\ U(B) \arrow[r, "U(v)"'] \arrow[ur, dashed, "h"] & U(Y) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "h'"] & Y \end{tikzcd}$$

Axiom S5 is proven in the same fashion as axiom S4 by using Lemma 5.97. The case of a spinal theory follows from the fact that the zoo of its underlying spinal category is the same as of its underlying vertebral category. Axiom S7 is proven in much the same way as axiom S0 or S6 by using Theorem 4.77. $\square$

Remark 5.100 (Towards homotopy theories)**.** Axioms S0; S1 and S7 are meant to be used for the construction of categories of fibrant objects (see Example 5.93) while axioms S0; S2; S3; S4; S5 and S7 are meant to be used for the construction of model categories (see Example 5.90, Example 5.91 and Example 5.92). The type of properties that Proposition 5.99 lacks are the factorisation axioms. In practice, the factorisation associated with categories of fibrant objects is given by Theorem 5.83 while the pair of factorisations associated with model categories are given by Corollary 5.86.

5.5.2. From quasi-small spinal theories to model categories.

5.5.2.1. *Combinatorial spinal theories.* Let $U : \mathcal{C} \to \hat{\mathcal{E}}^D$ be a standard quasi-small spinal theory such that $\mathcal{E}^D$ is cocomplete. We shall let I_{cof} denote the set of the seeds and stems of the vertebrae in $\mathtt{GCof}(U)$ and J_{cof} denote the set of the trivial stems of the vertebrae in $\mathtt{GCof}(U)$. For any non-zero limit ordinal κ, the spinal theory U will be said to be κ-*combinatorial* if it is refined and the category $\mathcal{E}^D$ is trivially κ-combinatorial (see section 5.4.5.4) over the sets I_{cof} and J_{cof} such that, for every morphism $f : X \to Y$ in $\mathcal{C}$, the two factorisations of $U(f)$ resulting from Corollary 5.86 lift along the functor $U : \mathcal{C} \to \mathcal{E}^D$ to two factorisations of f as follows.

$$\underbrace{X \xrightarrow{i(f)} R(f) \xrightarrow{p(f)} Y}_{\text{for } I_{\mathrm{cof}}} \qquad \underbrace{X \xrightarrow{j(f)} Q(f) \xrightarrow{q(f)} Y}_{\text{for } J_{\mathrm{cof}}} \tag{5.42}$$

Proposition 5.101. *In the case of a κ-combinatorial $U : \mathcal{C} \to \hat{\mathcal{E}}^D$ as defined above, the arrow $i(f)$ is a cofibration; the arrow $p(f)$ is an acyclic fibration; the arrow $j(f)$ is a trivial cofibration; the arrow $q(f)$ is an fibration;*

Proof. By Corollary 5.86, the image of $U(p(f))$ is in $\mathbf{rlp}(I_{\mathrm{cof}})$. Because, the system of vertebrae $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ is discrete for every object d in D, the class $\mathbf{rlp}(I_{\mathrm{cof}})$ corresponds to the class of morphisms that are trivial fibrations in $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ for every object d in D. By Lemma 5.98, the arrow $p(f)$ is an acyclic fibration for U. By Corollary 5.86, the image $U(i(f))$ is in $\mathbf{llp}(\mathbf{rlp}(I_{\mathrm{cof}}))$, which means that it has the llp with respect to the trivial fibrations in $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ for every object d in D and is hence a cofibration for U. By Corollary 5.86, the image of $U(q(f))$ is in $\mathbf{rlp}(J_{\mathrm{cof}})$. Because, the system of vertebrae $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ is discrete for every object d in D, the class $\mathbf{rlp}(J_{\mathrm{cof}})$ corresponds to the class of morphisms that are fibrations in $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ for every object d in D. By definition, the arrow $q(f)$ is hence a fibration for U. By Corollary 5.86, the image $U(j(f))$ is in $\mathbf{llp}(\mathbf{rlp}(J_{\mathrm{cof}}))$, which means that it has the llp with respect to the fibrations in $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ for every object d in D and is hence a trivial cofibration for U. □

5.5.2.2. *Well disposed combinatorial spinal theories.* Let κ denote a limit ordinal and $U : \mathcal{C} \to \hat{\mathcal{E}}^D$ be a κ-combinatorial spinal theory. By definition, for every morphism $f : X \to Y$ in $\mathcal{C}$, the arrow

$$U(j(f)) : U(X) \to U(Q(f))$$

that results from the leftmost factorisation of (5.42), which is generated with respect to the set of arrows J_{cof}, is the evaluation at the inequality $0 < \kappa$ of a sequential functor $G(f) : \mathbf{O}(\kappa + 1) \to \mathcal{E}^D$. Below, we shall regard the set J_{cof} as a small subcategory of $(\mathcal{E}^D)^2$. The spinal theory $U : \mathcal{C} \to \hat{\mathcal{E}}^D$ will be said to be *well disposed for surtractions (resp. intractions)* if

1) for every morphism $f : X \to Y$ in $\mathcal{E}^D$, the functor $G_f : \mathbf{O}(\kappa+1) \to \mathcal{E}^D$ is convergent with respect to the domain and codomain of every seed in $\mathtt{GCof}(U)$ (resp. of every stem in $\mathtt{GCof}(U)$).
2) for every small category $\mathtt{T}$, functor $\varphi : \mathtt{T} \to J_{\mathrm{cof}}$ that is compatible with the numbered category $(\mathcal{E}^D, \kappa)$ and pushout arrow $p : X \to Y$ of the arrow $\mathrm{col}_{\mathtt{T}}\varphi$ (see the diagram below, on the left), the image of p via the functor $\nabla_d : \mathcal{E}^D \to \mathcal{E}$ is a surtraction

(resp. intraction) in $\hat{\mathcal{E}}_d$ for every object d in D.

$$\begin{tikzcd} \mathrm{col}_{\mathtt{T}}\,\mathbf{A} \arrow[r, "x"] \arrow[d, "\mathrm{col}_{\mathtt{T}}\,\varphi"'] & X \arrow[d, "p"] \\ \mathrm{col}_{\mathtt{T}}\,\mathbf{B} \arrow[r] & P \end{tikzcd} \quad \Rightarrow \quad \begin{array}{c} \nabla_d(p) \textit{ surtraction} \\ \textit{(resp. intraction) in } \hat{\mathcal{E}}_d \end{array}$$

Proposition 5.102. *If $U : \mathcal{C} \to \hat{\mathcal{E}}^D$ is well disposed for surtractions (resp. intractions), then the morphism $\nabla_d \circ U(j(f))$ is a surtraction (resp. intraction) in $\hat{\mathcal{E}}_d$ for every object d of D.*

Proof. Let us deal with the case of surtractions, the case of intractions being similar. Denote by $\chi_k^n(f)$ the image of the inequality $k < n$ via $G(f) : \mathbf{O}(\kappa + 1) \to \mathcal{E}^D$. We are going to prove, by using a transfinite induction on n that the arrow $\chi_0^n(f)$ is a surtraction for every $1 \leq n \leq \kappa$. By definition of a trivially κ-combinatorial category and, more particularly, the construction of the functor $G(f)$ as the context functor of a strict narrative, any arrow $\chi_k^n(f) : G(f)(n) \to G(f)(n+1)$ (for every $n \in \kappa + 1$) is given by a pushout of the form below where $\varphi^n : \mathtt{T} \to (\mathcal{E}^D)^2$ is a certain functor[10] compatible with $(\mathcal{E}^D, \kappa)$, which factorises through the discrete category J_{cof}.

$$\begin{tikzcd} \mathrm{col}_{\mathtt{T}}\,\mathbf{A}^n \arrow[r, "x"] \arrow[d, "\mathrm{col}_{\mathtt{T}}\varphi^n"'] & G(f)(n) \arrow[d, "\chi_n^{n+1}(f)"] \\ \mathrm{col}_{\mathtt{T}}\,\mathbf{B}^n \arrow[r] & G(f)(n+1) \end{tikzcd}$$

Since U is well-disposed for surtractions, the image of the arrow $\chi_n^{n+1}(f) : G(f)(n) \to G(f)(n+1)$ via ∇_d is a surtraction in $\hat{\mathcal{E}}_d$ for every object d in D. In particular, this means that $\nabla_d(\chi_0^1(f))$ is a surtraction in $\hat{\mathcal{E}}_d$. Now, consider some successor ordinal n such that the arrow $\nabla_d(\chi_0^{n-1}(f))$ is a surtraction in $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$ for every object d in D. It follows from the equation

$$\chi_0^n(f) = \chi_{n-1}^n(f) \circ \chi_0^{n-1}(f)$$

and the fact that surtractions are composable in $\hat{\mathcal{E}}_d$ (see Proposition 4.52) that the arrow $\nabla_d(\chi_0^n(f))$ is a surtraction in $\hat{\mathcal{E}}_d$ for every object d in D.

Finally, consider some limit ordinal λ. For convenience, let us fix some object d in D. To prove that $\nabla_d(\chi_0^\lambda(f))$ is a surtraction in $\mathrm{Lan}_d(\hat{\mathcal{E}}_d)$, we are going to use the definition in terms of divisiblity with respect to the besoms[11] of the vertebrae of $\mathtt{GCof}_d(U)$. Let γ be an E_d-seed and consider a commutative diagram as given below on the left. By adjointness, we obtain the commutative diagram displayed on the right-hand side.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "x"] \arrow[d, "\gamma"'] & \nabla_d G(f)(0) \arrow[d, "\nabla_d \chi_0^\lambda(f)"] \\ \mathbb{D}_2 \arrow[r, "y"'] & \nabla_d G(f)(\lambda) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{Lan}_d(\mathbb{S}) \arrow[r, "x_*"] \arrow[d, "\mathrm{Lan}_d(\gamma)"'] & G(f)(0) \arrow[d, "\chi_0^\lambda(f)"] \\ \mathrm{Lan}_d(\mathbb{D}_2) \arrow[r, "y_*"'] & G(f)(\lambda) \end{tikzcd} \tag{5.43}$$

Since $G(f)$ is $\{\mathrm{Lan}_d(\mathbb{S}), \mathrm{Lan}_d(\mathbb{D})\}$-convergent in $\mathcal{E}^D$ by assumption, the functor $\mathbf{O}(\kappa+1) \to (\mathcal{E}^D)^2$ induced by the mapping $n \mapsto \chi_0^n(f)$ is uniformly $\mathrm{Lan}_d(\gamma)$-convergent in $\mathcal{E}^D$ (by definition). It follows from Remark 5.2 that this functor is $\{\mathrm{Lan}_d(\gamma)\}$-convergent in $(\mathcal{E}^D)^2$. This

[10] This functor is more precisely the composition of one of the elements of the set of events at rank n with the tome of the oeuvre at rank n.

[11] In this case, they are clearly given by the diskads of the vertebrae.

implies that there exists some $k \in \lambda$ for which diagram (5.43) factorises as follows.

$$\begin{tikzcd}
\mathrm{Lan}_d(\mathbb{S}) \arrow[r, "x'_*"] \arrow[d, "\mathrm{Lan}_d(\gamma)"'] & G(f)(0) \arrow[r, equal] \arrow[d, "\chi_0^k(f)"] & G(f)(0) \arrow[r, equal] \arrow[d, "\chi_0^{k+1}(f)"] & G(f)(0) \arrow[d, "\chi_0^\lambda(f)"] \\
\mathrm{Lan}_d(\mathbb{D}_2) \arrow[r, "y'_*"'] & G(f)(k) \arrow[r] & G(f)(k+1) \arrow[r] & G(f)(\lambda)
\end{tikzcd}$$

Since $\nabla_d \chi_0^k(f)$ is a surtraction in $\hat{\mathcal{E}}_d$, there exists a vertebra v in $\mathtt{GCof}_d(U)$ for which it is a surtraction. It is then easy to see that, after applying the adjunction $\mathrm{Lan}_d \dashv \nabla_d$ on the above diagram, the divisibility property satisfied by $\nabla_d \chi_0^k(f)$ with respect to the besom of v transfers to the arrow $\nabla_d \chi_0^\lambda(f)$, so that $\nabla_d \chi_0^\lambda(f)$ is surtraction for v. This shows that the arrow $\nabla_d \chi_0^\lambda(f)$ is a surtraction in $\hat{\mathcal{E}}_d$. The above induction then shows that $\nabla_d \circ U(j(f)) = \nabla_d(\chi_0^\kappa(f))$ is a surtraction in $\hat{\mathcal{E}}_d$ for every object d of D, which proves the statement. □

Remark 5.103. There exist other ways of proving that the arrow is an intraction $\nabla_d \circ U(j(f))$ in $\hat{\mathcal{E}}_d$ for every object d of D. For instance, when all the vertebrae of $\hat{\mathcal{E}}_d$ are reflexive with identity reflexive transitions, the homotopy contractions, α, used, define retractions with the trivial stems (i.e. $\alpha \circ \beta \circ \delta_1 = \mathrm{id}_{\mathbb{D}_1}$). These retractions may be used in the small object argument (in a fairly straightforward way) to show that there exists a morphism $r(f) : G(f)(\gamma) \to U(X)$ such that the equation $r(f) \circ U(j(f)) = \mathrm{id}_{U(X)}$ holds. It then follows from Lemma 2.52 that the arrow $\nabla_d \circ U(j(f))$ is an intraction in $\hat{\mathcal{E}}_d$ for every object d of D.

Theorem 5.104. *Let κ be a limit ordinal and $U : \mathcal{C} \to \hat{\mathcal{E}}^D$ be a fully faithful κ-combinatorial spinal theory that is well-disposed for intractions and surtractions. The category $\mathcal{C}$ admits a model structure with respect to its underlying weak equivalences, fibrations and cofibrations.*

Proof. Most of the properties of a model structure are proven by Proposition 5.99 (retract axiom; two-out-of-three property; coherent $\mathcal{C}$-classes; llp of cofibrations with respect to acyclic fibrations). The factorisation axioms are given by Proposition 5.102 and Proposition 5.101. There only remains to show that any acyclic cofibration is a trivial cofibration, which will prove by Proposition 5.99 that acyclic cofibrations have the llp with respect to fibrations. Let $h : A \to B$ be an acyclic cofibration in $\mathcal{C}$. By Proposition 5.102 and Proposition 5.101, we know that h may be factorised in terms of a composite $p \circ i$ where $i : A \to H$ is both a trivial cofibration and weak equivalence and $p : H \to B$ is a fibration.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "h"'] & H \arrow[d, "p"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

Because h and i are weak equivalences, the two-out-of-six property for weak equivalences implies that p is a weak equivalence and hence an acyclic fibration. It follows that the cofibration h has the llp with respect to p. This means that the above commutative square admits a lift $h' : B \to H$ that leads to a retraction as follows.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "h"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "p"] \\
B \arrow[r, dashed, "h'"] \arrow[rr, bend right, "\mathrm{id}_B"'] & H \arrow[r, "p"] & B
\end{tikzcd}$$

Since trivial cofibrations are stable under retracts by Proposition 5.99, the arrow h is a trivial cofibration, which finishes the proof. □

We will see in Chapter 6 how to provide a certain class of categories – which will be called *coheroids* – with spinal theory structures. In some cases, such as 1-groupoids or strict ∞-groupoids (it follows from the main result of [**5**] that strict ∞-groupoids are coheroids), it is straighforward – even though somewhat cumbersome – to check that these spinal theories are well-disposed for intractions and surtractions. In these particular cases, Theorem 5.104 retrieves the classical model structures on 1-groupoids or strict ∞-strict groupoids.

Chapter 6

Towards the Homotopy Hypothesis

6.1. Introduction

In this chapter, the category of Grothendieck's ∞-groupoids is equipped with the structure of a spinal category. The construction we give also works for topological spaces, strict n-categories for every $n \geq 1$, strict ω-categories and Maltsiniotis' ∞-categories. In much the same way as the definition of Grothendieck's ∞-groupoids is inductive, the definition of the spinal category will inherit an inductive pattern. Unfortunately, I have not found any proper way of reducing the length of the different inductive definitions involved as every case has its own particularity. Going through every construction simply seems to be the way to go.

To some extent, this chapter only consists in applying the constructions already discussed in the previous chapters and no actually new method is introduced – only variations of those seen before. The present introduction gives a concise description of the main ideas and goals of every section.

The chapter is divided in three main sections:

Section 6.2, called *vertebral structure*, gives the definitions of *coherator* and *coheroid*. Coherators are the sketches for the categories of Grothendieck's ∞-groupoids while coheroids give the domain of discourse in which the present chapter is written – the latter notion is more general than the former. The section ends with the construction of vertebral categories for coheroids.

Section 6.3, called *spines and their functorial framings*, defines all the tools needed to handle the notions of simple and extensive framings of spines. The main difficulty is to define the functoriality of these. In Chapter 3, 'simple framings' were framings compatible with the stems of the spines while 'extensive framings' were framings compatible with the diskad of the heads of the spines. Contrary to the type of functoriality discussed in Chapter 3, the functoriality considered in this chapter concerns the whole structure of the spines. Such framings will require an inductive definition on the dimension of the spines.

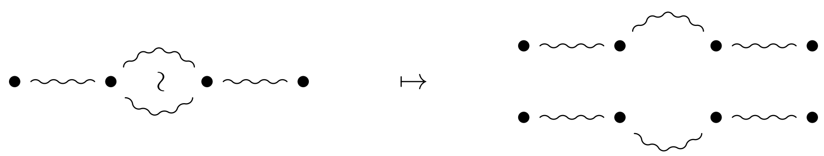
Section 6.4, called *spinal structure*, deals with the definition of spinal categories for coheroids. The main task is to define the notion of convergent conjugations. The conjugations will arise from the framings defined in section 6.3. The convergence will then be inductively defined by successively specifying every pair of mates that allows its realisation.

Our strategy for section 6.2 is the following. We first start with the definition of ω-globular and ω-spinal sketches. The globular world is that in which the sketch of ∞-groupoids lives while the spinal world is that in which the notion of homotopy lives. This first section therefore makes a systematic comparison between the two worlds. We introduce the notions of *ω-spinal* and *ω-globular objects* and show that they are related by a one-to-one correspondence when the ambient category has pushouts and initial objects. We then define the notion of *globular* and *spinal pre-extension*, the most meaningful being the latter. The pre-extensions followingly become *extensions* when all the *globular sums* encoding the concept of gluing of discs exist in the ambient category of the pre-extensions. Globular and spinal extensions are again compared. We pursue the discussion with the notion of parallelism in both globular and spinal extensions and compare the two notions. Then follows the notion of *coheroid*, which equips an extension with a class of parallel arrows for which certain lifting properties hold in the ambient category. The concept of coheroid will constitute the playground for the generating of spinal categories. We define the notion of *Grothendieck coherator* and briefly describe that of *Maltsiniotis coherator* by referring to [**35**] for complete details. We show that the categories of models for these sketches are endowed with canonical coheroids. Finally, we finish the section by equipping coheroids with sets of canonical vertebrae and show that these give rise to vertebral categories when the coheroids in question are endowed with certain parallel arrows. We shall come across two intermediate structures, namely:

→ *Reflexive spinal coheroids*, which will come along with reflexive vertebrae;

→ *Magmoidal spinal coheroids*, which will come along with framing of vertebrae;

Our strategy for section 6.3 goes as follows. We introduce the *kappa* and *tau constructions* $\kappa^m_{1,k}$, $\kappa^m_{2,k}$ and $\tau^m_{1,k}(\beta)$, $\tau^m_{2,k}(\beta)$ linking globular sums of some dimension to others of lower dimension. Specifically, using the analogy between the notion of cell in an ∞-groupoid and the notion of topological discs, the kappa and tau constructions will link a gluing of discs of the form given below, on the left, to the gluing of lower dimensional discs given by their respective borders, as shown on the right.



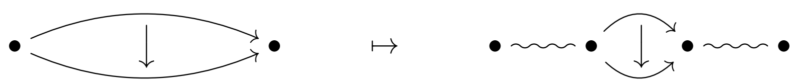
The kappa constructions $\kappa^m_{1,k}$, $\kappa^m_{2,k}$ are to handle gluings taking the form of whiskerings whose whiskers should be thought of as 'reversible' cells and whose middle parts should specifically be viewed as 'non-reversible' cells. This non-reversibility will concretely be charactirised by the use of seeds where we would most often use stems. For their part, tau constructions $\tau^m_{1,k}(\beta)$, $\tau^m_{2,k}(\beta)$ are to handle whiskerings whose cells should all be viewed as potentially 'reversible' cells. We then carry on with the notion of normality and transitivity, which realise the preceding gluings as actual compositional operations. These notions have inductive definitions and rely on the kappa and tau constructions.

→ $(k,0)$-*normal spinal coheroids* come along with a morphism π^0_k linking a 'non-reversible' cell of dimension k to a gluing of a 'non-reversible' cell of dimension k along two reversible ones of the same dimension and thus emulates a ternary composition of discs;

$$\bullet \xrightarrow{\qquad\qquad} \bullet \qquad \mapsto \qquad \bullet \rightsquigarrow \bullet \longrightarrow \bullet \rightsquigarrow \bullet$$

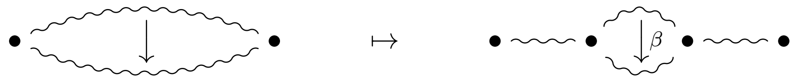
→ $(k,1)$-*normal spinal coheroids* come along with a morphism π^1_k linking a 'non-reversible' cell of dimension $k+1$ to a gluing of a 'non-reversible' cell of dimension

$k+1$ along two reversible ones of dimension k and thus emulates a ternary composition of discs. This notion will depend on the construction π_k^0, which will take care to compose the borders at dimension k.

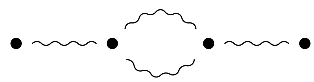


$\rightarrow$ $(k, m-k)$-*normal spinal coheroids* are then defined as above by induction.

The next notion is that of transitivity and copies the definition of normality by replacing the kappa constructions with the tau constructions. Broadly, a $(k, m-k)$-*transitive spinal coheroid* is equipped with morphisms $v_k^{m-k}(\beta)$ encoding the compositions of 'reversible' cells of dimension m along pairs of 'reversible' cells of dimension k.



The functoriality of these compositions will be ensured via the notion of closedness, which will require the compositions to be compatible with the spheres induced by the borders of the discs. Precisely, this requires the existence of gluings of spheres along any pair of 'reversible' cells.



Closedness will come along with canonical morphisms $d_{1,k}^m$, $d_{2,k}^m$, κ_k^m and $\tau_k^m(\beta)$ that factorises the kappa and tau constructions in canonical ways as follows.

$$\kappa_{i,k}^m = \kappa_k^m \circ d_{i,k}^m \qquad\qquad \tau_{i,k}^{m-1}(\beta) = \tau_k^m(\beta) \circ d_{i,k}^m$$

These morphisms will help define the functoriality of our framings, which will specifically follow from the commutative diagrams constructed in section 6.3.3.2, called *closedness and normality*, and section 6.3.3.3, called *closedness and transitivity*. Because the inductive developments given in these sections are quite cumbersome, I would now like to specify these commutative diagrams as well as give a short description of their roles (see below) – the reader is therefore excepted to come back here if they lose their way while reading these sections. Now, to resume our discussion, this so-called functoriality will follow from the canonical properties of the morphisms $d_{1,k}^m$, $d_{2,k}^m$, κ_k^m and $\tau_k^m(\beta)$ as follows:

$\triangleright$ diagrams (6.30), (6.33) and (6.38) of section 6.3.3.2 will be used to ensure the functoriality of the whiskerings of non-reversible cells along pairs of reversible cells *relative to* the whiskerings of the spheres, induced by the borders of the previous non-reversible cells, along the same pairs of reversible cells;

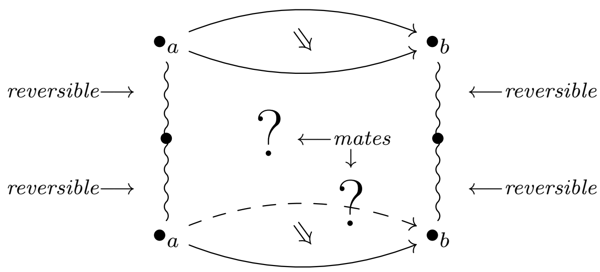
$\triangleright$ diagrams (6.32) and (6.37) of section 6.3.3.2, or, in fact, their lower parts, will be used to ensure the functoriality of the whiskerings of the spheres along pairs of reversible cells *relative to* the whiskerings of the non-reversible borders of these spheres along the same pairs of reversible cells;

$\triangleright$ diagrams (6.39) and (6.40) of section 6.3.3.3 will be used to ensure the functoriality of the whiskerings of reversible cells *relative to* the whiskerings of the spheres, induced by the borders of the middle cells, along the same pair of reversible cells;

$\triangleright$ all the other diagrams may be viewed as intermediate steps for the purpose of defining the functoriality of our framings.

We then finish the section by summarising the aformentioned constructions in the form of propositions expliciting the desired framings of spines and their functoriality.

Finally, our strategy regarding section 6.4 goes as follows. We first define two reflections of vertebrae via the notion of *symmetric spinal coheroid*. These reflections give a 'reversible' structure to the stems of our vertebrae and thereby allow us to define conjugations of spines. Once these conjugations are defined, the correspondences that result from them provide two pairs of parallel arrows. The notion of $(k, m-k)$-*coherent spinal coheroid* (see section 6.4.1.2) then provides these two pairs of parallel arrows with liftings giving rise to pairs of mates for the underlying correspondences. As in Chapter 3, the mates will be transferred throughout the whole spines via the notion of morphism of correspondences (see Proposition 3.60).



The transfer of the mates along the whole spines then enables us to frame the whole spines and carry along the information necessary to define the notion of $(k, m-k+1)$-*coherent spinal coheroid*. Defining mates by induction provides sequences of chainings whose ranks of definition gradually increase and eventually run over the degrees of all the involved spines (see Figure 1). This ultimately equips the memories of every conjugation with structures of convergent (and functorial) chaining of memories. In other words, we obtain convergent conjugations of spines. These conjugations are both simple and extensive since the involved framings are 'everywhere functorial'.

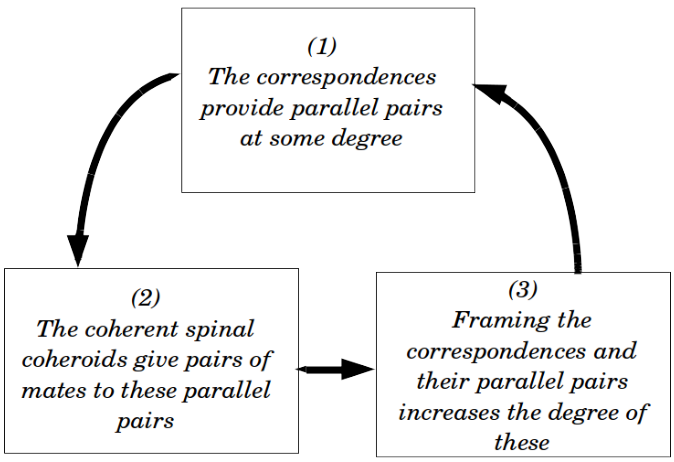


Figure 1. Iterative process to generate convergent chainings of memories

We end the chapter (see section 6.4.2) by defining the spinal structure of Grothendieck's ∞-groupoids, stemming from the convergent conjugations, and give a sketch of the proof of the Homotopy Hypothesis.

6.2. Vertebral structure

6.2.1. Globular and spinal extensions.

6.2.1.1. *Omega-globular sketches.* An *ω-globular sketch* is a small category whose structure is freely generated from a graph consisting of

1) objects $\mathbf{D}_k$ for every $k \in \omega$;
2) arrows $s_k : \mathbf{D}_k \to \mathbf{D}_{k+1}$ and $t_k : \mathbf{D}_k \to \mathbf{D}_{k+1}$ for every $k \in \omega$,

such that the following relations hold for every $i \in \omega$.

$$s_{k+1} \circ s_k = t_{k+1} \circ s_k \qquad\qquad s_{k+1} \circ t_k = t_{k+1} \circ t_k \tag{6.1}$$

All omega-globular sketches are obviously isomorphic and will be denoted by the symbol `Glob`. These categories may be presented by a diagram of the following form.

$$\begin{tikzcd}
\mathbf{D}_0 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & \mathbf{D}_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathbf{D}_2 \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & \dots \arrow[r, shift left, "s_{k-1}"] \arrow[r, shift right, "t_{k-1}"'] & \mathbf{D}_k \arrow[r, shift left, "s_k"] \arrow[r, shift right, "t_k"'] & \mathbf{D}_{k+1} \arrow[r, shift left, "s_{k+1}"] \arrow[r, shift right, "t_{k+1}"'] & \dots
\end{tikzcd}$$

6.2.1.2. *Omega-spinal sketches.* An *ω-spinal sketch* is a small category whose structure is freely generated from a graph consisting of

1) objects $\mathbf{D}_k$ for every $k \in \omega$;
2) objects $\mathbf{S}_{k-1}$ for every $k \in \omega$;
3) arrows $\gamma_k : \mathbf{S}_{k-1} \to \mathbf{D}_k$, $\delta_1^k : \mathbf{D}_k \to \mathbf{S}_k$ and $\delta_2^i : \mathbf{D}_k \to \mathbf{S}_k$ for every $k \in \omega$,

such that the following relation holds for every $k \in \omega$.

$$\delta_1^k \circ \gamma_k = \delta_2^k \circ \gamma_k$$

All omega-spinal sketches are obviously isomorphic and will be denoted by the symbol `Spine`. These categories may be presented by a diagram of the following form.

$$\begin{tikzcd}
& \mathbf{D}_0 \arrow[dr, "\delta_1^0"] & & \mathbf{D}_1 \arrow[dr, "\delta_1^1"] & & & & \mathbf{D}_k \arrow[dr, "\delta_1^k"] & & \\
\mathbf{S}_{-1} \arrow[ur, "\gamma_0"] \arrow[dr, "\gamma_0"'] & & \mathbf{S}_0 \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_1"'] & & \mathbf{S}_1 & \dots & \mathbf{S}_{k-1} \arrow[ur, "\gamma_k"] \arrow[dr, "\gamma_k"'] & & \mathbf{S}_k & \dots \\
& \mathbf{D}_0 \arrow[ur, "\delta_2^0"'] & & \mathbf{D}_1 \arrow[ur, "\delta_2^1"'] & & & & \mathbf{D}_k \arrow[ur, "\delta_2^k"'] & &
\end{tikzcd}$$

Remark 6.1. The object $\mathbf{S}_k$ is initial when $k = -1$ and is a pushout object for the following square for every $k \in \omega$.

$$\begin{tikzcd}
\mathbf{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbf{D}_k \arrow[d, "\delta_k^1"] \\
\mathbf{D}_k \arrow[r, "\delta_k^2"'] & \mathbf{S}_k
\end{tikzcd}$$

6.2.1.3. *Omega-spinal and omega-globular objects.* Let $\mathcal{C}$ be a category that has all pushouts and initial objects. Any functor of the form $F : \texttt{Glob} \to \mathcal{C}$ will be called an *ω-globular object in $\mathcal{C}$*. Similarly, the term *ω-spinal object* will be used to refer to any functor of the form $G : \texttt{Spine} \to \mathcal{C}$ that preserves the universal structure of the objects $\mathbf{S}_{k-1}$ pointed out by Remark 6.1 for every $k \in \omega$. Specifically, an ω-spinal object is a functor $G : \texttt{Spine} \to \mathcal{C}$ such that the object $G(\mathbf{S}_0)$ is initial in $\mathcal{C}$ and, for every $k > 0$, the following square is a pushout square in $\mathcal{C}$.

$$\begin{tikzcd} G(\mathbf{S}_{k-1}) \arrow[r, "G(\gamma_k)"] \arrow[d, "G(\gamma_k)"'] & G(\mathbf{D}_k) \arrow[d, "G(\delta_k^1)"] \\ G(\mathbf{D}_k) \arrow[r, "G(\delta_k^2)"'] & G(\mathbf{S}_k) \end{tikzcd}$$

Remark 6.2. Because the small categories `Glob` and `Spine` are free categories generated over some graphs, ω-globular and an ω-spinal objects in $\mathcal{C}$ may be identified with the image of the generating graph of `Glob` and `Spine` in $\mathcal{C}$, respectively.

There is a one-to-one correspondence (bijection) between ω-globular objects and ω-spinal objects in $\mathcal{C}$.

Proposition 6.3. *Any ω-globular object $F : \mathtt{Glob} \to \mathcal{C}$ in $\mathcal{C}$ gives rise to an ω-spinal object $\hat{F} : \mathtt{Spine} \to \mathcal{C}$. Conversely, any ω-spinal object $G : \mathtt{Spine} \to \mathcal{C}$ gives rise to ω-globular object $G^\dagger : \mathtt{Glob} \to \mathcal{C}$ in $\mathcal{C}$. These mappings define a one-to-one correspondence between ω-globular objects and ω-spinal objects in $\mathcal{C}$.*

Proof. Let F be a ω-gobular object such that the image of the generating graph of `Glob` via F is given by a diagram of the following form in $\mathcal{C}$.

$$\mathbb{D}_0 \overset{s_0}{\underset{t_0}{\rightrightarrows}} \mathbb{D}_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} \mathbb{D}_2 \overset{s_2}{\underset{t_2}{\rightrightarrows}} \cdots \overset{s_{k-1}}{\underset{t_{k-1}}{\rightrightarrows}} \mathbb{D}_k \overset{s_k}{\underset{t_k}{\rightrightarrows}} \mathbb{D}_{k+1} \overset{s_{k+1}}{\underset{t_{k+1}}{\rightrightarrows}} \cdots$$

Denote by $\mathbb{S}_{-1}$ the initial object of $\mathcal{C}$. Note that the following commutative diagram commute in $\mathcal{C}$.

$$\begin{tikzcd} \mathbb{S}_{-1} \arrow[r, "!"] \arrow[d, "!"'] & \mathbb{D}_0 \arrow[d, "s_0"] \\ \mathbb{D}_0 \arrow[r, "t_0"'] & \mathbb{D}_1 \end{tikzcd}$$

Forming the pushout of the span involved in the preceding square, which will be denoted by $\mathbb{S}_0$, leads to the existence of a canonical arrow $\gamma_1 : \mathbb{S}_0 \to \mathbb{D}_1$ making the following diagram commute.

$$\begin{tikzcd} \mathbb{S}_{-1} \arrow[r, "!"] \arrow[d, "!"'] & \mathbb{D}_0 \arrow[d, "\delta_1^0"] \arrow[dr, bend left, "s_0"] & \\ \mathbb{D}_0 \arrow[r, "\delta_2^0"'] \arrow[rr, bend right, "t_0"'] & \mathbb{S}_0 \arrow[ul, phantom, "\ulcorner"] \arrow[r, dashed, "\gamma_1"'] & \mathbb{D}_1 \end{tikzcd}$$

The canonical morphism $! : \mathbb{S}_{-1} \to \mathbb{D}_0$ will later be denoted by γ_0. Suppose that the following pushout square is defined for every $k \geq n$ for some $n \in \omega$.

$$\begin{tikzcd} \mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\delta_1^k"] \arrow[dr, bend left, "s_k"] & \\ \mathbb{D}_k \arrow[r, "\delta_2^k"'] \arrow[rr, bend right, "t_k"'] & \mathbb{S}_k \arrow[ul, phantom, "\ulcorner"] \arrow[r, dashed, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \end{tikzcd}$$

Equation (6.1) implies that pre-composing the preceding diagram with the arrows $s_{k+1} : \mathbb{D}_{k+1} \to \mathbb{D}_{k+2}$ and $t_{k+1} : \mathbb{D}_{k+1} \to \mathbb{D}_{k+2}$ provides two solutions for the same universal problem as follows.

$$
\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\delta_1^k"] \arrow[rr, bend left, "s_{k+1}\circ s_k"] & & \\
\mathbb{D}_k \arrow[r, "\delta_2^k"'] \arrow[rrr, bend right, "t_{k+1}\circ t_k"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_k \arrow[rr, dashed, "t_{k+1}\circ\gamma_{k+1}"'] & & \mathbb{D}_{k+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\delta_1^k"] \arrow[rr, bend left, "s_{k+1}\circ s_k"] & & \\
\mathbb{D}_k \arrow[r, "\delta_2^k"'] \arrow[rrr, bend right, "t_{k+1}\circ t_k"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_k \arrow[rr, dashed, "s_{k+1}\circ\gamma_{k+1}"'] & & \mathbb{D}_{k+1}
\end{tikzcd}
$$

By universality of the pushout $\mathbb{S}_{k+1}$, it follows that the following left diagram commutes. Forming the pushout over the span made from two copies of the arrow γ_{k+1} then provides the canonical morphism $\gamma_{k+2} : \mathbb{S}_{k+1} \to \mathbb{D}_{k+2}$.

$$
\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "s_{k+1}"] \\
\mathbb{D}_{k+1} \arrow[r, "t_{k+1}"'] & \mathbb{D}_{k+2}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_1^{k+1}"] \arrow[dr, bend left, "s_{k+1}"] & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_2^{k+1}"'] \arrow[rr, bend right, "t_{k+1}"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_{k+1} \arrow[r, dashed, "\gamma_{k+2}"'] & \mathbb{D}_{k+2}
\end{tikzcd}
$$

By induction, the ω-globular object F finally induces an ω-spinal object given by

- the object $\mathbb{D}_k$ for every $k \in \omega$;
- the object $\mathbb{S}_{k-1}$ for every $k \in \omega$;
- the arrows $\gamma_k : \mathbb{S}_{k-1} \to \mathbb{D}_k$, $\delta_1^k : \mathbb{D}_k \to \mathbb{S}_k$ and $\delta_2^i : \mathbb{D}_k \to \mathbb{S}_k$ for every $k \in \omega$.

This ω-spinal object will later be denoted by $\hat{F} : \texttt{Spine} \to \mathcal{C}$. Conversely, every ω-spinal object $G : \texttt{Spine} \to \mathcal{C}$ of the form

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{D}_0 \arrow[dr, "\delta_1^0"] & & \mathbb{D}_1 \arrow[dr, "\delta_1^1"] & & & & \mathbb{D}_k \arrow[dr, "\delta_1^k"] & & \\
\mathbb{S}_{-1} \arrow[ur, "\gamma_0"] \arrow[dr, "\gamma_0"'] & & \mathbb{S}_0 \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_1"'] & & \mathbb{S}_1 & \dots & \mathbb{S}_{k-1} \arrow[ur, "\gamma_k"] \arrow[dr, "\gamma_k"'] & & \mathbb{S}_k & \dots \\
 & \mathbb{D}_0 \arrow[ur, "\delta_2^0"'] & & \mathbb{D}_1 \arrow[ur, "\delta_2^1"'] & & & & \mathbb{D}_k \arrow[ur, "\delta_2^k"'] & &
\end{tikzcd}
$$

induces an obvious ω-globular object, say $G^\dagger$, given by the following diagram.

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, shift left, "\gamma_1\circ\delta_1^0"] \arrow[r, shift right, "\gamma_1\circ\delta_2^0"'] & \mathbb{D}_1 \arrow[r, shift left, "\gamma_2\circ\delta_1^1"] \arrow[r, shift right, "\gamma_2\circ\delta_2^1"'] & \mathbb{D}_2 \arrow[r, shift left, "\gamma_3\circ\delta_1^2"] \arrow[r, shift right, "\gamma_3\circ\delta_2^2"'] & \dots \arrow[r, shift left, "\gamma_k\circ\delta_1^{k-1}"] \arrow[r, shift right, "\gamma_k\circ\delta_2^{k-1}"'] & \mathbb{D}_k \arrow[r, shift left, "\gamma_{k+1}\circ\delta_1^k"] \arrow[r, shift right, "\gamma_{k+1}\circ\delta_2^k"'] & \mathbb{D}_{k+1} \arrow[r, shift left, "\gamma_{k+2}\circ\delta_1^{k+1}"] \arrow[r, shift right, "\gamma_{k+2}\circ\delta_2^{k+1}"'] & \dots
\end{tikzcd}
$$

It is not hard to check that the mapping $F \mapsto \hat{F}$ and $G \mapsto G^\dagger$ define a one-to-one correspondence. $\square$

Later on, for every ω-globular object $F : \texttt{Glob} \to \mathcal{C}$, the ω-spinal object $\hat{F} : \texttt{Spine} \to \mathcal{C}$ defined in Proposition 6.3 will be referred to as the *underlying ω-spinal object of F*.

Following Remark 6.2, any ω-spinal object given by a diagram of the form

$$\begin{tikzcd}
 & \mathbb{D}_0 \arrow[dr, "\delta_1^0"] & & \mathbb{D}_1 \arrow[dr, "\delta_1^1"] & & & \mathbb{D}_k \arrow[dr, "\delta_1^k"] & \\
\mathbb{S}_{-1} \arrow[ur, "\gamma_0"] \arrow[dr, "\gamma_0"'] & & \mathbb{S}_0 \arrow[ur, "\gamma_1"] \arrow[dr, "\gamma_1"'] & & \mathbb{S}_1 \ \dots & \mathbb{S}_{k-1} \arrow[ur, "\gamma_k"] \arrow[dr, "\gamma_k"'] & & \mathbb{S}_k \ \dots \\
 & \mathbb{D}_0 \arrow[ur, "\delta_2^0"'] & & \mathbb{D}_1 \arrow[ur, "\delta_2^1"'] & & & \mathbb{D}_k \arrow[ur, "\delta_2^k"'] &
\end{tikzcd}$$

will be denoted as a quadruple $(\mathbb{S}, \mathbb{D}, \gamma, \delta) : \mathtt{Spine} \to \mathcal{C}$, thereby referring to all the symbols of the diagram. The indexing will then follow the above conventions. In addition, for every pair of integers k and m satisfying the inequalities $-1 \leq k \leq m$, the arrow $\mathbb{S}_k \to \mathbb{S}_m$ defined by the following composite of arrows will be denoted by the symbol Γ_k^m.

$$\mathbb{S}_k \xrightarrow{\delta_1^{k+1} \circ \gamma_{k+1}} \mathbb{S}_{k+1} \xrightarrow{\delta_1^{k+2} \circ \gamma_{k+2}} \dots \xrightarrow{\delta_1^{m-1} \circ \gamma_{m-1}} \mathbb{S}_{m-1} \xrightarrow{\delta_1^{m} \circ \gamma_{m}} \mathbb{S}_m$$

By convention, when the equality $k = m$ holds, the arrow Γ_k^m will be the identity on $\mathbb{S}_k$. Similarly, an ω-globular object given by a diagram as follows will be denoted as a triple $(\mathbb{D}, s, t) : \mathtt{Glob} \to \mathcal{C}$.

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & \mathbb{D}_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathbb{D}_2 \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & \dots \arrow[r, shift left, "s_{k-1}"] \arrow[r, shift right, "t_{k-1}"'] & \mathbb{D}_k \arrow[r, shift left, "s_k"] \arrow[r, shift right, "t_k"'] & \mathbb{D}_{k+1} \arrow[r, shift left, "s_{k+1}"] \arrow[r, shift right, "t_{k+1}"'] & \dots
\end{tikzcd}$$

The indexing will then be deduced from the above conventions. In addition, for every pair of integers k and m satisfying the inequalities $0 \leq k < m$, the arrows $\mathbb{D}_k \to \mathbb{D}_m$ defined as the composites of the sequences of arrows

$$\mathbb{D}_k \xrightarrow{s_k} \mathbb{D}_{k+1} \xrightarrow{s_{k+1}} \dots \xrightarrow{s_{m-2}} \mathbb{D}_{m-1} \xrightarrow{s_{m-1}} \mathbb{D}_m$$

and

$$\mathbb{D}_k \xrightarrow{t_k} \mathbb{D}_{k+1} \xrightarrow{t_{k+1}} \dots \xrightarrow{t_{m-2}} \mathbb{D}_{m-1} \xrightarrow{t_{m-1}} \mathbb{D}_m$$

will be denoted by the symbols s_k^m and t_k^m, respectively.

6.2.1.4. *Globular and spinal pre-extensions.* A *globular pre-extension* consists of a category $\mathcal{C}$ equipped with an ω-globular object $F : \mathtt{Glob} \to \mathcal{C}$. Such a structure will be denoted by its associated ω-gobular object. A *spinal pre-extension* consists of a category $\mathcal{C}$ equipped with

1) an ω-spinal object $(\mathbb{S}, \mathbb{D}, \gamma, \delta) : \mathtt{Spine} \to \mathcal{C}$;
2) a class Ω_k of morphisms in $\mathcal{C}$ with domain $\mathbb{S}_k$ satisfying the following inclusion for every $k \in \omega$.
$$\Omega_k \subseteq \mathbf{llp}(\mathbf{rlp}(\{\gamma_m \mid m \in \omega\}))$$

The union of Ω_k with the singleton $\{\gamma_{k+1} : \mathbb{S}_k \to \mathbb{D}_{k+1}\}$ will be denoted by Ω_k°. Later on, such a structure will be denoted as an arrow of the following form.

$$(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to \mathcal{C}$$

Remark 6.4. Note that, for every category $\mathcal{C}$ that has pushouts and initial objects, any globular pre-extension $F : \mathtt{Glob} \to \mathcal{C}$ gives rise to a spinal pre-extension where

1) the spinal object is given by the underlying ω-spinal object $\hat{F} : \mathtt{Spine} \to \mathcal{C}$;
2) for every $k \in \omega$, the class Ω_k is the singleton consisting of $\gamma_{k+1} : \mathbb{S}_k \to \mathbb{D}_{k+1}$.

In this case, the class Ω_k is equal to the class Ω_k°. The above spinal pre-extension will later be referred to as the *underlying spinal pre-extension of F*.

Example 6.5 (Topological spaces)**.** The set of topological discs in **Top** define an obvious globular pre-extension given, for every integer $n \geq 0$, by the inclusions the n-discs $\mathbb{D}_n$ into the two hemispheres of the $(n+1)$-discs $\mathbb{D}_{n+1}$. For every integer $n \geq 0$, the object $\mathbb{S}_n$ associated with the underlying spinal pre-extension of the resulting globular object $(\mathbb{D}, s, t)$ is the topological sphere.

6.2.1.5. *Globular sums in globular pre-extensions.* Let $(\mathbb{D}, s, t) : \mathtt{Glob} \to \mathcal{C}$ be a spinal pre-extension and q be a non-negative integer. A *q-globular sum* in the pre-extension $(\mathbb{D}, s, t)$ consists of

- a table of non-negative integers

$$\left(\begin{array}{ccccccc} j_1 & & j_2 & & \dots & & j_{q+1} \\ & i_1 & & i_1 & \dots & i_q & \end{array}\right)$$

  where i_k is less than j_k and j_{k+1} for every $1 \leq k \leq q+1$;
- a colimit in $\mathcal{C}$ for the diagram of the form

$$\begin{tikzcd}
\mathbb{D}_{j_1} & & \mathbb{D}_{j_2} & & \mathbb{D}_{j_3} & \cdots & \mathbb{D}_{j_{q-1}} & & \mathbb{D}_{j_{q+1}} \\
& \mathbb{D}_{i_1} \arrow[ul, "s^{j_1}_{i_1}"] \arrow[ur, "t^{j_2}_{i_1}"'] & & \mathbb{D}_{i_2} \arrow[ul, "s^{j_2}_{i_2}"] \arrow[ur, "t^{j_3}_{i_2}"'] & & & & \mathbb{D}_{i_q} \arrow[ul, "s^{j_q}_{i_q}"] \arrow[ur, "t^{j_{q+1}}_{i_q}"'] &
\end{tikzcd}$$

Later on, the underlying colimit object associated with a q-globular sum will be referred to as its *universal object.* By definition, note that the object $\mathbb{D}_k$ defines a 0-globular sum of $(\mathbb{D}, s, t)$ equipped with the singleton table (k) for every $k \in \omega$ and will thus be referred to as the universal object of a 0-globular sum.

6.2.1.6. *Globular sums in spinal pre-extensions.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to \mathcal{C}$ be a spinal pre-extension and q be a non-negative integer. A *q-globular sum* in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ consists of

- a table of non-negative integers

$$\left(\begin{array}{ccccccc} j_1 & & j_2 & & \dots & & j_{q+1} \\ & i_1 & & i_1 & \dots & i_q & \end{array}\right)$$

  where i_k is less than or equal to j_k and j_{k+1} for every $1 \leq k \leq q+1$;
- a morphism $\beta_k : \mathbb{S}_{j_k} \to \mathbb{A}_k$ in the class Ω°_k for every integer $1 \leq k \leq q+1$;
- a colimit in $\mathcal{C}$ for the diagram of the form

$$\begin{tikzcd}
\mathbb{A}_1 & & \mathbb{A}_2 & & \mathbb{A}_3 & \cdots & \mathbb{A}_q & & \mathbb{A}_{q+1} \\
& \mathbb{D}_{i_1} \arrow[ul, "s^{j_1}_{i_1}"] \arrow[ur, "t^{j_2}_{i_1}"'] & & \mathbb{D}_{i_2} \arrow[ul, "s^{j_2}_{i_2}"] \arrow[ur, "t^{j_3}_{i_2}"'] & & & & \mathbb{D}_{i_q} \arrow[ul, "s^{j_q}_{i_q}"] \arrow[ur, "t^{j_{q+1}}_{i_q}"'] &
\end{tikzcd}$$

  where the following notations hold.

$$s^{j_1}_{i_1} := \beta_k \circ \Gamma^{j_k}_{i_k} \circ \delta^{i_k}_2 \qquad\qquad t^{j_1}_{i_1} := \beta_k \circ \Gamma^{j_k}_{i_k} \circ \delta^{i_k}_1$$

Later on, the underlying colimit object associated with a q-globular sum will be referred to as its *universal object.* By definition, note that the object $\mathbb{D}_k$ defines a 0-globular sum of $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ when equipped with the singleton table (k) for every $k \in \omega$ and will thus be referred to as the universal object of a 0-globular sum.

Remark 6.6. Let $\mathcal{C}$ be a category that has pushouts and initial objects. According to Remark 6.4, the globular sums of any globular pre-extension $F : \mathtt{Glog} \to \mathcal{C}$ corresponds to the globular sums of the underlying spinal pre-extension $\hat{F} : \mathtt{Spine} \to \mathcal{C}$.

6.2.1.7. *Globular extensions.* A globular pre-extension $(\mathbb{D}, s, t) : \mathtt{Glob} \to \mathcal{C}$ is called a *globular extension* if it admits all globular sums.

6.2.1.8. *Spinal extensions.* A spinal pre-extension $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega : \mathtt{Spine} \to \mathcal{C}$ is called a *spinal extension* if it admits the following globular sums:

- for every non-negative integers $i \leq j$, pair of arrows $\beta_\diamond : \mathbb{S}_i \to \mathbb{A}_\diamond$ and $\beta_\bullet : \mathbb{S}_i \to \mathbb{A}_\bullet$ in Ω_i and arrow $\beta : \mathbb{S}_j \to \mathbb{A}$ in Ω_j°, the diagram

$$\begin{tikzcd} \mathbb{A}_\diamond & & \mathbb{A} & & \mathbb{A}_\bullet \\ & \mathbb{D}_i \arrow[ul, "\beta_\diamond\circ\delta_2^i"] \arrow[ur, "t_i^j"'] & & \mathbb{D}_i \arrow[ul, "s_i^j"] \arrow[ur, "\beta_\bullet\circ\delta_1^i"'] & \end{tikzcd}$$

admits a colimit in $\mathcal{C}$ in which the following notations apply.

$$s_i^j := \beta \circ \Gamma_i^j \circ \delta_2^i \qquad t_i^j := \beta \circ \Gamma_i^j \circ \delta_1^i$$

The universal object associated with the above 3-globular sum will be denoted by $\mathbb{G}_i^j(\beta_\diamond,\beta,\beta_\bullet)$;

- for every non-negative integer i and pair of arrows $\beta : \mathbb{S}_i \to \mathbb{A}$ and $\beta' : \mathbb{S}_i \to \mathbb{A}'$ in Ω_i, the following diagram admits a colimit in $\mathcal{C}$.

$$\begin{tikzcd} \mathbb{A} & & \mathbb{A}' \\ & \mathbb{D}_i \arrow[ul, "\beta\circ\delta_2^i"] \arrow[ur, "\beta'\circ\delta_1^i"'] & \end{tikzcd}$$

The universal object associated with the previous 2-globular sum will be denoted by $\mathbb{E}_i(\beta,\beta')$.

Remark 6.7. Globular extensions are particular cases of spinal extensions when the underlying globular pre-extension is seen as a spinal pre-extension.

Remark 6.8. Let $\mathcal{C}$ be a category that has pushouts and initial objects. By Remark 6.6, any globular extension in $F : \mathtt{Glob} \to \mathcal{C}$ corresponds to a spinal extension $\hat{F} : \mathtt{Spine} \to \mathcal{C}$ in regard to the correspondence defined in Proposition 6.3.

Example 6.9 (Topological spaces)**.** Because the category **Top** is cocomplete, the topological globular pre-extension $(\mathbb{D}, s, t)$ defines in Example 6.5 gives rise to a globular extension. By Remark 6.8, its underlying spinal pre-extension define a spinal extension.

6.2.2. Parallelism in globular and spinal extensions.

6.2.2.1. *Parallel arrows in globular extensions.* Let $(\mathbb{D}, s, t) : \mathtt{Glob} \to \mathcal{C}$ be a globular extension and $\mathbb{B}$ be a universal object of a q-globular sum in $(\mathbb{D}, s, t)$ for some $q \geq 0$. For every $k \in \omega$, a pair of morphisms $f : \mathbb{D}_k \to \mathbb{B}$ and $g : \mathbb{D}_k \to \mathbb{B}$ will be said to be *k-parallel in* $(\mathbb{D}, s, t)$ if the integer k is equal to 0 or if the following two diagrams commute for $k \in \omega\backslash\{0\}$.

$$\begin{tikzcd} \mathbb{D}_{k-1} \arrow[r, "t_{k-1}"] \arrow[d, "t_{k-1}"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}_{k-1} \arrow[r, "s_{k-1}"] \arrow[d, "s_{k-1}"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd}$$

6.2.2.2. *Parallel arrows in spinal extensions.* Let $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega : \mathtt{Spine} \to \mathcal{C}$ be a globular extension and $\mathbb{B}$ be an object in $\mathcal{C}$ that may possibly be

1) the universal object of a q-globular sum in $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ for some $q \geq 0$;
2) the codomain of an arrow in Ω_n for some $n \in \omega$;

For every $k \in \omega$, a pair of morphisms $f : \mathbb{D}_k \to \mathbb{B}$ and $g : \mathbb{D}_k \to \mathbb{B}$ will be said to be *k-parallel in* $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ if the following diagram commutes.

(6.2)
$$\begin{tikzcd} \mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd}$$

The next proposition considers a globular extension $(\mathbb{D}, s, t) : \texttt{Glob} \to \mathcal{C}$ and denotes by $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to \mathcal{C}$ its underlying spinal extension mentioned in Remark 6.8. In particular, for every $k \in \omega$, the class Ω_k is the singleton containing the arrow $\gamma_{k+1} : \mathbb{S}_k \to \mathbb{D}_{k+1}$.

Proposition 6.10. *Suppose that $\mathcal{C}$ has pushouts and a initial object. Two arrows $f : \mathbb{D}_k \to \mathbb{B}$ and $g : \mathbb{D}_k \to \mathbb{B}$ are k-parallel in $(\mathbb{D}, s, t)$ if and only if they are k-parallel in the underlying spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$.*

Proof. Let $f : \mathbb{D}_k \to \mathbb{B}$ and $g : \mathbb{D}_k \to \mathbb{B}$ be two parallel arrows in the spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. Since f and g are obviously parallel in $(\mathbb{D}, s, t)$ when k is equal to 0, suppose that $k \in \omega \backslash \{0\}$. In this case, diagram (6.2) implies that the following square commutes.

$$\begin{tikzcd} \mathbb{D}_{k-1} \arrow[r, "\gamma_k \circ \delta_2^{k-1}"] \arrow[d, "\gamma_k \circ \delta_2^{k-1}"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}_{k-1} \arrow[r, "\gamma_k \circ \delta_1^{k-1}"] \arrow[d, "\gamma_k \circ \delta_1^{k-1}"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd}$$

This exactly means that f and g are parallel in the globular extension $(\mathbb{D}, s, t)$. Now, suppose that $f : \mathbb{D}_k \to \mathbb{B}$ and $g : \mathbb{D}_k \to \mathbb{B}$ are two parallel arrows in the globular extension $(\mathbb{D}, s, t)$. If the equality $k = 0$ holds, the following diagram commutes as $\mathbb{S}_{-1}$ is initial, which shows that f and g are parallel in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$.

$$\begin{tikzcd} \mathbb{S}_{-1} \arrow[r, "\gamma_{-1}"] \arrow[d, "\gamma_{-1}"'] & \mathbb{D}_0 \arrow[d, "f"] \\ \mathbb{D}_0 \arrow[r, "g"'] & \mathbb{B} \end{tikzcd}$$

Now, suppose that $k \in \omega \backslash \{0\}$. The fact that f and g are parallel in $(\mathbb{D}, s, t)$ means that the following diagrams commute.

(6.3)
$$\begin{tikzcd} \mathbb{D}_{k-1} \arrow[r, "\gamma_k \circ \delta_2^{k-1}"] \arrow[d, "\gamma_k \circ \delta_2^{k-1}"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}_{k-1} \arrow[r, "\gamma_k \circ \delta_1^{k-1}"] \arrow[d, "\gamma_k \circ \delta_1^{k-1}"'] & \mathbb{D}_k \arrow[d, "f"] \\ \mathbb{D}_k \arrow[r, "g"'] & \mathbb{B} \end{tikzcd}$$

These commutative diagrams then imply that the following universal problem has two solutions given by $f \circ \gamma_k : \mathbb{S}_{k-1} \to \mathbb{B}$ and $g \circ \gamma_k : \mathbb{S}_{k-1} \to \mathbb{B}$.

$$\begin{tikzcd} \mathbb{D}_{k-2} \arrow[r, "\gamma_{k-1}"] \arrow[d, "\gamma_{k-1}"'] & \mathbb{D}_{k-1} \arrow[d, "\delta_1^{k-1}"] \arrow[rd, bend left, "f \circ \gamma_k \circ \delta_1^{k-1}"] & \\ \mathbb{D}_{k-1} \arrow[r, "\delta_2^{k-1}"] \arrow[rr, bend right, "f \circ \gamma_k \circ \delta_2^{k-1}"'] & \mathbb{S}_{k-1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed] & \mathbb{B} \end{tikzcd}$$

By universality of $\mathbb{S}_{k-1}$, this implies that the following diagram must commute, which also proves that f and g are parallel in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "f"] \\
\mathbb{D}_k \arrow[r, "g"'] & \mathbb{B}
\end{tikzcd}$$

□

Remark 6.11. For any parallel pair $f : \mathbb{D}_k \to \mathbb{B}$ and $g : \mathbb{D}_k \to \mathbb{B}$ in a spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to \mathcal{C}$, the associated commutative square (6.2) gives rise to a canonical morphisms $\lceil f, g \rceil : \mathbb{S}_k \to \mathbb{B}$ making the following diagram commute.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\delta_1^k"] \arrow[dr, bend left, "g"] & \\
\mathbb{D}_k \arrow[r, "\delta_2^k"'] \arrow[ur, phantom, "\ulcorner"] \arrow[rr, bend right, "f"'] & \mathbb{S}_k \arrow[r, dashed, "\lceil f,g \rceil"] & \mathbb{B}
\end{tikzcd}$$

6.2.3. Globular and spinal coheroids.

6.2.3.1. *Globular coheroids.* A *globular coheroid* is a globular extension $(\mathbb{D}, s, t) : \texttt{Glob} \to \mathcal{C}$ equipped with a class $\mathcal{A}$ of pairs of parallel arrows in $(\mathbb{D}, s, t)$ such that for every $k \in \omega$ and k-parallel pair $f, g : \mathbb{D}_k \to \mathbb{B}$ in $\mathcal{A}$, there exists a morphism $h : \mathbb{D}_{k+1} \to \mathbb{B}$, called a *lift for* (f, g), making the following diagram commute.

$$\begin{tikzcd}
 & \mathbb{B} & \\
\mathbb{D}_k \arrow[ur, bend left, "f"] \arrow[r, "s_k"'] & \mathbb{D}_{k+1} \arrow[u, dashed, "h"] & \mathbb{D}_k \arrow[ul, bend right, "g"'] \arrow[l, "t_k"]
\end{tikzcd}$$

A pair (f, g) in $\mathcal{A}$ will be said to be *admissible* to distinguish it from any other choice of pair of parallel arrows in $(\mathbb{D}, s, t)$ that is not in $\mathcal{A}$. A globular coheroid such as above will be denoted as an arrow of the form $(\mathbb{D}, s, t) : \texttt{Glob} \to (\mathcal{C}, \mathcal{A})$.

Example 6.12 (Topological spaces)**.** The topological globular extension defined in Example 6.9 comes along with an obvious globular coheroid $(\mathbb{D}, s, t) : \texttt{Glob} \to (\mathbf{Top}, \mathcal{A})$ where $\mathcal{A}$ is the class of parallel arrows going from any topological disc to any globular sum for $(\mathbb{D}, s, t)$.

6.2.3.2. *Spinal coheroids.* A *spinal coheroid* is a spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to \mathcal{C}$ equipped with a class $\mathcal{A}$ of pairs of parallel arrows in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ such that for every $k \in \omega$ and k-parallel pair $f, g : \mathbb{D}_k \to \mathbb{B}$ in $\mathcal{A}$ for $k \in \omega$, there exists an arrow $\beta : \mathbb{S}_k \to \mathbb{A}$ in Ω_k° and a morphism $h : \mathbb{A} \to \mathbb{B}$, called a *lift for* (f, g), making the following diagram commute.

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, dashed, "h"] & \\
\mathbb{S}_k \arrow[u, "\beta"] \arrow[r, "\lceil f,g \rceil"'] & \mathbb{B}
\end{tikzcd}$$

A pair (f, g) in $\mathcal{A}$ will be said to be *admissible* to distinguish it from any other choice of pair of parallel arrows in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ that is not in $\mathcal{A}$. A spinal coheroid such as above will be denoted as an arrow of the form $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$.

Remark 6.13. It follows from Proposition 6.10 that if $\mathcal{C}$ has pushouts and an initial object, then any globular coheroid $(\mathbb{D}, s, t) : \mathtt{Glob} \to (\mathcal{C}, \mathcal{A})$ gives rise to a spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to (\mathcal{C}, \mathcal{A})$. This comes from the fact that the following left-hand commutative diagram implies the corresponding right-hand one.

$$\begin{tikzcd} & \mathbb{B} & \\ \mathbb{D}_k \arrow[ur, bend left, "f"] \arrow[r, "\gamma_{k+1}\circ\delta_2^k"'] & \mathbb{D}_{k+1} \arrow[u, dashed, "h"] & \mathbb{D}_k \arrow[ul, bend right, "g"'] \arrow[l, "\gamma_{k+1}\circ\delta_1^k"] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & \mathbb{B} & \\ \mathbb{D}_k \arrow[ur, bend left, "f"] \arrow[r, "\delta_2^k"'] & \mathbb{D}_{k+1} \arrow[u, dashed, "h\circ\gamma_{k+1}"] & \mathbb{D}_k \arrow[ul, bend right, "g"'] \arrow[l, "\delta_1^k"] \end{tikzcd}$$

By universality of the object $\mathbb{S}_k$, the rightmost commutative diagram implies that the following triangle commutes.

$$\begin{tikzcd} \mathbb{A} \arrow[dr, dashed, "h"] & \\ \mathbb{S}_k \arrow[u, "\gamma_{k+1}"] \arrow[r, "\lceil f,g \rceil"'] & \mathbb{B} \end{tikzcd}$$

Example 6.14 (Topological spaces)**.** According to Remark 6.13, because **Top** is cocomplete, the globular coheroid $(\mathbb{D}, s, t) : \mathtt{Glob} \to (\mathbf{Top}, \mathcal{A})$ of Example 6.12 gives rise to an obvious spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \mathtt{Spine} \to (\mathbf{Top}, \mathcal{A})$.

6.2.4. Grothendieck and Maltsiniotis coherators.

6.2.4.1. *Coherators.* A *coherator* is a globular coheroid $(\mathbb{D}, s, t) : \mathtt{Glob} \to (C, \mathcal{A})$ whose category C is the colimit of a sequence of functors between small categories as follows

$$\mathtt{Glob} \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \dots \longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow \dots \longrightarrow C \cong \mathrm{col}_{k \in \mathbf{O}(\omega)} C_k$$

where, for every $n \geq 0$, the composite $\mathtt{Glob} \to C_n$ is a globular extension equipped with a set of parallel arrows A_n so that

1) the functor $\mathtt{Glob} \to C_0$ is a free cocompletion of $\mathtt{Glob}$ by the globubar sums;
2) the composite $\mathtt{Glob} \to C_{n+1}$ is the universal globular extension of $\mathtt{Glob} \to C_n$ obtained by formally adding a lift for every pair of arrows in A_n.

A coherator as above is called

- *Grothendieck coherator* if $\mathcal{A}$ is the set of pairs of parallel arrows in C;
- *Maltsiniotis coherator* if $\mathcal{A}$ is the set of pairs of parallel arrows in C that are 'admissible' in the sense of [**35**, sec. 4.3, page 18] and, for every $n \geq 0$, the set A_n only consists of pair of parallel arrows that are 'admissible' in the sense of [**35**];

Proposition 6.15. *Let $F : C \to C'$ be a fully faithful functor where C' is a cocomplete category. For every coherator $(\mathbb{D}, s, t) : \mathtt{Glob} \to (C, \mathcal{A})$, the composite functor $F \circ (\mathbb{D}, s, t) : \mathtt{Glob} \to C'$ defines a globular coheroid when equipped with the image of the class $\mathcal{A}$ via F.*

Proof. First, the composite functor $F \circ (\mathbb{D}, s, t) : \mathtt{Glob} \to C'$ defines an obvious globular object by composition. Since C' is cocomplete, this functor actually defines a globular extension. Now, to show that it is a globular coheroid, take a pair (f, g) in $F(\mathcal{A})$. By definition, the arrows f and g go from an image $F(\mathbb{D}_k)$ to an image $F(\mathbb{B})$ where $\mathbb{D}_k$ and $\mathbb{B}$ are uniquely determined by injectivity of F. Because F is fully faithful, there exists a unique pair of arrows $f', g' : \mathbb{D}_k \to \mathbb{B}$ in C such that $F(f') = f$ and $F(g') = g$. By uniqueness, the pair (f', g') must belong to $\mathcal{A}$. The fully faithfulness of $F : C \to C'$ also implies that any commutative diagram in C' of the form given below on the left provides a non-dashed commutative diagram in C

as given on the right. By assumption, there exists a dashed lift $h : \mathbb{D}_{k+1} \to \mathbb{B}$ making the following right diagram commute.

$$\begin{tikzcd} & F(\mathbb{B}) & \\ F(\mathbb{D}_k) \arrow[ur, bend left, "f"] \arrow[r, "F(s_k)"'] & F(\mathbb{D}_{k+1}) & F(\mathbb{D}_k) \arrow[ul, bend right, "g"'] \arrow[l, "F(t_k)"] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & \mathbb{B} & \\ \mathbb{D}_k \arrow[ur, bend left, "f'"] \arrow[r, "s_k"'] & \mathbb{D}_{k+1} \arrow[u, dashed, "h"] & \mathbb{D}_k \arrow[ul, bend right, "g'"'] \arrow[l, "t_k"] \end{tikzcd}$$

Applying the functor F on the previous right diagram then provides a lift for the corresponding left diagram and proves the statement. □

The globular coheroid produced by Proposition 6.15 will later be denoted by $F(\mathbb{D}, s, t)$: $\texttt{Glob} \to (C', F(\mathcal{A}))$.

Remark 6.16. There exists two canonical choices for the functor $F : C \to C'$ of Proposition 6.15. The first one is the free cocompletion of C given by the Yoneda embedding $C \to \mathbf{Psh}(C)$ (see section 1.2.1.28) and the second one $C \to \mathbf{Mod}(C^{\mathrm{op}})$ is the lifting of the first one to the category of models for the sketch C when C is equipped with its globular sums as chosen colimits (see Proposition 1.18).

6.2.4.2. *Spinal coheroids for coherators.* Let $(\mathbb{D}, s, t) : \texttt{Glob} \to (C, \mathcal{A})$ be a coherator equipped with a fully faithful functor $F : C \to C'$ where C' is cocomplete. Since C' is complete, Remark 6.13 provides a spinal coheroid $F(\mathbb{D}, \mathbb{S}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (C', F(\mathcal{A}))$ stemming from the globular coheroid $F(\mathbb{D}, s, t) : \texttt{Glob} \to (C', F(\mathcal{A}))$ of Proposition 6.15. A *spinal coheroid for the coherator* $(\mathbb{D}, s, t)$ *along* F is given by any spinal coheroid made of

- the functor $F(\mathbb{D}, \mathbb{S}, \gamma, \delta) : \texttt{Spine} \to C'$;
- the class of pairs of parallel arrows $F(\mathcal{A})$;
- a class Ω' consisting of the image of Ω via F and, for every pair (f, g) in $F(\mathcal{A})$, an arrow $\beta \in \mathbf{llp}(\mathbf{rlp}(\{F(\gamma_m) \mid m \in \omega\}))$ factorising the canonical arrow $\lceil f, g \rceil$ into a composite $u \circ \beta$;

A structure as above defines a spinal coheroid by cocompleteness of C'.

Example 6.17 (Grothendieck coherators)**.** In the case where the globular extension $(\mathbb{D}, s, t)$: $\texttt{Glob} \to (C, \mathcal{A})$ is a Grothendieck coherator, a natural choice of class Ω' is the image class $F(\Omega)$.

Example 6.18 (Maltsiniotis coherators)**.** In the case where the globular extension $(\mathbb{D}, s, t)$: $\texttt{Glob} \to (C, \mathcal{A})$ is a Maltsiniotis coherator, a natural choice of class Ω' consists of the image of Ω via F and, for every pair (f, g) in $F(\mathcal{A})$, every arrow $\beta \in \mathbf{llp}(\mathbf{rlp}(\{F(\gamma_m) \mid m \in \omega\}))$ factorising the canonical arrow $\lceil f, g \rceil$ into a composite $u \circ \beta$ where $u \in \mathbf{rlp}(\{F(\gamma_m) \mid m \in \omega\})$. This spinal coheroid will however not define that required for the spinal structure and it will be necessary to extend the class $F(\mathcal{A})$ along the inductive process underlying its construction. It may be checked by the learned reader that the extension of the class $F(\mathcal{A})$ in C' will follow the same spirit as the definition of the arrows of $\mathcal{A}$ given in [**35**].

6.2.4.3. *Infinity-groupoids and infinity categories.* A *category of Grothendieck's* ∞*-groupoids* is the category of models for a Grothendieck coherator when equipped with its globular sums as chosen colimits. A *category of Maltsiniotis'* ∞*-categories* is the category of models for a Maltsiniotis coherator when equipped with its globular sums as chosen colimits.

Remark 6.19. By Remark 6.16, choosing F to be the embedding $C \to \mathbf{Mod}(C^{\mathrm{op}})$ in Example 6.17 and Example 6.18 provides the previous two (cocomplete) categories with (canonical) spinal coheroids.

6.2.5. Vertebrae for spinal coheroids.

6.2.5.1. *Reflexive spinal coheroids.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a spinal coheroid. Consider some element $k \in \omega$. Since the object $\mathbb{D}_k$ is a universal object of a 0-globular sum and the following diagram commutes in $\mathcal{C}$, the pair of arrows $(\mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k})$ is k-parallel.

$$\begin{tikzcd} \mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\mathrm{id}_{\mathbb{D}_k}"] \\ \mathbb{D}_k \arrow[r, "\mathrm{id}_{\mathbb{D}_k}"'] & \mathbb{D}_k \end{tikzcd}$$

The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will be said to be *reflexive* if the parallel pair $(\mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k})$ is admissible for every $k \in \omega$ so that there exists an arrow $\beta : \mathbb{S}_k \to \mathbb{A}$ in Ω_k and a morphism $\alpha : \mathbb{A} \to \mathbb{D}_k$ making the following diagram commute.

$$\begin{tikzcd} \mathbb{A} \arrow[dr, dashed, "\alpha"] & \\ \mathbb{S}_k \arrow[u, "\beta"] \arrow[r, "\ulcorner \mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k} \urcorner"'] & \mathbb{D}_k \end{tikzcd}$$

Note that the arrow β belongs to Ω_k and not necessarily to its augmentation Ω_k°.

Example 6.20 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \mathtt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is obviously reflexive by definition of $\mathcal{A}$.

Example 6.21 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is reflexive. The pair $(\mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k})$ is indeed admissible and the arrow β stands for the arrow γ_{k+1} contained in Ω'_k (see Example 6.17).

Example 6.22 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories is reflexive. The pair $(\mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k})$ is indeed admissible and the arrow β stands for the arrow γ_{k+1} contained in Ω'_k (see Example 6.18).

6.2.5.2. *Magmoidal spinal coheroid.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a spinal coheroid. Consider some element $k \in \omega$. Recall that for every pair of arrows $\beta : \mathbb{S}_k \to \mathbb{A}$ and $\beta' : \mathbb{S}_k \to \mathbb{A}'$ in the class Ω_k, the following pushout exists.

$$\begin{tikzcd} \mathbb{D}_k \arrow[r, "\beta \circ \delta_1^k"] \arrow[d, "\beta' \circ \delta_2^k"'] & \mathbb{A} \arrow[d, "\varepsilon_1"] \\ \mathbb{A}' \arrow[r, "\varepsilon_2"'] \arrow[ur, phantom, "\ulcorner" very near end] & \mathbb{E}_k(\beta, \beta') \end{tikzcd}$$

Pre-composing the preceding diagram with $\gamma_k : \mathbb{S}_{k-1} \to \mathbb{D}_k$ and using the relation $\delta_1^k \circ \gamma_k = \delta_2^k \circ \gamma_k$ provides a commutative diagram as follows.

$$\begin{tikzcd} \mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\varepsilon_1 \circ \beta \circ \delta_2^k"] \\ \mathbb{D}_k \arrow[r, "\varepsilon_2 \circ \beta' \circ \delta_1^k"'] & \mathbb{E}_k(\beta, \beta') \end{tikzcd}$$

The preceding commutative diagram exposes a pair of parallel arrows. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will be said to be *magmoidal* if the parallel pair

$$(\varepsilon_2 \circ \beta' \circ \delta_2^k, \varepsilon_1 \circ \beta \circ \delta_1^k)$$

is admissible for every $k \in \omega$ and pair $\beta, \beta' \in \Omega_k$ so that there exists an arrow $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ in Ω_k and a morphism $\eta : \mathbb{A}_\bullet \to \mathbb{B}$ making the following diagram commute.

$$\begin{tikzcd} \mathbb{A}_\bullet \arrow[drr, dashed, "\eta"] & & \\ \mathbb{S}_k \arrow[u, "\beta_\bullet"] \arrow[rr, "\ulcorner \varepsilon_2 \circ \beta' \circ \delta_2^k, \varepsilon_1 \circ \beta \circ \delta_1^k \urcorner"'] & & \mathbb{B} \end{tikzcd}$$

Note that the arrow β belongs to Ω_k and not to its augmentation Ω_k°.

Example 6.23 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \texttt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is magmoidal by definition of $\mathcal{A}$.

Example 6.24 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is magmoidal. The arrows β and β' must be equal to γ_{k+1} so that the pair $(\varepsilon_2 \circ \beta' \circ \delta_2^k, \varepsilon_1 \circ \beta \circ \delta_1^k)$ is admissible since $\mathbb{E}_k(\beta, \beta')$ is a globular sum in the coherator. The arrow $\beta_\bullet$ is then provided by γ_{k+1}, which is the only element of Ω'_k.

Example 6.25 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories becomes magmoidal when the class of parallel arrows is augmented by all the pairs of the form $(\varepsilon_2 \circ \beta' \circ \delta_2^k, \varepsilon_1 \circ \beta \circ \delta_1^k)$. Since the ambient category $\mathbf{Mod}(C^{\mathrm{op}})$ associated with the spinal coheroid is cocomplete (see Remark 6.19) and all the sequential functors in $\mathbf{Mod}(C^{\mathrm{op}})$ are convergent with respect to the representable models (see Example 5.8), Corollary 5.86 (small object argument) may be used to generate the arrow $\beta_\bullet$, which, in this case, satisfies all the properties required by Example 6.18 to belong to Ω'_k.

6.2.5.3. *Vertebrae for spinal coheroids.* Any spinal coheroid of the form

$$(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$$

may be associated with the canonical node of vertebrae $\|\gamma_k, \gamma_k\| \cdot \Omega_k$ for every $k \in \omega$ where the prevertebra $\|\gamma_k, \gamma_k\|$ is given by the following pushout square.

$$\begin{tikzcd} \mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\delta_1^k"] \\ \mathbb{D}_k \arrow[r, "\delta_2^k"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}_k \end{tikzcd}$$

Note that the commutative triangle

$$\begin{tikzcd} \mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[dl, "\mathrm{id}_{\mathbb{D}_k}"] \\ \mathbb{D}_k & \end{tikzcd}$$

provides $\|\gamma_k, \gamma_k\|$ with a reflexive prevertebra structure.

Proposition 6.26. *If the spinal coheroid* $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ *is reflexive, then the node of vertebrae* $\|\gamma_k, \gamma_k\| \cdot \Omega_k$ *is reflexive for every* $k \in \omega$*.*

Proof. We need to prove that the node of vertebrae $\|\gamma_k, \gamma_k\| \cdot \Omega_k$ contains a reflexive vertebra. Denote by $u_k : \mathbb{S}_k \to \mathbb{D}_k$ the canonical arrow making the following diagram commute.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\delta_1^k"] \arrow[dr, bend left, "\mathrm{id}_{\mathbb{D}_k}"] & \\
\mathbb{D}_k \arrow[r, "\delta_2^k"'] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{D}_k}"'] & \mathbb{S}_k \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "u_k"] & \mathbb{D}_k
\end{tikzcd}$$

The arrow u_k may be identified with the canonical arrow $\lceil \mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k} \rceil$. Since the pair of identities $(\mathrm{id}_{\mathbb{D}_k}, \mathrm{id}_{\mathbb{D}_k})$ belongs to $\mathcal{A}$ and the spinal coheroid is reflexive, the arrow $\alpha : \mathbb{S}_k \to \mathbb{D}_k$ may be factorised as follows for some $\beta : \mathbb{S}_k \to \mathbb{A}$ in Ω_k.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[rr, "u_k"] \arrow[dr, "\beta"'] & & \mathbb{D}_k \\
& \mathbb{A} \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

In other words, the vertebra $\|\gamma_k, \gamma_k\| \cdot \beta$ is reflexive when equipped with the arrow $\alpha : \mathbb{A} \to \mathbb{D}_k$. This shows the statement since it belongs to the node of vertebrae $\|\gamma_k, \gamma_k\| \cdot \Omega_k$. $\square$

Note that for every $k \in \omega$, the node of vertebrae $\|\gamma_k, \gamma_k\| \cdot \Omega_k$ communicates with itself since its coseed is equal to its seed.

Proposition 6.27. *If the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ is magmoidal, then the node of vertebrae $\|\gamma_k, \gamma_k\| \cdot \Omega_k$ frames two copies of itself for every $k \in \omega$.*

Proof. We need to show that every pair of vertebrae $\|\gamma_k, \gamma_k\| \cdot \beta$ and $\|\gamma_k, \gamma_k\| \cdot \beta'$ in $\|\gamma_k, \gamma_k\| \cdot \Omega_k$ is framed by a third one in $\|\gamma_k, \gamma_k\| \cdot \Omega_k$. First, notice that for every pair of vertebrae $\|\gamma_k, \gamma_k\| \cdot \beta$ and $\|\gamma_k, \gamma_k\| \cdot \beta'$ in $\|\gamma_k, \gamma_k\| \cdot \Omega_k$, the following pushout exists.

(6.4)
$$\begin{tikzcd}
\mathbb{D}_k \arrow[r, "\beta \circ \delta_1^k"] \arrow[d, "\beta' \circ \delta_2^k"'] & \mathbb{A} \arrow[d, "\varepsilon_1"] \\
\mathbb{A}' \arrow[r, "\varepsilon_2"'] & \mathbb{E}_k(\beta, \beta') \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Pre-composing the previous diagram with $\gamma_k : \mathbb{S}_{k-1} \to \mathbb{D}_k$ and using the diagrammatic relations defining the prevertebra $\|\gamma_k, \gamma_k\|$ provides a commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\varepsilon_2 \circ \beta' \circ \delta_1^k"] \\
\mathbb{D}_k \arrow[r, "\varepsilon_1 \circ \beta \circ \delta_2^k"'] & \mathbb{E}_k(\beta, \beta')
\end{tikzcd}$$

Since the pair $(\varepsilon_1 \circ \beta \circ \delta_2^k, \varepsilon_2 \circ \beta' \circ \delta_1^k)$ is in $\mathcal{A}$ and the spinal coheroid is magmoidal, there exists an arrow $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ in Ω_k factorising the canonical arrow $\lceil \varepsilon_1 \circ \beta \circ \delta_2^k, \varepsilon_2 \circ \beta' \circ \delta_1^k \rceil : \mathbb{S}_k \to \mathbb{E}_k(\beta, \beta')$ as follows.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[rr, "\lceil \varepsilon_1 \circ \beta \circ \delta_2^k, \varepsilon_2 \circ \beta' \circ \delta_1^k \rceil"] \arrow[dr, "\beta_\bullet"'] & & \mathbb{E}_k(\beta, \beta') \\
& \mathbb{A}_\bullet \arrow[ur, "\eta"'] &
\end{tikzcd}$$

In particular, this factorisation provides a commutative diagram as follows.

(6.5)
$$\begin{tikzcd}
\mathbb{S}_k & & \mathbb{D}_k \arrow[ll, "\delta_1^k"'] \arrow[dd, "\varepsilon_2\circ\beta'\circ\delta_1^k"] \\
& & \\
\mathbb{D}_k \arrow[uu, "\delta_2^k"] \arrow[rr, "\varepsilon_1\circ\beta\circ\delta_2^k"'] & & \mathbb{E}_k(\beta,\beta') \arrow[from=1-1, "\eta\circ\beta_\bullet"]
\end{tikzcd}$$

Finally, diagram (6.4) and diagram (6.5) show that the pair of vertebrae $\|\gamma_k,\gamma_k\|\cdot\beta$ and $\|\gamma_k,\gamma_k\|\cdot\beta'$ is framed by $\|\gamma_k,\gamma_k\|\cdot\beta_\bullet$ where the underlying cooperadic transition is given by $\eta:\mathbb{A}_\bullet\to\mathbb{E}_k(\beta,\beta')$. This induces a structure of framing of $\|\gamma_k,\gamma_k\|\cdot\Omega_k$ by itself. $\square$

6.2.5.4. *Underlying vertebral category of spinal coheroids.* Consider a reflexive magmoidal spinal coheroid $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega:\mathtt{Spine}\to(\mathcal{C},\mathcal{A})$ whose ω-spinal object is given by a faithful and injective-on-objects functor $\mathtt{Spine}\to\mathcal{C}$. This means that the objects and arrows of the spinal object are completely determined by their indexing. Denote by ν_k the node of vertebrae $\|\gamma_k,\gamma_k\|\cdot\Omega_k$ for every $k\in\omega$. The goal of this section is to define a vertebral category structure $(\mathcal{C},A',A,E)$ for the spinal coheroid $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ whose prolinear submodule is of the type presented in Remark 4.27, namely the two magmoids A and A' are equal, the prolinear map is given by the identity and the prolinear submodule consists of a submagmoid $(A,\circledcirc)\subseteq(\mathbf{Ally}(\mathcal{C}),\circledcirc)$ and a right A-submodule $(E,\eta,\odot,\circledcirc)\subseteq(\mathbf{Enov}(\mathcal{C}),\eta,\odot,\circledcirc)$. First, define the submagmoid A of $\mathbf{Ally}(\mathcal{C})$ whose object-class is given by the set of nodes of vertebrae

$$\mathrm{Obj}(A):=\{\nu_k\mid k\in\omega\}$$

and whose hom-classes are defined as follows.

$$A(\nu_k,\nu_m)=\left\{\begin{array}{ll}\emptyset & \text{if } k\neq m\\ \{\mathrm{id}_{\nu_k}\} & \text{if } k=m\end{array}\right.$$

It is not hard to check that it has a structure of magmoid for the composition $\circledcirc$ of $\mathbf{Ally}(\mathcal{C})$. Then, consider the A-submodule $(E,\eta,\odot,\circledcirc)$ of the right $\mathbf{Ally}(\mathcal{C})$-module $(\mathrm{Enov},\eta,\odot,\circledcirc)$ whose left and right object-classes are given by the sets

$$\mathrm{Obj}_L(E):=\{\gamma_k\mid k\in\omega\}\quad\text{and}\quad\mathrm{Obj}_R(E):=\{\nu_k\mid k\in\omega\}$$

and whose hom-classes are defined as follows.

$$E(\gamma_k,\nu_m)=\left\{\begin{array}{ll}\emptyset & \text{if } k\neq m\\ \{\nu_m\} & \text{if } k=m\end{array}\right.$$

This defines a right A-module E whose right object-class is equal to that of A. To define a vertebral algebra structure on (E,η'), take the source and target hinges to both be the metafunction η' that maps a nodes of vertebrae to its coseed. It follows that the graphs Σ_0E, Σ_1E and $\Sigma_\star E$ are equal to the same graph, say ΣE, defined as follows.

$$\Sigma E(\gamma_k,\gamma_m)=\left\{\begin{array}{ll}\emptyset & \text{if } k\neq m\\ \{\nu_k\} & \text{if } k=m\end{array}\right.$$

The graph ΣE has a structure of magmoid whose compositions are defined as follows for every $k\in\omega$.

$$\begin{array}{ccc}\Sigma E(\gamma_k,\gamma_k)\times\Sigma E(\gamma_k,\gamma_k) & \to & \Sigma E(\gamma_k,\gamma_k)\\ (\nu_k,\nu_k) & \mapsto & \nu_k\end{array}$$

By Proposition 6.27, this composition exactly gives a (semi-direct) vertebral algebra structure to (E,η'). Finally, it follows from Proposition 6.26 that for every $\gamma_k\in\mathrm{Obj}_L(E)$, the node of vertebrae ν_k is reflexive. Since the set $E(\gamma_k, \nu_k)$ is non-empty, the above discussion shows that the triple $(\mathcal{C}, A, E)$ defines a vertebral category.

6.3. Spines and their functorial framings

6.3.1. Kappa and tau constructions.

6.3.1.1. *Kappa constructions.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a spinal coheroid. By definition of the spinal extension associated with $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$, the following two double pushouts[1] must exist for every pair $m, k \in \omega$ where $m \geq k+1$ and pair of arrows $\beta_\diamond : \mathbb{S}_k \to \mathbb{A}_\diamond$ and $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ in Ω_k.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1}\circ\Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond\circ\delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{D}_{m+1} \arrow[drr, "\iota_{k,m}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,m}^\bullet"'] & & & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m}\circ\Gamma_k^{m-1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond\circ\delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m-1}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{D}_{m} \arrow[drr, "\iota_{k,m-1}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,m-1}^\bullet"'] & & & \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_{m}, \beta_\bullet)
\end{tikzcd}$$

Because the equation

$$\gamma_{m+1} \circ \Gamma_k^m = (\gamma_{m+1} \circ \delta_i^m) \circ \gamma_m \circ \Gamma_k^{m-1}$$

holds for every $i \in \{1, 2\}$, there exists a canonical arrow

$$\kappa_{i,k}^m : \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \to \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)$$

for every $i \in \{1, 2\}$ making the following three diagrams commute.

$$\begin{tikzcd}
\mathbb{A}_\diamond \arrow[r, "\iota_{k,m}^\diamond"] \arrow[dr, "\iota_{k,m-1}^\diamond"'] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \\
& \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "\kappa_{i,k}^m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}_\bullet \arrow[r, "\iota_{k,m}^\bullet"] \arrow[dr, "\iota_{k,m-1}^\bullet"'] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \\
& \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "\kappa_{i,k}^m"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[r, "\iota_{k,m}"] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \\
\mathbb{D}_m \arrow[u, "\gamma_{m+1}\circ\delta_i^m"] \arrow[r, "\iota_{k,m-1}"'] & \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "\kappa_{i,k}^m"']
\end{tikzcd}$$

Also, because the equation

$$\gamma_{m+2} \circ \delta_1^{m+1} \circ \gamma_{m+1} \circ \delta_i^m = \gamma_{m+2} \circ \delta_2^{m+1} \circ \gamma_{m+1} \circ \delta_i^m$$

[1]These pushouts are the pushouts required by the definition of section 3.3.2.4, which tells more about the geometric meaning of these.

holds in $\mathcal{C}$, the universality of the object $\mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet)$ provides the following equation for every $i \in \{1,2\}$.

$$\kappa_{1,k}^{m+1} \circ \kappa_{i,k}^{m} = \kappa_{2,k}^{m+1} \circ \kappa_{i,k}^{m} \tag{6.6}$$

6.3.1.2. *Tau constructions.* Let $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega : \mathtt{Spine} \to (\mathcal{C},\mathcal{A})$ be a spinal coheroid. Consider an integer $m \in \omega$ and an arrow $\beta : \mathbb{S}_{m+1} \to \mathbb{A}$ in Ω_{m+1}. By definition of the spinal extension associated with $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$, the following two pushouts must exist for every $k \in \omega$ where $m \geq k$ and pair of arrows $\beta_\diamond : \mathbb{S}_k \to \mathbb{A}_\diamond$ and $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ in Ω_k.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta\circ\Gamma_k^{m+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_1^k"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon_{k,m+1}^{\diamond}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{A} \arrow[dr, "\epsilon_{k,m+1}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon_{k,m+1}^{\bullet}"'] & & \mathbb{G}_k^{m+1}(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1}\circ\Gamma_k^{m}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_1^k"] & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m}^{\diamond}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{D}_{m+1} \arrow[dr, "\iota_{k,m}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota_{k,m}^{\bullet}"'] & & \mathbb{G}_k^{m}(\beta_\diamond,\gamma_{m+1},\beta_\bullet)
\end{tikzcd}$$

Because the equation

$$\beta \circ \Gamma_k^{m+1} = (\beta \circ \delta_i^{m+1}) \circ \gamma_{m+1} \circ \Gamma_k^m$$

holds for every $i \in \{1,2\}$, there exist a canonical arrow

$$\tau_{i,k}^m(\beta) : \mathbb{G}_k^m(\beta_\diamond,\gamma_{m+1},\beta_\bullet) \to \mathbb{G}_k^{m+1}(\beta_\diamond,\beta,\beta_\bullet)$$

for every $i \in \{1,2\}$ making the following three diagrams commute.

$$\begin{tikzcd}
\mathbb{A}_\diamond \arrow[r, "\epsilon_{k,m+1}^{\diamond}"] \arrow[dr, "\iota_{k,m}^{\diamond}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond,\beta,\beta_\bullet) \\
& \mathbb{G}_k^{m}(\beta_\diamond,\gamma_{m+1},\beta_\bullet) \arrow[u, dashed, "\tau_{i,k}^m(\beta)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}_\bullet \arrow[r, "\epsilon_{k,m+1}^{\bullet}"] \arrow[dr, "\iota_{k,m}^{\bullet}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond,\beta,\beta_\bullet) \\
& \mathbb{G}_k^{m}(\beta_\diamond,\gamma_{m+1},\beta_\bullet) \arrow[u, dashed, "\tau_{i,k}^m(\beta)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[r, "\epsilon_{k,m+1}"] & \mathbb{G}_k^{m+1}(\beta_\diamond,\beta,\beta_\bullet) \\
\mathbb{D}_m \arrow[u, "\beta\circ\delta_i^{m+1}"] \arrow[r, "\iota_{k,m}"'] & \mathbb{G}_k^{m}(\beta_\diamond,\gamma_{m+1},\beta_\bullet) \arrow[u, dashed, "\tau_{i,k}^m(\beta)"']
\end{tikzcd}$$

Also, because the equation

$$\beta \circ \delta_1^{m+1} \circ \gamma_{m+1} \circ \delta_i^m = \beta \circ \delta_2^{m+1} \circ \gamma_{m+1} \circ \delta_i^m$$

holds in $\mathcal{C}$, the universality of the object $\mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet)$ provides the following equation for every $i \in \{1,2\}$ when $k < m$.

$$\tau_{1,k}^m(\beta) \circ \kappa_{i,k}^m = \tau_{2,k}^m(\beta) \circ \kappa_{i,k}^m \tag{6.7}$$

6.3.2. Normal and transitive spinal coheroids. The goal of this section is to use the kappa and tau constructions – which may be seen as source and target maps between different types of gluing of discs – to define dimensionally coherent composition of cells.

On the one hand, normal spinal coheroids will permit the composition of three cells in the form of a whiskering (or cylinder of discs) where the middle cell (or base of the cylinder) should be regarded as non-reversible and where the whiskers (are sides of the cylinder) should be regarded as reversible. The term *normal* refers to the fact that the composition of a non-reversible cell along two others is what one would like to think as an elementary operation (in contrast to the next type of operation).

On the other hand, transitive spinal coheroids will permit the composition of three cells in the form of a whiskering (or cylinder of discs) where all the cells are considered reversible. The term *transitive* now refers to the fact that the composition of three reversible cells is reminiscent of a transitive property if one sees ∞-groupoids as generalisations of the notion of equivalence relation.

6.3.2.1. *Normal spinal coheroids.* Let $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega : \texttt{Spine} \to (\mathcal{C},\mathcal{A})$ be a spinal coheroid. Let us fix some $k \in \omega$ and consider the following pushout in the spinal extension $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ for some pair of arrows $\beta_\diamond : \mathbb{S}_k \to \mathbb{A}_\diamond$ and $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ in Ω_k.

(6.8)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond\circ\delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,k}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{D}_{k+1} \arrow[drr, "\iota_{k,k}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota^\bullet_{k,k}"'] & & & \mathbb{G}^k_k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)
\end{tikzcd}$$

Pre-composing the top-right and bottom-left commutative part of diagram (6.8) with the arrow $\gamma_k : \mathbb{S}_{k-1} \to \mathbb{D}_k$ and using the diagrammatic relations defining the prevertebra $\|\gamma_k,\gamma_k\|$ leads to the following commutative square.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\iota^\bullet_{k,k}\circ\beta_\bullet\circ\delta_1^k"] \\
\mathbb{D}_k \arrow[r, "\iota^\diamond_{k,k}\circ\beta_\diamond\circ\delta_2^k"'] & \mathbb{G}^k_k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)
\end{tikzcd}$$

This gives a new k-parallel pair in $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$. The spinal coheroid $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ will then be said to be $(k,0)$-*normal* if the parallel pair

$$(\iota^\diamond_{k,k}\circ\beta_\diamond\circ\delta_2^k, \iota^\bullet_{k,k}\circ\beta_\bullet\circ\delta_1^k)$$

is admissible for every pair $\beta_\diamond,\beta_\bullet \in \Omega_k$ so that there exists a morphism $\pi_k^0 : \mathbb{D}_{k+1} \to \mathbb{B}$ making the following diagram commute.

(6.9)
$$\begin{tikzcd}
\mathbb{D}_{k+1} \arrow[dr, dashed, "\pi^0_k"] & \\
\mathbb{S}_k \arrow[u, "\gamma_{k+1}"] \arrow[r, "\ulcorner\iota^\diamond_{k,k}\circ\beta_\diamond\circ\delta_2^k,\iota^\bullet_{k,k}\circ\beta_\bullet\circ\delta_1^k\urcorner"'] & \mathbb{G}^k_k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)
\end{tikzcd}$$

Note that, by definition of the canonical morphism $\lceil \iota^{\diamond}_{k,k} \circ \beta_{\diamond} \circ \delta^k_2, \iota^{\bullet}_{k,k} \circ \beta_{\bullet} \circ \delta^k_1 \rceil$, the previous factorisation may also be written as a commutative diagram of the following form.

(6.10)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_{\diamond}\circ\delta^k_2"] & \mathbb{A}_{\diamond} \arrow[dd, "\iota^{\diamond}_{k,k}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_{\bullet}\circ\delta^k_1"'] & \mathbb{D}_{k+1} \arrow[dr, "\pi^0_k"] & \\
\mathbb{A}_{\bullet} \arrow[rr, "\iota^{\bullet}_{k,k}"'] & & \mathbb{G}^k_k(\beta_{\diamond}, \gamma_{k+1}, \beta_{\bullet})
\end{tikzcd}$$

Post-composing diagram (6.9) with the kappa construction

$$\kappa^{k+1}_{i,k} : \mathbb{G}^k_k(\beta_{\diamond}, \gamma_{k+1}, \beta_{\bullet}) \to \mathbb{G}^{k+1}_k(\beta_{\diamond}, \gamma_{k+2}, \beta_{\bullet})$$

for some $i \in \{1, 2\}$ then provides the following commutative diagram.

(6.11)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_{\diamond}\circ\delta^k_2"] & \mathbb{A}_{\diamond} \arrow[dd, "\iota^{\diamond}_{k,k+1}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_{\bullet}\circ\delta^k_1"'] & \mathbb{D}_{k+1} \arrow[dr, "\kappa^{k+1}_{i,k}\circ\pi^0_k"] & \\
\mathbb{A}_{\bullet} \arrow[rr, "\iota^{\bullet}_{k,k+1}"'] & & \mathbb{G}^{k+1}_k(\beta_{\diamond}, \gamma_{k+2}, \beta_{\bullet})
\end{tikzcd}$$

The earlier diagram provides two different solutions for the same universal problem over $\mathbb{S}_k$ depending on the index $i \in \{1, 2\}$. It follows from the universality of $\mathbb{S}_k$ that the two solutions are equal, which may be expressed in terms of the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\kappa^{k+1}_{1,k}\circ\pi^0_k"] \\
\mathbb{D}_{k+1} \arrow[r, "\kappa^{k+1}_{2,k}\circ\pi^0_k"'] & \mathbb{G}^{k+1}_k(\beta_{\diamond}, \gamma_{k+2}, \beta_{\bullet})
\end{tikzcd}$$

Note that the preceding diagram provides a $(k+1)$-parallel pair of arrows in the spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. The spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will be said to be $(k, 1)$-*normal* if it is $(k, 0)$-normal and the parallel pair of arrows

$$(\kappa^{k+1}_{2,k} \circ \pi^0_k, \kappa^{k+1}_{1,k} \circ \pi^0_k)$$

is admissible so that there exists a morphism $\pi^1_k : \mathbb{D}_{k+2} \to \mathbb{G}^{k+1}_k(\beta_{\diamond}, \gamma_{k+2}, \beta_{\bullet})$ making the following diagram commute.

(6.12)
$$\begin{tikzcd}
\mathbb{D}_{k+2} \arrow[dr, dashed, "\pi^1_k"] & \\
\mathbb{S}_{k+1} \arrow[u, "\gamma_{k+2}"] \arrow[r, "\lceil \kappa^{k+1}_{2,k}\circ\pi^0_k, \kappa^{k+1}_{1,k}\circ\pi^0_k \rceil"'] & \mathbb{G}^{k+1}_k(\beta_{\diamond}, \gamma_{k+2}, \beta_{\bullet})
\end{tikzcd}$$

It follows from diagram (6.11) and the definition of the canonical arrow

$$\lceil \kappa^{k+1}_{2,k} \circ \pi^0_k, \kappa^{k+1}_{1,k} \circ \pi^0_k \rceil : \mathbb{S}_k \to \mathbb{G}^{k+1}_k(\beta_{\diamond}, \gamma_{k+2}, \beta_{\bullet})$$

that the previous factorisation gives rise to the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_2^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,k+1}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & \mathbb{D}_{k+1} \arrow[drr, "\pi_k^1"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota^\bullet_{k,k+1}"'] & & & \mathbb{G}_k^{k+1}(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)
\end{tikzcd}$$

with $\mathbb{S}_k \xrightarrow{\gamma_{k+2} \circ \Gamma_k^{k+1}} \mathbb{D}_{k+1}$.

$\boxed{\textit{Ind.}}$ ▷ The sequel extends the notion of normality by induction. To do so, suppose that for some $m \geq k+1$, the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ is $(k, m-k)$-normal. By construction, this means that we are given the next two commutative diagrams.

(6.13)
$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[dr, dashed, "\pi_k^{m-k}"] & \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[r, "{\lceil \kappa_{2,k}^m \circ \pi_k^{m-k-1}, \kappa_{1,k}^m \circ \pi_k^{m-k-1} \rceil}"'] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

(6.14)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1} \circ \Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_2^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,m}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & \mathbb{D}_{m+1} \arrow[drr, "\pi_k^{m-k}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota^\bullet_{k,m}"'] & & & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

Post-composing diagram (6.13) with the kappa construction

$$\kappa_{i,k}^{m+1} : \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \to \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)$$

gives the following factorisation.

(6.15)
$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[dr, dashed, "\kappa_{i,k}^{m+1} \circ \pi_k^{m-k}"] & \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[r, "{\kappa_{i,k}^{m+1} \circ \lceil \kappa_{2,k}^m \circ \pi_k^{m-k-1}, \kappa_{1,k}^m \circ \pi_k^{m-k-1} \rceil}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)
\end{tikzcd}$$

Note that the equation $\kappa_{2,k}^{m+1} \circ \kappa_{i,k}^m = \kappa_{1,k}^{m+1} \circ \kappa_{i,k}^m$ obtained in (6.6) implies that the following equalities hold in $\mathcal{C}$, where the symbol π_* stand for π_k^{m-k-1}.

$$\begin{aligned}
\kappa_{2,k}^{m+1} \circ \lceil \kappa_{2,k}^m \circ \pi_*, \kappa_{1,k}^m \circ \pi_* \rceil &= \lceil \kappa_{2,k}^{m+1} \circ \kappa_{2,k}^m \circ \pi_*, \kappa_{2,k}^{m+1} \circ \kappa_{1,k}^m \circ \pi_* \rceil \\
&= \lceil \kappa_{1,k}^{m+1} \circ \kappa_{2,k}^m \circ \pi_*, \kappa_{1,k}^{m+1} \circ \kappa_{1,k}^m \circ \pi_* \rceil \\
&= \kappa_{1,k}^{m+1} \circ \lceil \kappa_{2,k}^m \circ \pi_*, \kappa_{1,k}^m \circ \pi_* \rceil
\end{aligned}$$

This means that the two factorisations involved in diagram (6.15) are the factorisations of a same arrow. This therefore implies that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{D}_{m+1} \arrow[d, "\kappa^{m+1}_{1,k}\circ\pi^{m-k}_k"] \\
\mathbb{D}_{m+1} \arrow[r, "\kappa^{m+1}_{2,k}\circ\pi^{m-k}_k"'] & \mathbb{G}^{m+1}_k(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)
\end{tikzcd}$$

The preceding diagram exposes a parallel pair of arrows in $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega$ is then said to be $(k, m+1-k)$-*normal* if the parallel pair of arrows

$$(\kappa^{m+1}_{2,k} \circ \pi^{m-k}_k, \kappa^{m+1}_{1,k} \circ \pi^{m-k}_k)$$

is admissible so that there exists a morphism $\pi^{m-k+1}_k : \mathbb{D}_{m+2} \to \mathbb{G}^{m+1}_k(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)$ making the following diagram commute.

(6.16)
$$\begin{tikzcd}
\mathbb{D}_{m+2} \arrow[dr, dashed, "\pi^{m-k+1}_k"] & \\
\mathbb{S}_{m+1} \arrow[u, "\gamma_{m+2}"] \arrow[r, "\lceil \kappa^{m+1}_{2,k}\circ\pi^{m-k}_k, \kappa^{m+1}_{1,k}\circ\pi^{m-k}_k \rceil"'] & \mathbb{G}^{m+1}_k(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)
\end{tikzcd}$$

If we now post-compose diagram (6.14) with the kappa construction

$$\kappa^{m+1}_{i,k} : \mathbb{G}^m_k(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \to \mathbb{G}^{m+1}_k(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)$$

we obtain the following commutative diagram.

(6.17)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1}\circ\Gamma^m_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond\circ\delta^k_2"] & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,m+1}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet\circ\delta^k_1"'] & \mathbb{D}_{m+1} \arrow[dr, "\kappa^{m+1}_{i,k}\circ\pi^{m-k}_k"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota^\bullet_{k,m+1}"'] & & \mathbb{G}^{m+1}_k(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)
\end{tikzcd}$$

It follows from factorisation (6.16) that the following equations hold for every $i \in \{1, 2\}$.

$$\begin{aligned}
\kappa^{m+1}_{i,k} \circ \pi^{m-k}_k \circ \gamma_{m+1} \circ \Gamma^m_k &= \lceil \kappa^{m+1}_{2,k} \circ \pi^{m-k}_k, \kappa^{m+1}_{1,k} \circ \pi^{m-k}_k \rceil \circ \delta^{m+1}_i \circ \gamma_{m+1} \circ \Gamma^m_k \\
&= \pi^{m-k+1}_m \circ \gamma_{m+2} \circ \delta^{m+1}_i \circ \gamma_{m+1} \circ \Gamma^m_k \\
&= \pi^{m-k+1}_m \circ \gamma_{m+2} \circ \Gamma^{m+1}_k
\end{aligned}$$

The preceding equations together with diagram (6.17) finally show that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+2}\circ\Gamma^{m+1}_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond\circ\delta^k_2"] & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,m+1}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet\circ\delta^k_1"'] & \mathbb{D}_{m+1} \arrow[dr, "\pi^{m-k+1}_m"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota^\bullet_{k,m+1}"'] & & \mathbb{G}^{m+1}_k(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)
\end{tikzcd}$$

This last diagram together with factorisation (6.16) finishes the induction reasoning and thereby defines a notion of normality for any pair of integer $(k, m-k)$ where $m \geq k$. A spinal coheroid will later be said to be (k,ω)-*normal* for some $k \in \omega$ if it is $(k, m-k)$-normal for every $m \geq k$.

Example 6.28 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \texttt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is (k,ω)-normal by definition of $\mathcal{A}$.

Example 6.29 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is (k,ω)-normal for every $k \in \omega$ as factorisations (6.9), (6.12) and (6.16) hold by definition of the class Ω'_k and lifting condition for the parallel pairs in $\mathcal{A}$.

Example 6.30 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories becomes magmoidal when the class of parallel arrows is augmented by all the pairs involved in factorisations (6.9), (6.12) and (6.16). These parallel pairs depend on the morphisms β in $\mathbf{llp}(\mathbf{rlp}(\{F(\gamma_m) \mid m \in \omega\}))$ and the previously listed factorisations are therefore not straightforward. They may be obtained from canonical comparisons (using classical arguments) between the arrows γ_k and β expressed in terms of factorisations. These comparisons then allow to factorise the admissible parallel pairs through parallel pairs living in the initial set $F(\mathcal{A})$. Because these pairs factorise through the morphisms γ_k, the property is shown. The details are left to the reader.

6.3.2.2. *Transitive spinal coheroids.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a spinal coheroid. Let us fix some $k \in \omega$ and consider the following pushout for every triple of arrows $\beta_\diamond : \mathbb{S}_k \to \mathbb{A}_\diamond$, $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ and $\beta : \mathbb{S}_k \to \mathbb{A}$ in Ω_k.

(6.18)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_1^k"] & & \mathbb{A}_\diamond \arrow[ddd, "\epsilon^\diamond_{k,k}"] \\
& \mathbb{A} \arrow[ddrr, "\epsilon_{k,k}"] & & \\
\mathbb{D}_k \arrow[uu, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & & & \\
\mathbb{A}_\bullet \arrow[rrr, "\epsilon^\bullet_{k,k}"'] & & & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Case } 0}$ $\triangleright$ Pre-composing the top-right and bottom-left commutative part of diagram (6.18) with the arrow $\gamma_k : \mathbb{S}_{k-1} \to \mathbb{D}_k$ and using the diagrammatic relations defining the prevertebra $\|\gamma_k, \gamma_k\|$ leads to the following commutative square.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\epsilon^\bullet_{k,k} \circ \beta_\bullet \circ \delta_1^k"] \\
\mathbb{D}_k \arrow[r, "\epsilon^\diamond_{k,k} \circ \beta_\diamond \circ \delta_2^k"'] & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

This gives a new k-parallel pair in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will then be said to be $(k, 0)$-*transitive* if the parallel pair

$$(\epsilon^\diamond_{k,k} \circ \beta_\diamond \circ \delta_2^k, \epsilon^\bullet_{k,k} \circ \beta_\bullet \circ \delta_1^k)$$

is admissible for every pair $\beta_\diamond, \beta_\bullet \in \Omega_k$ so that there exist an arrow $\beta_* : \mathbb{S}_k \to \mathbb{A}_*$ in Ω_k and a morphism $\upsilon_k^0(\beta) : \mathbb{A}_* \to \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)$ making the following diagram commute.

(6.19)
$$\begin{tikzcd}
\mathbb{A}_* \arrow[drr, dashed, "\upsilon_k^0(\beta)"] & & \\
\mathbb{S}_k \arrow[u, "\beta_*"] \arrow[rr, "\ulcorner \epsilon^\diamond_{k,k} \circ \beta_\diamond \circ \delta_2^k, \epsilon^\bullet_{k,k} \circ \beta_\bullet \circ \delta_1^k \urcorner"'] & & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

Note that, by definition of the canonical morphism $\lceil\epsilon^\diamond_{k,k}\circ\beta_\diamond\circ\delta_2^k, \epsilon^\bullet_{k,k}\circ\beta_\bullet\circ\delta_1^k\rceil$, the preceding factorisation may also be written as a commutative diagram of the following form.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta_*"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon^\diamond_{k,k}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{D}_{k+1} \arrow[dr, "\upsilon_k^0(\beta)"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon^\bullet_{k,k}"'] & & \mathbb{G}_k^k(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Case } 1}$ $\triangleright$ The rest of the section extends the notion of transitivity by using the notion of normality. Fix some $k\in\omega$ and suppose that the spinal coheroid $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ is $(k,0)$-normal. By construction, the following diagram commutes for every pair of arrows $\beta_\diamond:\mathbb{S}_k\to\mathbb{A}_\diamond$ and $\beta_\bullet:\mathbb{S}_k\to\mathbb{A}_\bullet$ in Ω_k.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,k}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{D}_{k+1} \arrow[dr, "\pi_k^0"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota^\bullet_{k,k}"'] & & \mathbb{G}_k^k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)
\end{tikzcd}$$

Post-composing the previous diagram with the tau construction

$$\tau^k_{i,k}(\beta):\mathbb{G}_k^k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)\to\mathbb{G}_k^{k+1}(\beta_\diamond,\beta,\beta_\bullet)$$

gives the following commutative diagram for every $i\in\{1,2\}$.

(6.20)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon^\diamond_{k,k+1}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{D}_{k+1} \arrow[dr, "\tau^k_{i,k}(\beta)\circ\pi_k^0"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon^\bullet_{k,k+1}"'] & & \mathbb{G}_k^{k+1}(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

The previous commutative diagram means that the two composite arrows $\tau^k_{1,k}(\beta)\circ\pi_k^0:\mathbb{D}_{k+1}\to\mathbb{G}_k^{k+1}(\beta_\diamond,\beta,\beta_\bullet)$ and $\tau^k_{2,k}(\beta)\circ\pi_k^0:\mathbb{D}_{k+1}\to\mathbb{G}_k^{k+1}(\beta_\diamond,\beta,\beta_\bullet)$ are solutions of a same universal problem over the pushout $\mathbb{S}_k$. It follows from the universality of $\mathbb{S}_k$ that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\tau^k_{1,k}(\beta)\circ\pi_k^0"] \\
\mathbb{D}_{k+1} \arrow[r, "\tau^k_{2,k}(\beta)\circ\pi_k^0"'] & \mathbb{G}_k^{k+1}(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

Note that the preceding diagram provides a $(k+1)$-parallel pair of arrows in the spinal extension $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$. The spinal extension $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ will be said to be $(k,1)$-*transitive* if it is $(k,0)$-normal (as previously supposed) and the parallel pair of arrows

$$(\tau^k_{2,k}(\beta)\circ\pi^0_k, \tau^k_{1,k}(\beta)\circ\pi^0_k)$$

is admissible so that there exists an arrow $\beta_* : \mathbb{S}_{k+1}\to\mathbb{A}_*$ in Ω_{k+1} and a morphism $\upsilon^1_k(\beta) : \mathbb{A}_*\to\mathbb{G}^{k+1}_k(\beta_\diamond,\beta,\beta_\bullet)$ making the following diagram commute.

(6.21)
$$\begin{tikzcd}
\mathbb{A}_* \arrow[drr, dashed, "\upsilon^1_k(\beta)"] & & \\
\mathbb{S}_{k+1} \arrow[u, "\beta_*"] \arrow[rr, "\lceil\tau^k_{2,k}(\beta)\circ\pi^0_k,\tau^k_{1,k}(\beta)\circ\pi^0_k\rceil"'] & & \mathbb{G}^{k+1}_k(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

It follows from diagram (6.20) and the definition of the canonical arrow

$$\lceil\tau^k_{2,k}(\beta)\circ\pi^0_k, \tau^k_{1,k}(\beta)\circ\pi^0_k\rceil : \mathbb{S}_k\to\mathbb{G}^{k+1}_k(\beta_\diamond,\beta,\beta_\bullet)$$

that the previous factorisation gives rise to the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta_*\circ\Gamma^{k+1}_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond\circ\delta^k_2"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon^\diamond_{k,k+1}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet\circ\delta^k_1"'] & \mathbb{A}_* \arrow[dr, "\upsilon^1_k(\beta)"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon^\bullet_{k,k+1}"'] & & \mathbb{G}^{k+1}_k(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Ind.}}$ $\triangleright$ Now, suppose that the spinal coheroid $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega$ is $(k,m-k)$-normal for some $m\geq k+1$. By definition, the following two diagrams commute for every pair of arrows $\beta_\diamond : \mathbb{S}_k\to\mathbb{A}_\diamond$ and $\beta_\bullet : \mathbb{S}_k\to\mathbb{A}_\bullet$ in Ω_k.

(6.22)
$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[drr, dashed, "\pi^{m-k}_k"] & & \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[rr, "\lceil\kappa^m_{2,k}\circ\pi^{m-k-1}_k,\kappa^m_{1,k}\circ\pi^{m-k-1}_k\rceil"'] & & \mathbb{G}^m_k(\beta_\diamond,\gamma_{m+1},\beta_\bullet)
\end{tikzcd}$$

(6.23)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1}\circ\Gamma^m_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond\circ\delta^k_2"] & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,m}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet\circ\delta^k_1"'] & \mathbb{D}_{m+1} \arrow[dr, "\pi^{m-k}_k"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota^\bullet_{k,m}"'] & & \mathbb{G}^m_k(\beta_\diamond,\gamma_{m+1},\beta_\bullet)
\end{tikzcd}$$

Post-composing diagram (6.22) with the tau construction

$$\tau^m_{i,k}(\beta) : \mathbb{G}^m_k(\beta_\diamond,\gamma_{m+1},\beta_\bullet)\to\mathbb{G}^{m+1}_k(\beta_\diamond,\beta,\beta_\bullet)$$

gives the following factorisation for every $i\in\{1,2\}$.

(6.24)
$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[drr, dashed, "\tau^m_{i,k}(\beta)\circ\pi^{m-k}_k"] & & \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[rr, "\tau^m_{i,k}(\beta)\circ\lceil\kappa^m_{2,k}\circ\pi^{m-k-1}_k,\kappa^m_{1,k}\circ\pi^{m-k-1}_k\rceil"'] & & \mathbb{G}^{m+1}_k(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

Note that the equation $\tau_{2,k}^m(\beta) \circ \kappa_{i,k}^m = \tau_{1,k}^m(\beta) \circ \kappa_{i,k}^m$ obtained in (6.7) implies that the next equalities hold in $\mathcal{C}$, where the symbol π_* stand for π_k^{m-k-1}.

$$\begin{aligned}
\tau_{1,k}^m(\beta) \circ \lceil \kappa_{2,k}^m \circ \pi_*, \kappa_{1,k}^m \circ \pi_* \rceil &= \lceil \tau_{1,k}^m(\beta) \circ \kappa_{2,k}^m \circ \pi_*, \tau_{1,k}^m(\beta) \circ \kappa_{1,k}^m \circ \pi_* \rceil \\
&= \lceil \tau_{2,k}^m(\beta) \circ \kappa_{2,k}^m \circ \pi_*, \tau_{2,k}^m(\beta) \circ \kappa_{1,k}^m \circ \pi_* \rceil \\
&= \tau_{2,k}^m(\beta) \circ \lceil \kappa_{2,k}^m \circ \pi_*, \kappa_{1,k}^m \circ \pi_* \rceil
\end{aligned}$$

This means that the two factorisations involved in diagram (6.24) are the factorisations of a same arrow. This therefore implies that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{D}_{m+1} \arrow[d, "\tau_{1,k}^m(\beta)\circ\pi_k^{m-k}"] \\
\mathbb{D}_{m+1} \arrow[r, "\tau_{2,k}^m(\beta)\circ\pi_k^{m-k}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

The preceding diagram exposes a parallel pair of arrows in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is said to be $(k, m-k+1)$-*transitive* if the parallel pair of arrows

$$(\tau_{2,k}^m(\beta) \circ \pi_k^{m-k}, \tau_{1,k}^m(\beta) \circ \pi_k^{m-k})$$

is admissible so that there exist an arrow $\beta_* : \mathbb{S}_{m+1} \to \mathbb{A}_*$ in Ω_{m+1} and a morphism $\upsilon_k^{m-k+1}(\beta) : \mathbb{A}_* \to \mathbb{G}_k^{m+1}(\beta_\diamond, \beta, \beta_\bullet)$ making the following diagram commute.

(6.25)
$$\begin{tikzcd}
\mathbb{A}_* \arrow[dr, dashed, "\upsilon_k^{m-k+1}(\beta)"] & \\
\mathbb{S}_{m+1} \arrow[u, "\beta_*"] \arrow[r, "\lceil \tau_{2,k}^m(\beta)\circ\pi_k^{m-k}, \tau_{1,k}^m(\beta)\circ\pi_k^{m-k} \rceil"'] & \mathbb{G}_k^{m+1}(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

If we now post-compose diagram (6.23) with the tau construction

$$\tau_{i,k}^m(\beta) : \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \to \mathbb{G}_k^{m+1}(\beta_\diamond, \beta, \beta_\bullet)$$

we obtain the following commutative diagram.

(6.26)
$$\begin{tikzcd}
\mathbb{S}_k & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] \arrow[d, "\gamma_{m+1}\circ\Gamma_k^m"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon_{k,m+1}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{A}_* \arrow[dr, "\tau_{i,k}^m(\beta)\circ\pi_k^{m-k}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon_{k,m+1}^\bullet"'] & & \mathbb{G}_k^{m+1}(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

It follows from factorisation (6.25) that the following equations hold for every $i \in \{1, 2\}$.

$$\begin{aligned}
\tau_{i,k}^m(\beta) \circ \pi_k^{m-k} \circ \gamma_{m+1} \circ \Gamma_k^m &= \lceil \tau_{2,k}^m(\beta) \circ \pi_k^{m-k}, \tau_{1,k}^m(\beta) \circ \pi_k^{m-k} \rceil \circ \delta_i^{m+1} \circ \gamma_{m+1} \circ \Gamma_k^m \\
&= \upsilon_k^{m-k+1}(\beta) \circ \beta_* \circ \delta_i^{m+1} \circ \gamma_{m+1} \circ \Gamma_k^m \\
&= \upsilon_k^{m-k+1}(\beta) \circ \beta_* \circ \Gamma_k^{m+1}
\end{aligned}$$

The previous equations together with diagram (6.26) finally show that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_k & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] \arrow[d, "\beta_*\circ\Gamma_k^{m+1}"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon^\diamond_{k,m+1}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{A}_* \arrow[dr, "\upsilon_k^{m-k+1}(\beta)"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon^\bullet_{k,m+1}"'] & & \mathbb{G}_k^{m+1}(\beta_\diamond,\beta,\beta_\bullet)
\end{tikzcd}$$

A spinal coheroid will later be said to be (k,ω)-*transitive* for some $k\in\omega$ if it is $(k,m-k)$-transitive for every $m\geq k$. By definition, any (k,ω)-transitive spinal coheroid is (k,ω)-normal.

Example 6.31 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D},\mathbb{S},\delta,\gamma):\mathtt{Spine}\to(\mathbf{Top},\mathcal{A})$ is (k,ω)-transitive by definition of $\mathcal{A}$.

Example 6.32 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is (k,ω)-transitive for every $k\in\omega$ as factorisations (6.19), (6.21) and (6.25) hold by definition of $\mathcal{A}$.

Example 6.33 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories becomes magmoidal when the class of parallel arrows is augmented by all the pairs involved in factorisations (6.19), (6.21) and (6.25). The related factorisations are provided by Corollary 5.86 (small object argument).

6.3.3. Closed spinal coheroids. The idea behind closed spinal coheroids is that the spinal coheroid is closed under certain pushouts, namely those that one would like see as the spheres.

6.3.3.1. *Closed spinal coheroids.* A spinal coheroid $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega:\mathtt{Spine}\to(\mathcal{C},\mathcal{A})$ will be said to be *closed* if for every non-negative integers $k\leq m$ and pair of arrows $\beta_\diamond:\mathbb{S}_k\to\mathbb{A}_\diamond$ and $\beta_\bullet:\mathbb{S}_k\to\mathbb{A}_\bullet$ in Ω_k, the following diagram admits a colimit in $\mathcal{C}$.

$$\begin{tikzcd}
\mathbb{A}_\diamond & & \mathbb{S}_m & & \mathbb{A}_\bullet \\
& \mathbb{D}_k \arrow[ul, "\beta_\diamond\circ\delta_2^k"] \arrow[ur, "\Gamma_k^m\circ\delta_1^k"'] & & \mathbb{D}_k \arrow[ul, "\Gamma_k^m\circ\delta_2^k"] \arrow[ur, "\beta_\bullet\circ\delta_1^k"'] &
\end{tikzcd}$$

$\boxed{\textit{Def. } \kappa_k^k}$ $\triangleright$ The universal object associated with the above colimit will be denoted by the symbol $\mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet)$ and its universal cocone will be given by the following arrows.

$$\begin{tikzcd}
\mathbb{S}_k & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_1^k"] \arrow[d, "\Gamma_k^m"] & \mathbb{A}_\diamond \arrow[dd, "\jmath^\diamond_{k,m}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{S}_m \arrow[dr, "\jmath_{k,m}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath^\bullet_{k,m}"'] & & \mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

Recall that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is provided with the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{D}_{k+1} \arrow[drr, "\iota_{k,k}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,k}^\bullet"'] & & & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)
\end{tikzcd}$$

The universality of the pushout $\mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ then provides a canonical arrow

$$\kappa_k^k : \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \to \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)$$

making the following three diagrams commute.

$$\begin{tikzcd}
 & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \\
\mathbb{A}_\diamond \arrow[ur, "\iota_{k,k}^\diamond"] \arrow[r, "\jmath_{k,k}^\diamond"'] & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\kappa_k^k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \\
\mathbb{A}_\bullet \arrow[ur, "\iota_{k,k}^\bullet"] \arrow[r, "\jmath_{k,k}^\bullet"'] & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\kappa_k^k"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_{k+1} \arrow[r, "\iota_{k,k}"] & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \\
\mathbb{S}_k \arrow[u, "\gamma_{k+1}"] \arrow[r, "\jmath_{k,k}"'] & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\kappa_k^k"']
\end{tikzcd}$$

$\boxed{\textit{Def. } d_{i,k}^m, \kappa_k^m}$ ▷ Similarly, recall that, for every $m \geq k+1$, we are provided with the following two pushouts.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_m \circ \Gamma_k^{m-1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m-1}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{D}_m \arrow[drr, "\iota_k^{m-1}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,m-1}^\bullet"'] & & & \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1} \circ \Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{D}_{m+1} \arrow[drr, "\iota_k^m"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,m}^\bullet"'] & & & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

The universality of the pushouts $\mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ and $\mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet)$ provides a canonical factorisation

$$\begin{tikzcd}
\mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[r, "d_{i,k}^m"] \arrow[rr, bend right, "\kappa_{i,k}^m"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\kappa_k^m"] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

making the following three diagrams commute.

$$\begin{tikzcd}
 & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \\
\mathbb{A}_\diamond \arrow[ur, "\iota_{k,m}^\diamond"] \arrow[r, "\jmath_{k,m}^\diamond"] \arrow[dr, "\iota_{k,m-1}^\diamond"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\kappa_k^m"'] \\
 & \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "d_{i,k}^m"'] \arrow[uu, bend right=60, "\kappa_{i,k}^m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \\
\mathbb{A}_\bullet \arrow[ur, "\iota_{k,m}^\bullet"] \arrow[r, "\jmath_{k,m}^\bullet"] \arrow[dr, "\iota_{k,m-1}^\bullet"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\kappa_k^m"'] \\
 & \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "d_{i,k}^m"'] \arrow[uu, bend right=60, "\kappa_{i,k}^m"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[r, "\iota_{k,m}"] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[r, "\jmath_{k,m}"] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\kappa_k^m"'] \\
\mathbb{D}_m \arrow[u, "\delta_i^m"] \arrow[r, "\iota_{k,m-1}"'] & \mathbb{G}_k^{m-1}(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "d_{i,k}^m"'] \arrow[uu, bend right=60, "\kappa_{i,k}^m"']
\end{tikzcd}$$

Also, because the equation $\delta_1^{m+1} \circ \gamma_{m+1} = \delta_2^{m+1} \circ \gamma_{m+1}$ holds in $\mathcal{C}$, the universality of the object $\mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ provides the following two equations for every $i \in \{1, 2\}$.

$$d_{1,k}^{m+1} \circ \kappa_k^m = d_{2,k}^{m+1} \circ \kappa_k^m \qquad\qquad d_{1,k}^{m+1} \circ \kappa_{i,k}^m = d_{2,k}^{m+1} \circ \kappa_{i,k}^m \tag{6.27}$$

$\boxed{\textit{Def. } \tau_k^k}$ $\triangleright$ The rest of the section copies the above reasoning with respect to tau constructions. Recall that, for every arrow $\beta : \mathbb{S}_k \to \mathbb{A}$ in Ω_k, the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is provided with the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond \circ \delta_1^k"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{A} \arrow[dr, "\epsilon_{k,k}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon_{k,k}^\bullet"'] & & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

The universality of the pushout $\mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ then provides a canonical arrow

$$\tau_k^k(\beta) : \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \to \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)$$

making the following three diagrams commute.

$$\begin{tikzcd}
 & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet) \\
\mathbb{A}_\diamond \arrow[ur, "\epsilon_{k,k}^\diamond"] \arrow[r, "\jmath_{k,k}^\diamond"'] & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\tau_k^k(\beta)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet) \\
\mathbb{A}_\bullet \arrow[ur, "\epsilon_{k,k}^\bullet"] \arrow[r, "\jmath_{k,k}^\bullet"'] & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\tau_k^k(\beta)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_{k+1} \arrow[r, "\epsilon_{k,k}"] & \mathbb{G}^k_k(\beta_\diamond, \beta, \beta_\bullet) \\
\mathbb{S}_k \arrow[u, "\beta"] \arrow[r, "\jmath_{k,k}"'] & \mathbb{G}^k_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\tau^k_k(\beta)"']
\end{tikzcd}$$

$\boxed{\textit{Def. } \tau^m_k}$ ▷ Similarly, recall that, for every $m \geq k+1$ and arrow $\beta : \mathbb{S}_m \to \mathbb{A}$ in Ω_m, we are provided with the following two pushouts.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_m \circ \Gamma^{m-1}_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond \circ \delta^k_1"] & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,m-1}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet \circ \delta^k_2"'] & \mathbb{D}_m \arrow[dr, "\iota^{m-1}_k"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota^\bullet_{k,m-1}"'] & & \mathbb{G}^{m-1}_k(\beta_\diamond, \gamma_m, \beta_\bullet)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta \circ \Gamma^m_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond \circ \delta^k_1"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon^\diamond_{k,m}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet \circ \delta^k_2"'] & \mathbb{D}_{m+1} \arrow[dr, "\epsilon^m_k"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon^\bullet_{k,m}"'] & & \mathbb{G}^m_k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

The universality of the pushouts $\mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ and $\mathbb{G}^{m-1}_k(\beta_\diamond, \gamma_m, \beta_\bullet)$ provides a canonical factorisation

$$\begin{tikzcd}
\mathbb{G}^{m-1}_k(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[r, "d^m_{i,k}"] \arrow[rr, bend right, "\tau^{m-1}_{i,k}(\beta)"'] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\tau^m_k(\beta)"] & \mathbb{G}^m_k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

making the following three diagrams commute.

$$\begin{tikzcd}
 & \mathbb{G}^m_k(\beta_\diamond, \beta, \beta_\bullet) \\
\mathbb{A}_\diamond \arrow[ur, "\epsilon^\diamond_{k,m}"] \arrow[r, "\jmath^\diamond_{k,m}"] \arrow[dr, "\iota^\diamond_{k,m-1}"'] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\tau^m_k(\beta)"'] \\
 & \mathbb{G}^{m-1}_k(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "d^m_{i,k}"'] \arrow[uu, bend right=60, "\tau^{m-1}_{i,k}(\beta)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{G}^m_k(\beta_\diamond, \beta, \beta_\bullet) \\
\mathbb{A}_\bullet \arrow[ur, "\epsilon^\bullet_{k,m}"] \arrow[r, "\jmath^\bullet_{k,m}"] \arrow[dr, "\iota^\bullet_{k,m-1}"'] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\tau^m_k(\beta)"'] \\
 & \mathbb{G}^{m-1}_k(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "d^m_{i,k}"'] \arrow[uu, bend right=60, "\tau^{m-1}_{i,k}(\beta)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[r, "\epsilon_{k,m}"] & \mathbb{G}^m_k(\beta_\diamond, \beta, \beta_\bullet) \\
\mathbb{S}_m \arrow[u, "\beta"] \arrow[r, "\jmath_{k,m}"] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[u, dashed, "\tau^m_k(\beta)"'] \\
\mathbb{D}_m \arrow[u, "\delta^m_i"] \arrow[r, "\iota_{k,m-1}"'] & \mathbb{G}^{m-1}_k(\beta_\diamond, \gamma_m, \beta_\bullet) \arrow[u, dashed, "d^m_{i,k}"'] \arrow[uu, bend right=60, "\tau^{m-1}_{i,k}(\beta)"']
\end{tikzcd}$$

Example 6.34 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \texttt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is closed by cocompleteness of $\mathbf{Top}$.

Example 6.35 (Groupoids and categories)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids (i.e. the category of models $\mathbf{Mod}(C)$ for a Grothendieck coherator C; see section 6.2.4.3) or a category of Maltsiniotis ∞-categories is closed by cocompleteness of the category in which it lands (i.e. the category of models in question).

6.3.3.2. *Closedness and normality.* This section studies closed spinal coheroid under some normality conditions. Let $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega : \texttt{Spine} \to (\mathcal{C},\mathcal{A})$ be a closed spinal coheroid and fix some $k \in \omega$. Before using any normality condition, we shall proceed to a preliminary construction. It follows from the definition of a closed spinal coheroid that the following pushout exists for every pair of arrows $\beta_\diamond : \mathbb{S}_k \to \mathbb{A}_\diamond$ and $\beta_\bullet : \mathbb{S}_k \to \mathbb{A}_\bullet$ in Ω_k.

(6.28)
$$\begin{tikzcd}
\mathbb{S}_k & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_1^k"] & \mathbb{A}_\diamond \arrow[dd, "\jmath^\diamond_{k,k}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_2^k"'] & \mathbb{S}_k \arrow[ul, equal] \arrow[dr, "\jmath_{k,k}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath^\bullet_{k,k}"'] & & \mathbb{G}_k^k(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Case } 0}$ $\triangleright$ Pre-composing the top-right and bottom-left commutative parts of diagram (6.28) with the arrow $\gamma_k : \mathbb{S}_{k-1} \to \mathbb{D}_k$ and using the diagrammatic relations defining the prevertebra $\|\gamma_k,\gamma_k\|$ leads to the following commutative square.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\jmath^\bullet_{k,k}\circ\beta_\bullet\circ\delta_1^k"] \\
\mathbb{D}_k \arrow[r, "\jmath^\diamond_{k,k}\circ\beta_\diamond\circ\delta_2^k"'] & \mathbb{G}_k^k(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

Forming the pushout $\mathbb{S}_k$ in the previous diagram leads to the existence of a canonical arrow $v_k^0 : \mathbb{S}_k \to \mathbb{G}_k^k(\beta_\diamond,\mathrm{id},\beta_\bullet)$ making the following diagram commute.

(6.29)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[ddrr, "v_k^0"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\jmath^\diamond_{k,k}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath^\bullet_{k,k}"'] & & \mathbb{G}_k^k(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

Now, post-composing the preceding diagram with the canonical arrow

$$\kappa_k^k : \mathbb{G}_k^k(\beta_\diamond,\mathrm{id},\beta_\bullet) \to \mathbb{G}_k^k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)$$

provides the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[ddrr, "\kappa_k^k \circ \upsilon_k^0"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond \circ \delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & & \\
\mathbb{A}_\bullet \arrow[rr, "\iota_{k,k}^\bullet"'] & & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)
\end{tikzcd}$$

Recall that diagram (6.10) provides a similar commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[ddrr, "\pi_k^0 \circ \gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond \circ \delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & & \\
\mathbb{A}_\bullet \arrow[rr, "\iota_{k,k}^\bullet"'] & & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)
\end{tikzcd}$$

These last two diagrams give two solutions $\kappa_k^k \circ \upsilon_k^0 : \mathbb{S}_k \to \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)$ and $\pi_k^0 \circ \gamma_{k+1} : \mathbb{S}_k \to \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)$ for the same universal problem over the pushout $\mathbb{S}_k$. By universality, it follows that the following diagram must commute.

(6.30)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\upsilon_k^0"'] & \mathbb{D}_{k+1} \arrow[d, "\pi_k^0"] \\
\mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\kappa_k^k"'] & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Case } 1}$ $\triangleright$ Now, suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ is $(k, 0)$-normal. By construction, this means that we are given the commutative diagram:

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_2^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & \mathbb{D}_{k+1} \arrow[drr, "\pi_k^0"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,k}^\bullet"'] & & & \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)
\end{tikzcd}$$

Post-composing the previous diagram with the canonical arrow

$$d_{i,k}^{k+1} : \mathbb{G}_k^k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \to \mathbb{G}_k^{k+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet)$$

gives the following commutative diagram.

(6.31)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\jmath^\diamond_{k,k+1}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{D}_{k+1} \arrow[dr, "d^{k+1}_{i,k}\circ\pi^0_k"] & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath^\bullet_{k,k+1}"'] & & \mathbb{G}_k^{k+1}(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

By universality of the object $\mathbb{S}_k$, the following square must commute.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_k \arrow[d, "d^{k+1}_{1,k}\circ\pi^0_k"] \\
\mathbb{D}_k \arrow[r, "d^{k+1}_{2,k}\circ\pi^0_k"'] & \mathbb{G}_k^{k+1}(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

Forming the pushout $\mathbb{S}_{k+1}$ in the earlier diagram leads to the existence of a canonical arrow $\upsilon_k^1 : \mathbb{S}_{k+1} \to \mathbb{G}_k^{k+1}(\beta_\diamond,\mathrm{id},\beta_\bullet)$ making the following diagram commute.

(6.32)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_1^{k+1}"] \arrow[dr, bend left, "d^{k+1}_{1,k}\circ\pi^0_k"] & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_2^{k+1}"'] \arrow[rr, bend right, "d^{k+1}_{2,k}\circ\pi^0_k"'] & \mathbb{S}_{k+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\upsilon_k^1"] & \mathbb{G}_k^{k+1}(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

In particular, this last diagram turns diagram (6.31) into the following one.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\Gamma_k^{k+1}"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\jmath^\diamond_{k,k+1}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{S}_{k+1} \arrow[dr, "\upsilon_k^1"] & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath^\bullet_{k,k+1}"'] & & \mathbb{G}_k^{k+1}(\beta_\diamond,\mathrm{id},\beta_\bullet)
\end{tikzcd}$$

Now, post-composing diagram (6.32) with the canonical arrow

$$\kappa_k^{k+1} : \mathbb{G}_k^{k+1}(\beta_\diamond,\mathrm{id},\beta_\bullet) \to \mathbb{G}_k^{k+1}(\beta_\diamond,\gamma_{k+2},\beta_\bullet)$$

provides the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_1^{k+1}"] \arrow[dr, bend left, "\kappa^{k+1}_{1,k}\circ\pi^0_k"] & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_2^{k+1}"'] \arrow[rr, bend right, "\kappa^{k+1}_{2,k}\circ\pi^0_k"'] & \mathbb{S}_{k+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\kappa_k^{k+1}\circ\upsilon_k^1"] & \mathbb{G}_k^{k+1}(\beta_\diamond,\gamma_{k+2},\beta_\bullet)
\end{tikzcd}$$

By universality, the preceding diagram gives a factorisation of the canonical arrow $\lceil \kappa^{k+1}_{2,k} \circ \pi^0_k, \kappa^{k+1}_{1,k} \circ \pi^0_k \rceil : \mathbb{S}_{k+1} \to \mathbb{G}^{k+1}_k(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)$ as follows.

$$\begin{tikzcd}
\mathbb{S}_{k+1} \arrow[rr, "\lceil \kappa^{k+1}_{2,k}\circ\pi^0_k, \kappa^{k+1}_{1,k}\circ\pi^0_k \rceil"] \arrow[dr, "\upsilon^1_k"'] & & \mathbb{G}^{k+1}_k(\beta_\diamond, \gamma_{k+2}, \beta_\bullet) \\
& \mathbb{G}^{k+1}_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[ur, "\kappa^{k+1}_k"'] &
\end{tikzcd}$$

Recall that diagram (6.12) also provided another factorisation for the canonical arrow $\lceil \kappa^{k+1}_{2,k} \circ \pi^0_k, \kappa^{k+1}_{1,k} \circ \pi^0_k \rceil : \mathbb{S}_{k+1} \to \mathbb{G}^{k+1}_k(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)$, so that, if both expressions are combined, the diagram, below, commutes.

(6.33)
$$\begin{tikzcd}
\mathbb{S}_{k+1} \arrow[r, "\gamma_{k+2}"] \arrow[d, "\upsilon^1_k"'] & \mathbb{D}_{k+2} \arrow[d, "\pi^1_k"] \\
\mathbb{G}^{k+1}_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\kappa^{k+1}_k"'] & \mathbb{G}^{k+1}_k(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Ind.}}$ $\triangleright$ Now, suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ is $(k, m-k)$-normal for some $m \geq k+2$. By definition, this means that we are given the following two commutative diagrams.

(6.34)
$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[dr, dashed, "\pi^{m-k}_k"] & \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[r, "\lceil \kappa^m_{2,k}\circ\pi^{m-k-1}_k, \kappa^m_{1,k}\circ\pi^{m-k-1}_k \rceil"'] & \mathbb{G}^m_k(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

(6.35)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1}\circ\Gamma^m_k"] & \mathbb{D}_k \arrow[l, "\delta^k_2"'] \arrow[r, "\beta_\diamond\circ\delta^k_2"] & \mathbb{A}_\diamond \arrow[dd, "\iota^\diamond_{k,m}"] \\
\mathbb{D}_k \arrow[u, "\delta^k_1"] \arrow[d, "\beta_\bullet\circ\delta^k_1"'] & \mathbb{D}_{m+1} \arrow[dr, "\pi^{m-k}_k"] & \\
\mathbb{A}_\bullet \arrow[rr, "\iota^\bullet_{k,m}"'] & & \mathbb{G}^m_k(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

Post-composing diagram (6.34) with the canonical arrow

$$d^{m+1}_{i,k} : \mathbb{G}^m_k(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \to \mathbb{G}^{m+1}_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$$

gives the following factorisation.

(6.36)
$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[dr, dashed, "d^{m+1}_{i,k}\circ\pi^{m-k}_k"] & \\
\mathbb{S}_m \arrow[u, "\gamma_{m+1}"] \arrow[r, "d^{m+1}_{i,k}\circ\lceil \kappa^m_{2,k}\circ\pi^{m-k-1}_k, \kappa^m_{1,k}\circ\pi^{m-k-1}_k \rceil"'] & \mathbb{G}^{m+1}_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

Note that the equation $d_{2,k}^{m+1} \circ \kappa_{i,k}^{m} = d_{1,k}^{m+1} \circ \kappa_{i,k}^{m}$ obtained in (6.27) implies that the following equalities hold in $\mathcal{C}$, where the symbol π_* stands for π_k^{m-k-1}.

$$\begin{aligned} d_{2,k}^{m+1} \circ \lceil \kappa_{2,k}^{m} \circ \pi_*, \kappa_{1,k}^{m} \circ \pi_* \rceil &= \lceil d_{2,k}^{m+1} \circ \kappa_{2,k}^{m} \circ \pi_*, d_{2,k}^{m+1} \circ \kappa_{1,k}^{m} \circ \pi_* \rceil \\ &= \lceil d_{1,k}^{m+1} \circ \kappa_{2,k}^{m} \circ \pi_*, d_{1,k}^{m+1} \circ \kappa_{1,k}^{m} \circ \pi_* \rceil \\ &= d_{1,k}^{m+1} \circ \lceil \kappa_{2,k}^{m} \circ \pi_*, \kappa_{1,k}^{m} \circ \pi_* \rceil \end{aligned}$$

This means that the two factorisations involved in diagram (6.36) are factorisations of the same arrow. This implies that the following diagram commutes.

$$\begin{tikzcd} \mathbb{S}_m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{D}_{m+1} \arrow[d, "d_{1,k}^{m+1}\circ\pi_k^{m-k}"] \\ \mathbb{D}_{m+1} \arrow[r, "d_{2,k}^{m+1}\circ\pi_k^{m-k}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet) \end{tikzcd}$$

Forming the pushout $\mathbb{S}_{m+1}$ in the preceding diagram leads to the existence of a canonical arrow $\upsilon_k^{m-k+1} : \mathbb{S}_{m+1} \to \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ making the following diagram commute.

(6.37)
$$\begin{tikzcd} \mathbb{S}_m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{D}_{m+1} \arrow[d, "\delta_1^{m+1}"] \arrow[rd, bend left, "d_{1,k}^{m+1}\circ\pi_k^{m-k}"] & \\ \mathbb{D}_{m+1} \arrow[r, "\delta_2^{m+1}"'] \arrow[rr, bend right, "d_{2,k}^{m+1}\circ\pi_k^{m-k}"'] & \mathbb{S}_{m+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\upsilon_k^{m-k+1}"] & \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet) \end{tikzcd}$$

In particular, this last diagram turns diagram (6.35) into the following one.

$$\begin{tikzcd} \mathbb{S}_k & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond\circ\delta_2^k"] & & \mathbb{A}_\diamond \arrow[dd, "\jmath_{k,m+1}^\diamond"] \\ \mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{S}_{k+1} \arrow[from=ul, "\Gamma_k^{m+1}"] \arrow[rrd, "\upsilon_k^{m-k+1}"] & & \\ \mathbb{A}_\bullet \arrow[rrr, "\jmath_{k,m+1}^\bullet"'] & & & \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet) \end{tikzcd}$$

Finally, post-composing diagram (6.37) with the canonical arrow

$$\kappa_k^{m+1} : \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet) \to \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)$$

provides the commutative diagram:

$$\begin{tikzcd} \mathbb{S}_m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{D}_{m+1} \arrow[d, "\delta_1^{m+1}"] \arrow[rd, bend left, "\kappa_{1,m}^{m+1}\circ\pi_m^{m-k}"] & \\ \mathbb{D}_{m+1} \arrow[r, "\delta_2^{m+1}"'] \arrow[rr, bend right, "\kappa_{2,m}^{m+1}\circ\pi_m^{m-k}"'] & \mathbb{S}_{m+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\kappa_m^{m+1}\circ\upsilon_k^{m-k+1}"] & \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet) \end{tikzcd}$$

By universality, the preceding diagram gives a factorisation of the canonical arrow $\ulcorner \kappa_{2,m}^{m+1} \circ \pi_m^{m-k}, \kappa_{1,m}^{m+1} \circ \pi_m^{m-k} \urcorner : \mathbb{S}_{m+1} \to \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)$ as follows.

$$\begin{tikzcd}
\mathbb{S}_{m+1} \arrow[rr, "\ulcorner \kappa_{2,m}^{m+1} \circ \pi_m^{m-k}, \kappa_{1,m}^{m+1} \circ \pi_m^{m-k} \urcorner"] \arrow[dr, "\upsilon_k^{m-k+1}"'] & & \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet) \\
& \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[ur, "\kappa_k^{m+1}"'] &
\end{tikzcd}$$

Recall that diagram (6.12) also provided another factorisation for the canonical arrow $\ulcorner \kappa_{2,m}^{m+1} \circ \pi_m^{m-k}, \kappa_{1,m}^{m+1} \circ \pi_m^{m-k} \urcorner : \mathbb{S}_{m+1} \to \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)$, so that, if the two expressions are combined, the following diagram commutes.

(6.38)
$$\begin{tikzcd}
\mathbb{S}_{m+1} \arrow[r, "\gamma_{m+2}"] \arrow[d, "\upsilon_k^{m-k+1}"'] & \mathbb{D}_{m+2} \arrow[d, "\pi_k^{m-k+1}"] \\
\mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\kappa_k^{m+1}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond, \gamma_{m+2}, \beta_\bullet)
\end{tikzcd}$$

6.3.3.3. *Closedness and transitivity.* This section studies closed spinal coheroid under some transitivity conditions. Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a closed spinal coheroid and fix some $k \in \omega$. Recall that the beginning of section 6.3.3.2 gave the following commutative diagram (see diagram (6.29)).

$$\begin{tikzcd}
\mathbb{S}_k \arrow[ddrr, "\upsilon_k^0"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond \circ \delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\jmath_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath_{k,k}^\bullet"'] & & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Case } 0}$ $\triangleright$ Post-composing the previous diagram with the tau construction

$$\tau_k^k(\beta) : \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \to \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)$$

provides another commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[ddrr, "\tau_k^k(\beta) \circ \upsilon_k^0"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond \circ \delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon_{k,k}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon_{k,k}^\bullet"'] & & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

By universality of $\mathbb{S}_k$, the preceding diagram gives a factorisation of the canonical arrow $\ulcorner \epsilon^{\diamond}_{k,k} \circ \beta_\diamond \circ \delta_2^k, \epsilon^{\bullet}_{k,k} \circ \beta_\bullet \circ \delta_1^k \urcorner : \mathbb{S}_k \to \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)$ as follows.

$$\begin{tikzcd} \mathbb{S}_k \arrow[rr, "\ulcorner \epsilon^{\diamond}_{k,k}\circ\beta_\diamond\circ\delta_2^k, \epsilon^{\bullet}_{k,k}\circ\beta_\bullet\circ\delta_1^k \urcorner"] \arrow[dr, "\upsilon_k^0"'] & & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet) \\ & \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[ur, "\tau_k^k(\beta)"'] & \end{tikzcd}$$

If the spinal coheroid is supposed to be $(k, 0)$-transitive, then diagram (6.19) provides another factorisation for the canonical arrow $\ulcorner \epsilon^{\diamond}_{k,k} \circ \beta_\diamond \circ \delta_2^k, \epsilon^{\bullet}_{k,k} \circ \beta_\bullet \circ \delta_1^k \urcorner : \mathbb{S}_k \to \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)$ depending on some arrow $\beta_* : \mathbb{S}_k \to \mathbb{A}_*$ in Ω_k, so that the following diagram commutes when both factorisations are combined.

(6.39)
$$\begin{tikzcd} \mathbb{S}_k \arrow[r, "\beta_*"] \arrow[d, "\upsilon_k^0"'] & \mathbb{A}_* \arrow[d, "\upsilon_k^0(\beta)"] \\ \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\tau_k^k(\beta)"'] & \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet) \end{tikzcd}$$

$\boxed{\textit{Ind.}}$ $\triangleright$ From now on, suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega$ is $(k, m-k)$-transitive for some $m \geq k+1$. First, because the spinal coheroid is closed, diagram (6.37), which is recalled below, holds in the category $\mathcal{C}$.

$$\begin{tikzcd} \mathbb{S}_{m-1} \arrow[r, "\gamma_m"] \arrow[d, "\gamma_m"'] & \mathbb{D}_m \arrow[d, "\delta_1^m"] \arrow[dr, bend left, "d^m_{1,k}\circ\pi_k^{m-k-1}"] & \\ \mathbb{D}_m \arrow[r, "\delta_2^m"'] \arrow[rr, bend right, "d^m_{2,k}\circ\pi_k^{m-k-1}"'] & \mathbb{S}_m \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\upsilon_k^{m-k}"] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \end{tikzcd}$$

Post-composing the previous diagram with the tau construction

$$\tau_k^k(\beta) : \mathbb{G}_k^k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \to \mathbb{G}_k^k(\beta_\diamond, \beta, \beta_\bullet)$$

then provides another commutative diagram as follows.

$$\begin{tikzcd} \mathbb{S}_{m-1} \arrow[r, "\gamma_m"] \arrow[d, "\gamma_m"'] & \mathbb{D}_m \arrow[d, "\delta_1^m"] \arrow[dr, bend left, "\tau^{m-1}_{1,k}(\beta)\circ\pi_k^{m-k-1}"] & \\ \mathbb{D}_m \arrow[r, "\delta_2^m"'] \arrow[rr, bend right, "\tau^{m-1}_{2,k}(\beta)\circ\pi_k^{m-k-1}"'] & \mathbb{S}_m \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "\tau_k^m(\beta)\circ\upsilon_k^{m-k}"] & \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet) \end{tikzcd}$$

By universality of $\mathbb{S}_m$, the preceding diagram gives a factorisation as follows.

$$\begin{tikzcd} \mathbb{S}_m \arrow[rr, "\ulcorner \tau^{m-1}_{2,k}(\beta)\circ\pi_k^{m-k-1}, \tau^{m-1}_{1,k}(\beta)\circ\pi_k^{m-k-1} \urcorner"] \arrow[dr, "\upsilon_k^{m-k}"'] & & \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet) \\ & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[ur, "\tau_k^m(\beta)"'] & \end{tikzcd}$$

Since the spinal coheroid is now supposed to be $(k, m-k)$-transitive, diagram (6.25) provides another factorisation for the canonical arrow $\ulcorner \tau^{m-1}_{2,k}(\beta) \circ \pi_k^{m-k-1}, \tau^{m-1}_{1,k}(\beta) \circ \pi_k^{m-k-1} \urcorner : \mathbb{S}_m \to$

$\mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet)$ depending on some arrow $\beta_* : \mathbb{S}_k \to \mathbb{A}_*$ in Ω_m, so that the following diagram commutes when both factorisations are combined.

(6.40)
$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\beta_*"] \arrow[d, "\upsilon_k^{m-k}"'] & \mathbb{A}_* \arrow[d, "\upsilon_k^{m-k}(\beta)"] \\
\mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\tau_k^m(\beta)"'] & \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

6.3.4. Spines and their framings.

6.3.4.1. *Spines for spinal coheroids.* Let n be a non-negative integer. Any spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \mathtt{Spine} \to (\mathcal{C}, \mathcal{A})$ may be associated with a canonical node of spines of degree n consisting of the prevertebrae $p_0 := \|\gamma_0, \gamma_0\|$, $\dots$, $p_n := \|\gamma_n, \gamma_n\|$ and the class of arrows Ω_n. The prespine defined by the prevertebrae

$$\mathbb{S}_{-1} \xrightarrow{p_0} \mathbb{S}_0 \xrightarrow{p_1} \mathbb{S}_1 \xrightarrow{p_2} \dots \xrightarrow{p_{n-1}} \mathbb{S}_{n-1} \xrightarrow{p_n} \mathbb{S}_n$$

will later be denoted by P_n. The node of spines $P_n \cdot \Omega_n$ will be denoted by σ_n for every $n \in \omega$.

Proposition 6.36 (Local projectivity)**.** *For every $n \in \omega$, the node of spines σ_n is projective with respect to every surtraction of the vertebral category $(\mathcal{C}, A, E)$.*

Proof. Follows from the fact that $\mathbb{S}_{-1}$ is initial in the category $\mathcal{C}$. □

Proposition 6.37. *For every $k \in \omega$, $m \geq k$ and pair of vertebrae $v_\diamond := p_k \cdot \beta_\diamond$ and $v_\bullet := p_k \cdot \beta_\bullet$ in the node of vertebrae ν_k, if the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is*

1) *closed and $(k, m-k)$-normal, then the node of spines $p_k \cdot \Gamma_k^m$ frames itself along the pair of vertebrae $v_\diamond^{\mathrm{rv}}$ and $v_\bullet$ as follows;*
$$(p_k \cdot \Gamma_k^m, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot \Gamma_k^m$$
2) *$(k, m-k)$-normal, then the node of spines $p_k \cdot (\Gamma_k^n \gamma_{n+1})$ frames itself along the pair of vertebrae $v_\diamond^{\mathrm{rv}}$ and $v_\bullet$ as follows;*
$$(p_k \cdot (\Gamma_k^m \gamma_{m+1}), v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot (\Gamma_k^m \gamma_{m+1})$$
3) *$(k, m-k)$-transitive, then the node of spines $p_k \cdot (\Gamma_k^m \beta)$ frames itself along the pair of vertebrae $v_\diamond^{\mathrm{rv}}$ and $v_\bullet$ as follows;*
$$(p_k \cdot (\Gamma_k^m \beta), v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot (\Gamma_k^m \beta)$$

Proof. Recall that framings as given in the statement consist of pushout diagrams together with additional arrows, called cylinder transitions, making some other diagrams commute. Suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is closed and $(k, m-k)$-normal. The framing of item 1) is given by the pushout

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond \circ \delta_1^k"] & \mathbb{A}_\diamond \arrow[dd, "\jmath_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{S}_m \arrow[dr, "\jmath_{k,m}"] & \\
\mathbb{A}_\bullet \arrow[rr, "\jmath_{k,m}^\bullet"'] & & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

and the cylinder transition is given by the arrow $\upsilon_k^{m-k} : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ (defined by induction in section 6.3.3.2), which makes the following diagram commute.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_2^k"] & & \mathbb{A}_\diamond \arrow[dd, "\jmath_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & \mathbb{S}_m \arrow[drr, "\upsilon_k^{m-k}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\jmath_{k,m}^\bullet"'] & & & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

Suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is $(k, m-k)$-normal. The framing of item 2) is given by the pushout

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1} \circ \Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{D}_{m+1} \arrow[drr, "\iota_{k,m}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,m}^\bullet"'] & & & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

and the cylinder transition is given by the arrow $\pi_k^{m-k} : \mathbb{D}_{m+1} \to \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)$ (defined by induction in section 6.3.2.1), which makes the following diagram commute.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\gamma_{m+1} \circ \Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_2^k"] & & \mathbb{A}_\diamond \arrow[dd, "\iota_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_1^k"'] & \mathbb{D}_{m+1} \arrow[drr, "\pi_k^{m-k}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\iota_{k,m}^\bullet"'] & & & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

Finally, suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is $(k, m-k)$-transitive. The framing of item 3) is given by the pushout

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta \circ \Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[rr, "\beta_\diamond \circ \delta_1^k"] & & \mathbb{A}_\diamond \arrow[dd, "\epsilon_{k,m}^\diamond"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet \circ \delta_2^k"'] & \mathbb{A} \arrow[drr, "\epsilon_{k,m}"] & & \\
\mathbb{A}_\bullet \arrow[rrr, "\epsilon_{k,m}^\bullet"'] & & & \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

and the cylinder transition is given by the arrow $\upsilon_k^{m-k}(\beta) : \mathbb{A}_* \to \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet)$ (defined by induction in section 6.3.2.2), which makes the following diagram commute for some given

$\beta_* : \mathbb{S}_m \to \mathbb{A}_*$ in Ω_m.

$$\begin{tikzcd}
\mathbb{S}_k \arrow[dr, "\beta_*\circ\Gamma_k^m"] & \mathbb{D}_k \arrow[l, "\delta_2^k"'] \arrow[r, "\beta_\diamond\circ\delta_2^k"] & \mathbb{A}_\diamond \arrow[dd, "\epsilon^\diamond_{k,m}"] \\
\mathbb{D}_k \arrow[u, "\delta_1^k"] \arrow[d, "\beta_\bullet\circ\delta_1^k"'] & \mathbb{A}_* \arrow[dr, "\upsilon_k^{m-k}(\beta)"] & \\
\mathbb{A}_\bullet \arrow[rr, "\epsilon^\bullet_{k,m}"'] & & \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

This finishes the proof. □

6.3.4.2. *Functoriality of framings.* Let n be a non-negative integer and $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega : \mathtt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a (k,ω)-transitive and closed spinal coheroid. Consider some spine s of degree n in the node of spines σ_n. If the spine s is of the form $(p_k)\cdot\beta$, then the following diagram in $\mathbf{Vert}(\mathcal{C})$ will be denoted by Σ_s^k.

$$\begin{tikzcd}
p_k\cdot\Gamma_k^k & p_k\cdot(\Gamma_k^k\gamma_{k+1}) \arrow[l, "\gamma_{k+1}"'] & & \\
p_k\cdot(\Gamma_k^k\gamma_{k+1}) \arrow[u, "\gamma_{k+1}"] & p_k\cdot\Gamma_k^{k+1} \arrow[l, "\delta_2^{k+1}"] \arrow[u, "\delta_1^{k+1}"'] & & \\
& & \ddots & \\
& & p_k\cdot\Gamma_k^{n-1} & p_k\cdot(\Gamma_k^{n-1}\gamma_n) \arrow[l, "\gamma_n"'] \\
& & p_k\cdot(\Gamma_k^{n-1}\gamma_n) \arrow[u, "\gamma_n"] & p_k\cdot\Gamma_k^n \arrow[l, "\delta_2^n"] \arrow[u, "\delta_1^n"'] \\
& & & p_k\cdot(\Gamma_k^n\beta) \arrow[u, "\beta"]
\end{tikzcd}$$

Later on, the diagram Σ_s^k will be regarded as a functor $\mathtt{Z} \to \mathbf{Vert}(\mathcal{C})$ where $\mathtt{Z}$ is the free category over a graph of the preceding form where the squares commute. The prespine $(p_k)_{0\leq k\leq n}$ of degree n will be denoted by P_n.

Proposition 6.38. *For every $k \in \omega$, $n \geq k$ and spine $P_n\cdot\beta$ in σ_n, there exists a stem $\beta_* \in \Omega_n$ such that the functor $\Sigma^k_{P_n\cdot\beta_*}$ frames the functor $\Sigma^k_{P_n\cdot\beta}$ along any vertebra $\upsilon_\diamond^{\mathrm{rv}}$ in ν_k^{rv} and $\upsilon_\bullet$ in ν_k, which may be written as follows;*

$$(\Sigma^k_{P_n\cdot\beta}, \upsilon_\diamond^{\mathrm{rv}}, \upsilon_\bullet) \triangleright \Sigma^k_{P_n\cdot\beta_*}$$

Proof. The morphisms of framings of the form

$$(p_k\cdot(\Gamma_k^m\gamma_{m+1}), \upsilon_\diamond^{\mathrm{rv}}, \upsilon_\bullet) \triangleright p_k\cdot(\Gamma_k^m\gamma_{m+1}) \curvearrowright (p_k\cdot\Gamma_k^m, \upsilon_\diamond^{\mathrm{rv}}, \upsilon_\bullet) \triangleright p_k\cdot\Gamma_k^m$$

are provided by the pair $(\gamma_{m+1}, \gamma_{m+1})$ as well as diagram (6.30), diagram (6.33) and diagram (6.38), which may be compressed into the following one.

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\upsilon_k^{m-k}"'] & \mathbb{D}_{m+1} \arrow[d, "\pi_k^{m-k}"] \\
\mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\kappa_k^m"'] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)
\end{tikzcd}$$

The next morphisms of framings of the form

$$(p_k \cdot \Gamma_k^{m+1}, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot \Gamma_k^{m+1} \curvearrowright (p_k \cdot (\Gamma_k^m \gamma_{m+1}), v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot (\Gamma_k^m \gamma_{m+1})$$

are provided by the pairs $(\delta_1^{m+1}, \delta_1^{m+1})$ and $(\delta_2^{m+1}, \delta_2^{m+1})$ as well as diagram (6.32) and diagram (6.37), which give the following commutative diagram for every $i \in \{1, 2\}$.

$$\begin{tikzcd}
\mathbb{D}_{m+1} \arrow[r, "\delta_{i,k}^{m+1}"] \arrow[d, "\pi_k^{m-k}"'] & \mathbb{S}_{m+1} \arrow[d, "v_k^{m-k+1}"] \\
\mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) \arrow[r, "d_{i,k}^{m+1}"'] & \mathbb{G}_k^{m+1}(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

The equation $d_{1,k}^{m+1} \circ \kappa_k^m = d_{2,k}^{m+1} \circ \kappa_m^k$ obtained in (6.27) for every $k \in \omega$ and $m \geq k$ then implies that the following diagram commutes in $\mathbf{Vert}(\mathcal{C})$.

$$\begin{tikzcd}
p_k \cdot \Gamma_k^m \arrow[r, "\gamma_{m+1}"] \arrow[d, "\gamma_{m+1}"'] & p_k \cdot (\Gamma_k^m \gamma_{m+1}) \arrow[d, "\delta_1^{m+1}"] \\
p_k \cdot (\Gamma_k^m \gamma_{m+1}) \arrow[r, "\delta_2^{m+1}"'] & p_k \cdot \Gamma_k^{m+1}
\end{tikzcd}$$

Finally, the morphism of framings of the form

$$(p_k \cdot (\Gamma_k^m \beta), v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot (\Gamma_k^m \beta) \curvearrowright (p_k \cdot \Gamma_k^m, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k \cdot \Gamma_k^m$$

is provided by a pair (β, β_*), where β_* is given by transitivity and the commutative diagram, below, which follows from diagram (6.39) when $m = k$ and diagram (6.40) otherwise.

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\beta_*"] \arrow[d, "v_k^{m-k}"'] & \mathbb{A}_* \arrow[d, "v_k^{m-k}(\beta)"] \\
\mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[r, "\tau_k^m(\beta)"'] & \mathbb{G}_k^m(\beta_\diamond, \beta, \beta_\bullet)
\end{tikzcd}$$

□

By the axiom of choice, for every arrow $\beta : \mathbb{S}_n \to \mathbb{A}$ in Ω_k, we may choose an arrow $\psi_\beta : \mathbb{S}_n \to \mathbb{A}_*$ in Ω_k such that the following framing holds.

$$(\Sigma_{P_n \cdot \beta}^k, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright \Sigma_{P_n \cdot \psi_\beta}^k$$

These choices will be expressed in terms of a metafunction $\psi : \Omega_k \to \Omega_k$ mapping the arrow β to the choice ψ_β. For such any metafunction $\psi : \Omega_k \to \Omega_k$, the preceding collection of framings will be called a *framing of the spine* σ_n *by itself along the pair of vertebrae* $v_\diamond^{\mathrm{rv}}$ *and* $v_\bullet$ and will be referred to by the expression:

$$(\Sigma_{\sigma_n}^k, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright \Sigma_{\sigma_n}^k$$

The underlying metafunction $\psi : \Omega_k \to \Omega_k$ will then be referred to as a *framing gear* for the framing of the spine σ_n by itself along the pair of vertebrae $v_\diamond^{\mathrm{rv}}$ and $v_\bullet$.

6.4. Spinal structure

6.4.1. Symmetric and coherent spinal coheroids.

6.4.1.1. *Symmetric spinal coheroids.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a spinal coheroid and k be some integer in ω. By definition, any arrow $\beta : \mathbb{S}_k \to \mathbb{A}$ in Ω_k gives rise to a vertebra $p_k \cdot \beta$ in ν_k, for which the following diagram commutes.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, "\gamma_k"] \arrow[d, "\gamma_k"'] & \mathbb{D}_k \arrow[d, "\beta\circ\delta_2^k"] \\
\mathbb{D}_k \arrow[r, "\beta\circ\delta_1^k"'] & \mathbb{A}
\end{tikzcd}$$

This commutative square provides a parallel pair of arrows in $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will then be said to be *symmetric* if the parallel pair

$$(\beta \circ \delta_1^k, \beta \circ \delta_2^k)$$

is admissible for every $\beta \in \Omega_k$ so that there exist an arrow $\beta_* : \mathbb{S}_k \to \mathbb{A}_*$ in Ω_k and an morphism $\xi(\beta) : \mathbb{A}_* \to \mathbb{A}$ making the following diagram commute.

(6.41)
$$\begin{tikzcd}
\mathbb{A}_* \arrow[dr, dashed, "\xi(\beta)"] & \\
\mathbb{S}_k \arrow[u, "\beta_*"] \arrow[r, "\lceil\beta\circ\delta_1^k,\beta\circ\delta_2^k\rceil"'] & \mathbb{A}
\end{tikzcd}$$

For every $\beta \in \Omega_k$, we may choose a particular arrow $\beta_* \in \Omega_k$ so that we obtain a metafunction $\phi : \Omega_k \to \Omega_k$ mapping β to the choice β_*. Because diagram (6.42) commutes, this metafunction induces two alliances of nodes of vertebrae

$$(\mathrm{id}, \varkappa_k, \xi) : \nu_k \rightsquigarrow \nu_k^{\mathrm{rv}} \qquad \text{and} \qquad (\mathrm{id}, \varkappa_k, \xi) : \nu_k^{\mathrm{rv}} \rightsquigarrow \nu_k$$

where $\varkappa_k$ denotes the canonical symmetry $\mathbb{S}_k \to \mathbb{S}_k$ as shown below.

(6.42)
$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[rr, "\gamma_k"] \arrow[dr, equal] \arrow[dd, "\gamma_k"'] & & \mathbb{D}_k \arrow[dr, equal] \arrow[dd, "\delta_2^k" near end] & & & \\
& \mathbb{S}_{k-1} \arrow[rr, "\gamma_k"] \arrow[dd, "\gamma_k" near start] & & \mathbb{D}_k \arrow[dd, "\delta_1^k"] & & \\
\mathbb{D}_k \arrow[rr, "\delta_1^k"' near end] \arrow[dr, equal] & & \mathbb{S}_k \arrow[rr, "\phi(\beta)"' near start] \arrow[dr, dashed, "\varkappa_k"] & & \mathbb{A}_* \arrow[dr, "\xi(\beta)"] & \\
& \mathbb{D}_k \arrow[rr, "\delta_2^k"'] & & \mathbb{S}_k \arrow[rr, "\beta"'] & & \mathbb{A}
\end{tikzcd}$$

Example 6.39 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \texttt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is symmetric by definition of $\mathcal{A}$ (see Example 6.12, page 282).

Example 6.40 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is symmetric. This follows from the fact that the arrow β must be equal to γ_{k+1} so that the pair $(\beta \circ \delta_1^k, \beta \circ \delta_2^k)$ belongs to $\mathcal{A}$ (see section 6.2.4.2). Factorisation (6.41) follows by definition of a spinal coheroid.

Example 6.41 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories becomes symmetric when the class of parallel arrows is augmented by all the pairs involved in factorisations (6.41). The factorisation is provided by Corollary 5.86 (small object argument).

6.4.1.2. *Coherent spinal coheroids.* This section aims at defining a general notion of coherence in a spinal coheroid from the notion of conjugation of vertebrae. Let k be some integer in ω and $(\mathbb{S},\mathbb{D},\gamma,\delta)\cdot\Omega : \mathtt{Spine} \to (\mathcal{C},\mathcal{A})$ be a symmetric closed (k,ω)-transitive spinal coheroid. Let $v_\diamond := p_k \cdot \beta_\diamond$ and $v_\bullet := p_k \cdot \beta_\bullet$ be two vertebrae in the node of vertebrae ν_k and consider the two alliances of vertebrae

$$(\mathrm{id},\varkappa_k,\xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow p_k\cdot\phi(\beta_\diamond) \quad \text{and} \quad (\mathrm{id},\varkappa_k,\xi(\beta_\bullet)) : v_\bullet \rightsquigarrow p_k^{\mathrm{rv}}\cdot\phi(\beta_\bullet)$$

stemming from the pair of alliances of nodes of vertebrae $(\mathrm{id},\varkappa_k,\xi(\beta_\diamond)) : \nu_k^{\mathrm{rv}} \rightsquigarrow \nu_k$ and $(\mathrm{id},\varkappa_k,\xi(\beta_\diamond)) : \nu_k^{\mathrm{rv}} \rightsquigarrow \nu_k$. The two vertebrae $p_k\cdot\phi(\beta_\diamond)$ and $p_k\cdot\phi(\beta_\bullet)$, which belong to the node of vertebrae ν_k, will be denoted by $v_\flat$ and $v_\dagger$, respectively.

Remark 6.42. The alliances of prevertebrae $(\mathrm{id},\varkappa_k) : p_k^{\mathrm{rv}} \rightsquigarrow (p_k^{\mathrm{rv}})^{\mathrm{rv}}$ and $(\mathrm{id},\varkappa_k) : p_k \rightsquigarrow p_k^{\mathrm{rv}}$ are conjugable with the identity alliance on p_k along the identity morphism on $\mathbb{D}_k$ as shown by the following diagram of identity morphisms.

$$\begin{tikzcd}
\mathbb{S}_{k-1} \arrow[r, equal] \arrow[d, "\gamma_k"'] & \mathbb{S}_{k-1} \arrow[r, equal] \arrow[d, "\gamma_k"] & \mathbb{S}_{k-1} \arrow[d, "\gamma_k"] \\
\mathbb{D}_k \arrow[r, equal] & \mathbb{D}_k \arrow[r, equal] & \mathbb{D}_k
\end{tikzcd}$$

It follows that the alliance $(\mathrm{id},\varkappa_k,\xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow (v_\flat^{\mathrm{rv}})^{\mathrm{rv}}$ is conjugable with any identity alliance such that the base of its underlying vertebra is p_k. Similarly, the alliance $(\mathrm{id},\varkappa_k,\xi(\beta_\bullet)) : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$ is conjugable with any identity alliance such that the base of its underlying vertebra is p_k.

$\boxed{\textit{Case } 0}$ $\triangleright$ It follows from Remark 6.42 that the triple

$$\chi_k(\gamma_{k+1}) := (\quad p_k\cdot\gamma_{k+1},\quad \mathrm{id}_{p_k\cdot\gamma_{k+1}},\quad p_k\cdot\gamma_{k+1}\quad)$$

defines a conjugation of vertebrae along the following three pairs.

$$(v_\diamond^{\mathrm{rv}}, v_\bullet) \qquad ((\mathrm{id},\varkappa_k,\xi(\beta_\diamond)),(\mathrm{id},\varkappa_k,\xi(\beta_\bullet))) \qquad (v_\flat, v_\dagger^{\mathrm{rv}})$$

More specifically, the conjugation is given by

1) the pair of vertebrae $v_\diamond^{\mathrm{rv}} := p_k^{\mathrm{rv}}\cdot\beta_\diamond$ and $v_\bullet := p_k\cdot\beta_\bullet$;
2) the following two alliances of vertebrae;

$$(\mathrm{id},\varkappa_k,\xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow (v_\flat^{\mathrm{rv}})^{\mathrm{rv}} \qquad (\mathrm{id},\varkappa_k,\xi(\beta_\bullet)) : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$$

3) the following pair of framings of vertebrae.

$$\begin{cases} (p_k\cdot\gamma_{k+1}, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright p_k\cdot\gamma_{k+1} \\ (p_k\cdot\gamma_{k+1}, v_\flat^{\mathrm{rv}}, v_\dagger) \triangleright p_k\cdot\gamma_{k+1} \end{cases}$$

It follows from the construction of section 3.3.5.5 that the previous conjugation gives rise to a strong correspondence between two copies of the vertebra $p_k\cdot\gamma_{k+1}$ via the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$, which, in the present case, may be written as a commutative square

(6.43)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\iota_{k,k}"] \\
\mathbb{D}_{k+1} \arrow[r, "\zeta_{k,k}^0\circ\pi_k^0"'] & \mathbb{G}_k^k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)
\end{tikzcd}$$

where $\zeta^0_{k,k} : \mathbb{G}^k_k(\phi(\beta_\flat), \gamma_{k+1}, \phi(\beta_\dagger)) \to \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)$ is the universal morphism making the following diagrams commute.

$$\begin{tikzcd} \mathbb{A}_\flat \arrow[r, "\iota^\diamond_{k,k}\circ\xi(\beta_\diamond)"] \arrow[dr, "\iota^\diamond_{k,k}"'] & \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \\ & \mathbb{G}^k_k(\phi(\beta_\diamond), \gamma_{k+1}, \phi(\beta_\bullet)) \arrow[u, dashed, "\zeta^0_{k,k}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}_\dagger \arrow[r, "\iota^\bullet_{k,k}\circ\xi(\beta_\bullet)"] \arrow[dr, "\iota^\bullet_{k,k}"'] & \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \\ & \mathbb{G}^k_k(\phi(\beta_\diamond), \gamma_{k+1}, \phi(\beta_\bullet)) \arrow[u, dashed, "\zeta^0_{k,k}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{D}_{k+1} \arrow[r, "\pi^0_k"] \arrow[dr, "\iota_{k,k}"'] & \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \\ & \mathbb{G}^k_k(\phi(\beta_\diamond), \gamma_{k+1}, \phi(\beta_\bullet)) \arrow[u, dashed, "\zeta^0_{k,k}"'] \end{tikzcd}$$

In particular, diagram (6.43) provides a $(k+1)$-parallel pair of arrows in the spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta)\cdot\Omega$ will be said to be $(k,0)$-*coherent* if the parallel pair of arrows

$$(\zeta^0_{k,k}\circ\pi^0_k, \iota_{k,k})$$

is admissible for every $\beta_\diamond, \beta_\bullet \in \Omega_k$ so that there exist a morphism $\widehat{\beta}_0 : \mathbb{S}_k \to \widehat{\mathbb{A}}_0$ in Ω_k and a morphism $\varpi^0_k : \widehat{\mathbb{A}}_0 \to \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)$ making the following diagram commute.

$$\begin{tikzcd} \widehat{\mathbb{A}}_0 \arrow[dr, dashed, "\varpi^0_k"] & \\ \mathbb{S}_{k+1} \arrow[u, "\widehat{\beta}_0"] \arrow[r, "\lceil\zeta^0_{k,k}\circ\pi^0_k, \iota_{k,k}\rceil"'] & \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \end{tikzcd} \tag{6.44}$$

Example 6.43 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \mathtt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is $(k,0)$-coherent by definition of $\mathcal{A}$.

Example 6.44 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is $(k,0)$-coherent since the pair $(\zeta^0_{k,k}\circ\pi^0_k, \iota_{k,k})$ belongs to the image of $\mathcal{A}$ in the category of models for the coherator.

Example 6.45 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories becomes symmetric when the class of parallel arrows is augmented by all the pairs involved in factorisations (6.44). The factorisation is provided by Corollary 5.86 (small object argument).

$\boxed{\textit{Prep. }\mathtt{I}}$ $\triangleright$ Similarly, it is possible to use the framings of vertebrae given by Proposition 6.38 to define other conjugations. For every $m \geq k$, the two triples

$$\begin{cases} \chi_k(\Gamma^m_k) & := \ (\ p_k\cdot\Gamma^m_k,\ \ \mathrm{id}_{p_k\cdot\Gamma^m_k},\ \ p_k\cdot\Gamma^m_k\ ); \\ \chi_k(\Gamma^m_k\gamma_{m+1}) & := \ (\ p_k\cdot(\Gamma^m_k\gamma_{m+1}),\ \ \mathrm{id}_{p_k\cdot(\Gamma^m_k\gamma_{m+1})},\ \ p_k\cdot(\Gamma^m_k\gamma_{m+1})\ ), \end{cases}$$

define two other conjugations of vertebrae along the following three pairs.

$$(v^{\mathrm{rv}}_\diamond, v_\bullet) \qquad ((\mathrm{id}, \varkappa_k, \xi(\beta_\diamond)), (\mathrm{id}, \varkappa_k, \xi(\beta_\bullet))) \qquad (v_\flat, v^{\mathrm{rv}}_\dagger)$$

The conjugations $\chi_k(\Gamma^m_k)$ and $\chi_k(\Gamma^m_k\gamma_{m+1})$ are determined by the following respective pairs of framings of vertebrae.

$$\begin{cases} (p_k\cdot\Gamma^m_k, v^{\mathrm{rv}}_\diamond, v_\bullet) \triangleright p_k\cdot\Gamma^m_k \\ (p_k\cdot\Gamma^m_k, v^{\mathrm{rv}}_\flat, v_\dagger) \triangleright p_k\cdot\Gamma^m_k \end{cases} \qquad \begin{cases} (p_k\cdot(\Gamma^m_k\gamma_{m+1}), v^{\mathrm{rv}}_\diamond, v_\bullet) \triangleright p_k\cdot(\Gamma^m_k\gamma_{m+1}) \\ (p_k\cdot(\Gamma^m_k\gamma_{m+1}), v^{\mathrm{rv}}_\flat, v_\dagger) \triangleright p_k\cdot(\Gamma^m_k\gamma_{m+1}) \end{cases}$$

Recall that Proposition 6.38 provides the morphism of framings $(\gamma_{m+1}, \gamma_{m+1})$ of the following form.

$$(p_k\cdot(\Gamma^m_k\gamma_{m+1}), v^{\mathrm{rv}}_\diamond, v_\bullet) \triangleright p_k\cdot(\Gamma^m_k\gamma_{m+1}) \curvearrowright (p_k\cdot\Gamma^m_k, v^{\mathrm{rv}}_\diamond, v_\bullet) \triangleright p_k\cdot\Gamma^m_k$$

This morphism induces a morphism of conjugations

$$\Big((\gamma_{m+1},\gamma_{m+1}),(\gamma_{m+1},\gamma_{m+1})\Big):\chi_k(\Gamma_k^m\gamma_{m+1})\curvearrowright\chi_k(\Gamma_k^m)$$

since it determines a pair of morphisms of framings such that the second component of the first morphism $(\gamma_{m+1},\gamma_{m+1})$ is equal to the first component of the second morphism, which in the present case turns out to be the same pair $(\gamma_{m+1},\gamma_{m+1})$. This morphism of conjugations becomes, via the functor $\mathcal{S}_{\mathrm{cor}}:\mathbf{Conj}(\mathcal{C})\to\mathbf{Scov}(\mathcal{C})$, a morphism of strong correspondences of vertebrae

$$\underbrace{(\iota_{k,m},\zeta_{k,m}^0\circ\pi_k^{m-k})}_{p_k\cdot(\Gamma_k^m\gamma_{m+1})\overset{\sim}{\asymp}p_k\cdot(\Gamma_k^m\gamma_{m+1})}\Rightarrow\underbrace{(\jmath_{k,m},\zeta_{k,m}^0(\mathrm{id})\circ\upsilon_k^{m-k})}_{p_k\cdot\Gamma_k^m\overset{\sim}{\asymp}p_k\cdot\Gamma_k^m}\tag{6.45}$$

which must be encoded by a commutative diagram of the form

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet) \arrow[d, "\kappa_k^m"'] & \mathbb{S}_m \arrow[l, "\zeta_{k,m}^0(\mathrm{id})\circ\upsilon_k^{m-k}"'] \arrow[d, "\gamma_{m+1}"] \\
\mathbb{D}_{m+1} \arrow[r, "\iota_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond,\gamma_{m+1},\beta_\bullet) & \mathbb{D}_{m+1} \arrow[l, "\zeta_{k,m}^0\circ\pi_k^{m-k}"]
\end{tikzcd}$$

where the canonical arrows

$$\begin{array}{lcl}
\zeta_{k,m}^0(\mathrm{id}) & : & \mathbb{G}_k^m(\phi(\beta_\diamond),\mathrm{id},\phi(\beta_\bullet))\to\mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet)\\
\zeta_{k,m}^0 & : & \mathbb{G}_k^m(\phi(\beta_\diamond),\gamma_{m+1},\phi(\beta_\bullet))\to\mathbb{G}_k^m(\beta_\diamond,\gamma_{m+1},\beta_\bullet)
\end{array}$$

are determined by the functor $\mathcal{S}_{\mathrm{cor}}:\mathbf{Conj}(\mathcal{C})\to\mathbf{Scov}(\mathcal{C})$.

$\boxed{\textit{Prep. }\mathtt{II}}$ $\triangleright$ Now, suppose that the inequality $m\geq k+1$ holds. Notice that for every $i\in\{1,2\}$ and every non-negative integer q satisfying the inequalities $k\leq q\leq m-1$, Proposition 6.38 provides a morphism of framings $(\Gamma_{q+1}^m\circ\delta_i^{q+1},\Gamma_{q+1}^m\circ\delta_i^{q+1})$ of the following form.

$$(p_k\cdot\Gamma_k^m,v_\diamond^{\mathrm{rv}},v_\bullet)\triangleright p_k\cdot\Gamma_k^m\curvearrowright(p_k\cdot(\Gamma_k^q\gamma_{q+1}),v_\diamond^{\mathrm{rv}},v_\bullet)\triangleright p_k\cdot(\Gamma_k^q\gamma_{q+1})$$

This morphism induces a morphism of conjugations

$$\Big((\Gamma_{q+1}^m\circ\delta_i^{q+1},\Gamma_{q+1}^m\circ\delta_i^{q+1}),(\Gamma_{q+1}^m\circ\delta_i^{q+1},\Gamma_{q+1}^m\circ\delta_i^{q+1})\Big):\chi_k(\Gamma_k^m)\curvearrowright\chi_k(\Gamma_k^q\gamma_{q+1})$$

since it determines a pair of morphisms of framings such that the second component of the first morphism $(\Gamma_{q+1}^m\circ\delta_i^{q+1},\Gamma_{q+1}^m\circ\delta_i^{q+1})$ is equal to the first component of the second morphism, which in the present case turns out to be the same morphism $(\Gamma_{q+1}^m\circ\delta_i^{q+1},\Gamma_{q+1}^m\circ\delta_i^{q+1})$. This morphism of conjugations becomes, via the functor $\mathcal{S}_{\mathrm{cor}}:\mathbf{Conj}(\mathcal{C})\to\mathbf{Scov}(\mathcal{C})$, a morphism of strong correspondences of vertebrae

$$\underbrace{(\jmath_{k,m},\zeta_{k,m}^0(\mathrm{id})\circ\upsilon_k^{m-k})}_{p_k\cdot\Gamma_k^m\overset{\sim}{\asymp}p_k\cdot\Gamma_k^m}\Rightarrow\underbrace{(\iota_{k,q},\zeta_{k,q}^0\circ\pi_k^{q-k})}_{p_k\cdot(\Gamma_k^q\gamma_{q+1})\overset{\sim}{\asymp}p_k\cdot(\Gamma_k^q\gamma_{q+1})}\quad,\tag{6.46}$$

which must be encoded by a commutative diagram of the form

$$\begin{tikzcd}
\mathbb{D}_q \arrow[r, "\iota_{k,q}"] \arrow[d, "\Gamma_{q+1}^m\circ\delta_i^q"'] & \mathbb{G}_k^q(\beta_\diamond,\gamma_{q+1},\beta_\bullet) \arrow[d, "\kappa(\Gamma_{q+1}^m\circ\delta_i^q)"'] & \mathbb{D}_q \arrow[l, "\zeta_{k,q}^0\circ\pi_k^{q-k}"'] \arrow[d, "\Gamma_{q+1}^m\circ\delta_i^q"] \\
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet) & \mathbb{S}_m \arrow[l, "\zeta_{k,m}^0(\mathrm{id})\circ\upsilon_k^{m-k}"]
\end{tikzcd}\tag{6.47}$$

where the canonical arrow $\kappa(\Gamma_{q+1}^m\circ\delta_i^q):\mathbb{G}_k^q(\beta_\diamond,\gamma_{q+1},\beta_\bullet)\to\mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet)$ is determined by the definition of the functor $\mathcal{S}_{\mathrm{cor}}:\mathbf{Conj}(\mathcal{C})\to\mathbf{Scov}(\mathcal{C})$.

Remark 6.46. By universality, because the composite $\Gamma^m_{q+1} \circ \delta^q_i$ is an alternation of arrows of the form $\gamma_{r+1} : \mathbb{S}_r \to \mathbb{D}_{r+1}$ and $\delta^{r+1}_j : \mathbb{D}_{r+1} \to \mathbb{S}_{r+1}$ for every non-negative integer r satisfying the inequalities $q \leq r \leq m$ and any choice $j \in \{1,2\}$, the arrow $\kappa(\Gamma^m_{k+1} \circ \delta^k_i)$ is equal to the composite

$$\mathbb{G}^q_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \xrightarrow{d^q_{i,k}} \mathbb{G}^{q+1}_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \xrightarrow{\kappa^{q+1}_k} \dots \xrightarrow{d^m_{j,k}} \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$$

where the dots should be completed with an alternation of arrows of the form κ^{r+1}_k and $d^{r+1}_{j,k}$ for every $q \leq r \leq m$ and any choice $j \in \{1,2\}$.

$\boxed{\textit{Reminder}}$ $\triangleright$ Similarly, for every non-negative integer q and q' satisfying the inequalities $k \leq q \leq q' - 1$, Proposition 6.38 provides a morphism of framings

$$(\gamma_{q'+1} \circ \Gamma^{q'}_{q+1} \circ \delta^{q+1}_i, \gamma_{q'+1} \circ \Gamma^{q'}_{q+1} \circ \delta^{q+1}_i)$$

of the following form.

$$(p_k \cdot (\Gamma^{q'}_k \gamma_{q'+1}), v^{\mathrm{rv}}_\diamond, v_\bullet) \triangleright p_k \cdot (\Gamma^{q'}_k \gamma_{q'+1}) \curvearrowright (p_k \cdot (\Gamma^q_k \gamma_{q+1}), v^{\mathrm{rv}}_\diamond, v_\bullet) \triangleright p_k \cdot (\Gamma^q_k \gamma_{q+1})$$

Two copies of this morphism of framings induce a morphism of conjugations

$$\chi_k(\Gamma^{q'}_k \gamma_{q'+1}) \curvearrowright \chi_k(\Gamma^q_k \gamma_{q+1})$$

This morphism of conjugations becomes, via the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$, a morphism of strong correspondences of vertebrae

$$\underbrace{(\iota_{k,q'}, \zeta^0_{k,q'} \circ \pi^{q'-k}_k)}_{p_k \cdot (\Gamma^{q'}_k \gamma_{q'+1}) \cong p_k \cdot (\Gamma^{q'}_k \gamma_{q'+1})} \quad \Rightarrow \quad \underbrace{(\iota_{k,q}, \zeta^0_{k,q} \circ \pi^{q-k}_k)}_{p_k \cdot (\Gamma^q_k \gamma_{q+1}) \cong p_k \cdot (\Gamma^q_k \gamma_{q+1})} \quad , \tag{6.48}$$

which must be encoded by a commutative diagram of the form

$$\begin{tikzcd}
\mathbb{D}_q \arrow[r, "\iota_{k,q}"] \arrow[d, "\gamma_{q'+1} \circ \Gamma^{q'}_{q+1} \circ \delta^{q+1}_i"'] & \mathbb{G}^q_k(\beta_\diamond, \gamma_{q+1}, \beta_\bullet) \arrow[d, "\kappa(\gamma_{q'+1} \circ \Gamma^{q'}_{q+1} \circ \delta^{q+1}_i)"] & \mathbb{D}_q \arrow[l, "\zeta^0_{k,q} \circ \pi^{q-k}_k"'] \arrow[d, "\gamma_{q'+1} \circ \Gamma^{q'}_{q+1} \circ \delta^{q+1}_i"] \\
\mathbb{D}_{q'+1} \arrow[r, "\iota_{k,q'}"'] & \mathbb{G}^{q'}_k(\beta_\diamond, \gamma_{q'+1}, \beta_\bullet) & \mathbb{D}_{q'+1} \arrow[l, "\zeta^0_{k,q'} \circ \pi^{q'-k}_k"]
\end{tikzcd} \tag{6.49}$$

where the canonical arrow

$$\kappa(\gamma_{q'+1} \circ \Gamma^{q'}_{q+1} \circ \delta^{q+1}_i) : \mathbb{G}^q_k(\beta_\diamond, \gamma_{q+1}, \beta_\bullet) \to \mathbb{G}^{q'}_k(\beta_\diamond, \gamma_{q'+1}, \beta_\bullet)$$

is determined by the definition of the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$. In fact, the definition of $\mathcal{S}_{\mathrm{cor}}$ requires this arrow to be equal to the following composite.

$$\mathbb{G}^q_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \xrightarrow{\kappa(\Gamma^{q'}_{q+1} \circ \delta^{q+1}_i)} \mathbb{G}^{q'}_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) \xrightarrow{\kappa^{k+1}_k} \mathbb{G}^{q'}_k(\beta_\diamond, \gamma_{q'+1}, \beta_\bullet)$$

$\boxed{\textit{Mates for } \mathtt{I}}$ $\triangleright$ We are now going to use the previous constructions to build a pair of mates for the memory induced by the morphism of correspondences of (6.45). First, because the inequality $m \geq k+1$ holds, it follows from Proposition 3.76 that the morphism of strong correspondences

$$\underbrace{(\iota_{k,m}, \zeta^0_{k,m} \circ \pi^{m-k}_k)}_{p_k \cdot (\Gamma^m_k \gamma_{m+1}) \cong p_k \cdot (\Gamma^m_k \gamma_{m+1})} \quad \Rightarrow \underbrace{(\jmath_{k,m}, \zeta^0_{k,m}(\mathrm{id}) \circ v^{m-k}_k)}_{p_k \cdot \Gamma^m_k \cong p_k \cdot \Gamma^m_k}$$

may be seen as morphism of correspondences as follows.

$$\underbrace{(\iota_{k,m}, \zeta^0_{k,m} \circ \pi^{m-k}_k)}_{p_{k+1}\cdot(\Gamma^m_{k+1}\gamma_{m+1}) \asymp p_{k+1}\cdot(\Gamma^m_{k+1}\gamma_{m+1})} \Rightarrow \underbrace{(\jmath_{k,m}, \zeta^0_{k,m}(\mathrm{id}) \circ \upsilon^{m-k}_k)}_{p_{k+1}\cdot\Gamma^m_{k+1} \asymp p_{k+1}\cdot\Gamma^m_{k+1}}$$

In the same spirit as Proposition 3.60, we may use the morphisms of correspondences given in (6.46) to provide the domain and codomain of the preceding morphism of correspondences with two mates stemming from the following vertebra induced by factorisation (6.44).

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta^k_1"] \arrow[rrd, bend left, "\iota_{k,k}"] & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta^k_2"] \arrow[rrr, bend right, "\zeta^0_{k,k}\circ\pi^0_k"'] & \mathbb{S}_{k+1} \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[r, "\varpi^0_k"] & \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet)
\end{tikzcd}$$

To do so, post-compose the earlier vertebra with the morphism

$$\kappa(\Gamma^m_{k+1} \circ \delta^k_i) : \mathbb{G}^k_k(\beta_\diamond, \gamma_{k+1}, \beta_\bullet) \to \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$$

for every $i \in \{1, 2\}$, which leads to the following diagram by using the relations of diagram (6.47).

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta^k_1"] \arrow[rrd, bend left, "\jmath_{k,m}\circ\Gamma^m_{k+1}\circ\delta^k_i"] & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta^k_2"] \arrow[rrr, bend right, "\zeta^0_{k,m}(\mathrm{id})\circ\upsilon^{m-k}_k\circ\Gamma^m_{k+1}\circ\delta^k_i"'] & \mathbb{S}_{k+1} \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[r, "\kappa(\Gamma^m_{k+1}\circ\delta^k_i)\circ\varpi^0_k"] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

The pair of vertebrae given by the previous diagram for $i = 1$ and $i = 2$ does define a pair of mates, but at this stage, no framing may be defined along this pair. To rectify this, we need to use the fact that the spinal coheroid is symmetric and consider the following pair of mates.

(6.50)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta^k_2"] \arrow[rrd, bend left, "\jmath_{k,m}\circ\Gamma^m_{k+1}\circ\delta^k_2"] & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta^k_1"] \arrow[rrr, bend right, "\zeta^0_{k,m}(\mathrm{id})\circ\upsilon^{m-k}_k\circ\Gamma^m_{k+1}\circ\delta^k_2"'] & \mathbb{S}_{k+1} \arrow[r, "\phi(\widehat{\beta}_0)"] & \widehat{\mathbb{A}} \arrow[r, "\kappa(\Gamma^m_{k+1}\circ\delta^k_2)\circ\varpi^0_k\circ\xi(\widehat{\beta}_0)"] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

(6.51)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta^k_1"] \arrow[rrd, bend left, "\jmath_{k,m}\circ\Gamma^m_{k+1}\circ\delta^k_1"] & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta^k_2"] \arrow[rrr, bend right, "\zeta^0_{k,m}(\mathrm{id})\circ\upsilon^{m-k}_k\circ\Gamma^m_{k+1}\circ\delta^k_1"'] & \mathbb{S}_{k+1} \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[r, "\kappa(\Gamma^m_{k+1}\circ\delta^k_1)\circ\varpi^0_k"] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)
\end{tikzcd}$$

Notice that the vertebra appearing in diagram (6.50) belongs to ν^{rv}_{k+1} while the vertebra appearing in diagram (6.51) belongs to ν_{k+1}. In the sequel, the vertebra $p^{\mathrm{rv}}_{k+1} \cdot \phi(\widehat{\beta}_0)$ of diagram (6.50) will be denoted by $\upsilon^0_\diamond$ while the vertebra $p_{k+1} \cdot \widehat{\beta}_0$ of diagram (6.51) will be denoted by $v^0_\bullet$. The pair of vertebra $(v^0_\diamond, v^0_\bullet)$ then induces a pair of mates $\mu^{0,m}_k(\mathrm{id})$ for the codomain of the following morphism of correspondences.

$$\underbrace{\underbrace{(\iota_{k,m}, \zeta^0_{k,m} \circ \pi^{m-k}_k)}_{p_{k+1}\cdot(\Gamma^m_{k+1}\gamma_{m+1}) \asymp p_{k+1}\cdot(\Gamma^m_{k+1}\gamma_{m+1})}}_{=:c^{0,m}_k} \Rightarrow \underbrace{\underbrace{(\jmath_{k,m}, \zeta^0_{k,m}(\mathrm{id}) \circ v^{m-k}_k)}_{p_{k+1}\cdot\Gamma^m_{k+1} \asymp p_{k+1}\cdot\Gamma^m_{k+1}}}_{=:c^{0,m}_k(\mathrm{id})} \tag{6.52}$$

By Proposition 3.60, the pair of mates is also transferred to the domain of the preceding morphism in the form of another pair of mates $\mu^{0,m}_k$. By Proposition 6.38, the framing $(\Sigma^k_{\sigma_m}, v^0_\diamond, v^0_\bullet) \rhd \Sigma^k_{\sigma_m}$ exists. In particular, this framing involves the following two framings.

$$\begin{cases} (p_{k+1} \cdot \Gamma^m_{k+1}, v^0_\diamond, v^0_\bullet) & \rhd \;\; p_{k+1} \cdot \Gamma^m_{k+1} \\ (p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1}), v^0_\diamond, v^0_\bullet) & \rhd \;\; p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1}) \end{cases}$$

These two framings of vertebrae imply that

1) the vertebra $p_{k+1} \cdot \Gamma^m_{k+1}$ frames the pair $(c^{0,m}_k(\mathrm{id}), \mu^{0,m}_k(\mathrm{id}))$;
2) the vertebra $p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1})$ frames the pair $(c^{0,m}_k, \mu^{0,m}_k)$;

By the definitions of section 3.3.4.6 and Proposition 3.61, the preceding framings induce

1) a span of correspondences $((c^{0,m}_k(\mathrm{id}), \mu^{0,m}_k(\mathrm{id})), c^{1,m}_k(\mathrm{id}))$ where $c^{1,m}_k(\mathrm{id})$ is a strong correspondence of vertebrae of the form

$$(\mathrm{id}_{p_{k+1}}, \jmath_{k,m}, u^{1,m}_k(\mathrm{id})) \vdash p_{k+1} \cdot \Gamma^m_{k+1} \underset{\sim}{\asymp} p_{k+1} \cdot \Gamma^m_{k+1}$$

consisting of the identity alliance of prevertebrae over p_{k+1}, the canonical arrow $\jmath_{k,m} : \mathbb{S}_m \to \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ and the canonical arrow $u^{1,m}_k(\mathrm{id}) : \mathbb{S}_m \to \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ encoded by the following pushout arrow.

$$\ulcorner \underbrace{\kappa(\Gamma^m_{k+1} \circ \delta^k_2) \circ \varpi^0_k \circ \xi(\widehat{\beta}_0)}_{\textit{first mate}} \left(\zeta^0_{k,m}(\mathrm{id}) \circ v^{m-k}_k\right) \underbrace{\kappa(\Gamma^m_{k+1} \circ \delta^k_1) \circ \varpi^0_k}_{\textit{second mate}} \urcorner$$

2) a span of correspondences $((c^{0,m}_k, \mu^{0,m}_k), c^{1,m}_k)$ where $c^{1,m}_k$ is a strong correspondence of vertebrae of the form

$$(\mathrm{id}_{p_{k+1}}, \iota_{k,m}, u^{1,m}_k) \vdash p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1}) \underset{\sim}{\asymp} p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1})$$

consisting of the identity alliance of prevertebrae over p_{k+1}, the canonical arrow $\iota_{k,m} : \mathbb{D}_{m+1} \to \mathbb{G}^m_k(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)$ and the canonical arrow $u^{1,m}_k : \mathbb{D}_{m+1} \to \mathbb{G}^m_k(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)$ encoded by the following pushout arrow.

$$\ulcorner \underbrace{\kappa^m_k \circ \kappa(\Gamma^m_{k+1} \circ \delta^k_2) \circ \varpi^0_k \circ \xi(\widehat{\beta}_0)}_{\textit{first mate}} \left(\zeta^0_{k,m} \circ \pi^{m-k}_k\right) \underbrace{\kappa^m_k \circ \kappa(\Gamma^m_{k+1} \circ \delta^k_1) \circ \varpi^0_k}_{\textit{second mate}} \urcorner$$

Recall that the conventions on correspondences were made so that the preceding two spans define two framings of correspondences as follows.

$$(c^{0,m}_k(\mathrm{id}), \mu^{0,m}_k(\mathrm{id})) \rhd c^{1,m}_k(\mathrm{id}) \qquad\qquad (c^{0,m}_k, \mu^{0,m}_k) \rhd c^{1,m}_k$$

The morphism of correspondences involved in diagram (6.52) also induces a morphism of framings of correspondences

$$(c^{0,m}_k, \mu^{0,m}_k) \rhd c^{1,m}_k \Rightarrow (c^{0,m}_k(\mathrm{id}), \mu^{0,m}_k(\mathrm{id})) \rhd c^{1,m}_k(\mathrm{id})$$

when equipped with the morphism of framing $(\gamma_{m+1}, \gamma_{m+1})$ of the following form.

$$(p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1}), v^0_\diamond, v^0_\bullet) \rhd p_{k+1} \cdot (\Gamma^m_{k+1}\gamma_{m+1}) \curvearrowright (p_{k+1} \cdot \Gamma^m_{k+1}, v^0_\diamond, v^0_\bullet) \rhd p_{k+1} \cdot \Gamma^m_{k+1}$$

By the definitions of section 3.3.4.9 and Proposition 3.62, the above data induces a morphism of spans of correspondences as follows.

$$((c_k^{0,m},\mu_k^{0,m}),c_k^{1,m}) \Rightarrow ((c_k^{0,m}(\mathrm{id}),\mu_k^{0,m}(\mathrm{id})),c_k^{1,m}(\mathrm{id}))$$

In particular, the previous reasoning gave a morphism of strong correspondences

$$\underbrace{\underbrace{(\iota_{k,m},u_k^{1,m})}_{p_{k+1}\cdot(\Gamma^m_{k+1}\gamma_{m+1})\overset{\approx}{\frown}p_{k+1}\cdot(\Gamma^m_{k+1}\gamma_{m+1})}}_{=:c_k^{1,m}} \quad\Rightarrow\quad \underbrace{\underbrace{(\jmath_{k,m},u_k^{1,m}(\mathrm{id}))}_{p_{k+1}\cdot\Gamma^m_{k+1}\overset{\approx}{\frown}p_{k+1}\cdot\Gamma^m_{k+1}}}_{=:c_k^{1,m}(\mathrm{id})}$$

that is encoded by a commutative diagram of the following form.

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{G}_k^m(\beta_\diamond,\mathrm{id},\beta_\bullet) \arrow[d, "\kappa_k^m"'] & \mathbb{S}_m \arrow[l, "u_k^{1,m}(\mathrm{id})"'] \arrow[d, "\gamma_{m+1}"] \\
\mathbb{D}_{m+1} \arrow[r, "\iota_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond,\gamma_{m+1},\beta_\bullet) & \mathbb{D}_{m+1} \arrow[l, "u_k^{1,m}"]
\end{tikzcd}$$

$\boxed{\textit{Mates for } \texttt{II}}$ $\triangleright$ We are now going to build a pair of mates for the memory induced by the morphism of correspondences (6.46). In the case where the inequalities $q \geq k+1$ and $m \geq k+1$ holds, Proposition 3.76 implies that the morphism of strong correspondences

$$\underbrace{(\jmath_{k,m},\zeta^0_{k,m}(\mathrm{id})\circ\upsilon_k^{m-k})}_{p_k\cdot\Gamma_k^m\overset{\approx}{\frown}p_k\cdot\Gamma_k^m} \Rightarrow \underbrace{(\iota_{k,q},\zeta^0_{k,q}\circ\pi_k^{q-k})}_{p_k\cdot(\Gamma_k^q\gamma_{q+1})\overset{\approx}{\frown}p_k\cdot(\Gamma_k^q\gamma_{q+1})}$$

may be seen as a morphism of correspondences as follows.

$$\underbrace{(\jmath_{k,m},\zeta^0_{k,m}(\mathrm{id})\circ\upsilon_k^{m-k})}_{p_{k+1}\cdot\Gamma_{k+1}^m\overset{\approx}{\frown}p_{k+1}\cdot\Gamma_{k+1}^m} \Rightarrow \underbrace{(\iota_{k,q},\zeta^0_{k,q}\circ\pi_k^{q-k})}_{p_{k+1}\cdot(\Gamma_{k+1}^q\gamma_{q+1})\overset{\approx}{\frown}p_{k+1}\cdot(\Gamma_{k+1}^q\gamma_{q+1})}$$

In the same spirit as Proposition 3.60, we may use the morphisms of correspondences given in (6.48) to provide the domain and codomain of the earlier morphism of correspondences with two mates stemming from the following vertebra induced by factorisation (6.44).

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_1^k"] \arrow[rrr, bend left, "\iota_{k,k}"] & & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_2^k"] \arrow[rrrr, bend right, "\zeta^0_{k,k}\circ\pi_k^0"'] & \mathbb{S}_{k+1} \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[rr, "\varpi_k^0"] & & \mathbb{G}_k^k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)
\end{tikzcd}$$

To do so, post-compose the preceding vertebra with the morphism

$$\kappa_k^q\circ\kappa(\Gamma^q_{k+1}\circ\delta_i^k):\mathbb{G}_k^k(\beta_\diamond,\gamma_{k+1},\beta_\bullet)\to\mathbb{G}_k^q(\beta_\diamond,\gamma_{q+1},\beta_\bullet)$$

for every $i \in \{1,2\}$, which leads to the following diagram by using the relations of diagram (6.49).

$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_1^k"] \arrow[rrr, bend left, "\iota_{k,q}\circ\gamma_{q+1}\circ\Gamma^q_{k+1}\circ\delta_i^k"] & & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_2^k"] \arrow[rrrr, bend right, "\zeta^0_{k,q}\circ\upsilon_k^{q-k}\circ\gamma_{q+1}\circ\Gamma^q_{k+1}\circ\delta_i^k"'] & \mathbb{S}_{k+1} \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[rr, "\kappa_k^q\circ\kappa(\Gamma^q_{k+1}\circ\delta_i^k)\circ\varpi_k^0"] & & \mathbb{G}_k^q(\beta_\diamond,\gamma_{q+1},\beta_\bullet)
\end{tikzcd}$$

The pair of vertebrae given by the previous diagram for $i = 1$ and $i = 2$ does define a pair of mates, but at this stage, no framing may be defined along this pair. To rectify this, we need to use the fact that the spinal coheroid is symmetric and consider the following pair of mates.

(6.53)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_2^k"] \arrow[rrd, bend left, "\iota_{k,q}\circ\gamma_{q+1}\circ\Gamma_{k+1}^q\circ\delta_2^k"] & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_1^k"] \arrow[rrr, bend right, "\zeta_{k,q}^0\circ\upsilon_k^{q-k}\circ\gamma_{q+1}\circ\Gamma_{k+1}^q\circ\delta_2^k"'] & \mathbb{S}_{k+1} \arrow[r, "\phi(\widehat{\beta}_0)"] & \widehat{\mathbb{A}} \arrow[r, "\kappa_k^q\circ\kappa(\Gamma_{k+1}^q\circ\delta_2^k)\circ\varpi_k^0\circ\xi(\widehat{\beta}_0)"] & \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)
\end{tikzcd}$$

(6.54)
$$\begin{tikzcd}
\mathbb{S}_k \arrow[r, "\gamma_{k+1}"] \arrow[d, "\gamma_{k+1}"'] & \mathbb{D}_{k+1} \arrow[d, "\delta_1^k"] \arrow[rrd, bend left, "\iota_{k,q}\circ\gamma_{q+1}\circ\Gamma_{k+1}^q\circ\delta_1^k"] & & \\
\mathbb{D}_{k+1} \arrow[r, "\delta_2^k"] \arrow[rrr, bend right, "\zeta_{k,q}^0\circ\upsilon_k^{q-k}\circ\gamma_{q+1}\circ\Gamma_{k+1}^q\circ\delta_1^k"'] & \mathbb{S}_{k+1} \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[r, "\kappa_k^q\circ\kappa(\Gamma_{k+1}^q\circ\delta_1^k)\circ\varpi_k^0"] & \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)
\end{tikzcd}$$

Notice that the vertebra appearing in diagram (6.53) is exactly $\upsilon_\diamond^0$ while the vertebra appearing in diagram (6.54) is exactly $\upsilon_\bullet^0$. The pair of mates $\mu_k^{0,q}$ associated with $(\upsilon_\diamond^0, \upsilon_\bullet^0)$ is therefore a pair of mates for the codomain of the following morphism of correspondences.

(6.55)
$$\underbrace{\underbrace{(\jmath_{k,m}, \zeta_{k,m}^0(\mathrm{id}) \circ \upsilon_k^{m-k})}_{p_{k+1}\cdot\Gamma_{k+1}^m \asymp p_{k+1}\cdot\Gamma_{k+1}^m}}_{=:c_k^{0,m}(\mathrm{id})} \Rightarrow \underbrace{\underbrace{(\iota_{k,q}, \zeta_{k,q}^0 \circ \pi_k^{q-k})}_{p_{k+1}\cdot(\Gamma_{k+1}^q\gamma_{q+1}) \asymp p_{k+1}\cdot(\Gamma_{k+1}^q\gamma_{q+1})}}_{=:c_k^{0,q}}$$

By Proposition 3.60, the pair of mates $\mu_k^{0,q}$ is transferred to the domain of the above morphism in the form of the pair of mates $\mu_k^{0,m}(\mathrm{id})$. The framing $(\Sigma_{\sigma_m}^k, \upsilon_\diamond^0, \upsilon_\bullet^0) \rhd \Sigma_{\sigma_m}^k$ given by Proposition 6.38 involves the following two framings.

$$\begin{cases} (p_{k+1} \cdot \Gamma_{k+1}^m, \upsilon_\diamond^0, \upsilon_\bullet^0) & \rhd \quad p_{k+1} \cdot \Gamma_{k+1}^m \\ (p_{k+1} \cdot (\Gamma_{k+1}^q\gamma_{q+1}), \upsilon_\diamond^0, \upsilon_\bullet^0) & \rhd \quad p_{k+1} \cdot (\Gamma_{k+1}^q\gamma_{q+1}) \end{cases}$$

These two framings of vertebrae imply that

1) the vertebra $p_{k+1} \cdot \Gamma_{k+1}^m$ frames the pair $(c_k^{0,m}(\mathrm{id}), \mu_k^{0,m}(\mathrm{id}))$;
2) the vertebra $p_{k+1} \cdot (\Gamma_{k+1}^q\gamma_{q+1})$ frames the pair $(c_k^{0,q}, \mu_k^{0,q})$;

By the definitions of section 3.3.4.6 and Proposition 3.61, the above framings induce

1) a span of correspondences $((c_k^{0,m}(\mathrm{id}), \mu_k^{0,m}(\mathrm{id})), c_k^{1,m}(\mathrm{id}))$ where $c_k^{1,m}(\mathrm{id})$ is a strong correspondence of vertebrae of the form
$$(\mathrm{id}_{p_{k+1}}, \jmath_{k,m}, u_k^{1,m}(\mathrm{id})) \vdash p_{k+1} \cdot \Gamma_{k+1}^m \stackrel{\sim}{\asymp} p_{k+1} \cdot \Gamma_{k+1}^m$$
consisting of the identity alliance of prevertebrae over p_{k+1}, the canonical arrow $\jmath_{k,m} : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ and the canonical arrow $u_k^{1,m}(\mathrm{id}) : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ encoded by the following pushout arrow.
$$\ulcorner \underbrace{\kappa(\Gamma_{k+1}^m \circ \delta_2^k) \circ \varpi_k^0 \circ \xi(\widehat{\beta}_0)}_{\textit{first mate}} \left(\zeta_{k,m}^0(\mathrm{id}) \circ \upsilon_k^{m-k}\right) \underbrace{\kappa(\Gamma_{k+1}^m \circ \delta_1^k) \circ \varpi_k^0}_{\textit{second mate}} \urcorner$$

2) a span of correspondences $((c_k^{0,q}, \mu_k^{0,q}), c_k^{1,q})$ where $c_k^{1,q}$ is a strong correspondence of vertebrae of the form

$$(\mathrm{id}_{p_{k+1}}, \iota_{k,q}, u_k^{1,q}) \vdash p_{k+1} \cdot (\Gamma_{k+1}^q \gamma_{q+1}) \approx p_{k+1} \cdot (\Gamma_{k+1}^q \gamma_{q+1})$$

consisting of the identity alliance of prevertebrae over p_{k+1}, the canonical arrow $\iota_{k,q} : \mathbb{D}_{q+1} \to \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)$ and the canonical arrow $u_k^{1,q} : \mathbb{D}_{q+1} \to \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)$ encoded by the following pushout arrow.

$$\ulcorner \underbrace{\kappa_k^q \circ \kappa(\Gamma_{k+1}^q \circ \delta_2^k) \circ \varpi_k^0 \circ \xi(\widehat{\beta}_0)}_{\textit{first mate}} \left(\zeta_{k,q}^0 \circ \pi_k^{q-k}\right) \underbrace{\kappa_k^q \circ \kappa(\Gamma_{k+1}^q \circ \delta_1^k) \circ \varpi_k^0}_{\textit{second mate}} \urcorner$$

The conventions on correspondences were made so that the preceding two spans define two framings of correspondences as follows.

$$(c_k^{0,m}(\mathrm{id}), \mu_k^{0,m}(\mathrm{id})) \triangleright c_k^{1,m}(\mathrm{id}) \qquad (c_k^{0,q}, \mu_k^{0,q}) \triangleright c_k^{1,q}$$

The morphism of correspondences involved in diagram (6.55) also induces a morphism of framings of correspondences

$$(c_k^{0,m}(\mathrm{id}), \mu_k^{0,m}(\mathrm{id})) \triangleright c_k^{1,m}(\mathrm{id}) \Rightarrow (c_k^{0,q}, \mu_k^{0,q}) \triangleright c_k^{1,q}$$

when equipped with the morphism of framing $(\Gamma_{q+1}^m \circ \delta_i^{q+1}, \Gamma_{q+1}^m \circ \delta_i^{q+1})$ of the following form.

$$(p_{k+1} \cdot \Gamma_{k+1}^m, v_\diamond^0, v_\bullet^0) \triangleright p_{k+1} \cdot \Gamma_{k+1}^m \curvearrowright (p_{k+1} \cdot (\Gamma_{k+1}^q \gamma_{q+1}), v_\diamond^0, v_\bullet^0) \triangleright p_{k+1} \cdot (\Gamma_{k+1}^q \gamma_{q+1})$$

By the definitions of section 3.3.4.9 and Proposition 3.62, the above data induces a morphism of spans of correspondences as follows.

$$((c_k^{0,m}(\mathrm{id}), \mu_k^{0,m}(\mathrm{id})), c_k^{1,m}(\mathrm{id})) \qquad ((c_k^{0,q}, \mu_k^{0,q}), c_k^{1,q})$$

In particular, the above reasoning produces a morphism of strong correspondences

$$\underbrace{\underbrace{(\jmath_{k,m}, u_k^{1,m}(\mathrm{id}))}_{p_{k+1}\cdot\Gamma_{k+1}^m \approx p_{k+1}\cdot\Gamma_{k+1}^m}}_{=:c_k^{1,m}(\mathrm{id})} \Rightarrow \underbrace{\underbrace{(\iota_{k,q}, u_k^{1,q})}_{p_{k+1}\cdot(\Gamma_{k+1}^q\gamma_{q+1}) \approx p_{k+1}\cdot(\Gamma_{k+1}^q\gamma_{q+1})}}_{=:c_k^{1,q}}$$

that is encoded by a commutative diagram of the following form.

$$\begin{tikzcd}
\mathbb{D}_q \arrow[r, "\iota_{k,q}"] \arrow[d, "\Gamma_{q+1}^m \circ \delta_i^q"'] & \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet) \arrow[d, "\kappa(\Gamma_{q+1}^m \circ \delta_i^q)"'] & \mathbb{D}_q \arrow[l, "u_k^{1,q}"'] \arrow[d, "\Gamma_{q+1}^m \circ \delta_i^q"] \\
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) & \mathbb{S}_m \arrow[l, "u_k^{1,m}(\mathrm{id})"]
\end{tikzcd}$$

$\boxed{\textit{Case } 1}$ Let us now look at the correspondence $(\iota_{k,q}, u_k^{1,q})$ in the case where the equality $q = k+1$ holds. In this case, we have the following strong correspondence of vertebrae.

$$(\iota_{k,k+1}, u_k^{1,k+1}) \vdash p_{k+1} \cdot \gamma_{k+2} \approx p_{k+1} \cdot \gamma_{k+2}$$

This correspondence is, by definition, equipped with a commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{S}_{k+1} \arrow[r, "\gamma_{k+2}"] \arrow[d, "\gamma_{k+2}"'] & \mathbb{D}_{k+2} \arrow[d, "\iota_{k,k+1}"] \\
\mathbb{D}_{k+2} \arrow[r, "u_k^{1,k+1}"'] & \mathbb{G}_k^{k+1}(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)
\end{tikzcd}$$

The previous commutative square provides a $(k+2)$-parallel pair of arrows in the spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will be said to be $(k, 1)$-*coherent* if the parallel pair of arrows

$$(u_k^{1,k+1}, \iota_{k,k+1})$$

is admissible for every $\beta_\diamond, \beta_\bullet \in \Omega_k$ so that there exist a morphism $\widehat{\beta}_1 : \mathbb{S}_{k+2} \to \widehat{\mathbb{A}}_1$ in Ω_{k+2} and a morphism $\varpi_k^1 : \widehat{\mathbb{A}}_1 \to \mathbb{G}_k^{k+1}(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)$ making the following diagram commute.

(6.56)
$$\begin{tikzcd}
\widehat{\mathbb{A}}_1 \arrow[rrd, dashed, "\varpi_k^1"] & & \\
\mathbb{S}_{k+2} \arrow[u, "\widehat{\beta}_1"] \arrow[rr, "\lceil u_k^{1,k+1}{,}\iota_{k,k+1} \rceil"'] & & \mathbb{G}_k^{k+1}(\beta_\diamond, \gamma_{k+2}, \beta_\bullet)
\end{tikzcd}$$

$\boxed{\textit{Ind.}}$ $\triangleright$ The rest of this section adapts the previous constructions to define, inductively, the notion of coherency. Consider two integers m and n satisfying the inequalities $m > n \geq k+1$. Suppose that the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ is $(k, n-k)$-coherent. By construction, we are given the parallel pair

(6.57)
$$\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\iota_{k,n}"] \\
\mathbb{D}_{n+1} \arrow[r, "u_k^{n-k,m}"'] & \mathbb{G}_k^n(\beta_\diamond, \gamma_{n+1}, \beta_\bullet)
\end{tikzcd}$$

for which there exist a morphism $\widehat{\beta}_{n-k} : \mathbb{S}_n \to \widehat{\mathbb{A}}_{n-k}$ in Ω_k and a morphism $\varpi_k^{n-k} : \widehat{\mathbb{A}}_{n-k} \to \mathbb{G}_k^n(\beta_\diamond, \gamma_{n+1}, \beta_\bullet)$ making the following diagram commute.

(6.58)
$$\begin{tikzcd}
\widehat{\mathbb{A}}_{n-k} \arrow[rrd, dashed, "\varpi_k^{n-k}"] & & \\
\mathbb{S}_{n+1} \arrow[u, "\widehat{\beta}_{n-k}"] \arrow[rr, "\lceil u_k^{n-k,m}{,}\iota_{k,n} \rceil"'] & & \mathbb{G}_k^n(\beta_\diamond, \gamma_{n+1}, \beta_\bullet)
\end{tikzcd}$$

In addition, we are provided with

- a morphism of strong correspondences of vertebrae

(6.59)
$$\underbrace{\underbrace{(\iota_{k,m}, u_k^{n-k,m})}_{p_n \cdot (\Gamma_n^m \gamma_{m+1}) \overset{\sim}{\asymp} p_n \cdot (\Gamma_n^m \gamma_{m+1})}}_{=: c_k^{n-k,m}} \Rightarrow \underbrace{\underbrace{(\jmath_{k,m}, u_k^{n-k,m}(\mathrm{id}))}_{p_n \cdot \Gamma_n^m \overset{\sim}{\asymp} p_n \cdot \Gamma_n^m}}_{=: c_k^{n-k,m}(\mathrm{id})}$$

  that is encoded by a commutative diagram of the following form.

$$\begin{tikzcd}
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[d, "\kappa_k^m"'] & \mathbb{S}_m \arrow[l, "u_k^{n-k,m}(\mathrm{id})"'] \arrow[d, "\gamma_{m+1}"] \\
\mathbb{D}_{m+1} \arrow[r, "\iota_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) & \mathbb{D}_{m+1} \arrow[l, "u_k^{n-k,m}"]
\end{tikzcd}$$

- a morphism of strong correspondences of vertebrae

(6.60)
$$\underbrace{\underbrace{(\jmath_{k,m}, u_k^{n-k,m}(\mathrm{id}))}_{p_n \cdot \Gamma_n^m \overset{\sim}{\asymp} p_n \cdot \Gamma_n^m}}_{=: c_k^{n-k,m}(\mathrm{id})} \Rightarrow \underbrace{\underbrace{(\iota_{k,q}, u_k^{n-k,q})}_{p_n \cdot (\Gamma_n^q \gamma_{q+1}) \overset{\sim}{\asymp} p_n \cdot (\Gamma_n^q \gamma_{q+1})}}_{=: c_k^{n-k,q}}$$

for every non-negative integer q satisfying the inequality $n \leq q \leq m-1$, which is encoded by a commutative diagram of the following form for every $i \in \{1,2\}$.

$$\begin{tikzcd}
\mathbb{D}_q \arrow[r, "\iota_{k,q}"] \arrow[d, "\Gamma^m_{q+1}\circ\delta^q_i"'] & \mathbb{G}^q_k(\beta_\diamond, \gamma_{q+1}, \beta_\bullet) \arrow[d, "\kappa(\Gamma^m_{q+1}\circ\delta^q_i)"'] & \mathbb{D}_q \arrow[l, "u^{n-k,q}_k"'] \arrow[d, "\Gamma^m_{q+1}\circ\delta^q_i"] \\
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"'] & \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet) & \mathbb{S}_m \arrow[l, "u^{n-k,m}_k(\mathrm{id})"]
\end{tikzcd}$$

- A fortiori, by composing the previous two morphisms for general m's and q's, a morphism of strong correspondences of vertebrae

(6.61)
$$\underbrace{\underbrace{(\iota_{k,q'}, u^{n-k,q'}_k)}_{p_n\cdot(\Gamma^{q'}_n\gamma_{q'+1})\,\tilde{\asymp}\,p_n\cdot(\Gamma^{q'}_n\gamma_{q'+1})}}_{=:c^{n-k,q'}_k} \quad\Rightarrow\quad \underbrace{\underbrace{(\iota_{k,q}, u^{n-k,q}_k)}_{p_n\cdot(\Gamma^{q}_n\gamma_{q+1})\,\tilde{\asymp}\,p_n\cdot(\Gamma^{q}_n\gamma_{q+1})}}_{=:c^{n-k,q}_k}$$

for every non-negative integers q and q' satisfying the inequality $n \leq q \leq q'-1$, which is encoded by a commutative diagram of the following form.

$$\begin{tikzcd}
\mathbb{D}_q \arrow[r, "\iota_{k,q}"] \arrow[d, "\gamma_{q'+1}\circ\Gamma^{q'}_{q+1}\circ\delta^{q+1}_i"'] & \mathbb{G}^q_k(\beta_\diamond, \gamma_{q+1}, \beta_\bullet) \arrow[d, "\kappa(\gamma_{q'+1}\circ\Gamma^{q'}_{q+1}\circ\delta^{q+1}_i)"'] & \mathbb{D}_q \arrow[l, "u^{n-k,q}_k"'] \arrow[d, "\gamma_{q'+1}\circ\Gamma^{q'}_{q+1}\circ\delta^{q+1}_i"] \\
\mathbb{D}_{q'+1} \arrow[r, "\iota_{k,q'}"'] & \mathbb{G}^{q'}_k(\beta_\diamond, \gamma_{q'+1}, \beta_\bullet) & \mathbb{D}_{q'+1} \arrow[l, "u^{n-k,q'}_k"]
\end{tikzcd}$$

We are now going to use the above constructions to build a pair of mates for the memory induced by the morphism of correspondences (6.59). First, because the inequality $m \geq n+1$ holds, it follows from Proposition 3.76 that the morphism of strong correspondences

$$\underbrace{(\iota_{k,m}, u^{n-k,m}_k)}_{p_n\cdot(\Gamma^m_n\gamma_{m+1})\,\tilde{\asymp}\,p_n\cdot(\Gamma^m_n\gamma_{m+1})} \quad\Rightarrow \underbrace{(\jmath_{k,m}, u^{n-k,m}_k(\mathrm{id}))}_{p_n\cdot\Gamma^m_n\,\tilde{\asymp}\,p_n\cdot\Gamma^m_n}$$

may be seen as morphism of correspondences as follows.

$$\underbrace{(\iota_{k,m}, u^{n-k,m}_k)}_{p_{n+1}\cdot(\Gamma^m_{n+1}\gamma_{m+1})\asymp p_{n+1}\cdot(\Gamma^m_{n+1}\gamma_{m+1})} \quad\Rightarrow\quad \underbrace{(\jmath_{k,m}, u^{n-k,m}_k(\mathrm{id}))}_{p_{n+1}\cdot\Gamma^m_{n+1}\asymp p_{n+1}\cdot\Gamma^m_{n+1}}$$

In the same spirit as Proposition 3.60, we may use the morphisms of correspondences given in (6.60) to provide the domain and codomain of the earlier morphism of correspondences with two mates stemming from the following vertebra induced by factorisation (6.58).

$$\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta^n_1"] \arrow[rrd, bend left, "\iota_{k,n}"] & & \\
\mathbb{D}_{n+1} \arrow[r, "\delta^n_2"] \arrow[rrr, bend right, "u^{n-k,m}_k"'] & \mathbb{S}_{n+1} \arrow[r, "\widehat{\beta}_{n-k}"] & \widehat{\mathbb{A}} \arrow[r, "\varpi^{n-k}_k"] & \mathbb{G}^n_k(\beta_\diamond, \gamma_{n+1}, \beta_\bullet)
\end{tikzcd}$$

To do so, post-compose the preceding vertebra with the morphism

$$\kappa(\Gamma^m_{n+1}\circ\delta^n_i) : \mathbb{G}^n_k(\beta_\diamond, \gamma_{n+1}, \beta_\bullet) \to \mathbb{G}^m_k(\beta_\diamond, \mathrm{id}, \beta_\bullet)$$

for every $i \in \{1, 2\}$, which leads to the following diagram by using the relations of diagram (6.47).

$$
\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta_1^n"] \arrow[rrr, bend left, "\jmath_{k,m}\circ\Gamma_{n+1}^m\circ\delta_i^n"] & & & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \\
\mathbb{D}_{n+1} \arrow[r, "\delta_2^n"] \arrow[rrrru, bend right, "u_k^{n-k,m}\circ\Gamma_{n+1}^m\circ\delta_i^n"'] & \mathbb{S}_{n+1} \arrow[r, "\widehat{\beta}_{n-k}"] & \widehat{\mathbb{A}} \arrow[rru, "\kappa(\Gamma_{n+1}^m\circ\delta_i^n)\circ\varpi_k^{n-k}"] & &
\end{tikzcd}
$$

The pair of vertebrae given by the previous diagram for $i = 1$ and $i = 2$ does define a pair of mates, but at this stage, no framing may be defined along this pair. To rectify this, we need to use the fact that the spinal coheroid is symmetric and consider the following pair of mates.

(6.62)
$$
\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta_2^n"] \arrow[rrr, bend left, "\jmath_{k,m}\circ\Gamma_{n+1}^m\circ\delta_2^n"] & & & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \\
\mathbb{D}_{n+1} \arrow[r, "\delta_1^n"] \arrow[rrrru, bend right, "u_k^{n-k,m}\circ\Gamma_{n+1}^m\circ\delta_2^n"'] & \mathbb{S}_{n+1} \arrow[r, "\phi(\widehat{\beta}_{n-k})"] & \widehat{\mathbb{A}} \arrow[rru, "\kappa(\Gamma_{n+1}^m\circ\delta_2^n)\circ\varpi_k^{n-k}\circ\xi(\widehat{\beta}_{n-k})"] & &
\end{tikzcd}
$$

(6.63)
$$
\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta_1^n"] \arrow[rrr, bend left, "\jmath_{k,m}\circ\Gamma_{n+1}^m\circ\delta_1^n"] & & & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \\
\mathbb{D}_{n+1} \arrow[r, "\delta_2^n"] \arrow[rrrru, bend right, "u_k^{n-k,m}\circ\Gamma_{n+1}^m\circ\delta_1^n"'] & \mathbb{S}_{n+1} \arrow[r, "\widehat{\beta}_{n-k}"] & \widehat{\mathbb{A}} \arrow[rru, "\kappa(\Gamma_{n+1}^m\circ\delta_1^n)\circ\varpi_k^{n-k}"] & &
\end{tikzcd}
$$

Notice that the vertebra appearing in diagram (6.62) belongs to ν_{n+1}^{rv} while the vertebra appearing in diagram (6.51) belongs to ν_{n+1}. In the sequel, the vertebra $p_{n+1}^{\mathrm{rv}} \cdot \phi(\widehat{\beta}_{n-k})$ of diagram (6.62) will be denoted by $v_\diamond^{n-k}$ while the vertebra $p_{n+1} \cdot \widehat{\beta}_{n-k}$ of diagram (6.51) will be denoted by $v_\bullet^{n-k}$. The pair of vertebra $(v_\diamond^{n-k}, v_\bullet^{n-k})$ then induces a pair of mates $\mu_k^{n-k,m}(\mathrm{id})$ for the codomain of the following morphism of correspondences.

(6.64)
$$
\underbrace{\underbrace{(\iota_{k,m}, u_k^{n-k,m})}_{p_{n+1}\cdot(\Gamma_{n+1}^m\gamma_{m+1}) \asymp p_{n+1}\cdot(\Gamma_{n+1}^m\gamma_{m+1})}}_{=c_k^{n-k,m}} \Rightarrow \underbrace{\underbrace{(\jmath_{k,m}, u_k^{n-k,m}(\mathrm{id}))}_{p_{n+1}\cdot\Gamma_{n+1}^m \asymp p_{n+1}\cdot\Gamma_{n+1}^m}}_{=c_k^{n-k,m}(\mathrm{id})}
$$

By Proposition 3.60, the pair of mates is also transferred to the domain of the preceding morphism giving another pair of mates $\mu_k^{n-k,m}$. By Proposition 6.38, the framing

$$(\Sigma_{\sigma_m}^k, v_\diamond^{n-k}, v_\bullet^{n-k}) \triangleright \Sigma_{\sigma_m}^k$$

exists. In particular, this framing involves the following two framings.

$$
\begin{cases}
(p_{n+1} \cdot \Gamma_{n+1}^m, v_\diamond^0, v_\bullet^0) & \triangleright \quad p_{n+1} \cdot \Gamma_{n+1}^m \\
(p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1}), v_\diamond^0, v_\bullet^0) & \triangleright \quad p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1})
\end{cases}
$$

These two framings of vertebrae imply that

1) the vertebra $p_{n+1} \cdot \Gamma_{n+1}^m$ frames the pair $(c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id}))$;
2) the vertebra $p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1})$ frames the pair $(c_k^{n-k,m}, \mu_k^{n-k,m})$;

By the definitions of section 3.3.4.6 and Proposition 3.61, the preceding framings induce

1) a span of correspondences $((c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})), c_k^{n-k+1,m}(\mathrm{id}))$ where the object $c_k^{n-k+1,m}(\mathrm{id})$ is a strong correspondence of vertebrae of the form
$$(\mathrm{id}_{p_{n+1}}, \jmath_{k,m}, u_k^{n-k+1,m}(\mathrm{id})) \vdash p_{n+1} \cdot \Gamma_{n+1}^m \stackrel{\sim}{\asymp} p_{n+1} \cdot \Gamma_{n+1}^m$$
consisting of the identity alliance of prevertebrae over p_{n+1}, the canonical arrow $\jmath_{k,m} : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ and the arrow $u_k^{n-k+1,m}(\mathrm{id}) : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ encoded by the following pushout arrow.
$$\ulcorner \underbrace{\kappa(\Gamma_{n+1}^m \circ \delta_2^n) \circ \varpi_k^{n-k} \circ \xi(\widehat{\beta}_{n-k})}_{\textit{first mate}} \left(u_k^{n-k,m}(\mathrm{id})\right) \underbrace{\kappa(\Gamma_{n+1}^m \circ \delta_1^n) \circ \varpi_k^{n-k}}_{\textit{second mate}} \urcorner$$

2) a span of correspondences $((c_k^{n-k,m}, \mu_k^{n-k,m}), c_k^{n-k+1,m})$ where the object $c_k^{n-k+1,m}$ is a strong correspondence of vertebrae of the form
$$(\mathrm{id}_{p_{n+1}}, \iota_{k,m}, u_k^{n-k+1,m}) \vdash p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1}) \stackrel{\sim}{\asymp} p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1})$$
consisting of the identity alliance of prevertebrae over p_{n+1}, the canonical arrow $\iota_{k,m} : \mathbb{D}_{m+1} \to \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)$ and the arrow $u_k^{n-k+1,m} : \mathbb{D}_{m+1} \to \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet)$ encoded by the following pushout arrow.
$$\ulcorner \underbrace{\kappa_k^m \circ \kappa(\Gamma_{n+1}^m \circ \delta_2^n) \circ \varpi_k^{n-k} \circ \xi(\widehat{\beta}_{n-k})}_{\textit{first mate}} \left(u_k^{n-k,m}\right) \underbrace{\kappa_k^m \circ \kappa(\Gamma_{n+1}^m \circ \delta_1^n) \circ \varpi_k^{n-k}}_{\textit{second mate}} \urcorner$$

Recall that the conventions on correspondences were made so that the preceding two spans define two framings of correspondences as follows.
$$(c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})) \triangleright c_k^{n-k+1,m}(\mathrm{id}) \qquad (c_k^{n-k,m}, \mu_k^{n-k,m}) \triangleright c_k^{n-k+1,m}$$
The morphism of correspondences involved in diagram (6.64) also induces a morphism of framings of correspondences
$$(c_k^{n-k,m}, \mu_k^{n-k,m}) \triangleright c_k^{n-k+1,m} \Rightarrow (c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})) \triangleright c_k^{n-k+1,m}(\mathrm{id})$$
when equipped with the morphism of framing $(\gamma_{m+1}, \gamma_{m+1})$ of the following form.
$$\begin{array}{c} (p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1}), v_\diamond^{n-k}, v_\bullet^{n-k}) \triangleright p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1}) \\ \curvearrowright \\ (p_{n+1} \cdot \Gamma_{n+1}^m, v_\diamond^{n-k}, v_\bullet^{n-k}) \triangleright p_{n+1} \cdot \Gamma_{n+1}^m \end{array}$$
By the definitions of section 3.3.4.9 and Proposition 3.62, the previous data induces a morphism of spans of correspondences as follows.
$$((c_k^{n-k,m}, \mu_k^{n-k,m}), c_k^{n-k+1,m}) \Rightarrow ((c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})), c_k^{n-k+1,m}(\mathrm{id}))$$
In particular, the previous reasoning gives a morphism of strong correspondences
$$\underbrace{\underbrace{(\iota_{k,m}, u_k^{n-k+1,m})}_{p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1}) \stackrel{\sim}{\asymp} p_{n+1} \cdot (\Gamma_{n+1}^m \gamma_{m+1})}}_{=: c_k^{n-k+1,m}} \Rightarrow \underbrace{\underbrace{(\jmath_{k,m}, u_k^{n-k+1,m}(\mathrm{id}))}_{p_{n+1} \cdot \Gamma_{n+1}^m \stackrel{\sim}{\asymp} p_{n+1} \cdot \Gamma_{n+1}^m}}_{=: c_k^{n-k+1,m}(\mathrm{id})}$$
that is encoded by a commutative diagram of the following form.
$$\begin{tikzcd} \mathbb{S}_m \arrow[r, "\jmath_{k,m}"] \arrow[d, "\gamma_{m+1}"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) \arrow[d, "\kappa_k^m"'] & \mathbb{S}_m \arrow[l, "u_k^{n-k+1,m}(\mathrm{id})"'] \arrow[d, "\gamma_{m+1}"] \\ \mathbb{D}_{m+1} \arrow[r, "\iota_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond, \gamma_{m+1}, \beta_\bullet) & \mathbb{D}_{m+1} \arrow[l, "u_k^{n-k+1,m}"] \end{tikzcd}$$

We are now going to build a pair of mates for the memory induced by the morphism of correspondences (6.60). In the case where the inequalities $q \geq n+1$ and $m \geq n+1$ holds, Proposition 3.76 implies that the morphism of strong correspondences

$$\underbrace{(\jmath_{k,m}, u_k^{n-k,m}(\mathrm{id}))}_{p_n \cdot \Gamma_n^m \stackrel{\simeq}{\leftarrow} p_n \cdot \Gamma_n^m} \Rightarrow \underbrace{(\iota_{k,q}, u_k^{n-k,q})}_{p_n \cdot (\Gamma_n^q \gamma_{q+1}) \stackrel{\simeq}{\leftarrow} p_n \cdot (\Gamma_n^q \gamma_{q+1})}$$

may be seen as morphism of correspondences as follows.

$$\underbrace{(\jmath_{k,m}, u_k^{n-k,m}(\mathrm{id}))}_{p_{n+1} \cdot \Gamma_{n+1}^m \stackrel{\simeq}{\leftarrow} p_{n+1} \cdot \Gamma_{n+1}^m} \Rightarrow \underbrace{(\iota_{k,q}, u_k^{n-k,q})}_{p_{n+1} \cdot (\Gamma_{n+1}^q \gamma_{q+1}) \stackrel{\simeq}{\leftarrow} p_{n+1} \cdot (\Gamma_{n+1}^q \gamma_{q+1})}$$

In the same spirit as Proposition 3.60, we may use the morphisms of correspondences given in (6.61) to provide the domain and codomain of the earlier morphism of correspondences with two mates stemming from the vertebra

$$\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta_1^n"] \arrow[rrd, bend left, "\iota_{k,n}"] & & \\
\mathbb{D}_{n+1} \arrow[r, "\delta_2^n"] \arrow[rrr, bend right, "u_k^{n-k,m}"'] & \mathbb{S}_{n+1} \arrow[r, "\widehat{\beta}_{n-k}"] & \widehat{\mathbb{A}} \arrow[r, "\varpi_k^{n-k}"] & \mathbb{G}_k^n(\beta_\diamond, \gamma_{n+1}, \beta_\bullet)
\end{tikzcd}$$

induced by factorisation (6.58). To do so, post-compose the preceding vertebra with the morphism

$$\kappa_k^q \circ \kappa(\Gamma_{n+1}^q \circ \delta_i^n) : \mathbb{G}_k^n(\beta_\diamond, \gamma_{n+1}, \beta_\bullet) \to \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)$$

for every $i \in \{1, 2\}$, which leads to the diagram, below, by using the relations of diagram (6.49).

$$\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta_1^n"] \arrow[rrd, bend left, "\iota_{k,q}\circ\gamma_{q+1}\circ\Gamma_{n+1}^q\circ\delta_i^n"] & & \\
\mathbb{D}_{n+1} \arrow[r, "\delta_2^n"] \arrow[rrr, bend right, "u_k^{n-k,q}\circ\gamma_{q+1}\circ\Gamma_{n+1}^q\circ\delta_i^n"'] & \mathbb{S}_{n+1} \arrow[r, "\widehat{\beta}_{n-k}"] & \widehat{\mathbb{A}} \arrow[r, "\kappa_k^q\circ\kappa(\Gamma_{n+1}^q\circ\delta_i^n)\circ\varpi_k^{n-k}"] & \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)
\end{tikzcd}$$

The pair of vertebrae given by the previous diagram for $i = 1$ and $i = 2$ does define a pair of mates, but at this stage, no framing may be defined along this pair. To rectify this, we need to use the fact that the spinal coheroid is symmetric and consider the following pair of mates.

(6.65)
$$\begin{tikzcd}
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & \mathbb{D}_{n+1} \arrow[d, "\delta_2^n"] \arrow[rrd, bend left, "\iota_{k,q}\circ\gamma_{q+1}\circ\Gamma_{n+1}^q\circ\delta_2^n"] & & \\
\mathbb{D}_{n+1} \arrow[r, "\delta_1^n"] \arrow[rrr, bend right, "u_k^{n-k,q}\circ\gamma_{q+1}\circ\Gamma_{n+1}^q\circ\delta_2^n"'] & \mathbb{S}_{n+1} \arrow[r, "\phi(\widehat{\beta}_{n-k})"] & \widehat{\mathbb{A}} \arrow[r, "\kappa_k^q\circ\kappa(\Gamma_{n+1}^q\circ\delta_2^n)\circ\varpi_k^{n-k}\circ\xi(\widehat{\beta}_{n-k})"] & \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[d, "\gamma_{n+1}"'] \& \mathbb{D}_{n+1} \arrow[d, "\delta_1^n"] \arrow[rr, bend left, "\iota_{k,q}\circ\gamma_{q+1}\circ\Gamma^q_{n+1}\circ\delta_1^n"] \& \& \\
\mathbb{D}_{n+1} \arrow[r, "\delta_2^n"] \arrow[rrr, bend right, "u_k^{n-k,q}\circ\gamma_{q+1}\circ\Gamma^q_{n+1}\circ\delta_1^n"'] \& \mathbb{S}_{n+1} \arrow[r, "\widehat{\beta}_{n-k}"] \& \widehat{\mathbb{A}} \arrow[r, "\kappa_k^q\circ\kappa(\Gamma^q_{n+1}\circ\delta_1^n)\circ\varpi_k^{n-k}"] \& \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)
\end{tikzcd} \tag{6.66}$$

Notice that the vertebra appearing in diagram (6.65) is exactly $v_\diamond^{n-k}$ while the vertebra appearing in diagram (6.66) is exactly $v_\bullet^{n-k}$. The pair of mates $\mu_k^{n-k,q}$ associated with $(v_\diamond^{n-k}, v_\bullet^{n-k})$ is therefore a pair of mates for the codomain of the following morphism of correspondences.

$$\underbrace{\underbrace{(\jmath_{k,m}, u_k^{n-k,m}(\mathrm{id}))}_{p_{n+1}\cdot\Gamma^m_{n+1}\asymp p_{n+1}\cdot\Gamma^m_{n+1}}}_{=:c_k^{n-k,m}(\mathrm{id})} \Rightarrow \underbrace{\underbrace{(\iota_{k,q}, u_k^{n-k,q})}_{p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1})\asymp p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1})}}_{=:c_k^{n-k,q}} \tag{6.67}$$

By Proposition 3.60, the pair of mates $\mu_k^{n-k,q}$ is transferred to the domain of the preceding morphism in the form of the pair of mates $\mu_k^{n-k,m}(\mathrm{id})$. The framing $(\Sigma^k_{\sigma_m}, v_\diamond^{n-k}, v_\bullet^{n-k}) \triangleright \Sigma^k_{\sigma_m}$ given by Proposition 6.38 involves the following two framings.

$$\left\{\begin{array}{lcl} (p_{n+1}\cdot\Gamma^m_{n+1}, v_\diamond^{n-k}, v_\bullet^{n-k}) & \triangleright & p_{n+1}\cdot\Gamma^m_{n+1} \\ (p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1}), v_\diamond^{n-k}, v_\bullet^{n-k}) & \triangleright & p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1}) \end{array}\right.$$

These two framings of vertebrae imply that

1) the vertebra $p_{n+1}\cdot\Gamma^m_{n+1}$ frames the pair $(c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id}))$;
2) the vertebra $p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1})$ frames the pair $(c_k^{n-k,q}, \mu_k^{n-k,q})$;

By section 3.3.4.6 and Proposition 3.61, the preceding framings induce

1) a span of correspondences $((c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})), c_k^{n-k+1,m}(\mathrm{id}))$ where the object $c_k^{n-k+1,m}(\mathrm{id})$ is a strong correspondence of vertebrae of the form

$$(\mathrm{id}_{p_{n+1}}, \jmath_{k,m}, u_k^{n-k+1,m}(\mathrm{id})) \vdash p_{n+1}\cdot\Gamma^m_{n+1} \overset{\sim}{\asymp} p_{n+1}\cdot\Gamma^m_{n+1}$$

   consisting of the identity alliance of prevertebrae over p_{n+1}, the canonical arrow $\jmath_{k,m} : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ and the arrow $u_k^{n-k+1,m}(\mathrm{id}) : \mathbb{S}_m \to \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet)$ encoded by the pushout arrow:

$$\lceil \underbrace{\kappa(\Gamma^m_{n+1}\circ\delta_2^n)\circ\varpi_k^{n-k}\circ\xi(\widehat{\beta}_{n-k})}_{\textit{first mate}} \Big(u_k^{n-k,m}(\mathrm{id})\Big) \underbrace{\kappa(\Gamma^m_{n+1}\circ\delta_1^n)\circ\varpi_k^{n-k}}_{\textit{second mate}} \rceil$$

2) a span of correspondences $((c_k^{n-k,q}, \mu_k^{n-k,q}), c_k^{n-k+1,q})$ where the object $c_k^{n-k+1,q}$ is a strong correspondence of vertebrae of the form

$$(\mathrm{id}_{p_{n+1}}, \iota_{k,q}, u_k^{n-k+1,q}) \vdash p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1}) \overset{\sim}{\asymp} p_{n+1}\cdot(\Gamma^q_{n+1}\gamma_{q+1})$$

   consisting of the identity alliance of prevertebrae over p_{n+1}, the canonical arrow $\iota_{k,q} : \mathbb{D}_{q+1} \to \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)$ and the arrow $u_k^{n-k+1,q} : \mathbb{D}_{q+1} \to \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet)$ encoded by the pushout arrow:

$$\lceil \underbrace{\kappa_k^q\circ\kappa(\Gamma^q_{n+1}\circ\delta_2^n)\circ\varpi_k^{n-k}\circ\xi(\widehat{\beta}_{n-k})}_{\textit{first mate}} \Big(u_k^{n-k,q}\Big) \underbrace{\kappa_k^q\circ\kappa(\Gamma^q_{n+1}\circ\delta_1^n)\circ\varpi_k^{n-k}}_{\textit{second mate}} \rceil$$

The conventions on correspondences were made so that the preceding two spans define two framings of correspondences as follows.

$$(c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})) \triangleright c_k^{n-k+1,m}(\mathrm{id}) \qquad (c_k^{n-k,q}, \mu_k^{n-k,q}) \triangleright c_k^{n-k+1,q}$$

The morphism of correspondences involved in diagram (6.67) also induces a morphism of framings of correspondences

$$(c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})) \triangleright c_k^{n-k+1,m}(\mathrm{id}) \Rightarrow (c_k^{n-k,q}, \mu_k^{n-k,q}) \triangleright c_k^{n-k+1,q}$$

when equipped with the morphism of framinsg $(\Gamma_{q+1}^m \circ \delta_i^{q+1}, \Gamma_{q+1}^m \circ \delta_i^{q+1})$ of the following form (note that the symbol of the arrows is displayed between the two given framings).

$$\begin{gathered}(p_{n+1} \cdot \Gamma_{n+1}^m, v_\diamond^{n-k}, v_\bullet^{n-k}) \triangleright p_{n+1} \cdot \Gamma_{n+1}^m \\ \curvearrowright \\ (p_{n+1} \cdot (\Gamma_{n+1}^q \gamma_{q+1}), v_\diamond^{n-k}, v_\bullet^{n-k}) \triangleright p_{n+1} \cdot (\Gamma_{n+1}^q \gamma_{q+1})\end{gathered}$$

By the definitions of section 3.3.4.9 and Proposition 3.62, the above data induces a morphism of spans of correspondences as follows.

$$((c_k^{n-k,m}(\mathrm{id}), \mu_k^{n-k,m}(\mathrm{id})), c_k^{n-k+1,m}(\mathrm{id})) \qquad ((c_k^{n-k,q}, \mu_k^{n-k,q}), c_k^{n-k+1,q})$$

In particular, the above reasoning produces a morphism of strong correspondences

$$\underbrace{\underbrace{(\jmath_{k,m}, u_k^{n-k+1,m}(\mathrm{id}))}_{p_{n+1}\cdot\Gamma_{n+1}^m \,\widetilde{\asymp}\, p_{n+1}\cdot\Gamma_{n+1}^m}}_{=:c_k^{n-k+1,m}(\mathrm{id})} \Rightarrow \underbrace{\underbrace{(\iota_{k,q}, u_k^{n-k+1,q})}_{p_{n+1}\cdot(\Gamma_{n+1}^q\gamma_{q+1}) \,\widetilde{\asymp}\, p_{n+1}\cdot(\Gamma_{n+1}^q\gamma_{q+1})}}_{=:c_k^{n-k+1,q}}$$

that is encoded by a commutative diagram of the following form.

$$\begin{tikzcd}
\mathbb{D}_q \arrow[r, "\iota_{k,q}"] \arrow[d, "\Gamma_{q+1}^m\circ\delta_i^q"'] & \mathbb{G}_k^q(\beta_\diamond, \gamma_{q+1}, \beta_\bullet) \arrow[d, "\kappa(\Gamma_{q+1}^m\circ\delta_i^q)"'] & \mathbb{D}_q \arrow[l, "u_k^{n-k+1,q}"'] \arrow[d, "\Gamma_{q+1}^m\circ\delta_i^q"] \\
\mathbb{S}_m \arrow[r, "\jmath_{k,m}"'] & \mathbb{G}_k^m(\beta_\diamond, \mathrm{id}, \beta_\bullet) & \mathbb{S}_m \arrow[l, "u_k^{n-k+1,m}(\mathrm{id})"]
\end{tikzcd}$$

Let us now look at the correspondence $(\iota_{k,q}, u_k^{n-k+1,q})$ in the case where the equality $q = n+1$ holds. In this case, we have the following strong correspondence of vertebrae.

$$(\iota_{k,n+1}, u_k^{n-k+1,n+1}) \vdash p_{n+1} \cdot \gamma_{n+2} \,\widetilde{\asymp}\, p_{n+1} \cdot \gamma_{n+2}$$

This correspondence is, by definition, equipped with a commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{S}_{n+1} \arrow[r, "\gamma_{n+2}"] \arrow[d, "\gamma_{n+2}"'] & \mathbb{D}_{n+2} \arrow[d, "\iota_{k,n+1}"] \\
\mathbb{D}_{n+2} \arrow[r, "u_k^{n-k+1,n+1}"'] & \mathbb{G}_k^{n+1}(\beta_\diamond, \gamma_{n+2}, \beta_\bullet)
\end{tikzcd}$$

The previous commutative square provides a $(n+2)$-parallel pair of arrows in the spinal extension $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. The spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$ will be said to be $(k, n-k+1)$-*coherent* if the parallel pair of arrows

$$(u_k^{n-k+1,n+1}, \iota_{k,n+1})$$

is admissible for every $\beta_\diamond, \beta_\bullet \in \Omega_k$ so that there exist a morphism $\widehat{\beta}_{n-k+1} : \mathbb{S}_k \to \widehat{\mathbb{A}}_{n-k+1}$ in Ω_{n+2} and a morphism $\varpi_k^{n-k+1} : \widehat{\mathbb{A}}_{n-k+1} \to \mathbb{G}_k^{n+1}(\beta_\diamond, \gamma_{n+2}, \beta_\bullet)$ making the following diagram commute.

$$\begin{tikzcd}
\widehat{\mathbb{A}}_{n-k+1} \arrow[drr, dashed, "\varpi_k^{n-k+1}"] & & \\
\mathbb{S}_{n+2} \arrow[u, "\widehat{\beta}_{n-k+1}"] \arrow[rr, "\ulcorner u_k^{n-k+1,n+1}{,}\iota_{k,n+1} \urcorner"'] & & \mathbb{G}_k^{n+1}(\beta_\diamond, \gamma_{n+2}, \beta_\bullet)
\end{tikzcd}$$

A spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ will be said to be (k, ω)-*coherent* if it is (k, m)-coherent for every $m \in \omega$.

Example 6.47 (Topological spaces)**.** The topological spinal coheroid $(\mathbb{D}, \mathbb{S}, \delta, \gamma) : \texttt{Spine} \to (\mathbf{Top}, \mathcal{A})$ is (k, ω)-coherent by definition of $\mathcal{A}$.

Example 6.48 (Grothendieck's ∞-groupoids)**.** The spinal coheroid of a category of Grothendieck ∞-groupoids is (k, ω)-coherent by definition of $\mathcal{A}$.

Example 6.49 (Maltsiniotis' categories)**.** The spinal coheroid of a category of Maltsiniotis ∞-categories becomes (k, ω)-coherent when the class of parallel arrows is augmented by the pairs of parallel arrows $(u_k^{n-k,n}, \iota_{k,n})$ for every $k \leq n$. The underlying factorisations are again provided by Corollary 5.86 (small object argument).

6.4.2. Towards the Homotopy Hypothesis.

6.4.2.1. *Underlying spinal categories of spinal coheroids.* Let $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega : \texttt{Spine} \to (\mathcal{C}, \mathcal{A})$ be a spinal coheroid that is

1) reflexive;
2) magmoidal;
3) (k, ω)-coherent for every $k \in \omega$ (which includes (k, ω)-normality for every $k \in \omega$; (k, ω)-transitivity for every $k \in \omega$; closedness and symmetry),

and whose ω-spinal object is given by a faithful and injective-on-objects functor $\texttt{Spine} \to \mathcal{C}$, which means that the objects and arrows of the spinal object are completely determined by their indexing. The goal of this section is to define the structure of a spinal category $(\mathcal{C}, A_\_, E_\_, T)$ for the underlying vetebral category $(\mathcal{C}, A', A, E)$ of the spinal coheroid $(\mathbb{S}, \mathbb{D}, \gamma, \delta) \cdot \Omega$. To define the local echelon, first define the collection of subgraphs $\{A_n\}_{n \in \omega}$ of $(\mathbf{Ally}(\mathcal{C}), \circledcirc)$. For every $n \in \omega$, the object-class of A_n is given by the singleton set

$$\mathrm{Obj}(A_n) := \{\sigma_n\}$$

while its hom-set $A_n(\sigma_n, \sigma_n)$ is defined as the singleton containing the identity alliance on σ_n. It is not hard to check that there is an obvious collection of morphisms of graphs

$$\tau^n : \left[\begin{array}{ccc} A_n & \Rightarrow & A \\ \sigma_n & \mapsto & \nu_n \end{array} \right]$$

that is jointly surjective on objects. Then, define the A_n-subprecompass

$$\{(E_n, \eta_n)\}_{n \in \omega}$$

of the $\mathbf{Ally}(\mathcal{C})$-precompass $(\mathrm{Enov}, \eta, \odot, \circledcirc)$. For every $n \in \omega$, the left and right object-classes of (E_n, η) are given by the singletons

$$\mathrm{Obj}_L(E_n) := \{P_{n-1} \cdot \gamma_n\} \quad \text{and} \quad \mathrm{Obj}_R(E_n) := \{\sigma_n\}$$

and its hom-set $E_n(P_{n-1} \cdot \gamma_n, \sigma_n)$ is defined as the singleton containing the obvious extended nodes of spines $P_{n-1} \cdot \gamma_n \overset{\mathrm{EX}}{\rightsquigarrow} \sigma_n$. It is not hard to check that there is an obvious collection of morphism of precompasses as follows that is jointly surjective on the left object-class.

$$\pi^n : \left[\begin{array}{ccc} E_n & \Rightarrow & E \\ P_{n-1} \cdot \gamma_n & \mapsto & \gamma_n \\ \sigma_n & \mapsto & \nu_n \end{array} \right]$$

For every $n \in \omega$, the arrows τ^n and π^n define a morphisms of precompasses over graphs. In addition, it is straightforward to check that the A_n-precompass (E_n, η) is echeloned for the following trivial cograded graph.

$$\begin{array}{ccccccccccccc} O^n_{\_} & : & \{\sigma_n\} & \longrightarrow & \{\sigma_{n-1}\} & \longrightarrow & \{\sigma_{n-2}\} & \longrightarrow & \dots & \longrightarrow & \{\sigma_1\} & \longrightarrow & \{\sigma_0\} \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & & & \downarrow\downarrow & & \downarrow\downarrow \\ S^n_{\_} & & \{\gamma_n\} & & \{\gamma_{n-1}\} & & \{\gamma_{n-2}\} & & \dots & & \{\gamma_1\} & & \{\gamma_0\} \end{array}$$

In other words, the A-precompass (E, η) is locally echeloned under the collection of A_n-precompass (E_n, η) for every $n \in \omega$. Since the morphisms τ^n and π^n are obvious fibrations (over identity alliances), the resulting local A-echelon $[\pi, \tau](E, \eta)$ is regular. To define a whiskered structure on E, define two subgraphs $T_\ltimes$ and $T_\rtimes$ of $\mathbf{Sev}(\mathcal{C})$ whose object-classes are given by the set

$$\mathrm{Obj}(T_\ltimes) = \{\gamma_k \mid k \in \omega\} = \mathrm{Obj}(T_\rtimes)$$

and whose hom-classes are given by the following equations.

$$T_\ltimes(\gamma_k, \gamma_m) = \left\{ \begin{array}{ll} \emptyset & \text{if } k \neq m \\ \nu_k^{\mathrm{rv}} & \text{if } k = m \end{array} \right.$$

$$T_\rtimes(\gamma_k, \gamma_m) = \left\{ \begin{array}{ll} \emptyset & \text{if } k \neq m \\ \nu_k & \text{if } k = m \end{array} \right.$$

The pregraphs $T_\ltimes$ and $T_\rtimes$ define two whiskering bundles of E above the singleton $\{\gamma_k\}$ for every $k \in \omega$. This is obvious for $T_\rtimes$ since the derivation $d\nu_k$ may be identified with ν_k itself and the extended node of vertebrae encoded by ν_k may therefore be identified with the following extended node of vertebrae.

$$\gamma_k \xrightarrow{d\nu_k} \nu_k \xrightarrow{\mathrm{id}_{\nu_k}} \nu_k$$

To see this for $T_\ltimes$, it suffices to notice that a node of vertebrae ν_k^{rv} is equal to the node of vertebrae given by the following composite (see section 6.4.1.1).

$$\gamma_k \xrightarrow{d\nu_k} \nu_k \xrightarrow{(\mathrm{id}, \varkappa_k, \xi(\_), \phi)} \nu_k^{\mathrm{rv}}$$

The pair $(T_\ltimes, T_\rtimes)$ will later be denoted as T. The quadruple $(\mathcal{C}, A, E, T)$ is then equipped with a structure of spinal category if

1) **(framing i)** for every pair of integers q and n such that $0 \leq q \leq n$ and 3-tuple of the form

$$(v_\diamond, \sigma_n, v_\bullet) \in T_\ltimes(\gamma_q, \gamma_q) \times O^n_q(\gamma_q, \gamma_q) \times T_\rtimes(\gamma_q, \gamma_q),$$

the T-whiskering $(v_\diamond \ltimes \sigma_n \rtimes v_\bullet)^A_q$ is given by the nodes of spines σ_n, which implies a simple framing of node of spines of the following form by Proposition 6.38;

$$(\sigma_n, v_\diamond, v_\bullet) \triangleright \sigma_n$$

2) **(framing ii)** for every pair of integers q and n such that $0 \leq q \leq n-1$ and 3-tuple of the form

$$(v_\diamond, P_{n-1} \cdot \gamma_n, v_\bullet) \in T_\ltimes(\gamma_q, \gamma_q) \times \partial O^n_q(\gamma_q, \gamma_q) \times T_\rtimes(\gamma_q, \gamma_q),$$

the T-whiskering $(v_\diamond \ltimes P_{n-1}\cdot\gamma_n \rtimes v_\bullet)^A_q$ is given by the spine $P_{n-1}\cdot\gamma_n$, which implies a simple framing of node of spines of the following form by Proposition 6.38.

$$(P_{n-1}\cdot\gamma_n, v_\diamond, v_\bullet) \triangleright P_{n-1}\cdot\gamma_n$$

The next two propositions show that the quadruple $(\mathcal{C}, A, E, T)$ indeed defines a spinal category.

Proposition 6.50. *Every 3-tuple of the form*

$$(v_\diamond, \sigma_n, v_\bullet) \in T_\ltimes(\gamma_q,\gamma_q) \times O^n_q(\gamma_q,\gamma_q) \times T_\rtimes(\gamma_q,\gamma_q)$$

where $0 \leq q \leq n$ is associated with a structure of convergent conjugation of nodes of spines $(\sigma_n, \mathrm{id}_{\sigma_n}, \sigma_n)$ in $\mathcal{C}$ where the canonical alliance of nodes of spines $\mathfrak{all}_0(\sigma_n, \mathrm{id}_{\sigma_n}, \sigma_n)$ defined in section 3.3.8.1 is equal to σ_n.

Proof. Let $v_\diamond := p_q\cdot\beta_\diamond$ and $v_\bullet := p_q\cdot\beta_\bullet$ be two vertebrae in the node of vertebrae ν_q and consider the two alliances of vertebrae

$$(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow p_q\cdot\phi(\beta_\diamond) \quad\text{and}\quad (\mathrm{id}, \varkappa_q, \xi(\beta_\bullet)) : v_\bullet \rightsquigarrow p_q^{\mathrm{rv}}\cdot\phi(\beta_\bullet)$$

stemming from the pair of alliances of nodes of vertebrae $(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : \nu_q^{\mathrm{rv}} \rightsquigarrow \nu_q$ and $(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : \nu_q^{\mathrm{rv}} \rightsquigarrow \nu_q$. The two vertebrae $p_q\cdot\phi(\beta_\diamond)$ and $p_q\cdot\phi(\beta_\bullet)$, which belong to the node of vertebrae ν_k, will be denoted by $v_\flat$ and $v_\dagger$, respectively. The structure of conjugation is given by

1) the pair of vertebrae $v_\diamond^{\mathrm{rv}} := p_q^{\mathrm{rv}}\cdot\beta_\diamond$ and $v_\bullet := p_q\cdot\beta_\bullet$;
2) the following two alliances of vertebrae;
$$(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow (v_\flat^{\mathrm{rv}})^{\mathrm{rv}} \qquad (\mathrm{id}, \varkappa_q, \xi(\beta_\bullet)) : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$$
3) the following pair of simple q-framings of spines induced by Proposition 6.38.
$$\begin{cases} (\sigma_n, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright \sigma_n \\ (\sigma_n, v_\flat^{\mathrm{rv}}, v_\dagger) \triangleright \sigma_n \end{cases}$$

The preceding pair of framings comprises the following two framings of vertebrae.

$$\begin{cases} (p_q\cdot\Gamma^n_q, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright^V_q p_q\cdot\Gamma^n_q \\ (p_q\cdot\Gamma^n_q, v_\flat^{\mathrm{rv}}, v_\dagger) \triangleright^V_q p_q\cdot\Gamma^n_q \end{cases}$$

It follows from the construction of section 3.3.5.5 that the earlier conjugation gives rise to a strong correspondence between two copies of the vertebra $p_q\cdot\Gamma^n_q$ via the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$, which also defines a memory of nodes of spines between σ_n and itself. This memory is, by definition, given by the correspondence $c^{0,n}_q(\mathrm{id}) \vdash p_{q+1}\cdot\Gamma^n_{q+1} \asymp p_{q+1}\cdot\Gamma^n_{q+1}$. The following sequence of framings of correspondences, defined throughout section 6.4.1.2, then exactly says that the conjugation is convergent.

$$(c^{0,n}_q(\mathrm{id}), \mu^{0,n}_q(\mathrm{id})) \triangleright_{q+1} \dots (c^{1,n}_q(\mathrm{id}), \mu^{1,n}_q(\mathrm{id})) \triangleright_{q+2} \triangleright_n c^{n-q,n}_q(\mathrm{id})$$

This finishes the proof since the last framing involved in the previous sequences of framings forces $\mathfrak{all}_0(\sigma_n, \mathrm{id}_{\sigma_n}, \sigma_n)$ to be σ_n by definition. □

Proposition 6.51. *Every T-whiskering 3-tuple*

$$(v_\diamond, P_{n-1}\cdot\gamma_n, v_\bullet) \in T_\ltimes(\gamma_q,\gamma_q) \times \partial O^n_q(\gamma_q,\gamma_q) \times T_\rtimes(\gamma_q,\gamma_q)$$

where $0 \leq q \leq n-1$ is associated with a structure of convergent extended conjugation of nodes of spines $(P_{n-1}\cdot\gamma_n, \sigma_n, \sigma_n)$ in $\mathcal{C}$ where the closure of the conjugation is equal to σ_n.

Proof. Let $v_\diamond := p_q \cdot \beta_\diamond$ and $v_\bullet := p_q \cdot \beta_\bullet$ be two vertebrae in the node of vertebrae ν_q and consider the two alliances of vertebrae

$$(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow p_q \cdot \phi(\beta_\diamond) \quad \text{and} \quad (\mathrm{id}, \varkappa_q, \xi(\beta_\bullet)) : v_\bullet \rightsquigarrow p_q^{\mathrm{rv}} \cdot \phi(\beta_\bullet)$$

stemming from the pair of alliances of nodes of vertebrae $(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : \nu_q^{\mathrm{rv}} \rightsquigarrow \nu_q$ and $(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : \nu_q^{\mathrm{rv}} \rightsquigarrow \nu_q$. The two vertebrae $p_q \cdot \phi(\beta_\diamond)$ and $p_q \cdot \phi(\beta_\bullet)$, which belong to the node of vertebrae ν_k, will be denoted by $v_\flat$ and $v_\dagger$, respectively. The structure of conjugation is given by

1) the pair of vertebrae $v_\diamond^{\mathrm{rv}} := p_q^{\mathrm{rv}} \cdot \beta_\diamond$ and $v_\bullet := p_q \cdot \beta_\bullet$;
2) the following two alliances of vertebrae;
$$(\mathrm{id}, \varkappa_q, \xi(\beta_\diamond)) : v_\diamond^{\mathrm{rv}} \rightsquigarrow (v_\flat^{\mathrm{rv}})^{\mathrm{rv}} \qquad (\mathrm{id}, \varkappa_q, \xi(\beta_\bullet)) : v_\bullet \rightsquigarrow v_\dagger^{\mathrm{rv}}$$
3) the following pair of simple q-framings of spines induced by Proposition 6.38.
$$\begin{cases} (P_{n-1} \cdot \gamma_n, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright P_{n-1} \cdot \gamma_n \\ (\sigma_n, v_\flat^{\mathrm{rv}}, v_\dagger) \triangleright \sigma_n \end{cases}$$

The preceding pair of framings comprises the following two framings of vertebrae.

$$\begin{cases} (p_q \cdot \Gamma_q^{n-1}, v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright_q^V p_q \cdot \Gamma_q^{n-1} \\ (p_q \cdot \Gamma_q^{n-1}, v_\flat^{\mathrm{rv}}, v_\dagger) \triangleright_q^V p_q \cdot \Gamma_q^{n-1} \end{cases} \qquad \begin{cases} (p_q \cdot (\Gamma_q^{n-1}\gamma_n), v_\diamond^{\mathrm{rv}}, v_\bullet) \triangleright_q^V p_q \cdot (\Gamma_q^{n-1}\gamma_n) \\ (p_q \cdot (\Gamma_q^{n-1}\gamma_n), v_\flat^{\mathrm{rv}}, v_\dagger) \triangleright_q^V p_q \cdot (\Gamma_q^{n-1}\gamma_n) \end{cases}$$

It follows from the construction of section 3.3.5.5 that the earlier conjugations give rise to a strong correspondence between two copies of the vertebra $p_q \cdot \Gamma_q^{n-1}$ and a strong correspondence between two copies of the vertebra $p_q \cdot (\Gamma_q^{n-1}\gamma_n)$ via the functor $\mathcal{S}_{\mathrm{cor}} : \mathbf{Conj}(\mathcal{C}) \to \mathbf{Scov}(\mathcal{C})$. These correspondences define a strong memory of extended nodes of spines between $P_{n-1} \cdot \gamma_n$ and σ_n. If $q = n-1$, then the following diagram defines a mate for the underlying recollection.

$$\begin{tikzcd}
\mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[d, "\delta_1^{n-1}"] \arrow[drr, bend left, "\iota_{q,n-1}"] & & \\
\mathbb{D}_n \arrow[r, "\delta_2^{n-1}"] \arrow[rrr, bend right, "\zeta_{q,q}^0 \circ \pi_q^0"'] & \mathbb{S}_n \arrow[r, "\widehat{\beta}_0"] & \widehat{\mathbb{A}} \arrow[r, "\varpi_q^0"] & \mathbb{G}_q^{n-1}(\beta_\diamond, \gamma_n, \beta_\bullet)
\end{tikzcd}$$

Because the spinal coheroid is magmoidal, the previous vertebra and the head of σ_n admit a framing, which is equal to σ_n. The conjugation is therefore convergent and the closure is equal to σ_n. If $q < n-1$, this memory is, by definition, given by the two correspondences $c_q^{0,n-1}(\mathrm{id}) \vdash p_{q+1} \cdot \Gamma_{q+1}^{n-1} \asymp p_{q+1} \cdot \Gamma_{q+1}^{n-1}$ and $c_q^{0,n-1} \vdash p_{q+1} \cdot (\Gamma_{q+1}^{n-1}\gamma_n) \asymp p_{q+1} \cdot (\Gamma_{q+1}^{n-1}\gamma_n)$ and the obvious morphism between them. The following two sequences of framings of correspondences, defined throughout section 6.4.1.2, then exactly say that the conjugation is convergent.

$$(c_q^{0,n-1}(\mathrm{id}), \mu_q^{0,n-1}(\mathrm{id})) \triangleright_{q+1} (c_q^{1,n-1}(\mathrm{id}), \mu_q^{1,n-1}(\mathrm{id})) \triangleright_{q+2} \cdots \triangleright_{n-1} c_q^{n-1-q,n-1}(\mathrm{id})$$

$$(c_q^{0,n-1}, \mu_q^{0,n-1}) \triangleright_{q+1} (c_q^{1,n-1}, \mu_q^{1,n-1}) \triangleright_{q+2} \cdots \triangleright_{n-1} c_q^{n-1-q,n-1}$$

This finishes the proof since the last correspondence, which defines a recollection, involve a mate

$$\begin{tikzcd}
\mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[d, "\delta_1^{n-1}"] \arrow[drr, bend left, "\iota_{q,n-1}"] & & \\
\mathbb{D}_n \arrow[r, "\delta_2^{n-1}"] \arrow[rrr, bend right, "u_q^{n-1-q,n-1}"'] & \mathbb{S}_n \arrow[r, "\widehat{\beta}_{n-1-q}"] & \widehat{\mathbb{A}} \arrow[r, "\varpi_q^{n-1-q}"] & \mathbb{G}_q^{n-1}(\beta_\diamond, \gamma_n, \beta_\bullet)
\end{tikzcd}$$

along which it is possible to frame the head of the node of spine σ_n (since the spinal coheroid is magmoidal). Again, the closure is equal to σ_n. □

6.4.2.2. *Homotopy Hypothesis.* The goal of this section is to take the results proven earlier and rearrange them to set up the foundations for a proof of the Homotopy hypothesis for Grothendieck's ∞-groupoids.

Theorem 6.52. *The spinal coheroid associated with a category of Grothendieck ∞-groupoids admits a (quasi-small refined) spinal structure.*

Proof. This follows from the examples entitled 'Grothendieck's ∞-groupoids' as well as section 6.2.5.4 for the vertebral category structure; Proposition 6.36 for the local projectivity; section 6.4.2.1 for the whiskering axioms; Proposition 6.50 and Proposition 6.51 for the framing and conjugation axioms. The fact that the spinal category is quasi-small and refined is obvious as the classes Ω are singletons and the refinement follows from Proposition 4.61. □

Proposition 6.53. *If the spinal structure of a category of Grothendieck's ∞-groupoids is well-disposed for surtractions, then its defines a model structure for the underlying weak equivalences, fibrations and cofibrations.*

Proof. The statement follows from *(1)* Theorem 6.52 when the spinal structure of the category of ∞-groupoids is seen as an obvious fully faithful spinal theory (see Example 5.90) and *(2)* an obvious variation of Theorem 5.104 that replaces the assumption that the spinal structure must be well-disposed for intractions with the result of that Remark 5.103. This second point is possible since *(2.a)* any sequential functor in $\mathbf{Mod}(C^{\mathrm{op}})$ is convergent with respect to the representable models (see Example 5.8) and *(2.b)* all vertebrae involved in the spinal structure of a category of Grothendieck's ∞-groupoids are reflexive with trivial reflexive transitions (see Remark 5.103). □

Checking the assumption of Proposition 6.53 requires the computation of a colimit of Grothendieck's ∞-groupoids. This kind of calculation amounts to giving an elementary description of the reflection functor $\mathbf{Mod}(C^{\mathrm{op}}) \to \mathbf{Psh}(C)$, which was exactly the purpose of Chapter 5. It only remains to apply the construction thereof to the models of some coherator C and use this construction to show by induction that any category of Grothendieck's ∞-groupoids is well-disposed for surtractions.

Then, a strategy to show the Homotopy Hypothesis is to combine Proposition 6.53 together with Remark 4.96. Specifically, there is a functor $i : C \to \mathbf{Top}$ constructed in [**35**] whose free extensions $i^{\triangledown} : C^{\triangledown} \to \mathbf{Top}$ are 0- and 1-regular. Then, by copying our previous reasoning by which every category of Grothendieck's ∞-groupoids has been equipped with a spinal category structure, one may show that the free cocompletion $C^{\triangledown} = \mathbf{Psh}(C)$ has a spinal category structure too (see Remark 6.16). Finally, we could try to conclude by using Remark 4.96, which requires us to show that the unit of the adjunction

$$\int^{g\in C} \nabla_g(\_) \otimes i(g) \vdash \mathbf{Top}(i(\_), \_) \tag{6.68}$$

(defined thereof) is a componentwise weak equivalence in $\mathbf{Mod}(C^{\mathrm{op}})$. This would imply that the functor $\mathbf{Top}(i(\_), \_) : \mathbf{Top} \to \mathbf{Mod}(C^{\mathrm{op}})$ is a covertebral equivalence, which by Proposition 1.48 and Proposition 4.95, would prove the Homotopy Hypothesis. But this argument happens to fail in some cases and therefore needs to be refined. Indeed, it simply suffices to notice that the argument does not work for strict groupoids. For instance, the group with two elements $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ may be seen as an obvious one-object strict ∞-groupoid, let us call it $B(\mathbb{Z}/2\mathbb{Z})$, and hence a Grothendieck ∞-groupoid by the result of [**5**]. But the image of $B(\mathbb{Z}/2\mathbb{Z})$ via the left adjoint functor of adjunction (6.68) is a terminal topological space since the relation $1+1=0$ forces any path in $B(\mathbb{Z}/2\mathbb{Z})$ to contract to the topological realisation of the point 0.

$$\int^{g\in C} B(\mathbb{Z}/2\mathbb{Z})(g)\otimes i(g)=\{0\}$$

We thus conclude that the unit of adjunction (6.68) at the Grothendieck ∞-groupoid $B(\mathbb{Z}/2\mathbb{Z})$ is the canonical map $! : B(\mathbb{Z}/2\mathbb{Z}) \to \mathbf{1}$, which is far from being a weak equivalence in $\mathbf{Mod}(C^{\mathrm{op}})$. Instead, we should rather seek to prove the following property: For every object $X : C^{\mathrm{op}} \to \mathbf{Set}$ in $\mathbf{Mod}(C^{\mathrm{op}})$ for which there exists a topological space Y and a morphisms $f : X(\_) \Rightarrow \mathbf{Top}(i(\_), Y)$ in $\mathbf{Mod}(C^{\mathrm{op}})$, the unit of (6.68) at the object X is a weak equivalence in $\mathbf{Mod}(C^{\mathrm{op}})$. At least, this property does not hold for the Grothendieck ∞-groupoid $B(\mathbb{Z}/2\mathbb{Z})$, since it is too strict for being mapped to an image of $\mathbf{Top}(i(\_),\_)$, and it does not contradict Quillen's criteria for Quillen equivalences (see Proposition 1.48).

6.4.3. Link with Ara's work. In his manuscript of 1983 (see [**24**]), Grothendieck gave a definition of weak equivalence different from that given in this thesis. Specifically, for every ∞-groupoid X and 0-cell $p \in X$, he describes a collection of 'homotopy groups'

$$\{\pi_0(X)\}\cup\{\pi_n(X,p)\}_{n\geq 1},$$

whose definitions are very similar to those defined for topological spaces (see [**4**, section 4.17] or [**3**, section 4.3]). For every morphism of ∞-groupoids $f : X \to Y$ and 0-cell $p \in X$, every homotopy group induce functions of the form $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ or $\pi_n(f,p) : \pi_n(X,p) \to \pi_n(Y, f(p))$ for $n > 0$. He then goes on defining a weak equivalence as a morphism of ∞-groupoids $f : X \to Y$ whose images $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ and $\pi_n(f,p) : \pi_n(X,p) \to \pi_n(Y, f(p))$ are isomorphisms for every $n \geq 1$.

As seen in Proposition 3.20 and Proposition 3.22, the notion of weak equivalence given in the present thesis may be described as a morphism of ∞-groupoids $f : X \to Y$ such that for every pair of parallel n-cells u and v in X, a certain morphism of groups of the following form (described in [**4**, section 4.11] or [**3**, section 4.4.8]) is an isomorphism.

$$\begin{cases} \pi_0(f) : \pi_0(X) \to \pi_0(Y) & (\text{if } n=0) \\ \pi_n(f,u,v) : \pi_n(X,u,v) \to \pi_n(Y, f(u), f(v)) & (\text{if } n>0) \end{cases}$$

It is a result of D. Ara (see [**4**, Theorem 4.18] or [**3**, section 4.3.5]) that the two previous notions are equivalent.

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A

List of Symbols

$\ulcorner$, $\lrcorner$	used to indicate pushout and pullback objects
$\mathbb{N}$	the set of non-negative integers
ω	the least infinite ordinal
Set	category of sets
Top	category of topological spaces
$\mathbf{Mod}_R$	category of left R-modules
$\mathbf{Ch}_R$	category of non-negatively chain complexes of left R-modules
$\mathbf{Aff}(k)$	category of affine varieties over a field k
Sch	category of schemes (Algebraic Geometry)
$\mathbf{Cat}(n)$	category of (small) n-categories
$\mathcal{C}^D$	category of functors $D \to \mathcal{C}$ from a small category
$\mathbf{O}(\kappa)$	preordered category of ordinals less than κ
$\mathbf{1}$	any terminal object
∇_d	evaluation functor $\mathcal{C}^D \to \mathcal{C}$ at an object d in D
Δ_D	pre-composition functor from $\mathcal{C}$ to $\mathcal{C}^D$ mapping an object $\mathbf{1} \to \mathcal{C}$ to the composite $D \to \mathbf{1} \to \mathcal{C}$
$[\mathcal{D}, \mathcal{C}]$	category of functors $\mathcal{D} \to \mathcal{C}$ between categories
$\mathbf{Mod}_{\mathcal{C}}(\mathtt{S})$	category of models over a sketch $\mathtt{S}$
Lan_d	left Kan extension along the constant functor $d : \mathbf{1} \to D$
$\mathrm{Ho}(\mathcal{C})$	homotopy category of $\mathcal{C}$
$\\|\gamma, \gamma'\\|$	prevertebra of seed γ and coseed γ'
$p \cdot \beta$	vertebra of base p and stem β
$p \cdot \Omega$	node of vertebrae of base p and class of stems Ω
$P \cdot \beta$	spine of base P and stem β
$P \cdot \Omega$	node of spines of base P and class of stems Ω
V_s^k, E_s^k, V_σ^k, E_σ^k	functors of vertebrae associated with a spine s and a node of vertebrae σ, respectively
$\Gamma_k(P)$	k-th central cord of a prespine P
`Prev`	sketch for prevertebrae, p. 52

`Vert` .... sketch for vertebrae, p. 55

p^{rv}, v^{rv}, ν^{rv} .. dual of a prevertebra p, vertebra v and node of vertebrae ν

$\mathbf{disk}(v)$ ...... the diskad of a vertebra v, p. 55

$\mathbf{seed}(\mathfrak{a})$ ...... the commutative square between the seeds of an alliance $\mathfrak{a}$, p. 55

$\mathbf{triv}(\mathfrak{a})$ .... the commutative square between the trivial stems of an alliance $\mathfrak{a}$, p. 55

$\mathbf{bste}(\mathfrak{a})$ ...... the biased commutative square between the stems of an alliance $\mathfrak{a}$, p. 55

$\mathbf{rlp}(\mathcal{A})$ .... class of arrows that have the right lifting property with respect to the arrows of a class $\mathcal{A}$

$\mathbf{llp}(\mathcal{A})$ .... class of arrows that have the left lifting property with respect to the arrows of a class $\mathcal{A}$

$\langle x, y\rangle$ .... universal arrow associated with two parallel paths x and y

$(\_, \mathfrak{v}_\diamond, \mathfrak{v}_\bullet) \triangleright \_$ .. framing along two semi-extended vertebrae $\mathfrak{v}_\diamond$ and $\mathfrak{v}_\bullet$

$\mathcal{C}(v, X)(x, y)$, $\mathcal{C}(\nu, X)(x, y)$ the classes of paths, in an object X, going from x to y

$\lceil e_\diamond h e_\bullet \rceil$ ...... universal arrow associated with a framing (hom-language)

∂P, $\partial \mathfrak{p}_\_$ ...... derived prespine, derived alliance of prespines

$\asymp$, $\tilde{\asymp}$ .... correspondences

$\asymp_q$, $\tilde{\asymp}_q$, $\frown_q$, $\tilde{\frown}_q$ q-memories

$\smile$, $\tilde{\smile}$ .... recollections

$\mathfrak{all}_0(\sigma, \mathfrak{a}, \overline{\sigma})$, $\mathfrak{all}_1(\sigma, \mathfrak{a}, \overline{\sigma})$ the two canonical alliances resulting from a convergent conjugation of node of spines $(\sigma, \mathfrak{a}, \overline{\sigma})$

$\coprod$ .... coproduct

$\lim_A$, $\lim_d$ .... limits over a category A or shifted by a functor d

col_A, col_d .... colimits over a category A or shifted by a functor d

$\int_{a \in \mathtt{K}}$, $\int^{a \in \mathtt{K}}$ ... ends and coends

$\mathtt{K}^{\triangledown}$ .... free cocompletion of a colimit sketch $\mathtt{K}$

$\hat{\mathcal{C}}$ .... short notation for a system of vertebrae of ambient category $\mathcal{C}$

`Glob` .... sketch of globular sets

`Spine` .... sketch of spinal sets

$\mathbf{Ally}(\mathcal{C})$ ...... category of nodes of vertebrae and alliances of these in $\mathcal{C}$, p. 58

$\mathbf{Vert}(\mathcal{C})$ ...... category of vertebrae and morphisms of vertebrae in $\mathcal{C}$, p. 107

$\mathbf{Fov}(\mathcal{C})$ ...... category of framings of vertebrae in $\mathcal{C}$, p. 110

$\mathbf{Corov}(\mathcal{C})$ .... category of correspondences of vertebrae in $\mathcal{C}$, p. 116

$\mathbf{Scov}(\mathcal{C})$ ..... category of strong correspondences of vertebrae in $\mathcal{C}$, p. 116

$\mathbf{Mcov}(\mathcal{C})$ .... category of correspondences of vertebrae equipped with pairs of mates in $\mathcal{C}$, p. 117

$\mathbf{Focov}(\mathcal{C})$ .... category of framings of correspondences of vertebrae in $\mathcal{C}$, p. 120

$\mathbf{Alov}(\mathcal{C})$ ..... category of alliances of vertebrae and morphisms of alliances of vertebrae in $\mathcal{C}$, p. 122

$\mathbf{Aos}(\mathcal{C}, n)$ .... category of spines of degree n in $\mathcal{C}$, p. 158

$\mathbf{Anos}(\mathcal{C}, n)$ .. category of nodes of spines of degree n in $\mathcal{C}$, p. 158

$\mathbf{Enov}(\mathcal{C})$ ..... span of extended nodes of vertebrae in $\mathcal{C}$, p. 164

$\mathbf{Enos}_n(\mathcal{C})$ .... span of extended nodes of spines of degree n in $\mathcal{C}$, p. 164

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