Sketches in Higher Category Theories and the Homotopy Hypothesis

Rémy Tuyéras

Macquarie University

August 8th, 2016

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation $\Rightarrow$ Pattern

Models and Theories

All mathematical theories are born from **inductive** reasoning:

Observation $\Rightarrow$ Pattern $\Rightarrow$ Hypothesis

Models and Theories

All mathematical theories are born from **inductive** reasoning:

Observation $\Rightarrow$ Pattern $\Rightarrow$ Hypothesis $\Rightarrow$ Theory

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation ⇒ Pattern ⇒ Hypothesis ⇒ Theory

Once a theory is obtained, one then uses **deductive** reasonings to explain other observations:

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation $\Rightarrow$ Pattern $\Rightarrow$ Hypothesis $\Rightarrow$ Theory

Once a theory is obtained, one then uses **deductive** reasonings to explain other observations:

Theory

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation $\Rightarrow$ Pattern $\Rightarrow$ Hypothesis $\Rightarrow$ Theory

Once a theory is obtained, one then uses **deductive** reasonings to explain other observations:

Theory $\Rightarrow$ Hypothesis

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation ⇒ Pattern ⇒ Hypothesis ⇒ Theory

Once a theory is obtained, one then uses **deductive** reasonings to explain other observations:

Theory ⇒ Hypothesis ⇒ Observation

Models and Theories

All mathematical theories are born from **inductive** reasonings:

Observation ⇒ Pattern ⇒ Hypothesis ⇒ Theory

Once a theory is obtained, one then uses **deductive** reasonings to explain other observations:

Theory ⇒ Hypothesis ⇒ Observation ⇒ Confirmation

Refinement of Theories

When a theory does not implify an observation ...

$$\boxed{\text{Theory}} \not\Rightarrow \boxed{\text{Confirmation}}$$

Refinement of Theories

When a theory does not imply an observation ...

$$\boxed{\text{Theory}} \not\Rightarrow \boxed{\text{Confirmation}}$$

... One usually decides to **refine** the theory by adding more axioms ...

Refinement of Theories

When a theory does not imply an observation ...

Theory $\not\Rightarrow$ Confirmation

... One usually decides to **refine** the theory by adding more axioms ...

Possible Theories $\rightleftarrows$ Covered Examples

... One thus gets a **back-and-forth** between

Refinement of Theories

When a theory does not imply an observation ...

$$\boxed{\text{Theory}} \not\Rightarrow \boxed{\text{Confirmation}}$$

... One usually decides to **refine** the theory by adding more axioms ...

$$\boxed{\text{Possible Theories}} \rightleftarrows \boxed{\text{Covered Examples}}$$

... One thus gets a **back-and-forth** between

- ▷ the poset of theories (ordered by the **level of abstraction**)

Refinement of Theories

When a theory does not imply an observation ...

Theory $\not\Rightarrow$ Confirmation

... One usually decides to **refine** the theory by adding more axioms ...

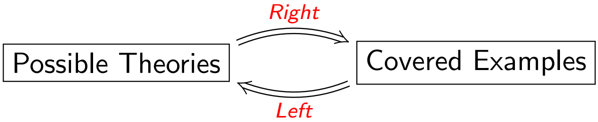
Possible Theories $\rightleftarrows$ Covered Examples

... One thus gets a **back-and-forth** between

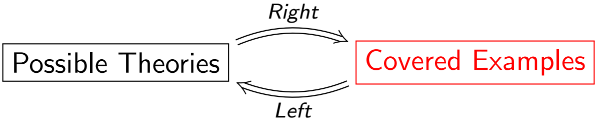
- ▷ the poset of theories (ordered by the **level of abstraction**)
- ▷ the poset of classes of Examples (ordered by the **inclusion**)

Refinement of Theories

Informally, we have an adjunction of order-preserving relations



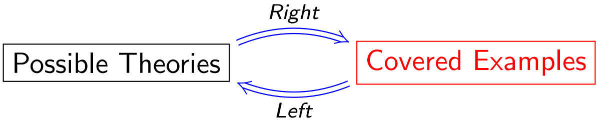
Informally, we have an adjunction of order-preserving relations



For the Examples:

Refinement of Theories

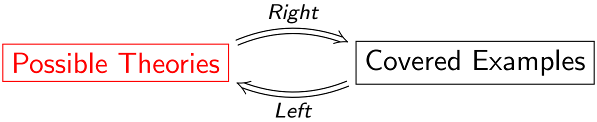
Informally, we have an adjunction of order-preserving relations



For the Examples: We want to find a greastest fixed point

Refinement of Theories

Informally, we have an adjunction of order-preserving relations

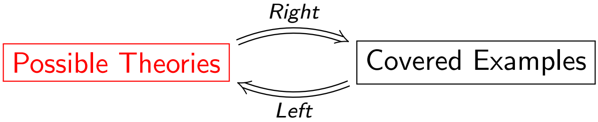


For the Examples: We want to find a greastest fixed point

For the Theories:

Refinement of Theories

Informally, we have an adjunction of order-preserving relations

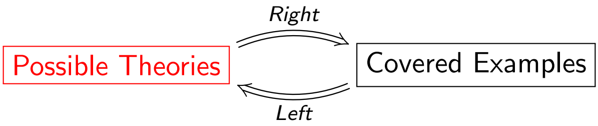


For the Examples: We want to find a greastest fixed point

For the Theories: We want to find the right level of abstraction

Refinement of Theories

Informally, we have an adjunction of order-preserving relations



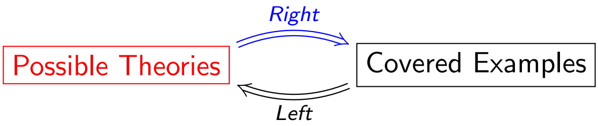
For the Examples: We want to find a greastest fixed point

For the Theories: We want to find the right level of abstraction

... the refinement of a theory can be done in two ways:

Refinement of Theories

Informally, we have an adjunction of order-preserving relations



For the Examples: We want to find a greastest fixed point

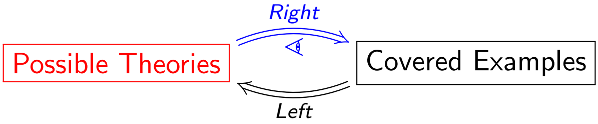
For the Theories: We want to find the right level of abstraction

... the refinement of a theory can be done in two ways:

- ▷ Either one **piles up** axioms, and we get a long list of properties

Refinement of Theories

Informally, we have an adjunction of order-preserving relations



For the Examples: We want to find a greastest fixed point

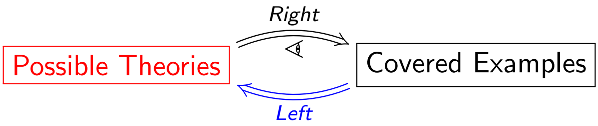
For the Theories: We want to find the right level of abstraction

... the refinement of a theory can be done in two ways:

- ▷ Either one **piles up** axioms, and we get a long list of properties

Refinement of Theories

Informally, we have an adjunction of order-preserving relations



For the Examples: We want to find a greastest fixed point

For the Theories: We want to find the right level of abstraction

... the refinement of a theory can be done in two ways:

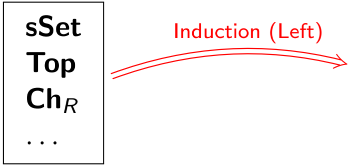
- Either one **piles up** axioms, and we get a long list of properties
- Or one can **starts over** the inductive process with the hope of finding a better set of axioms

Quillen's Model Categories

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.

sSet
Top
$\mathbf{Ch}_R$
...

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.



Quillen's Model Categories

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.

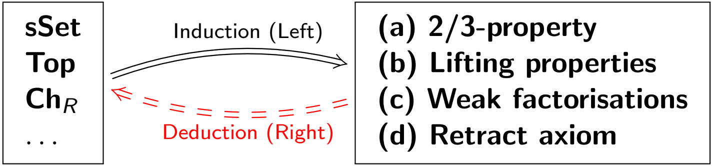
sSet
Top
$\mathbf{Ch}_R$
...

Induction (Left) $\Rightarrow$

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom

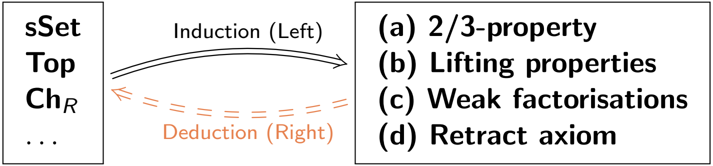
Quillen's Model Categories

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.



But, using the concept of model category is generally **not enough** to retrieve the combinatorial properties of these examples; e.g.

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.

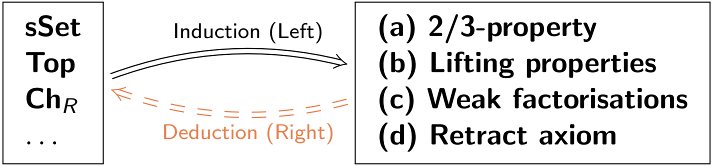


But, using the concept of model category is generally **not enough** to retrieve the combinatorial properties of these examples; e.g.

- ▷ Left or right Properness;

Quillen's Model Categories

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.

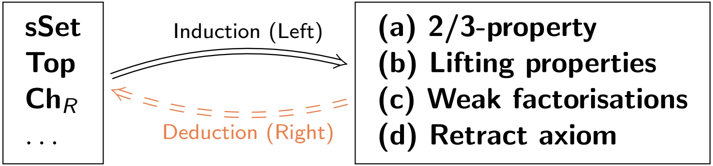


But, using the concept of model category is generally **not enough** to retrieve the combinatorial properties of these examples; e.g.

- ▷ Left or right Properness;
- ▷ The idea of smallness;

Quillen's Model Categories

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.

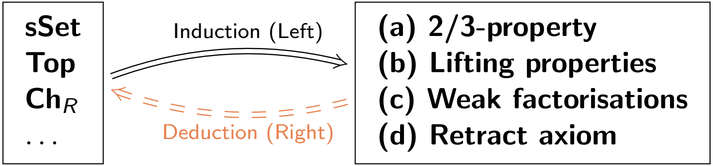


But, using the concept of model category is generally **not enough** to retrieve the combinatorial properties of these examples; e.g.

- ▷ Left or right Properness;
- ▷ The idea of smallness;
- ▷ The concept of CW-complex;

Quillen's Model Categories

For instance, in 1967, **D. Quillen** came up with a set of axioms for the concept of **model category** on the base of a particular set of examples.



But, using the concept of model category is generally **not enough** to retrieve the combinatorial properties of these examples; e.g.

- ▷ Left or right Properness;
- ▷ The idea of smallness;
- ▷ The concept of CW-complex;
- ▷ The Whitehead theorem ...

Cofibrantly Generated Model Categories

So, we add **more axioms** on the theoretical side (without losing too many examples though)

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) . . .

Cofibrantly Generated Model Categories

So, we add **more axioms** on the theoretical side (without losing too many examples though)

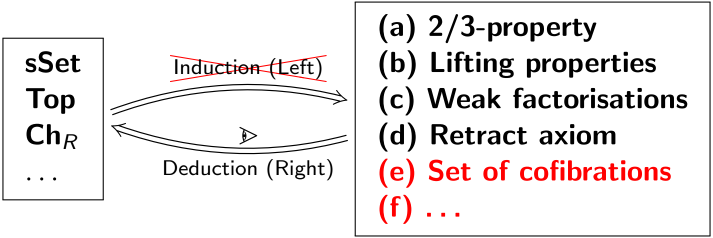
sSet
Top
$\mathbf{Ch}_R$
...

⟸ Deduction (Right)

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) ...

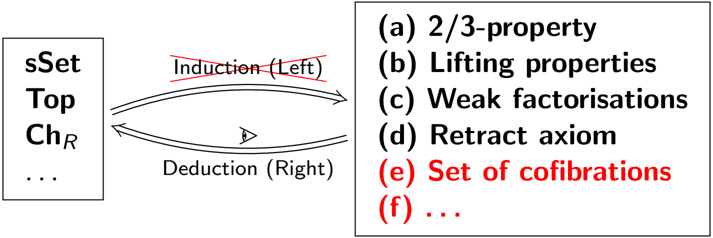
Cofibrantly Generated Model Categories

So, we add **more axioms** on the theoretical side (without losing too many examples though)



Cofibrantly Generated Model Categories

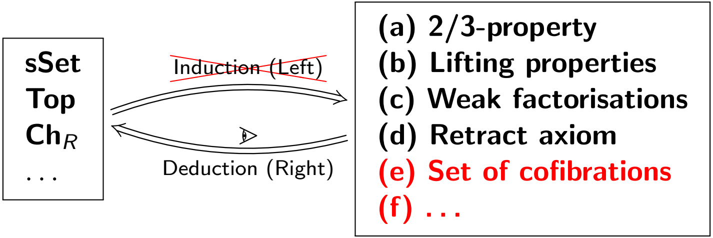
So, we add **more axioms** on the theoretical side (without losing too many examples though)



This need to further restrict our theoretical setting suggests that we have not reached the **right level of abstraction** ...

Cofibrantly Generated Model Categories

So, we add **more axioms** on the theoretical side (without losing too many examples though)



This need to further restrict our theoretical setting suggests that we have not reached the **right level of abstraction** ...

... and thus have **failed** to see the existence of a more fundamental setting...

A More Fundamental Setting

For instance, in **cofibrantly generated model categories**, we distinguish between

A More Fundamental Setting

For instance, in **cofibrantly generated model categories**, we distinguish between

- the genuine structure (i.e. the given **classes** of arrows);

For instance, in **cofibrantly generated model categories**, we distinguish between

- the genuine structure (i.e. the given **classes** of arrows);
- the generating structure (i.e. some **sets** of cofibrations);

A More Fundamental Setting

For instance, in **cofibrantly generated model categories**, we distinguish between

- the genuine structure (i.e. the given **classes** of arrows);
- the generating structure (i.e. some **sets** of cofibrations);

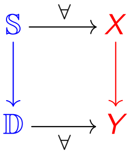
But the genuine structure is entirely determined by the generating structure via lifting properties:

A More Fundamental Setting

For instance, in **cofibrantly generated model categories**, we distinguish between

- the genuine structure (i.e. the given **classes** of arrows);
- the generating structure (i.e. some **sets** of cofibrations);

But the genuine structure is entirely determined by the generating structure via lifting properties:

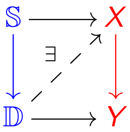


A More Fundamental Setting

For instance, in **cofibrantly generated model categories**, we distinguish between

- the genuine structure (i.e. the given **classes** of arrows);
- the generating structure (i.e. some **sets** of cofibrations);

But the genuine structure is entirely determined by the generating structure via lifting properties:



A More Fundamental Setting

Similarly, in **tractable model categories**, we can characterise the weak equivalences of the model category from the generating structure ...

A More Fundamental Setting

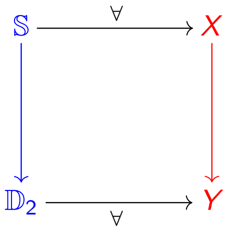
Similarly, in **tractable model categories**, we can characterise the weak equivalences of the model category from the generating structure ...

These can be characterised by using a lifting property with respect to commutative squares of generating structure as follows:

A More Fundamental Setting

Similarly, in **tractable model categories**, we can characterise the weak equivalences of the model category from the generating structure ...

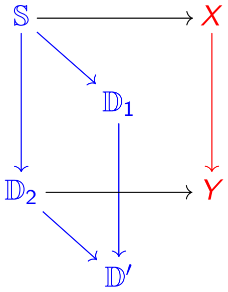
These can be characterised by using a lifting property with respect to commutative squares of generating structure as follows:



A More Fundamental Setting

Similarly, in **tractable model categories**, we can characterise the weak equivalences of the model category from the generating structure …

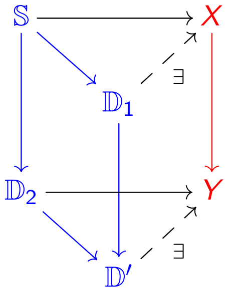
These can be characterised by using a lifting property with respect to commutative squares of generating structure as follows:



A More Fundamental Setting

Similarly, in **tractable model categories**, we can characterise the weak equivalences of the model category from the generating structure ...

These can be characterised by using a lifting property with respect to commutative squares of generating structure as follows:



Generating Structure = Internal Logic

What really happens is that these generating arrows form an **internal logic** of the model category.

Generating Structure = Internal Logic

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Generating Structure = Internal Logic

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Long list of Axioms

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) . . .

Generating Structure = Internal Logic

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Long list of Axioms

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) ...

Deduction $\Longrightarrow$

Internal Logic

Generating Structure = Internal Logic

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Long list of Axioms

(a) 2/3-property

(b) Lifting properties

(c) Weak factorisations

(d) Retract axiom

(e) Set of cofibrations

(f) ...

Deduction $\Longrightarrow$

Internal Logic

Generating arrows

Generating Structure = Internal Logic

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Long list of Axioms

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) ...

Deduction $\Longrightarrow$

Internal Logic

Generating arrows
Commutative squares

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Long list of Axioms

(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) ...

Deduction $\Longrightarrow$

Internal Logic

Generating arrows
Commutative squares
Commutativity relations

What really happens is that these generating arrows form an **internal logic** of the model category.

... and as in every logic, the commutativity relations satisfied by these generating arrows form the **axioms** of the internal logic...

Long list of Axioms
(a) 2/3-property
(b) Lifting properties
(c) Weak factorisations
(d) Retract axiom
(e) Set of cofibrations
(f) ...

Deduction $\Longrightarrow$

Internal Logic
Generating arrows
Commutative squares
Commutativity relations
...

Opposite Point of View

So, to avoid the Long List of Axioms ...

Opposite Point of View

So, to avoid the Long List of Axioms ...

... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...

Opposite Point of View

So, to avoid the Long List of Axioms ...

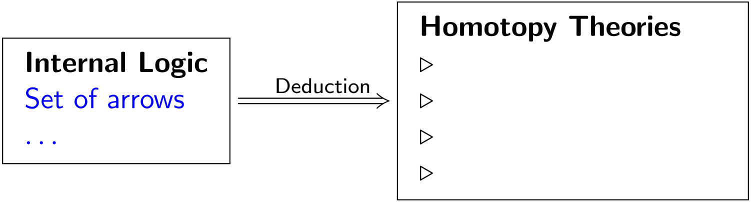
... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...

Internal Logic
Set of arrows
. . .

Opposite Point of View

So, to avoid the Long List of Axioms ...

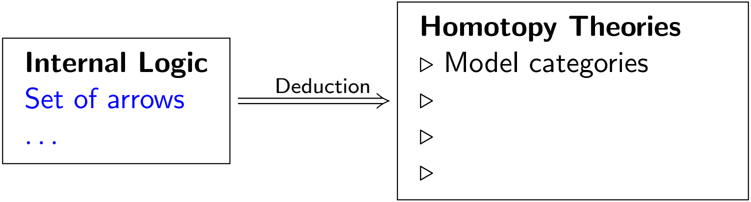
... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...



Opposite Point of View

So, to avoid the Long List of Axioms ...

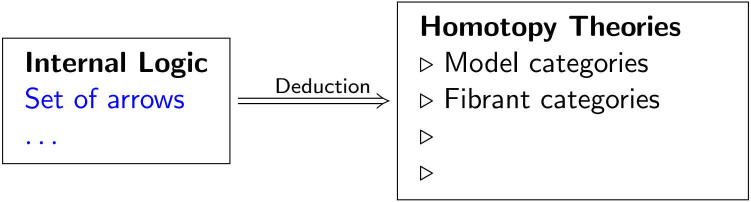
... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...



Opposite Point of View

So, to avoid the Long List of Axioms ...

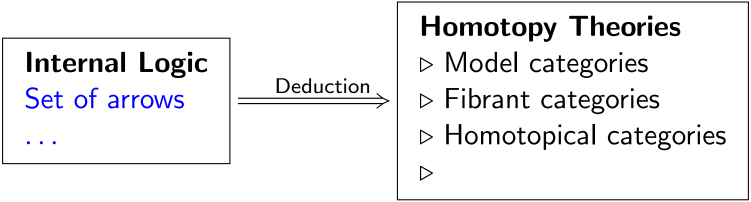
... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...



Opposite Point of View

So, to avoid the Long List of Axioms ...

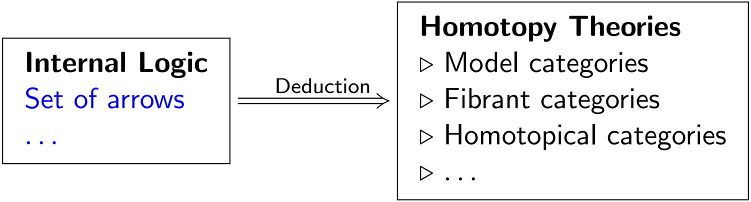
... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...



Opposite Point of View

So, to avoid the Long List of Axioms ...

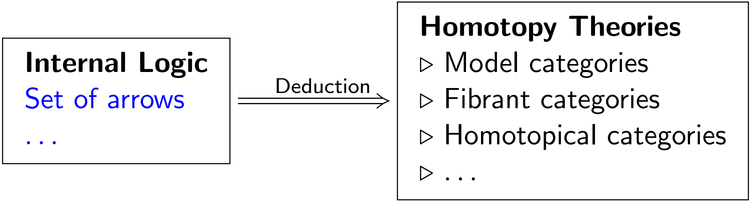
... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...



Opposite Point of View

So, to avoid the Long List of Axioms ...

... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...

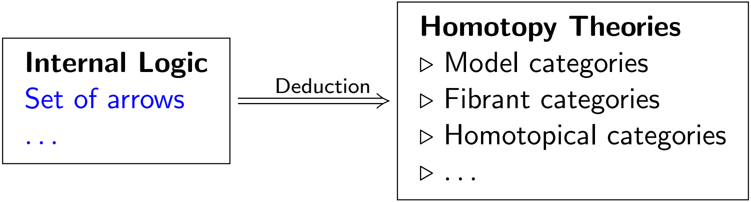


But what kind of **axioms** should we set for this new Homotopy logic?

Opposite Point of View

So, to avoid the Long List of Axioms ...

... we could take inspiration on these internal logics and take their associated **internal language** as a starting point for a brand new Homotopy Theory ...



But what kind of **axioms** should we set for this new Homotopy logic?

...well, this will be determined by **Induction** from the world of Examples.

Loading .

... After looking into how the internal logics work ...

Loading ..

... After looking into how the internal logics work ...

... we notice that a very elementary structure appears constantly ...

Loading ...

... After looking into how the internal logics work ...

... we notice that a very elementary structure appears constantly ...

... in any formulation of the internal axioms ...

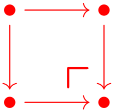
Elementary Structure for Homotopy Theory

This elementary structure is made of :

Elementary Structure for Homotopy Theory

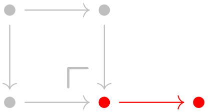
This elementary structure is made of :

- a pushout square;



This elementary structure is made of :

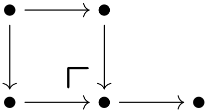
- a pushout square;
- and an additional arrow as follows.



Elementary Structure for Homotopy Theory

This elementary structure is made of :

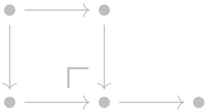
- a pushout square;
- and an additional arrow as follows.



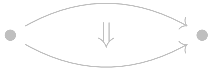
Elementary Structure for Homotopy Theory

This elementary structure is made of :

- a pushout square;
- and an additional arrow as follows.



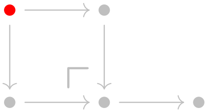
This diagram encodes nothing but the internal logic of a **cell** or **disc**:



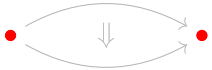
Elementary Structure for Homotopy Theory

This elementary structure is made of :

- ▷ a pushout square;
- ▷ and an additional arrow as follows.



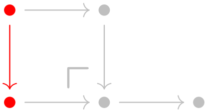
This diagram encodes nothing but the internal logic of a **cell** or **disc**:



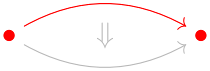
Elementary Structure for Homotopy Theory

This elementary structure is made of :

- ▷ a pushout square;
- ▷ and an additional arrow as follows.



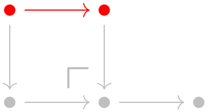
This diagram encodes nothing but the internal logic of a **cell** or **disc**:



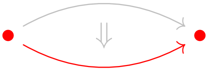
Elementary Structure for Homotopy Theory

This elementary structure is made of :

- a pushout square;
- and an additional arrow as follows.



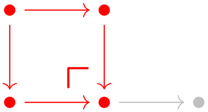
This diagram encodes nothing but the internal logic of a **cell** or **disc**:



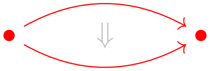
Elementary Structure for Homotopy Theory

This elementary structure is made of :

- a pushout square;
- and an additional arrow as follows.



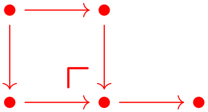
This diagram encodes nothing but the internal logic of a **cell** or **disc**:



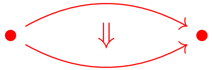
Elementary Structure for Homotopy Theory

This elementary structure is made of :

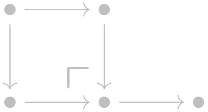
- a pushout square;
- and an additional arrow as follows.



This diagram encodes nothing but the internal logic of a **cell** or **disc**:

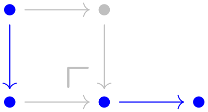


We can also retrieve the representatives of the so-called **generating structure** seen previously as follows:



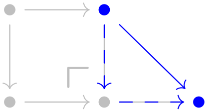
We can also retrieve the representatives of the so-called **generating structure** seen previously as follows:

- generating cofibrations;



We can also retrieve the representatives of the so-called **generating structure** seen previously as follows:

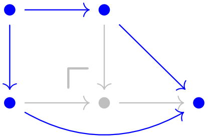
- ▷ generating cofibrations;
- ▷ generating trivial cofibrations;



Elementary Structure for Homotopy Theory

We can also retrieve the representatives of the so-called **generating structure** seen previously as follows:

- ▷ generating cofibrations;
- ▷ generating trivial cofibrations;
- ▷ generating squares;



In the category of topological spaces **Top**:

$$
\begin{array}{ccccc}
\mathbb{S}^{n-1} & \longrightarrow & \mathbb{D}^{n} & & \\
\big\downarrow & & \big\downarrow & & \\
\mathbb{D}^{n} & \xrightarrow{\ulcorner} & \mathbb{S}^{n} & \longrightarrow & \mathbb{D}^{n+1}
\end{array}
$$

In the category of simplicial sets **sSet**:

$$
\begin{array}{ccccc}
\partial\Delta_{n-1} & \longrightarrow & \Lambda_{n}^{k} & & \\
\big\downarrow & & \big\downarrow & & \\
\Delta_{n-1} & \xrightarrow{\ulcorner} & \partial\Delta_{n} & \longrightarrow & \Delta_{n}
\end{array}
$$

In the category of topological spaces **Top**:

$$
\begin{array}{ccccc}
\mathbb{S}^{n-1} & \longrightarrow & \mathbb{D}^{n} & & \\
\downarrow & & \downarrow & & \\
\mathbb{D}^{n} & \longrightarrow & \mathbb{S}^{n} & \longrightarrow & \mathbb{D}^{n+1}
\end{array}
$$

In the category of simplicial sets **sSet**:

$$
\begin{array}{ccccc}
\partial\Delta_{n-1} & \longrightarrow & \Lambda_n^k & & \\
\downarrow & & \downarrow & & \\
\Delta_{n-1} & \longrightarrow & \partial\Delta_{n} & \longrightarrow & \Delta_{n}
\end{array}
$$

Examples

In the category of topological spaces **Top**:

$$
\begin{array}{ccccc}
\mathbb{S}^{n-1} & \longrightarrow & \mathbb{D}^{n} & & \\
\downarrow & & \downarrow & & \\
\mathbb{D}^{n} & \longrightarrow & \mathbb{S}^{n} & \longrightarrow & \mathbb{D}^{n+1}
\end{array}
$$

In the category of simplicial sets **sSet**:

$$
\begin{array}{ccccc}
\partial\Delta_{n-1} & \longrightarrow & \Lambda_n^k & & \\
\downarrow & & \downarrow & & \\
\Delta_{n-1} & \longrightarrow & \partial\Delta_n & \longrightarrow & \Delta_n
\end{array}
$$

Examples

In the category of topological spaces **Top**:

$$\begin{array}{ccccc} \mathbb{S}^{n-1} & \longrightarrow & \mathbb{D}^{n} & & \\ \downarrow & & \downarrow & & \\ \mathbb{D}^{n} & \longrightarrow & \mathbb{S}^{n} & \longrightarrow & \mathbb{D}^{n+1} \end{array}$$

In the category of simplicial sets **sSet**:

$$\begin{array}{ccccc} \partial\Delta_{n-1} & \longrightarrow & \Lambda_n^k & & \\ \downarrow & & \downarrow & & \\ \Delta_{n-1} & \longrightarrow & \partial\Delta_{n} & \longrightarrow & \Delta_{n} \end{array}$$

Examples

In the topos $\mathbf{Sh}(C^\infty\mathbf{Ring}^{\mathrm{op}})$:

$$\begin{array}{ccccc} \{0\} & \longrightarrow & D & & \\ \downarrow & & \downarrow & & \\ D & \longrightarrow & D(2) & \longrightarrow & D \end{array}$$

Examples

In the topos $\mathbf{Sh}(C^\infty\mathbf{Ring}^{\mathrm{op}})$:

$$\begin{array}{ccccc}
\{0\} & \longrightarrow & D & & \\
\downarrow & & \downarrow & & \\
D & \longrightarrow & \color{red}{D(2)} & \longrightarrow & D
\end{array}$$

Examples

In the topos $\mathbf{Sh}(C^\infty\mathbf{Ring}^{\mathrm{op}})$:

$$\begin{array}{ccccc} \{0\} & \longrightarrow & D & & \\ \downarrow & & \downarrow & & \\ D & \longrightarrow & D(2) & \longrightarrow & D \end{array}$$

... many other examples exist.

A Mini-Homotopy Theory

By taking inspiration on the (previously seen) characterisations of

A Mini-Homotopy Theory

By taking inspiration on the (previously seen) characterisations of

- ▷ fibrations;

A Mini-Homotopy Theory

By taking inspiration on the (previously seen) characterisations of

- fibrations;
- trivial fibrations;

A Mini-Homotopy Theory

By taking inspiration on the (previously seen) characterisations of

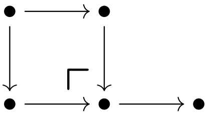
- fibrations;
- trivial fibrations;
- weak equivalences;

A Mini-Homotopy Theory

By taking inspiration on the (previously seen) characterisations of

- ▷ fibrations;
- ▷ trivial fibrations;
- ▷ weak equivalences;

we can define a **mini-homotopy theory** for each structure of the form



By taking inspiration on the (previously seen) characterisations of

- ▷ fibrations;
- ▷ trivial fibrations;
- ▷ weak equivalences;

we can define a **mini-homotopy theory** for each structure of the form

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & & \downarrow & & \\ & & \ulcorner & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

A Mini-Homotopy Theory

So, for every structure

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

we say that a morphism $f : X \to Y$ is ...

A Mini-Homotopy Theory

So, for every structure

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

we say that a morphism $f : X \to Y$ is ...

---

a **fibration** if

So, for every structure

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

we say that a morphism $f : X \to Y$ is ...

---

a **fibration** if

$$\begin{array}{ccc} \mathbb{D}_1 & \xrightarrow{\forall} & X \\ \downarrow & & \downarrow f \\ \mathbb{D}' & \xrightarrow[\forall]{} & Y \end{array}$$

So, for every structure

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

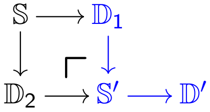
we say that a morphism $f : X \to Y$ is ...

---

a **fibration** if

$$\begin{array}{ccc} \mathbb{D}_1 & \xrightarrow{\forall} & X \\ \downarrow & & \downarrow f \\ \mathbb{D}' & \xrightarrow[\forall]{} & Y \end{array}$$

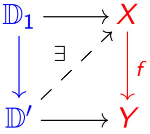
So, for every structure



we say that a morphism $f : X \to Y$ is ...

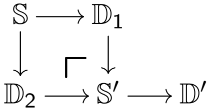
---

a **fibration** if



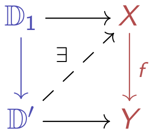
A Mini-Homotopy Theory

So, for every structure



we say that a morphism $f : X \to Y$ is ...

a **fibration** if



a **trivial fibration** if

A Mini-Homotopy Theory

So, for every structure

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

we say that a morphism $f : X \to Y$ is ...

---

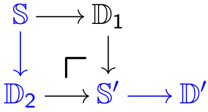
a **fibration** if

$$\begin{array}{ccc} \mathbb{D}_1 & \longrightarrow & X \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow f \\ \mathbb{D}' & \longrightarrow & Y \end{array}$$

a **trivial fibration** if

$$\begin{array}{ccc} \bullet & \xrightarrow{\forall} & X \\ \downarrow & & \downarrow f \\ \bullet & \xrightarrow[\forall]{} & Y \end{array}$$

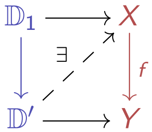
So, for every structure



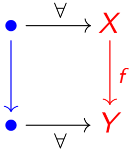
we say that a morphism $f : X \to Y$ is ...

---

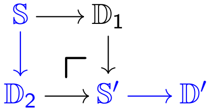
a **fibration** if



a **trivial fibration** if



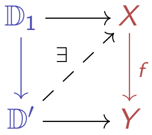
So, for every structure



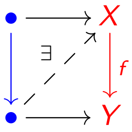
we say that a morphism $f : X \to Y$ is …

---

a **fibration** if

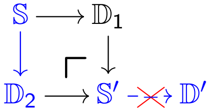


a **trivial fibration** if



A Mini-Homotopy Theory

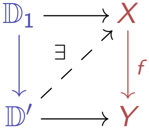
So, for every structure



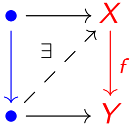
we say that a morphism $f : X \to Y$ is ...

---

a **fibration** if



a ~~trivial~~ **pseudo** **fibration** if



A Mini-Homotopy Theory

So, for every structure

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

we say that a morphism $f : X \to Y$ is ...

---

a **fibration** if

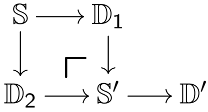
$$\begin{array}{ccc} \mathbb{D}_1 & \longrightarrow & X \\ \downarrow & \exists \nearrow & \downarrow f \\ \mathbb{D}' & \longrightarrow & Y \end{array}$$

a ~~**trivial**~~ **fibration** if
pseudo

$$\begin{array}{ccc} \bullet & \longrightarrow & X \\ \downarrow & \exists \nearrow & \downarrow f \\ \bullet & \longrightarrow & Y \end{array}$$

a **surtraction** if

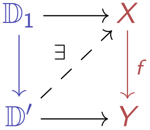
So, for every structure



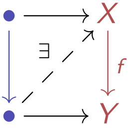
we say that a morphism $f : X \to Y$ is ...

---

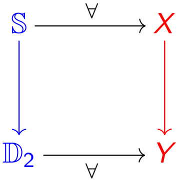
a **fibration** if



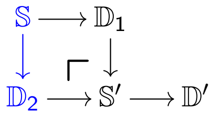
a ~~trivial~~ **fibration** if
pseudo



a **surtraction** if



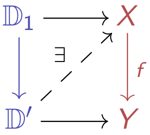
So, for every structure



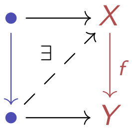
we say that a morphism $f : X \to Y$ is ...

---

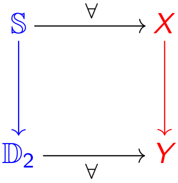
a **fibration** if



a ~~**trivial**~~ **fibration** if
pseudo

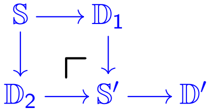


a **surtraction** if



A Mini-Homotopy Theory

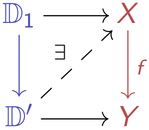
So, for every structure



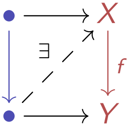
we say that a morphism $f : X \to Y$ is ...

---

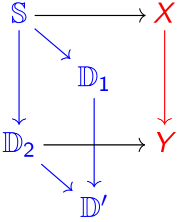
a **fibration** if



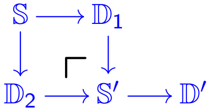
a ~~**trivial**~~ **pseudo** **fibration** if



a **surtraction** if



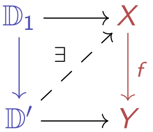
So, for every structure



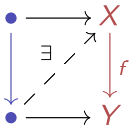
we say that a morphism $f : X \to Y$ is …

---

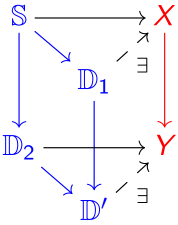
a **fibration** if



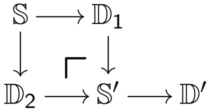
a ~~**trivial**~~ **pseudo** **fibration** if



a **surtraction** if



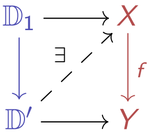
So, for every structure



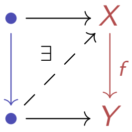
we say that a morphism $f : X \to Y$ is ...

---

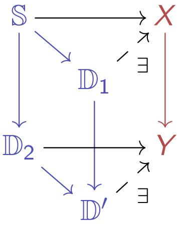
a **fibration** if



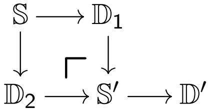
a ~~**trivial**~~ **fibration** if
pseudo



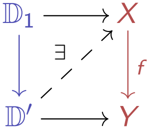
a **surtraction** if



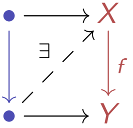
an **intraction** if

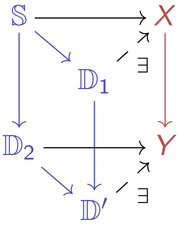
So, for every structure  we say that a morphism $f : X \to Y$ is ...

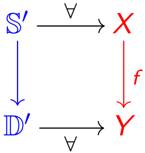
---

a **fibration** if 

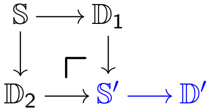
a ~~**trivial**~~ **fibration** if
pseudo



a **surtraction** if 

an **intraction** if 

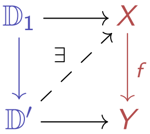
So, for every structure



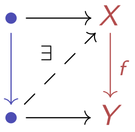
we say that a morphism $f : X \to Y$ is ...

---

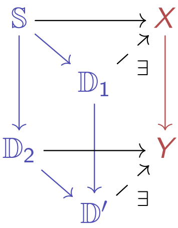
a **fibration** if



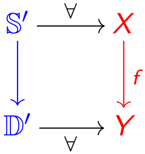
a ~~**trivial**~~ **fibration** if
pseudo



a **surtraction** if

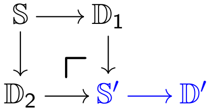


an **intraction** if



A Mini-Homotopy Theory

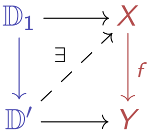
So, for every structure



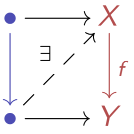
we say that a morphism $f : X \to Y$ is ...

---

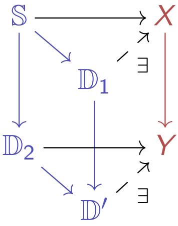
a **fibration** if



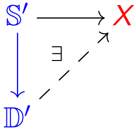
a ~~trivial~~ **fibration** if
pseudo



a **surtraction** if



an **intraction** if



weak equivalence $=$ **surtraction** $+$ **intraction**

And now, let the magic happen...

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;
- f and g intractions $\Rightarrow$ $f \circ g$ intraction;

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;
- f and g intractions $\Rightarrow$ $f \circ g$ intraction;
- $f \circ g$ surtraction and f intraction $\Rightarrow$ g surtraction;

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;
- f and g intractions $\Rightarrow$ $f \circ g$ intraction;
- $f \circ g$ surtraction and f intraction $\Rightarrow$ g surtraction;
- isomorphisms are * fibrations;

$* =$ pseudo / trivial / $\emptyset$

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;
- f and g intractions $\Rightarrow$ $f \circ g$ intraction;
- $f \circ g$ surtraction and f intraction $\Rightarrow$ g surtraction;
- isomorphisms are * fibrations;
- * fibrations are stable under pullbacks;

* $=$ pseudo / trivial / $\emptyset$

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;
- f and g intractions $\Rightarrow$ $f \circ g$ intraction;
- $f \circ g$ surtraction and f intraction $\Rightarrow$ g surtraction;
- isomorphisms are * fibrations;
- * fibrations are stable under pullbacks;
- * fibrations/surtractions/intractions are stable under retracts;

* $=$ pseudo / trivial / $\emptyset$

A Mini-Homotopy Theory

And now, let the magic happen... It follows from the definitions that:

- isomorphisms are intractions;
- $f \circ g$ intraction $\Rightarrow$ g intraction;
- f and g intractions $\Rightarrow$ $f \circ g$ intraction;
- $f \circ g$ surtraction and f intraction $\Rightarrow$ g surtraction;
- isomorphisms are * fibrations;
- * fibrations are stable under pullbacks;
- * fibrations/surtractions/intractions are stable under retracts;

* $=$ pseudo / trivial / $\emptyset$

...

A Mini-Homotopy Theory

- trivial fibrations $\subseteq$ pseudo fibrations

A Mini-Homotopy Theory

- trivial fibrations $\subseteq$ pseudo fibrations
- fibrations $\cap$ surtractions $\subseteq$ pseudo fibrations

A Mini-Homotopy Theory

- trivial fibrations $\subseteq$ pseudo fibrations
- fibrations $\cap$ surtractions $\subseteq$ pseudo fibrations
- trivial fibrations $\subseteq$ fibrations $\cap$ intractions

A Mini-Homotopy Theory

- trivial fibrations $\subseteq$ pseudo fibrations
- fibrations $\cap$ surtractions $\subseteq$ pseudo fibrations
- trivial fibrations $\subseteq$ fibrations $\cap$ intractions

The last two items remind of:

$$\boxed{\textbf{fibrations} \cap \textbf{weak equivalence} = \textbf{trivial fibrations}}$$

- trivial fibrations $\subseteq$ pseudo fibrations
- fibrations $\cap$ surtractions $\subseteq$ pseudo fibrations
- trivial fibrations $\subseteq$ fibrations $\cap$ intractions

The last two items remind of:

$$\boxed{\textbf{fibrations} \cap \textbf{weak equivalence} = \textbf{trivial fibrations}}$$

But we cannot really prove **more** if we do not require **more**...

- trivial fibrations $\subseteq$ pseudo fibrations
- fibrations $\cap$ surtractions $\subseteq$ pseudo fibrations
- trivial fibrations $\subseteq$ fibrations $\cap$ intractions

The last two items remind of:

$$\boxed{\textbf{fibrations} \cap \textbf{weak equivalence} = \textbf{trivial fibrations}}$$

But we cannot really prove **more** if we do not require **more**...

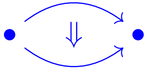
It is possible to obtain **more** properties if we add give some axioms to our internal logic.

Axioms

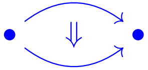
Recall that

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

Axioms

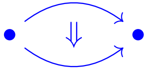
Recall that $\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$ $=$  so if we encode:

Axioms

Recall that
$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array} \quad = $$

so if we encode:

▷ an **identity cell**  we obtain:

Axioms

Recall that $\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$ $=$  so if we encode:

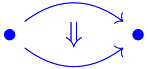
▷ an **identity cell**  we obtain:

- If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.

Axioms

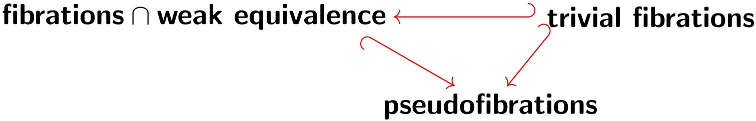
Recall that

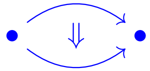
$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array} \quad = $$



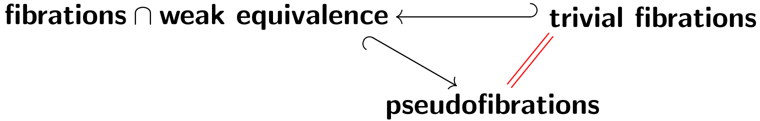
so if we encode:

▷ an **identity cell**  we obtain:

- If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.
- 

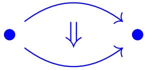
Recall that
$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array} \quad = $$

so if we encode:

▷ an **identity cell**  we obtain:

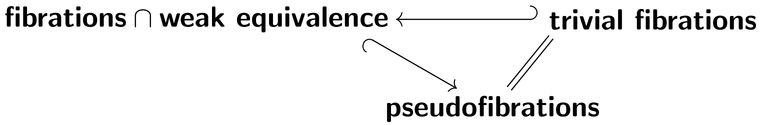
- If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.
- 

Recall that

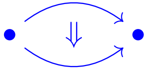
$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$$

=  so if we encode:

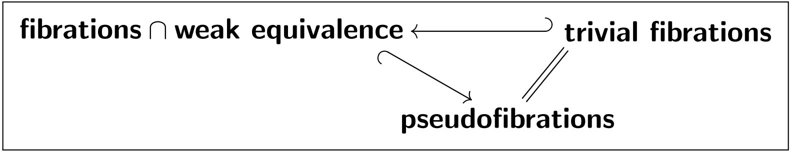
▷ an **identity cell**  we obtain:

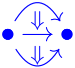
- If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.
- 

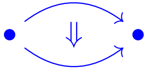
Axioms

Recall that $\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$ $=$  so if we encode:

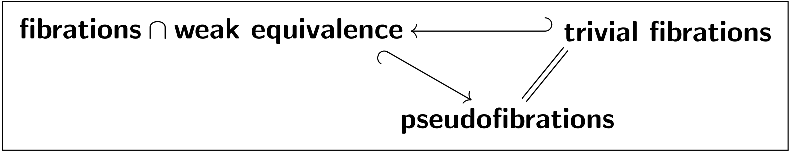
▷ an **identity cell**  we obtain:

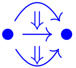
- If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.
- 

▷ a **composition of cells**  we obtain:

Recall that $\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathbb{D}_1 & & \\ \downarrow & \ulcorner & \downarrow & & \\ \mathbb{D}_2 & \longrightarrow & \mathbb{S}' & \longrightarrow & \mathbb{D}' \end{array}$ $=$  so if we encode:

▷ an **identity cell**  we obtain:

- If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.
- 

▷ a **composition of cells**  we obtain:

- weak equivalences are stable under composition.

Axioms

Also, to be able to finish proving the 2/3-property, we need to "draw" more complex axioms than **identities** or **compositions**

Axioms

Also, to be able to finish proving the 2/3-property, we need to "draw" more complex axioms than **identities** or **compositions**

In fact we need to draw and fully understand the internal logic of **higher coherence theory**

Also, to be able to finish proving the 2/3-property, we need to “draw” more complex axioms than **identities** or **compositions**

In fact we need to draw and fully understand the internal logic of **higher coherence theory**

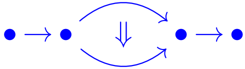
Basically, the main operations that I define are:

Axioms

Also, to be able to finish proving the 2/3-property, we need to "draw" more complex axioms than **identities** or **compositions**

In fact we need to draw and fully understand the internal logic of **higher coherence theory**

Basically, the main operations that I define are:

- **whiskerings**  in all dimensions;

Axioms

Also, to be able to finish proving the 2/3-property, we need to "draw" more complex axioms than **identities** or **compositions**

In fact we need to draw and fully understand the internal logic of **higher coherence theory**

Basically, the main operations that I define are:

- **whiskerings** $\bullet \to \bullet \underset{\to}{\overset{\to}{\Downarrow}} \bullet \to \bullet$ in all dimensions;
- a very weak notion of **invertible** cell $\bullet \xrightarrow{\cong} \bullet$ in all dimensions.

How do we encode dimensions?

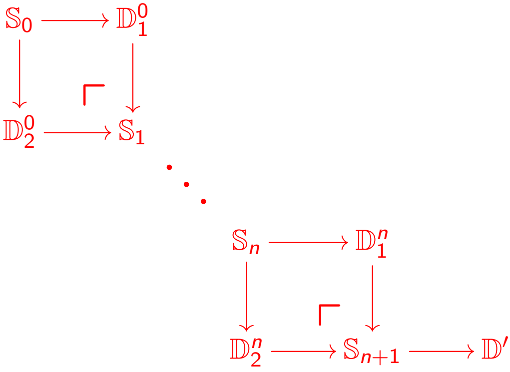
... and we need to extend the elementary structures

$$\begin{array}{ccccc} \mathbb{S} & \rightarrow & \mathbb{D}_1 & & \\ \downarrow & & \downarrow & & \\ \mathbb{D}_2 & \rightarrow & \mathbb{S}' & \rightarrow & \mathbb{D}' \end{array}$$

to longer ones made out of the elementary ones:

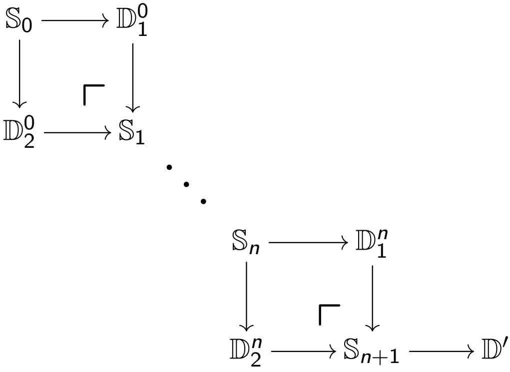
How do we encode dimensions?

... and we need to extend the elementary structures $\begin{array}{ccccc} \mathbb{S} & \to & \mathbb{D}_1 & & \\ \downarrow & & \downarrow & & \\ \mathbb{D}_2 & \to & \mathbb{S}' & \to & \mathbb{D}' \end{array}$ to longer ones made out of the elementary ones:



How do we encode dimensions?

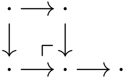
... and we need to extend the elementary structures $\begin{array}{ccccc} \mathbb{S} & \to & \mathbb{D}_1 & & \\ \downarrow & & \downarrow & & \\ \mathbb{D}_2 & \to & \mathbb{S}' & \to & \mathbb{D}' \end{array}$ to longer ones made out of the elementary ones:



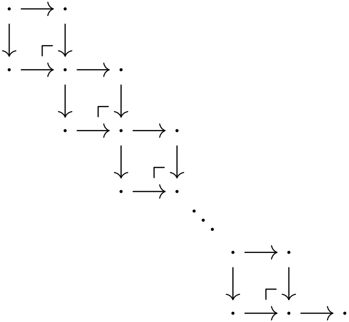
... this encodes the dimension of a cell!

Vertebrae & Spines

I call every elementary structure



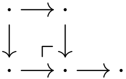
a **vertebra** and I call every structure



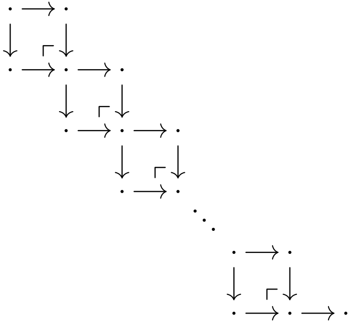
a **spine**

Vertebrae & Spines

I call every elementary structure

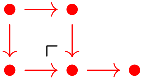


a **vertebra** and I call every structure

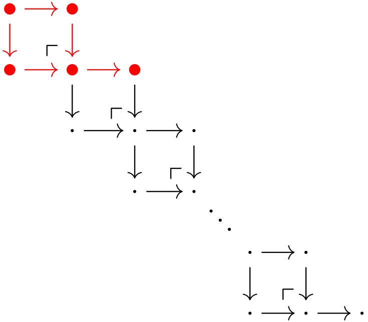


a **spine** (which is made of **vertebrae**)

Vertebrae & Spines

I call every elementary structure  a **vertebra** and I call

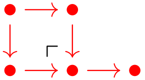
every structure



a **spine** (which is made of **vertebrae**)

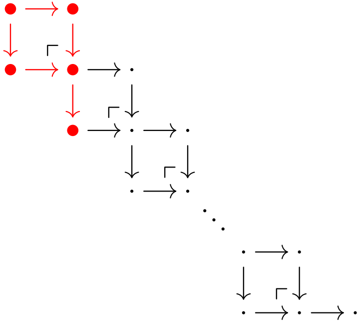
Vertebrae & Spines

I call every elementary structure



a **vertebra** and I call

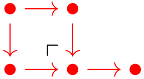
every structure



a **spine** (which is made of **vertebrae**)

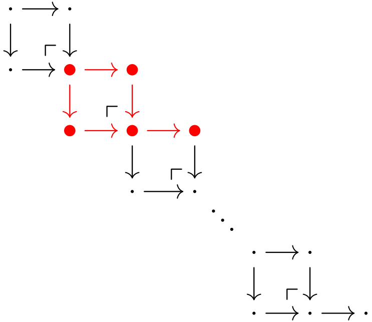
Vertebrae & Spines

I call every elementary structure



a **vertebra** and I call

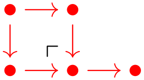
every structure



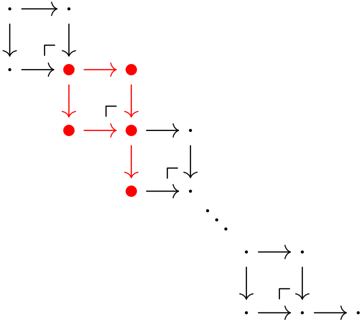
a **spine** (which is made of **vertebrae**)

Vertebrae & Spines

I call every elementary structure



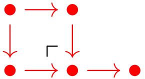
a **vertebra** and I call every structure



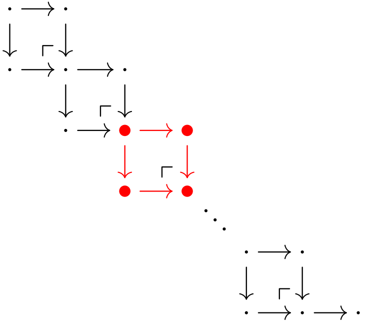
a **spine** (which is made of **vertebrae**)

Vertebrae & Spines

I call every elementary structure

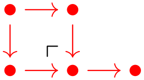


a **vertebra** and I call every structure

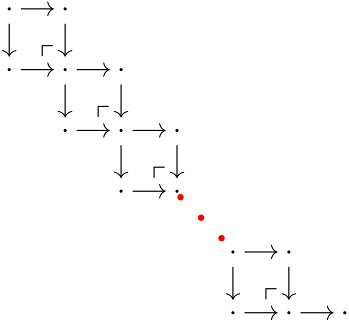


a **spine** (which is made of **vertebrae**)

Vertebrae & Spines

I call every elementary structure  a **vertebra** and I call

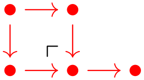
every structure



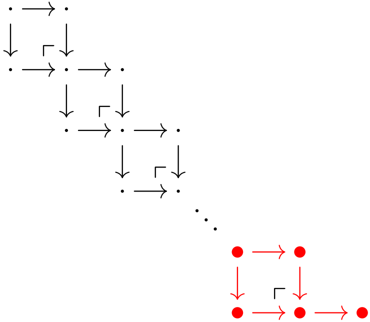
a **spine** (which is made of **vertebrae**)

Vertebrae & Spines

I call every elementary structure



a **vertebra** and I call every structure



a **spine** (which is made of **vertebrae**)

Spinal Categories

If you now take

Spinal Categories

If you now take

- a category $\mathcal{C}$;

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms** I previously described on the vertebrae and spines, we get the notion of

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms** I previously described on the vertebrae and spines, we get the notion of

SPINAL CATEGORY

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms I** previously described on the vertebrae and spines, we get the notion of

SPINAL CATEGORY

equipped with three classes of morphisms (defined vertebra-wise)

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms I** previously described on the vertebrae and spines, we get the notion of

SPINAL CATEGORY

equipped with three classes of morphisms (defined vertebra-wise)

- weak equivalences;

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms** I previously described on the vertebrae and spines, we get the notion of

SPINAL CATEGORY

equipped with three classes of morphisms (defined vertebra-wise)

- weak equivalences;
- fibrations

Spinal Categories

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms** I previously described on the vertebrae and spines, we get the notion of

SPINAL CATEGORY

equipped with three classes of morphisms (defined vertebra-wise)

- weak equivalences;
- fibrations
- trivial fibrations

If you now take

- a category $\mathcal{C}$;
- a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures;

$$+$$

the **axioms** I previously described on the vertebrae and spines, we get the notion of

SPINAL CATEGORY

equipped with three classes of morphisms (defined vertebra-wise)

- weak equivalences;
- fibrations $\rightsquigarrow$ trivial cofibrations;
- trivial fibrations $\rightsquigarrow$ cofibrations;

"acyclic fibrations" $=$ cofibrations $+$ weak equivalences "acyclic fibrations" $=$ cofibrations $+$ weak equivalences

Theorem

If pushouts of acyclic cofibrations are weak equivalences (and some smallness condition holds), then a **spinal category** *is a* **model category***.*

Spinal Categories $\rightsquigarrow$ Model Categories

"acyclic fibrations" $=$ cofibrations $+$ weak equivalences

Theorem

If pushouts of acyclic cofibrations are weak equivalences (and some smallness condition holds), then a **spinal category** *is a* **model category**.

The proof does not even need the **Jeff Smith's criteria** as all the desired properties come from my **definitions**.

More General Structures

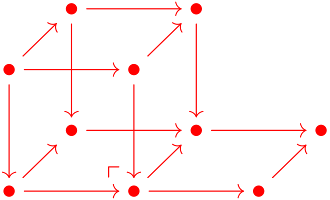
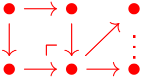
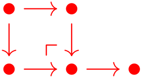
... more general structures are sometimes needed ...

More General Structures

... more general structures are sometimes needed ...

Local (e.g. **stacks**)	non-algebraic	algebraic

... more general structures are sometimes needed ...

Local (e.g. **stacks**)	non-algebraic	algebraic
		

Homotopy Hypothesis

Can we apply our theory to the category of Grothendieck's ∞-groupoids ∞-**Grp**?

Homotopy Hypothesis

Can we apply our theory to the category of Grothendieck's ∞-groupoids ∞-**Grp**?

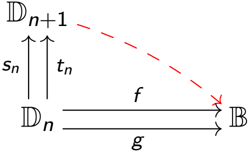
It follows from the definition of a Grothendieck's ∞-groupoid that any pair of parallel arrows in ∞-**Grp** ...

$$\mathbb{D}_n \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{B}$$

Homotopy Hypothesis

Can we apply our theory to the category of Grothendieck's ∞-groupoids ∞-**Grp**?

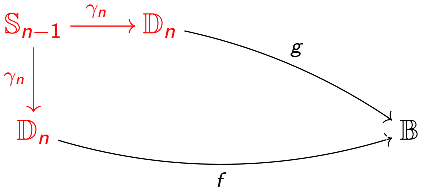
It follows from the definition of a Grothendieck's ∞-groupoid that any pair of parallel arrows in ∞-**Grp** ...



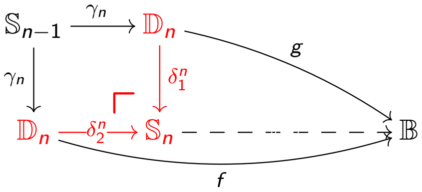
...can be filled by a homotopy.

Homotopy Hypothesis

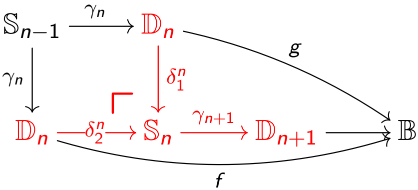
Equivalently, this amount to saying that ...



Equivalently, this amount to saying that ...



Equivalently, this amount to saying that ...



Homotopy Hypothesis

On the one hand, the operations on ∞-groupoids come from the **existence of a certain homotopy** filling of a certain pair of **parallel arrows** going to a certain globular sum.

Homotopy Hypothesis

On the one hand, the operations on ∞-groupoids come from the **existence of a certain homotopy** filling of a certain pair of **parallel arrows** going to a certain globular sum.

On the other hand, all the operations defined on vertebrae and spines come from **existence of a certain vertebra** filling a certain given commutative square of **parallel arrows** going to a certain globular sum.

Homotopy Hypothesis

On the one hand, the operations on ∞-groupoids come from the **existence of a certain homotopy** filling of a certain pair of **parallel arrows** going to a certain globular sum.

On the other hand, all the operations defined on vertebrae and spines come from **existence of a certain vertebra** filling a certain given commutative square of **parallel arrows** going to a certain globular sum.

Theorem

The category of Grothendieck's ∞-groupoids admits a **spinal structure***.*

Conclusion

Key points of this talk:

Conclusion

Key points of this talk:

- internal logic of homotopy theory $\rightsquigarrow$ Spinal categories;

Conclusion

Key points of this talk:

- internal logic of homotopy theory $\rightsquigarrow$ Spinal categories;
- Category of Grothendieck's ∞-groupoids admits a **spinal structure**;

Conclusion

Key points of this talk:

- internal logic of homotopy theory $\rightsquigarrow$ Spinal categories;
- Category of Grothendieck's ∞-groupoids admits a **spinal structure**;

What remains to be proven:

Conclusion

Key points of this talk:

- internal logic of homotopy theory $\rightsquigarrow$ Spinal categories;
- Category of Grothendieck's ∞-groupoids admits a **spinal structure**;

What remains to be proven:

- pushouts of acyclic cofibrations in ∞-**Grp** are weak equivalences;

Conclusion

Key points of this talk:

- internal logic of homotopy theory $\rightsquigarrow$ Spinal categories;
- Category of Grothendieck's ∞-groupoids admits a **spinal structure**;

What remains to be proven:

- pushouts of acyclic cofibrations in ∞-**Grp** are weak equivalences;
- Consider the canonical adjunction

$$\mathbf{Top} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \infty\text{-}\mathbf{Grp}$$

For every X such that there exists some Y for which ∞-$\mathbf{Grp}(X, U(Y)) \neq \emptyset$, the unit $X \Rightarrow FU(X)$ is a component-wise weak equivalence.

The End

Thank you!

References:

[1] R. Tuyéras, *Sketches in Higher Category Theory*, version 1.01, http://www.normalesup.org/~tuyeras/th/thesis.html.

[2] R. Tuyéras, *Elimination of quotients in various localisation of models into premodels*, arXiv:1511.09332