SKETCHES IN HIGHER CATEGORY THEORIES AND THE HOMOTOPY HYPOTHESIS

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1. Preamble

The goal of this talk[1] is to present the results of my PhD thesis, which builds a categorical theory briding *Logic* and *Homotopy theory*. As a by product, this theory implies a major part of the so-called *Homotopy Hypothesis* for Grothendieck's ∞-groupoids – the last remaining properties are in the process of being written up. The slides that are meant to be used along with this paper are available on my webpage: `http://www.normalesup.org/~tuyeras/th/thesis.html`. Throughout the present text, the reader will be assumed to know the basics of model categories (and their axioms) as well as to be familiar with Algebraic Topology.

2. Introduction

As metioned in the preamble of this text, my goal is to bridge the domain of Logic with that of Homotopy Theory. To some extent, this amounts to answering the question whether it is possible to capture the internal logic of topological spaces. I will begin this text with general facts about the relationship between theories and models – this will allow me to recall certain important points about how to axiomatise theories and develop their associated logic. I shall then gradually switch from theories and logics to Abstract Homotopy Theory, the goal being to introduce what I would like to call the internal logic of Homotopy Theory. I shall finally finish with an account on the status of the proof of the Homotopy Hypothesis for Grothendieck's ∞-Groupoids.

3. Models, theories and logic

3.1. **Models and theories.** Recall that, in pure Mathematics, theories are born from **inductive** reasonings. To put it simply, this inductive process follows the following steps: one starts with some observations, one then tries to see the patterns, one makes some hypothesis about these and one finally constructs a theory from those. Once one has obtained one's theory, one might want to apply it to see what one can **deduce** about other observations, so one starts describing the set of observations in the form of basic properties and sees if these follow from the axioms of the theory. But when this fails, one usually realises that one has to refine the initial set of axioms. This process of refining the set of axioms involves a back-and-forth between the poset of possible theories, ordered by the level of abstraction, and the poset of classes of examples, ordered by the inclusion.

$$\begin{tikzcd} \boxed{\text{Possible Theories}} \arrow[r, bend left, "Right"] & \boxed{\text{Covered Examples}} \arrow[l, bend left, "Left"] \end{tikzcd}$$

[1]This is the paper version of a talk presented at the international conference Category Theory 2016, which was held in Halifax (Nova Scotia, Canada) in August 2016; `http://mysite.science.uottawa.ca/phofstra/CT2016/CT2016.htm`.

Informally, we can see this back-and-forth as an adjunction of order-preserving relations. On the example side, one usually searches what one can see as a greatest fixed point, while, on the theory side, one usually searches the *right* level of abstraction, which might require to refine the initial set of axioms. The subtlety, here, is that this refinement can be done in two ways:

- ▷ either one piles up axioms and try to not worry to much about the possible redundancies;
- ▷ or one does worry and prefers to start the inductive investigation of our theory again.

The next section gives an example.

3.2. **Quillen's model categories.** For instance, in 1967, D. Quillen [3] came up with a set of axioms for the concept of model category on the base of a particular set of examples. From this set, he used an inductive reasoning – by looking at the common patterns – and brought out the well-known set of axioms for model categories.

The problem is that when one wants to use these axioms to deduce some of the combinatorial properties satisfied by certain examples of the initial set of examples, one realises that one does not have enough. For examples, not much can be said – from the concept of *model category* only – about properness properties; the idea of smallness; the concept of CW-complex and to some extent the Whitehead Theorem.

The usual attitude regarding this lack of properties is to add more axioms to the set of axioms of model categories, but this process of forcing the desired properties by adding what one needs from what one sees is more of a deductive and passive reasoning – as one does not make the attempt of starting the inductive construction over again. Somehow, the fact that one piles up axioms even suggests that one has not reached the right level of abstraction to express what one wants to really talk about and one might have failed to see another language hidden behind the concept of these restricted version of model categories.

3.3. **Cofibrantly generated model categories.** To see what I am referring to, let us consider the example of **cofibrantly generated** model categories (see [1]). In general, in these structures, one distinguishes between

- ▷ the genuine structure of the model category;
- ▷ the generating structure (as in *generating cofibrations*).

This so-called genuine structure is entirely determined by the generating structure. For instance, fibrations and acyclic fibrations are determined by lifting properties as shown below, where the generating structure is given on the left, in blue, while the genuine structure is given on the right, in red.

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\forall"] \arrow[d] & X \arrow[d] \\
\mathbb{D} \arrow[r, "\forall"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{D} \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}
$$

Similarly, in **tractable** model categories (see [4]), we can characterise the weak equivalences from the generating structure, where, here, the generating elements take the form of squares as show below: a weak equivalence (in red) is an arrow such that for every commutative square as given below, on the left, there exists one of those generating squares (in blue) that factorises the square (see corresponding diagram).

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\forall"] \arrow[d] & X \arrow[d] \\
\mathbb{D}_2 \arrow[r, "\forall"'] & Y
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
\mathbb{S} \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dd] \\
& \mathbb{D}_1 \arrow[ur, dashed, "\exists"'] \arrow[dd] & \\
\mathbb{D}_2 \arrow[rr] \arrow[dr] & & Y \\
& \mathbb{D}' \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

If one thinks about it for some time, one realises that this generating arrows form an **internal logic** of the model category, and as in every logic, this logic admits axioms, which are given by the commutative relations that the generating arrows satisfy.

3.4. **Generating structure = Internal logic.** Let us summarise the ideas of the previous section: I started with the axioms of some cofibrantly generated model category and deduced or detected from those an internal logic whose domain of discourse is given by the generating arrows, some additional commutative squares and the underlying commutative relations satisfied by these.

Now, what I propose to do is to go the other way around, where I would take the internal language previously detected as a starting point and deduce all sorts of homotopy theories including model categories; categories of fibrant objects, homotopical categories; etc.

The **question** is: *what kind of axioms does this internal logic need to satisfy?* Well, in my thesis [6], this is answered by induction from the examples, but one can quickly see that these axioms strongly resemble what we would like to see as the axioms for a theory of ∞-groupoids.

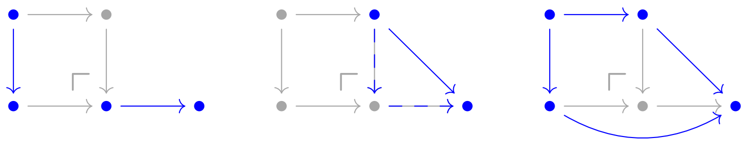
4. Internal logic of Homotopy Theory

4.1. **Elementary structures for Homotopy Theory.** After looking into how the internal logics of our examples work, we notice that a very elementary structure appears constantly in every axioms of the internal logic. Precisely, this "elementary structure" – this how I will call it for now – is made of a pushout square and an additional arrow sticking out of the square as show below, on the left.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

This diagram encodes nothing but the internal logic of a cell (see diagram on the right): precisely, the source and target objects of the cell are given by the top-left object in the diagram; the source arrow of the cell is given by the leftmost vertical arrow; the target arrow of the cell is given by the top vertical arrow; the pair of source and target arrows together is encoded by the pushout and the cell arrow is given by the arrow that sticks out of the pushout square.

This structure is also made of representatives of the generating structures that one may usually find in cofibrantly model categories. Specifically, the blue arrows in the leftmost diagram, below, should be thought of as representatives of generating cofibrations; the composite blue arrow in the diagram given in the middle should be thought of as a generating tivial cofibration and the outer square generated by the diagram, as shown on the right, should be thought of as one these commutative square that characterises weak equivalences in tractable model categories.



Now, to give some more intuition, here are a few examples, which may be found in the category of topological space, for the leftmost one, in the category of simplicial sets, for the middle one, and in the Dubuc topos, for the rightmost one.

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d] & \\ \mathbb{D}^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}^n \arrow[r] & \mathbb{D}^{n+1} \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta_{n-1} \arrow[r] \arrow[d] & \Lambda^k_n \arrow[d] & \\ \Delta_{n-1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \partial\Delta_n \arrow[r] & \Delta_n \end{tikzcd} \qquad \begin{tikzcd} \{0\} \arrow[r] \arrow[d] & D \arrow[d] & \\ D \arrow[r] & D(2) \arrow[r] & D \end{tikzcd}$$

More examples may be found in [6].

4.2. **A mini-homotopy theory.** Now, by taking inspiration on the characterisations of fibrations; trivial fibrations and weak equivalences seen earlier in the case of cofibrantly generated model categories, I am going to associated every "elementary structure" with a mini-homotopy theory. For convenience, I shall use the following notations.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\gamma'"] \arrow[d, "\gamma"'] & \mathbb{D}_1 \arrow[d, "\delta_1"] & \\ \mathbb{D}_2 \arrow[r, "\delta_2"'] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}' \arrow[r, "\beta"] & \mathbb{D}' \end{tikzcd}$$

So, for every such structure, we shall say that a morphism $f : X \to Y$ is

1) a *fibration* if every commutative square as given below, on the left, admits a lift as shown on the right;

$$\begin{tikzcd} \mathbb{D}_2 \arrow[r, "\forall"] \arrow[d, "\beta\delta_1"'] & X \arrow[d, "f"] \\ \mathbb{D}' \arrow[r, "\forall"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{D}_2 \arrow[r] \arrow[d, "\beta\delta_1"'] & X \arrow[d, "f"] \\ \mathbb{D}' \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

2) a *trivial fibration* if every commutative square as given below, on the left, where the blue arrow is either γ or β, admits a lift as shown on the right;

$$\begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d] & X \arrow[d, "f"] \\ \bullet \arrow[r, "\forall"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

3) a *pseudo fibration* if every commutative square as given below, on the left, admits a lift as shown on the right;

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\forall"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\ \mathbb{D}_2 \arrow[r, "\forall"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\ \mathbb{D}_2 \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

4) a *surtraction* if every commutative square as given below, on the left, admits a factorisation as shown on the right;

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\forall"] \arrow[d] & X \arrow[d, "f"] \\ \mathbb{D}_2 \arrow[r, "\forall"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S} \arrow[rr] \arrow[dd] \arrow[dr] & & X \arrow[dd, "f"] \\ & \mathbb{D}_1 \arrow[ur, dashed, "\exists"'] \arrow[dd] & \\ \mathbb{D}_2 \arrow[rr] \arrow[dr] & & Y \\ & \mathbb{D}' \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

5) an *intraction* if every commutative square as given below, on the left, admits a upper lift as shown on the right;

$$\begin{tikzcd} \mathbb{S}' \arrow[r, "\forall"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ \mathbb{D}' \arrow[r, "\forall"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbb{S}' \arrow[r] \arrow[d, "\beta"'] & X \\ \mathbb{D}' \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

6) a *weak equivalence* if it is both a surtraction and an intraction.

From this definition, it is now a relatively easy (or formal) exercise to show the following propositions.

Proposition 4.1. *All isomorphisms are intractions.*

Proof. Straightforward. □

Proposition 4.2. *If a composite arrow $f \circ g$ is an intraction, then so is g.*

Proof. Straightforward. □

Proposition 4.3. *If two composable arrows f and g are intractions, so is $f \circ g$.*

Proof. Straightforward. □

Proposition 4.4. *If a composite arrow $f \circ g$ is an surtraction and f is an intraction, then g is a surtraction.*

Proof. First use the fact that $f \circ g$ is an surtraction, then use the universal property of the pushout associated with the elementary structure and finally use the fact that f is an intraction. □

Proposition 4.5. *All isomorphisms are (pseudo/trivial) fibrations.*

Proof. Straightforward. □

Proposition 4.6. *(Pseudo/Trivial) fibrations are stable under pullbacks.*

Proof. Straightforward. □

Proposition 4.7. *(Pseudo/Trivial) fibrations/surtractions/intractions are stable under retracts.*

Proof. Straightforward. □

Proposition 4.8. *All trivial fibrations are pseudo fibrations.*

Proof. Straightforward. □

Proposition 4.9. *All fibrations that are surtractions are also pseudo fibrations.*

Proof. First use the definition of a surtraction and then that of a fibration. □

Proposition 4.10. *All trivial fibrations are both fibrations and intractions.*

Proof. Straightforward. □

Note that the last two propositions remind us of the well-known equality between the class of fibrations that are weak equivalences and the class of trivial fibrations. We have now reached a point where we cannot really prove more, unless we equip our elementary structure with some axioms. First of all, recall that our elementary structures can be seen as (categorical) cells (see the beginning of section 4.1). This comparison will enable me to describe the axioms that I am going to put on the elementary structures without 'doing maths'. First, if one encodes what it means to be an identity cell, it turns out that one may prove the following interesting property.

Proposition 4.11. *If $r \circ i = \mathrm{id}_X$ where r is an intraction, then i is a weak equivalence.*

One can also prove the following commutative triangle of inclusions (see picture below), the good news being that, in certain cases, this triangle turns out to be degenerate on the right (i.e. the symbol $\subseteq$ turns out to be an identity $=$), which means that one retrieves the fact that the fibrations that are weak equivalences are exactly the trivial fibrations.

$$\begin{tikzcd}
\textbf{fibrations} \cap \textbf{weak equivalence} \arrow[dr, hook] & & \textbf{trivial fibrations} \arrow[ll, hook'] \arrow[dl, "\subseteq"] \\
& \textbf{pseudofibrations} &
\end{tikzcd}$$

If one now encodes what it means to have a composition of cells, one can prove that weak equivalences are stable under compositions. You can find these properties and their proofs (in a more general setting than the one given here) in Chapter 2 of [6].

4.3. **Higher coherence theory.** I will now say a few words on how to prove the 2/3-property, which requires more than identities and horizontal compositions. In fact, we need to draw the axioms of the internal logic of what one could call *higher coherence theory*. Basically, these require two operations that one could see as:

- ▷ ternary whiskerings of cells in all dimensions;
- ▷ a notion of weakly invertible cell in all dimensions (which comes along with the whiskerings);

As can be seen above, the previous two items use the word *dimension*. Note that this makes sense with the whiskerings, which, in the present case, could be thought of as horizontal compositions of a cell of dimension of $n+1$ together with two cells of dimension n on each side.

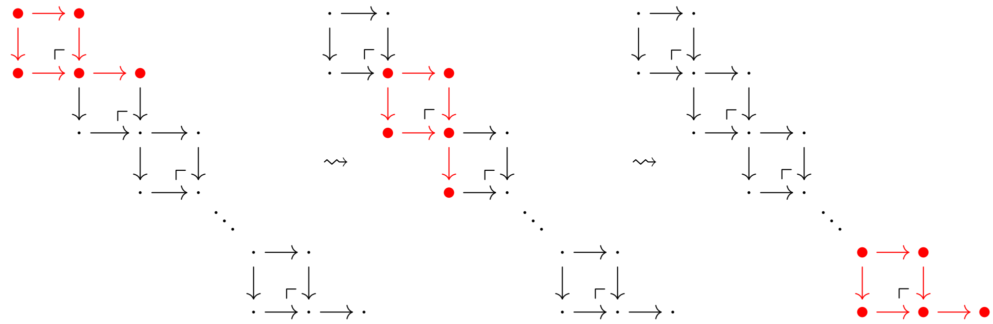
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

In order for the previous operations to be well-defined, we need to extend the elementary structures into the type of structure given below, on the right.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[ul, phantom, "\ulcorner" very near start] & \bullet
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] & & & & & \\
\cdot \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] & \cdot \arrow[d] & & & & \\
& \cdot \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] & \cdot \arrow[d] & & & \\
& & \cdot \arrow[r] & \cdot \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, phantom, "\ddots"] & & & \\
& & & & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\
& & & & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ul, phantom, "\ulcorner" very near start] & \bullet
\end{tikzcd}
\tag{4.1}$$

This tower of pushout squares allows us to encode the dimension of our elementary structure located at the tip of the column (bottom-right).

4.4. **Spinal Categories.** The goal of this section is to give a formalisation of the *internal logic of Homotopy Theory*. From now on, I will refer to an elementary structure as a *vertebra* (see left diagram of (4.1)) while the longer one will referred to as a *spine* (see right diagram of (4.1)). This terminology comes from the fact that spines are made of vertebrae as shown below.



As mentioned at the beginning of this presentation, I want to formalise the internal logic of homotopy theories. This is realised through the notion of *spinal category*, which informally consists of

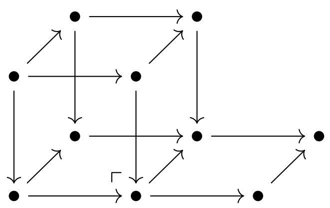
- ▷ a category $\mathcal{C}$;
- ▷ a bunch of vertebrae in $\mathcal{C}$ equipped with spine structures

such that the vertebrae (or, in fact, the spines) satisfy all the previously described axioms[2] in a 'suitable way'. Interestingly, a spinal category is equipped with weak equivalences, fibrations and trivial fibrations in much the same way we defined them for an isolated vertebra. Here, the difference is that they are defined vertebra-wise[3]. Naturally, from fibrations and trivial fibrations, we also obtain the notions of cofibration and trivial cofibration (via right lifting property)

Now, if we call *acyclic fibration* any cofibration that is a weak equivalence, we have the following theorem from Chapter 4 of [6].

Theorem 4.12. *If pushouts of acyclic cofibrations are weak equivalences (and if some smallness condition holds on the objects of the vertebrae), then a* ***spinal category*** *is a* ***model category****.*

Note that the proof does not make use of the Jeff Smith's theorem, as all the desired properties follow from our definitions. Before moving on to the Homotopy Hypothesis, I would like to briefly show to what extent the notion of vertebra can be extended. In the case of categories that contain objects possessing some local information (stacks, sheaves, etc.) vertebra will take the form of 3-dimensional structures as follows.



The 3-dimensionality of this structures precisely encode the fact of being local. In the case of categories that contain objects that one would usually qualify to be *non-algebraic*, the

[2]Recall that these axioms mainly consists in encoding identity cells, vertical compositions, whiskerings and weakly invertible cells

[3]Note that the full abstraction of spinal categories leads to more subtle definitions of weak equivalences, fibrations and trivial fibrations (see [6])

lack of operations in these categories forces us to generalise the structure of vertebra to the following form.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d, dotted, no head] \\
\bullet \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \bullet \arrow[r] \arrow[ur] & \bullet
\end{tikzcd}$$

The fact that a multitude of arrows can stick out from the pushout square comes from the fact that we are forcing these operations to exist by adding them to the vertebra. On the other hand, categories whose objects are completely *algebraic* only need the structure of vertebra.

5. Homotopy Hypothesis

5.1. **Homotopy Hypothesis.** In this last section, I would like to address the question: Can we apply our theory to the category of Grothendieck's ∞-groupoids. As John Bourke explained it to us this morning in his talk, it follows from the definition of a Grothendieck ∞-groupoid that any pair of parallel arrows going to a *glogular sum* $\mathbb{B}$ can be filled with a homotopy (or cell) as shown by the following implication.

$$\begin{tikzcd}
\mathbb{D}_n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{B}
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{D}_{n+1} \arrow[dr, dashed, bend left] & \\
\mathbb{D}_n \arrow[u, shift left, "s_n"] \arrow[u, shift right, "t_n"'] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{B}
\end{tikzcd}
\tag{5.1}$$

The fact that the arrows f and g form a "pair of parallel arrows" amounts to saying that the following lefthand diagram commutes, where the top-left part (in red) specifically encodes the parallelism of f and g.

$$\begin{tikzcd}
\mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[dr, bend left, "g"] & \\
\mathbb{D}_n \arrow[rr, bend right, "f"'] & & \mathbb{B}
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
\mathbb{S}_{n-1} \arrow[r, "\gamma_n"] \arrow[d, "\gamma_n"'] & \mathbb{D}_n \arrow[d, "\delta_1^n"] \arrow[drr, bend left, "g"] & & \\
\mathbb{D}_n \arrow[r, "\delta_2^n"] \arrow[rrr, bend right, "f"'] & \mathbb{S}_n \arrow[r, "\gamma_{n+1}"] \arrow[ul, phantom, "\ulcorner"] & \mathbb{D}_{n+1} \arrow[r] & \mathbb{B}
\end{tikzcd}$$

Then, of course, we can form the pushout of the red part and expose a canonical arrow from the pushout $\mathbb{S}'_n$ and the globular sum $\mathbb{B}$. Condition (5.1) is then equivalent to saying that the obtained canonical arrow $\mathbb{S}'_n \to \mathbb{B}$ may be factorised as shown above, on the right. As we can see, the red arrows in the previous row of diagrams form a vertebra. We finally conclude by the following two observations:

▷ On the one hand, the operations on ∞-groupoids come from the **existence of a certain homotopy** filling of a certain pair of **parallel arrows** going to a certain globular sum.

▷ On the other hand, all the operations defined on vertebrae and spines come from **existence of a certain vertebra** filling a certain given commutative square of **parallel arrows** going to a certain globular sum.

In the end, these characterisations of the operations in spinal categories and categories[4] of Grothendieck's ∞-groupoids and the previous diagrammatic reformulation in terms vertebrae leads us straightforwardly[5] to the following theorem.

Theorem 5.1. *The category of Grothendieck's ∞-groupoids admits a* ***spinal structure****.*

Proof. By noticing that the definitional properties of Grothendieck ∞-groupoids may be reformulated in terms of vertebrae. □

[4]The use of a plural is more adequate as there exist more than one theory for the definition of a Grothendieck ∞-groupoid: see [2].

[5]The proof however involves lengthy inductive explanations

The *Homotopy Hypothesis* says that the category of Grothendieck's ∞-groupoids should admit a model structure and that this model structure should be equivalent to the model category of topological spaces. By Theorem 5.1 and Theorem 4.12, the only step that separates us from a half of the hypothesis is to prove that pushouts of acyclic cofibrations in the category of Grothendieck's ∞-groupoids are weak equivalences. The other half would follow if one could prove the following: Consider the canonical adjunction

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \infty\text{-}\mathbf{Grp} \arrow[l, shift right, "F"'] \end{tikzcd}$$

where the right adjoint homs a topological space to its fundamental ∞-groupoid (see [2]). For every X such that there exists some Y for which $\infty\text{-}\mathbf{Grp}(X, U(Y)) \neq \emptyset$, the unit $X \Rightarrow FU(X)$ should be a component-wise weak equivalence.

5.2. **Proof in progress.** In 2015, I put on the arXiv an article [5] whose goal is to facilitate the description of colimits in the category of Grothendieck's ∞-groupoids. I intend to use the tools developed in this article to show the first missing half of the proof, namely pushouts of acyclic cofibrations in the category of Grothendieck's ∞-groupoids are weak equivalences. Here, the only difficult part is the description of colimits as the definition of weak equivalences is quite clear and simple. The details of the proof are currently in the process of being written up. The other missing half of the hypothesis can also be described in terms of colimits of ∞-groupoids, while it is initially expressed as a problem about topological colimits. This transfer from topological spaces to the 'logical models' greatly eases the combinatorics, provided that one has the right tools to handle colimits of ∞-groupoids.

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