Isoperimetric profile of subgroups and probability of return of random walks on geometrically elementary solvable groups

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Abstract

We introduce a large class of amenable locally compact groups containing all solvable algebraic groups over a local field and their discrete subgroups. We show that the isoperimetric profile of these groups is in some sense optimal among amenable groups. We use this fact to compute the probability of return of symmetric random walks, and to derive various other geometric properties which seem to be only satisfied by these groups.

Contents

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1 Introduction

We introduce a notion of large-scale foliation for metric measure spaces and we prove that if X is large-scale foliated by Y, and if Y satisfies a Sobolev inequality at large-scale, then so does X. In particular the L^p-isoperimetric profile of Y grows faster than the one of X. A special case is when $Y = H$ is a closed subgroup of a locally compact group $X = G$. The class of geometrically elementary solvable groups is a class of amenable locally compact groups, stable under quasi-isometries, and containing all quotients of unimodular closed compactly generated subgroups of the group of upper triangular matrices $T(d,k)$ for every $d \in \mathbf{N}$ and every local field k. If G is a geometrically elementary solvable group with exponential growth, we prove that the L^p-isoperimetric profile of G satisfies $j_{G,p}(t) \approx \log t$, for every $1 \leq p \leq \infty$. As a consequence, the probability of return of symmetric random walks on such groups decreases like $e^{-n^{1/3}}$. We obtain a stronger result when the group is a quotient of a solvable algrebraic group over a q-adic field (q is a prime), namely, such a group has linear isoperimetric profiles inside balls. Among other consequences, we obtain that these groups have trivial reduced cohomology with values in the left regular representation on $L^p(G)$, for all $1 < p < \infty$.

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2 Results

2.1 Basic definitions and notation

Let G be a locally compact, compactly generated group equipped with a left-invariant Haar measure μ. Let S be a compact symmetric generating subset of G, i.e. $\bigcup_{n\in\mathbf{N}} S^n = G$. Equip G with the left-invariant word metric[1] associated to S, i.e. $d_S(g,h) = \inf\{n,\ g^{-1}h \in S^n\}$. The closed ball of center g and of radius r is denoted by $B(g,r)$ and its volume by $V(r)$.

L^p-isoperimetric profiles

Let λ be the action of G by left-translations on functions on G, i.e. $\lambda(g)f(x) = f(g^{-1}x)$. Restricted to elements of $L^p(G)$, λ is called the left regular representation of G on $L^p(G)$.

For any $1 \leq p \leq \infty$, and any subset A of G, define

$$J_p(A) = \sup_f \frac{\|f\|_p}{\sup_{s\in S}\|f - \lambda(s)f\|_p},$$

where f runs over locally bounded functions in $L^p(G)$, supported in A.

We define

- the L^p-isoperimetric profile (see for instance [Cou3]),

$$j_{G,p}(v) = \sup_{\mu(A)\leq v} J_p(A);$$

- and the L^p-isoperimetric profile inside balls [T2],

$$J^b_{G,p}(r) = J_p(B(1,r)).$$

[1] To have a real metric we must assume that S is symmetric. However, this assumption does not play any role in the sequel

Asymptotic behavior

Let $f, g : \mathbf{R}_+ \to \mathbf{R}_+$ be two monotonic functions. We write respectively $f \preceq g$, $f \prec g$ if there exists $C > 0$ such that $f(t) = O(g(Ct))$, resp. $f(t) = o(g(Ct))$ when $t \to \infty$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. The asymptotic behavior of f is its class modulo the equivalence relation $\approx$.

2.2 Monotonicity of the isoperimetric profiles

In [T4], we proved that

Theorem 1. *Let (G, S) and (H, T) be two compactly generated, locally compact groups, equipped with symmetric generating subsets S and T respectively. Then, the asymptotic behaviours of $j_{G,p}$, $J^b_{G,p}$, for any $1 \leq p \leq \infty$ does not depend on S. Moreover, if G and H are both unimodular, and if G is quasi-isometric to H, then*

$$j_{G,p} \approx j_{H,p},$$

and

$$J^b_{G,p} \approx J^b_{H,p}.$$

Here we prove

Theorem 2. (see Theorem 6.2) *Let H be a closed, compactly generated subgroup of G and let $1 \leq p \leq \infty$. Then,*

- *if H is unimodular, then*
$$j_{G,p} \preceq j_{H,p};$$
- *if H is not distorted in G, then*
$$J^b_{G,p} \preceq J^b_{H,p}.$$

If the groups are finitely generated, these statements are much easier to prove [E].

We also show

Theorem 3. (see Theorem 6.2) *Let G and Q be two compactly generated locally compact groups and let $\pi : G \to Q$ be a surjective continuous homomorphism. Let $1 \leq p \leq \infty$. Then, $j_{G,p} \preceq j_{Q,p}$, and $J^b_{G,p} \preceq J^b_{H,p}$.*

2.3 Geometrically elementary solvable groups

By a theorem of Coulhon and Saloff-Coste [CS2], if G is a compactly generated, locally compact group with exponential growth, then $j_{G,p}(t) \prec \log t$. On the other hand it is very easy to see that $J_{G,p}(t) \leq 2t$. Our main result is to prove that the converse inequalities are true for certain classes of groups. Note that $J_{G,p}(r) \leq j_{G,p}(V(r))$. So, in particular, if the group has exponential growth, $J^b_{G,p}(t) \succeq t$ implies $j_{G,p} \succeq \log t$.

Definition 2.1. The class of elementary solvable groups ES is the class of all quotients of compactly generated closed subgroups of finite products of groups of triangular matrices $T(d,k)$ for any integer d and any local field k.

Note that the class ES does not only contain linear groups as shown by the following example due to Hall [H]. Fix a prime q and consider the group of upper triangular 3 by 3 matrices:

$$G = \left\{ \begin{pmatrix} q^n & 0 & x & z \\ 0 & q^{-n} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; x, y, z \in Z[1/q]; n \in \mathbf{Z} \right\}.$$

Taking the quotient by the central infinite cyclic subgroup of unipotent matrices $I + mE_{1,4}$ where $m \in \mathbf{Z}$, we obtain an elementary solvable group which is non-residually finite since its center is isomorphic to $\mathbf{Z}[1/q]/\mathbf{Z}$.

Definition 2.2. The class GES of geometrically elementary solvable groups is the smallest class of locally compact groups

- containing unimodular closed compactly generated subgroups of the group $T(d,k)$, for any integer d and any local field k;
- stable under taking finite products, quotients, and unimodular closed compactly generated subgroups;
- stable under quasi-isometry.

The class GES contains the (non necessarily solvable) lamplighter groups $F \wr \mathbf{Z} = F^{(\mathbf{Z})} \rtimes \mathbf{Z}$, where F is a finite groups. Namely, such a group is trivially quasi-isometric to any $F' \wr \mathbf{Z}$ where F' has same cardinality as F. If F' is a product of $\mathbf{Z}/q$ for finitely many primes q, then $F' \wr \mathbf{Z}$ is a subgroup in a finite product of lamplighter groups $\mathbf{Z}/q \wr \mathbf{Z}$. On the other hand, one can easily embed $\mathbf{Z}/q \wr \mathbf{Z}$ as a discrete subgroup of the algebraic unimodular group $(k_q \times k_q) \rtimes k_q^*$ over the local field $k_q = \mathbf{Z}/q((X))$.

Theorem 4. *Let G be a geometrically elementary solvable group. Then, for every $1 \leq p \leq \infty$,*

$$j_{G,p}(t) \succeq \log t.$$

This result was known for connected amenable Lie groups [Pit], for the lamplighter and other particular examples [CG2]. To prove Theorem 4, we establish a stronger result for the group of triangular matrices $T(d,k)$ over a local field k, i.e. that $J^b_{G,p}(t) \succeq t$. The stability under finite product is trivial, and we obtain all geometric elementary solvable groups using Theorems 1, 2 and 3.

Restricting to groups with exponential growth, we obtain

Corollary 5. *Let G be an geometrically elementary solvable group with exponential growth. Then, for every $1 \leq p \leq \infty$,*

$$j_{G,p}(t) \approx \log t.$$

Let q be a prime integer and let $1 \leq p \leq \infty$. By a theorem of Mustapha [Mu1], if k is a q-adic field, then closed subgroups of $T(d,k)$ whose Zariski closure is compactly generated are non-distorted. We therefore obtain

Theorem 6. *Let k be a q-adic field. Let G be a quotient of a closed compactly generated subgroups of $T(d,k)$, whose Zariski closure is compactly generated. Then, for every $1 \leq p \leq \infty$,*

$$J^b_{G,p}(t) \approx t.$$

In particular these groups have controlled Følner sequences (see [T3]).

In [T2], we proved it for connected amenable Lie groups, lamplighter groups, and solvable Baumslag-Solitar groups.

2.4 Random walks

Let G be a locally compact, compactly generated group. Let (X,μ) be a quasi-transitive G-space, i.e. a locally compact Borel measure space on which G acts measurably, co-compactly, properly, and almost preserving the measure μ, i.e.

$$\sup_{g\in G}\sup_{x\in X}\frac{d(g\cdot\mu)}{d\mu}(x) < \infty.$$

For every $x \in X$, let ν_x be a probability measure on X which is absolutely continuous with respect to μ. We assume that there exist $S \subset S'$, two compact generating subsets of G, and a compact subset of X satisfying $GK = X$, such that for every $x \in X$, the support of ν_x is contained in $gS'K$, for some $g \in G$ such that $x \in gSK$. Let us also suppose that $\nu_x(y)$ is larger than a constant $c > 0$ for y in gSK. Denote by P the Markov operator on $L^2(X)$ defined by

$$Pf(x) = \int f(gy) d\nu_x(y).$$

Definition 2.3. With the previous notation, we call (X, P) a quasi-G-transitive random walk. Moreover, if P is self-adjoint, then (X, P) is called a symmetric quasi-G-transitive random walk.

By a slight abuse of notation, we write $dP_x(y) = d\nu_x(y) = p_x(y)d\mu(y)$ and $dP_x^n(y) = p_x^n(y)d\mu(y)$.

Using [T4, Theorem 8.1] and [T4, Theorem 9.2] (which is a straightforward generalization of [Cou3, Theorem 7.1]), we obtain the following result.

Theorem 7. (see Theorem 7.3) *Let G be a geometric elementary solvable group with exponential growth. Then for every symmetric quasi-G-transitive random walk (X, P), we have*

$$\sup_{x \in X} p_x^{2n}(x) \approx e^{-n^{1/3}}.$$

Classically, we define a random walk directly on the group by taking P to be the convolution by a compactly supported, symmetric probability ν. For instance, take ν to be the uniform probability on S: when the group is discrete, it coincides with the simple random walk on the Cayley graph of G associated to S. In this context, the fact that $p_x^n(x) \approx e^{-n^{1/3}}$ was known for connected amenable Lie groups [Pit], for finitely generated torsion-free solvable groups with finite Prüfer rank [PS], and for the lamplighter group $F \wr \mathbf{Z}$, where F is a finite group in [CG2]. Using a probabilistic approach, Mustapha [Mu2] was able to prove that $p_x^n(x) \approx e^{-n^{1/3}}$ for analytic p-adic unimodular groups (which are particular cases of geometric elementary solvable groups).

2.5 L^p-compression

Equivariant L^p-compression

Recall that the equivariant L^p-compression rate of a locally compact compactly generated group is the supremum of $0 \leq \alpha \leq 1$ such that there exists a proper isometric affine action σ on some L^p-spaces satisfying, for all $g \in G$,

$$\|\sigma(g).0\|_p \geq C^{-1}|g|_S^{\alpha} - C,$$

for some constant $C < \infty$, $|g|_S$ being the word length of g with respect to a compact generating subset S.

It follows from [T2, Corollary 13], that a group with linear L^p-isoperimetric profile inside balls have equivariant L^p-compression $B_p(G)=1$. Hence, we obtain

Theorem 8. *Let k be a q-adic field. Let G be a quotient of a closed compactly generated subgroups of $T(d,k)$, whose Zariski closure is compactly generated. Then, $B_p(G)=1$ for any $1\leq p\leq\infty$.*

Non-equivariant L^p-compression

Recall that the L^p-compression rate of a metric space (X,d) is the supremum of all $0\leq\alpha\leq 1$ such that there exists a map F from X to some L^p-space satisfying, for all $x,y\in X$,

$$C^{-1}d(x,y)^{\alpha}-C\leq\|F(x)-F(y)\|_p\leq d(x,y),$$

for some constant $C<\infty$.

Another theorem of Mustapha [Mu1] says that an algebraic compactly generated subgroup of $GL(d,k)$, where k is a q-adic field, is non-distorted in $GL(d,k)$. As $T(d,k)$ is co-compact in $GL(d,k)$ and satisfies $B_p(T(d,k))=1$, we obtain

Theorem 9. *Let k be a q-adic field. Let G be an algebraic compactly generated subgroups of $GL(d,k)$. Then, the L^p-compression rate of G satisfies $R_p(G)=1$ for any $1\leq p\leq\infty$.*

2.6 L^p-cohomology

Recall that the first reduced cohomology of a compactly generated locally compact group G with values in a representation π on some Banach space E, is the space of affine actions of G, with linear part π, modulo those actions σ which admit a sequence v_n of almost fixed points, i.e.

$$\|\sigma(g)v_n-v_n\|\to 0,$$

uniformly on compact subsets.

In [T3, Theorem 1], we proved that groups with controlled Følner sequences (i.e. $J^b_{G,1}(t)\approx t$) have trivial reduced cohomology with values in the left regular representation on $L^p(G)$, for $1<p<\infty$. We therefore obtain

Theorem 10. *Let k be a p-adic field. Let G be a quotient of a closed compactly generated subgroups of $T(d,k)$, whose Zariski closure is compactly generated. Then for every $1<p<\infty$, $\overline{H}^1(G,\lambda_{G,p})=0$.*

2.7 Questions

- We conjecture that all (geometrically) elementary solvable groups have controlled Følner pairs (see Section 4.7), and therefore satisfy $J_{G,p}(t) \approx t$.
- Is the class ES, (resp. GES, of groups having controlled Følner pairs) stable under extension?
- Is every group satisfying $j_{G,p} \succeq \log t$ geometrically elementary solvable? Or better: is it quasi-isometric to an elementary solvable group? Note also that all the known amenable groups with equivariant Hilbert compression rate $B(G) = 1$ are quasi-isometric to elementary solvable groups.

  Gromov remarked (see [CTV]) that if G is amenable, then $B_2(G) = R_2(G)$. In particular, $B_2(G)$ is invariant under quasi-isometry among amenable groups.

  Question 2.4. Do we have $B_p(G) = R_p(G)$ for amenable groups, and for all $1 \leq p, \infty$?

  Question 2.5. Let H be a closed compactly generated subgroup of G. Do we have $B_p(G) \leq B_p(H)$ (resp. $R_p(G) \leq R_p(H)$) for all $1 \leq p, \infty$?

  This would be especially interesting for $p = 2$ and for amenable groups, as $B_2(G)$ could be interpreted as a geometric measurement of the amenability of G.

3 Organization of the paper

- In Section 4, we briefly recall the notions of Sobolev inequalities at scale h and the results of [T4] that we need here.
- In Section 5, we prove our main result about "large-scale foliations" of metric measure spaces.
- In Section 6, we prove the case of closed subgroups and quotients.
- Finally, in Section 7, we prove Theorems 4, 6 and 7.

4 Preliminaries: functional analysis at a given scale

The purpose of this section is to briefly recall the notions introduced in [T4]. By metric measure space (X, d, μ), we mean a metric space (X, d) equipped with a locally finite Borel measure μ supported on X. The volume of the closed ball $B(x, r)$ is denoted by $V(x, r)$.

4.1 The locally doubling property

The metric measure spaces that we will consider satisfy a very weak property of bounded geometry introduced in [CS1] in the context of Riemannian manifolds.

Definition 4.1. We say[2] that a space X is locally doubling at scale $r > 0$ if there exists a constant C_r such that

$$\forall x \in X, \quad V(x, 2r) \leq C_r V(x, r).$$

If it is locally doubling at every scale $r > 0$, then we just say that X is locally doubling.

Example 4.2. *Let X be a connected graph with degree bounded by d, equipped with the counting measure. The volume of balls of radius r satisfies*

$$\forall x \in X, \quad 1 \leq V(x, r) \leq d^r.$$

In particular, X is locally doubling.

Example 4.3. *Let (X, d, μ) be a metric measure space and let G be a locally compact group acting by measure-preserving isometries. If G acts co-compactly, then X is locally doubling.*

4.2 Local norm of gradient at scale h

The purpose of this section is to define a notion of "local norm of gradient" (whose infinitesimal analogue is the modulus of gradient on a Riemannian manifold), which captures the geometry at a certain scale –say h– of a metric measure space (X, d, μ).

Consider a family $P = (P_x)_{x \in X}$ of probability measures on X. Then for every $p \in [1, \infty]$, we define an operator $|\nabla|_{P,p}$ on $L^\infty(X)$ by

$$\forall f \in L^\infty(X), \quad |\nabla f|_{P,p}(x) = \|f - f(x)\|_{P_x,p} = \left(\int |f(y) - f(x)|^p dP_x(y) \right)^{1/p},$$

[2] In [CS1] and in [T1], the local doubling property is denoted $(DV)_{loc}$.

if $p < \infty$; and for $p = \infty$, we decide that

$$|\nabla f|_{P,\infty}(x) = \|f - f(x)\|_{P_x,\infty} = \sup\{|f(y) - f(x)|,\ y \in \mathrm{Supp}(P_x)\}.$$

To simplify the notation, we will write $|\nabla f|_h$ instead of $|\nabla f|_{P,\infty}$, when P is the uniform distribution on $B(x, h)$.

Definition 4.4. A family of probabilities $P = (P_x)_{x\in X}$ on X is called a viewpoint at scale $h > 0$ on X if there exist a "large" constant $1 \leq A < \infty$ and a "small" constant $c > 0$ such that for (μ-almost) every $x \in X$:

- $P_x \ll \mu$;
- $p_x = dP_x/d\mu$ is supported in $B(x, Ah)$;
- p_x is larger than c on $B(x, h)$.

This notion of local norm of the gradient has the following advantages:

- **Large scale:** first, it allows to do some analysis at large scale on a metric measure space, forgetting completely the local structure of the space;
- **Flexibility:** second, it allows different choices of local norms of gradients, some of them being more adapted to specific contexts: for example (see Section 4.5), taking the local L^2-norm with respect to a viewpoint which is the probability transition of a symmetric random walk is very convenient to study the relations between Sobolev inequalities and probability of return of the random walk;
- **Robustness:** finally, if X has the local doubling property, all these different local norms of gradient are essentially equivalent (see Remark 4.9 and Theorem 4.21).

Example 4.5. *A basic example of viewpoint at scale h is given by*

$$P_x = \frac{1}{V(x,h)} 1_{B(x,h)}, \quad \forall x \in X,$$

where $V(x, r)$ denotes the volume of the ball centered at x of radius r. We denote the associated L^p-gradient by $|\nabla|_{h,p}$.

Remark 4.6. (see [T4] for more details) We can also define a Laplacian w.r.t. a viewpoint $P = (P_x)_{x\in X}$ by

$$\Delta_P f(x) = (P - id)f(x).$$

If P is self-adjoint with respect to the scalar product associated to μ, then we have the usual relation

$$\langle \Delta_P f, f\rangle = \int\int |f(y) - f(x)|^2 p_x(y) d\mu(x) d\mu(y) = \||\nabla_{P,2} f|\|_2^2.$$

4.3 Sobolev inequalities

Let $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing function and let $p \in [1, \infty]$. The following formulation of Sobolev inequality was first introduced in [Cou2] in the context of Riemannian manifolds.

Definition 4.7. One says that X satisfies a Sobolev inequality (S^p_φ) at scale (at least) $h > 0$ if there exist $C, C' > 0$ such that

$$\|f\|_p \leq C\varphi(C'|\Omega|)\||\nabla f|_h\|_p$$

where Ω ranges over all compact subsets of X, $|\Omega|$ denotes the measure $\mu(\Omega)$, and $f \in L^\infty(\Omega)$, $L^\infty(\Omega)$ being the set of elements of $L^\infty(X)$ with support in Ω.

Definition 4.8. We say that X satisfies a large-scale Sobolev inequality (S^p_φ) if it satisfies (S^p_φ) at some scale h (equivalently, for h large enough).

Crucial remark 4.9. *Note that these definitions could be written replacing $|\nabla f|_h$ by $|\nabla f|_{P,q}$ for any $1 \leq q < \infty$ and any viewpoint P at scale h. The point is that if X is locally doubling, then up to changing the scale and the constants, the validity of some Sobolev inequality does not depend either on this choice, nor actually on the choice of the viewpoint (see [T4, Section 7] for a precise statement).*

4.4 Isoperimetric profiles

Let A be a measurable subset of X. For every $p > 0$, every viewpoint P at scale h, and every $1 \leq q \leq \infty$, we define

$$J_{(P,q),p}(A) = \sup_f \frac{\|f\|_p}{\||\nabla f|_{P,q}\|_p}$$

where the supremum is taken over functions $f \in L^\infty(A)$. Note that if $p = q = 2$, and if P is self-adjoint, this is just the square root of the inverse of the first eigenvalue of the Laplacian Δ_P acting on square-integrable functions supported by A.

Definition 4.10. [T4] Let P be a viewpoint at scale h on X, and let $1 \leq q \leq \infty$. The L^p-isoperimetric profile $j_{X,(P,q),p}$ (resp. inside balls: $J^b_{X,(P,q),p}$) is a nondecreasing function defined by

$$j_{X,(P,q),p}(v) = \sup_{|A| \leq v} J_{(P,q),p}(A).$$

(resp. $J^b_{X,(P,q),p}(t) = \sup_{x \in X} J_{(P,q),p}(B(x,t))$.)

To simplify the notation, we will simply write $j_{X,h,p}(A)$ (resp. $J^b_{X,h,p}(A)$) if the local norm of the gradient is $|\nabla f|_h$.

For $p = 1$, the isoperimetric profiles $j_{X,h,1}$ and $J^b_{X,h,1}$ have geometric interpretations (see [T4, Proposition 4.2]).

Finally, one can check that $j_{X,p} \preceq j_{X,q}$ is always true when $p \le q < \infty$. Moreover, in most cases (e.g. all known examples of groups), $j_{X,p} \approx j_{X,q}$.

Link between Sobolev inequalities and isoperimetric profiles

Sobolev inequalities (S^p_φ) can also be interpreted as L^p-isoperimetric inequalities. Clearly, the space X always satisfies the Sobolev inequality (S^p_φ) with $\varphi = j_{X,p}$. Conversely, if X satisfies (S^p_φ) for a function φ, then

$$j_{X,p} \succeq \varphi.$$

4.5 L^2-profile and probability of return of random walks

The case $p = 2$ is of particular interest since it contains some probabilistic information on the space X. It is proved in [CG2] that for manifolds with bounded geometry, upper bounds of the large-time on-diagonal behavior of the heat kernel are equivalent to some Sobolev inequality (S^2_φ). In the survey [Cou3], a similar statement is proved for the standard random walk on a weighted graph. In [T4, Theorem 9.1], we give a discrete-time version of this theorem for general metric measure spaces.

Definition 4.11. Let (X, d, μ) be a metric measure space and consider some $h > 0$. A view-point $P = (P_x)_{x\in X}$ at scale h on X is called symmetric if one of the following equivalent statement holds.

- The random walk whose probability of transition is P is reversible with respect to the measure μ.
- The associated operator on $L^2(X, \mu)$ defined by

$$Pf(x) = \int_X f(y)dP_x(y)$$

  is self-adjoint.
- For a.e. $x, y \in X$, $p_x(y) = p_y(x)$.

Definition 4.12. We call a reversible random walk at scale h a random walk whose probability transition is a symmetric view-point at scale h.

Example 4.13. *Let (X, d, μ) be a metric measure space. Consider the standard viewpoint at scale h of density $p_x = 1_{B(x,h)}/V(x, h)$ with respect to μ. In general, this is not a symmetric viewpoint, i.e. the random walk of probability transition $dP_x(y) = p_x(y)d\mu(y)$ is not reversible with respect to μ. However, it is reversible with respect to the measure μ' defined by*

$$d\mu'(x) = V(x, h)d\mu(x).$$

It is easy to check that if (X, d, μ) is locally doubling, then so is (X, d, μ'). Moreover, if $x \mapsto V(x, h)$ is bounded from above and from below, then P defines a symmetric viewpoint on (X, d, μ').

We will need the following particular case of [T4, Theorem 9.2], which is a straightforward adaptation of [Cou3, Theorem 7.1].

Theorem 4.14. *Let $X = (X, d, \mu)$ be a metric measure space. Then, the large-scale isoperimetric profile satisfies*

$$j_{X,2}(t) \approx \log t,$$

if and only if for any reversible random walk at scale large enough we have

$$\sup_{x\in X} p_x^{2n}(x) \approx e^{-n^{1/3}} \quad \forall n \in \mathbf{N}.$$

4.6 The case of groups: left and right translations

Let G be a locally compact, compactly generated group, and let S be a generating set. Let $g \in G$ and let $f \in L^p(G)$ for some $1 \leq p \leq \infty$. We have

$$|\nabla f|_1(g) = \sup_{s\in S} |f(gs) - f(g)|.$$

In other word, if ρ is the action of G by right-translation on functions, i.e. $\rho(g)f(x) = f(xg)$, the isoperimetric profile should be defined as

$$j_{G,p} = \sup_{|A|\leq m} \sup_{f} \frac{\|f\|_p}{\sup_{s\in S} \|f - \rho(s)f\|_p}.$$

Note that this is coherent with the fact that $d(gs, g) \leq 1$.

So one may wonder why we chose to define the isoperimetric profiles with left-translations in the introduction (in general, $d(s^{-1}g, g)$ is not bounded). Here are the reasons of this choice

- if the group G is unimodular, then the isoperimetric profiles are the same, defined by left-translations, or by right-translations;

- if the group G is non-unimodular, then $j_{G,p}$ is not an interesting quantity: indeed, if we define it with a left-translations and if the group is amenable, then one checks that $j_{G,p} = \infty$, and if we define it with right-translations, then $j_{G,p}$ bounded;

- if G is non-unimodular and amenable, and if $J^b_{G,p}$ is defined with right-translations, then it is bounded and therefore not interesting. But if we define it with left-translations, then it is interesting, as it gives rise to a non-bounded increasing function.

4.7 Følner pairs and isoperimetric profile

Definition 4.15. [T2] Let G be a locally compact compactly generated group and let S be a compact symmetric generating subset of G. A sequence (F_n, F'_n) of pairs of compact subsets is a sequence of controlled Følner pairs if there is a constant $C < \infty$ such that for all $n \in \mathbf{N}$,

- $\mu(F'_n) \leq C\mu(F_n)$;
- $S^n F_n \subset F'_n$.

We will need the following easy fact.

Proposition 4.16. *[T2, Proposition 4.9] If G has a sequence of controlled Følner pairs, then for all $1 \leq p \leq \infty$, $J^b_{G,p}(t) \succeq t$. Moreover, if G is unimodular, then*

$$j_{G,p}(t) \succeq \log t.$$

4.8 Large-scale equivalence between metric measure spaces

Definition 4.17. Let (X, d, μ) and (X', d', μ) two spaces satisfying the locally doubling property. Let us say that X and X' are large-scale equivalent if there is a function F from X to X' with the following properties

(a) for every sequence of pairs $(x_n, y_n) \in (X^2)^{\mathbb{N}}$

$$(d(F(x_n), F(y_n)) \to \infty) \Leftrightarrow (d(x_n, y_n) \to \infty).$$

(b) F is almost onto, i.e. there exists a constant C such that $[F(X)]_C = X'$.

(c) For $r > 0$ large enough, there is a constant $C_r > 0$ such that for all $x \in X$

$$C_r^{-1} V(x, r) \leq V(F(x), r) \leq C_r V(x, r).$$

Remark 4.18. Note that being large-scale equivalent is an equivalence relation between metric measure spaces with locally doubling property.

Remark 4.19. If X and X' are quasi-geodesic, then (a) and (b) imply that F is roughly bi-Lipschitz: there exists $C \geq 1$ such that

$$C^{-1}d(x,y) - C \leq d(F(x), F(y)) \leq Cd(x,y) + C.$$

This is very easy and left to the reader. In this case, (a) and (b) correspond to the classical definition of a *quasi-isometry*.

Example 4.20. *Consider the subclass of metric measure spaces including graphs with bounded degree, equipped with the countable measure; Riemannian manifolds with Ricci curvature bounded from below and sectional curvature bounded from above, equipped with the Riemannian measure; compactly generated, locally compact groups equipped with a left Haar measure and a word metric associated to a compact, generating subset. In this class, quasi-isometries are always large-scale equivalences.*

Theorem 4.21. *[T4, Theorem 8.1] Let $F : X \to X'$ be a large-scale equivalence between two spaces X and X' satisfying the locally doubling property. Assume that for $h > 0$ fixed, the space X satisfies a Sobolev inequality (S^p_φ) at scale h, then there exists h', only depending on h and on the constants of F such that X' satisfies (S^p_φ) at scale h'. In particular, large-scale Sobolev inequalities are invariant under large scale equivalence.*

5 Large-scale foliation of a metric measure space and monotonicity of the isoperimetric profile

Definition 5.1. Let $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \lambda)$ be two metric measure spaces satisfying the locally doubling property. We say that X is large-scale foliated (resp. normally large-scale foliated) by Y if it admits a measurable partition $X = \sqcup_{z \in Z} Y_z$ satisfying the conditions (i) and (ii) (resp. the following three conditions).

(i) There exists a measure ν on Z and a measure λ_z on ν-almost every Y_z such that for every continuous compactly supported function f on X,

$$\int_X f(x)d\mu(x) = \int_Z \left(\int_{Y_z} f(t)d\lambda_z(t) \right) d\nu(z).$$

The subsets Y_z are called the leaves, and the space Z is called the base of the foliation.

(ii) For ν-almost every z in Z, $Y_z = (Y_z, d_X, \mu)$ is large-scale equivalent to (Y, d_Y, λ) uniformly with respect to $z \in Z$.

(iii) Here, we impose a normalization condition on the measures λ_z: there exists a constant $1 \leq C < \infty$ such that for every $z \in Z$ and every $x \in Y_z$,

$$C^{-1}V_X(x,1) \leq V_{Y_z}(x,1) \leq CV_X(x,1).$$

Recall that the compression of a map F between two metric space X and Y is the function ρ defined by

$$\forall t > 0, \quad \rho(t) = \inf_{d_X(x,x') \geq t} d_Y(F(x), F(x')).$$

Definition 5.2. We call the compression of a large-scale foliation of X by Y the function

$$\rho(t) = \inf_{z \in Z} \rho_z(t)$$

where ρ_z is the compression function of the large-scale equivalence $Y \to Y_z$.

A crucial example that we will consider in some details in the next Section is the case when $Y = H$ is a closed subgroup of a locally compact group $G = X$ such that G/H carries a G-invariant measure. In [E, Lemma 4], it is proved that if H is finitely generated subgroup of a finitely generated group G, then $j_H \preceq j_G$. Here is a generalization of this easy result.

Theorem 5.3. *Let $X = (X, d, \mu)$ and $Y = (Y, \delta, \lambda)$ be two metric measure spaces satisfying the locally doubling property. Assume that X is normally large-scale foliated by Y. Then if Y satisfies a Sobolev inequality (S^p_φ) at scale h, then X satisfies (S^p_φ) at scale h', for h' large enough. In other words, if $j_{X,p}$ and $j_{Y,p}$ denote respectively the L^p-isoperimetric profiles of X and Y at scale h and h', then*

$$j_{Y,p} \succeq j_{X,p}.$$

Moreover, if ρ is the compression of the large-scale equivalence, then

$$J^b_{Y,p} \succeq J^b_{X,p} \circ \rho.$$

The latter result is true under the weaker assumption that X is merely large-scale foliated by Y.

The main difficulty comes from the fact that we need to control the measure of the support of the restriction to a leaf of a function defined on X. On the contrary, due to the definition of ρ, the control on the diameter of this support is trivial (and does not requires Condition (iii)), and so the inequality $J^b_{Y,p} \succeq J^b_{X,p} \circ \rho$ is easy and left to the reader.

Definition 5.4. A subset A of a metric space is called h-thick if it is a reunion of closed balls of radius h.

Roughly speaking, the following lemma says that we can restrict our attention to functions with thick support.

Lemma 5.5. *(see [T4, Proposition 8.3]) Let $X = (X, d, \mu)$ be a metric measure space. Fix some $h > 0$ and some $p \in [1, \infty]$. There exists a constant $C > 0$ such that for any $f \in L^\infty(X)$, there is a function $\tilde{f} \in L^\infty(X)$ whose support is included in a $h/2$-thick subset Ω such that*

$$\mu(\Omega) \leq \mu(\mathrm{Supp}(f)) + C$$

and for every $p \in [1, \infty]$,

$$\frac{\| |\nabla \tilde{f}|_{h/2} \|_p}{\|\tilde{f}\|_p} \leq C \frac{\| |\nabla f|_h \|_p}{\|f\|_p}.$$

On the other hand, the locally doubling property "extends" to thick subsets in the following sense (this lemma follows from a standard covering argument).

Lemma 5.6. *Let X be a metric measure space satisfying the locally doubling property. Fix two positive numbers u and v. There exists a constant $C = C(u, v) < \infty$ such that for any u-thick subset $A \subset X$, we have*

$$\mu([A]_v) \leq C\mu(A).$$

Finally, we need

Lemma 5.7. *Assume that X is normally large-scale foliated by Y. For every $z \in Z$, let $[Y_z]_1$ be the 1-neighborhood of Y_z in X. The inclusion map $Y_z \to [Y_z]_1$ is a large-scale equivalence, uniformly w.r.t. z.*

Proof of Lemma 5.7. The two metric conditions (a) and (b) for being a large-scale equivalence (see Definition 4.17) are trivially satisfied here, the uniformity w.r.t. z resulting from the one of $Y \to Y_z$. It remains to compare the volume of balls of fixed radius. But this is done by Condition (iii) of Definition 5.1. ■

Proof of Theorem 5.3. All along the proof, C will possibly different positive constants. Assume that Y satisfies the Sobolev inequality (S^p_φ). Let Ω be a compact subset of X and $f \in L^\infty(\Omega)$. We want to prove that f satisfies (S^p_φ) at some scale h'. By Lemma 5.5, we can assume that Ω is 1-thick. For every $z \in Z$, denote by f_z the restriction of f to Y_z and $\Omega_z = \Omega \cap Y_z$.

Claim 5.8. *There exists $C < \infty$ such that for every $z \in Z$ $\lambda_z(\Omega_z) \leq C\mu(\Omega)$.*

Proof: As Ω is 1-thick, the claim follows from the previous lemma and Lemma 5.6. ■

By Theorem 4.21, there exists $h' > 0$ such that Y_z satisfies (S^p_φ) at scale h', uniformly with respect to $z \in Z$. So for every $z \in Z$,

$$\|f_z\|_p \leq C\varphi(C\lambda_z(\Omega_z))\||\nabla f_z|_{h'}\|_p.$$

Since $\lambda_z(\Omega_z) \leq C\mu(\Omega)$ and φ is nondecreasing, we have

$$\|f_z\|_p \leq C\varphi(C\mu(\Omega))\||\nabla f_z|_{h'}\|_p.$$

Moreover, we have

$$\|f\|_p^p = \int_Z \|f_z\|_p^p d\nu(z)$$

and

$$\||\nabla f|_{h'}\|_p^p = \int_Z \||\nabla f_z|_{h'}\|_p^p d\nu(z).$$

Clearly, since Y_z is equipped with the induced distance, for every $z \in Z$ and every $x \in Y_z$,

$$|\nabla f|_{h'}(x) \geq |\nabla f_z|_{h'}(x).$$

Therefore,

$$\||\nabla f|_{h'}\|_p^p \geq \int_Z \||\nabla f_z|_{h'}\|_p^p d\nu(z).$$

We then have

$$\|f\|_p \leq C\varphi(C\mu(\Omega))\||\nabla f|_{h'}\|_p,$$

and we are done. ■

6 Application to locally compact groups

As we already explained, the definition of isoperimetric profile that we adopted for groups,

$$j_{G,p}(v) = \sup_{|A|\leq v} \sup_f \frac{\|f\|_p}{\sup_{s\in S} \|f - \lambda(s)f\|_p}$$

(defined with left-translations) is not equivalent to the "geometric one" that we introduced in Section 4.4 where $\sup_{s\in S} \|f - \lambda(s)f\|_p$ is replaced by $\||\nabla f|_1\|_p = \sup_{s\in S} \|f - \rho(s)f\|_p$, where ρ is the action of G on $L^p(G)$ by right-translations. In the following sections, we will note change our notation but rather indicate whether we consider a "left-profile" or a "right-profile" on G.

6.1 Closed subgroups

Proposition 6.1. *Let H be a closed compactly generated subgroup of a locally compact compactly generated group G. Assume that the quotient G/H carries a G-invariant Borel measure, then G is normally large-scale foliated by H.*

Proof: Let ν be a G-invariant σ-finite measure on the quotient $Z = G/H$. Since ν is G-invariant, up to normalize it, one can assume that for every continuous compactly supported function f on G,

$$\int_G f(g)d\mu(x) = \int_Z \left(\int_H f(gh)d\lambda(h)\right) d\nu(gH).$$

We claim that the partition $G = \sqcup_{gH\in Z} gH$ satisfies Conditions (i) to (iii) of Definition 4.17. Clearly, (i) follows from the above decomposition of μ. For every $g \in G$, the left-translation by g is an isometry on G. On the other hand, since H is a closed subgroup, the inclusion map $H \to G$ is a uniform embedding, i.e. satisfies condition (i) of Definition 4.17. This proves (ii). Finally, condition (iii) follows from the left-invariance of both ν and μ. Namely, the left-invariance of ν implies that, for every $g \in G$, the mesure λ_g on gH is the direct image of λ under the map $h \to gh$ from H to gH. ■

Corollary 6.2. *Let H be a closed, compactly generated subgroup of G and let $1 \leq p \leq \infty$. Assume that G/H carries a G-invariant measure. Then,*

- *The right-profiles satisfy $j_{G,p} \preceq j_{H,p}$;*
- *A weaker conclusion holds for the right-profiles inside balls: $J^b_{G,p} \preceq J^b_{H,p} \circ \rho$, where ρ is the compression of the injection $H \hookrightarrow G$.* ■

Remark 6.3. Corollary 6.2 holds for example if G and H are both unimodular. Actually this is the only interesting situation since, by [T4, Lemma 11.10], a non-unimodular group always satisfies the "best" Sobolev inequality at large scale for the right-profile: $\||\nabla f|_h\|_p \geq c_p\|f\|_p$ for every $p \geq 1$ and $h \geq 1$. On the other hand, if H is non-unimodular and if G is unimodular and amenable, then, by [T4, Proposition 11.11] all the conclusions of Corollary 6.2 are false[3].

[3] For example, consider the non-unimodular group H of positive affine transformations of $\mathbf{R}$: this group, equipped with its left-invariant Riemannian metric is isometric to the Hyperbolic plane. In particular, it has a bounded isoperimetric profile. On the other hand, it is a closed subgroup of the solvable unimodular Lie group Sol, whose isoperimetric profile $j_{G,p}$ is asymptotically equivalent to $\log t$.

Extension of Corollary 6.2 for left-profiles in balls to any subgroup

Proposition 6.4. *Let H be a closed, compactly generated subgroup of a compactly generated locally compact group G and let $1 \leq p \leq \infty$. If we define $J^b_{G,p}$ with a gradient on the left, then $J^b_{G,p} \preceq J^b_{H,p} \circ \rho$, where ρ is the compression of the injection $H \hookrightarrow G$.*

Proof: Let ν be an almost right-G-invariant σ-finite measure on the quotient $Z = H \setminus G$. Let α be the corresponding cocycle. Then, up to a multiplicative constant, for every continuous compactly supported function f on G,

$$\int_G f(g)d\mu(g) = \int_Z \left(\int_H f(g^{-1}h)d\lambda(h)\right)\alpha(g)^{-1}d\nu(Hg).$$

To check this easy fact, one just have to show that the expression

$$\Lambda(f) = \int_Z \left(\int_H f(g^{-1}h)d\lambda(h)\right)\alpha(g)^{-1}d\nu(Hg)$$

defines a left-G-invariant functional. If we apply it to $f_y = \lambda(y)f$ where $y \in G$, we obtain (by a change of variable $g = gy$)

$$\begin{aligned}
\Lambda(f_y) &= \int_Z \left(\int_H f_y(g^{-1}h)\alpha(g)^{-1}d\lambda(h)\right)d\nu(Hg)\\
&= \int_Z \left(\int_H f(y^{-1}g^{-1}h)d\lambda(h)\right)\alpha(g)^{-1}d\nu(Hg)\\
&= \int_Z \left(\int_H f(y^{-1}g^{-1}h)d\lambda(h)\right)\alpha(g)^{-1}d\nu(Hg)\\
&= \int_Z \left(\int_H f(g^{-1}h)d\lambda(h)\right)\alpha(y)^{-1}\alpha(gy^{-1})^{-1}d\nu(Hg)\\
&= \int_Z \left(\int_H f(g^{-1}h)d\lambda(h)\right)\alpha(g)^{-1}d\nu(Hg)
\end{aligned}$$

Now, consider a continuous function f, supported in a ball (for the word metric associated to S) of radius $\rho(n)$ for some $n \in \mathbf{N}$. For every $x \in G$, denote $f_x(h) = f(x^{-1}h)$. Assume that $T = S \cap H$ generates H. The support of f_x is contained in a ball (for the metric associated to T in H) of radius n. Hence, the proposition follows from the fact that for every $t \in T$,

$$\begin{aligned}
\|f - \lambda(t)f\|_p^p &= \int_Z \left(\int_H |f(g^{-1}h) - (f(g^{-1}th)|^p\alpha(g)^{-1}d\lambda(h)\right)d\nu(Hg)\\
&= \int_Z \|f_g - \lambda(t)f_g\|_p^p d\alpha(g)^{-1}d\nu(Hg).
\end{aligned}$$

The end of the argument is straightforward and similar to the end the proof of Proposition 5.3. ■

6.2 Quotients

Proposition 6.5. *Let $Q = G/H$ be the quotient of a locally compact, compactly generated group G by a closed normal subgroup H. Then for all $1 \leq p \leq \infty$, the left-profiles satisfy $j_{G,p} \preceq j_{Q,p}$ and the left-profiles in balls satisfy $J^b_{G,p} \preceq J^b_{G,p}$.*

Proof: We denote by π the projection on G/H. Let us equip G and H with left Haar measures μ and ν normalized so that the projection decreases the measure. Take a Haar measure λ on H such that for every continuous compactly supported function f on G,

$$\int_G f(g)d\mu(g) = \int_Q \left(\int_H f(gh)d\lambda(h) \right) d\nu(gH).$$

Let S be a symmetric compact generating subset of G and consider its image T by the projection on Q. The projection π is therefore 1-Lipschitz between (G,S) and (Q,T). For every $1 \leq p < \infty$, consider the application $\Psi : \ C_0(G) \to C_0(Q)$ defined by

$$\Psi(f)(gH) = \left(\int_H |f(gh)|^p d\lambda(h) \right)^{1/p}.$$

Clearly, the support of $\Psi(f)$ is the projection of the support of f. Moreover, Ψ preserves the L^p-norm. Take $s \in S$ and $t = \pi(s)$, $g \in G$ and $q = \pi(g)$,

$$\begin{aligned} |\Psi(f)(t^{-1}q) - \Psi(f)(q)| &= \left(\int_H |f(s^{-1}gh)|^p d\lambda(h) \right)^{1/p} - \left(\int_H |f(gh)|^p d\lambda(h) \right)^{1/p} \\ &\leq \left(\int_H |f(s^{-1}gh) - f(gh)|^p d\lambda(h) \right)^{1/p}. \end{aligned}$$

Therefore,

$$\|\lambda(t)\Psi(f) - \Psi(f)\|_p \leq \|\lambda(s)f - f\|_p,$$

and we are done. ■

7 Geometrically elementary solvable groups

The proofs of Theorems 4 and 6 follow, on the one hand, from the stability results: Proposition 6.5, Theorem 6.2 and Theorem 6.4 (and the result of Mustapha recalled in the introduction), and on the other hand from the case of $T(d,k)$, treated in the following section.

7.1 The case of $T(d,k)$

Theorem 7.1. *Let k be a local field, then $G = T(d,k)$ has a sequence of Følner pairs. In particular (see Proposition 4.16), it satisfies $J^b_{G,p}(t) \approx t$.*

Proof: The case $k = \mathbf{R}$ (or $\mathbf{C}$) has been treated in [T2]. So here we will consider the case of a non-archimedean local field and we assume that $d \geq 2$ (the case $d = 1$ being trivial). Let v be a valuation on k, and for every $x \in k$, let $|x| = e^{-v(x)}$ be the corresponding norm. We have

$$|x + y| \leq \max\{|x|, |y|\},$$

and

$$|xy| = |x||y|.$$

Let k_n be the (compact) subring of k consisting of elements $y \in k$ of norm $|y| \leq n$. We fix $x_0 \in k$ such that $|x_0| = 1$. We have

$$x_0 k_n = k_{n+1}.$$

Let U be the subgroup of G consisting of unipotent elements, and let $T \simeq (k^*)^d$ be the subgroup of diagonal elements. We have a semidirect product

$$G = T \ltimes U.$$

For every $n \in \mathbf{N}$, let U_n be the compact normal subgroup of U consisting of unipotent matrices such that for $1 \leq i < j \leq d$, the (i,j)'th coefficient lies in $k_{(j-i)n}$. We also consider the compact subset of T defined by $T_n = (k_n \setminus \{0\})^d$.

Let us identify G with the cartesian product $T \times U$, where the group law is given by

$$(t, u)(s, v) = (ts, u^s v),$$

where $u^s = s^{-1}us$. We define a compact subset S of G by

$$S = T_1 \cup U_0.$$

Let $t_0 = (x_0, x_0, \ldots, x_0) \in T_1$. An easy computation shows that for every $n \in \mathbf{N}$,

$$U_{n+1} = t_0^{-1} U_n t_0. \tag{7.1}$$

Note that, as $G = \bigcup_n T_n \times U_n$, this implies that S is a generating subset of G. On the other hand, we deduce from (7.1) that

$$U_n \subset S^{2n+1}.$$

As $T_n \subset S^n$, we have

$$T_n \times U_n \subset S^{3n+1}.$$

Claim 7.2. *For all $n \geq 1$, $S^n \subset T_n \times U_n$.*

Proof: This is true for $n = 1$. Now, assume that this is true for $n \geq 1$, and take an element $g = (t, u)$ in $T_n \times U_n$, and an element h of S. Let us check that $gh \in T_{n+1} \times U_{n+1}$. First, assume that $h = (s, 1) \in T_1$. Then,

$$gh = (ts, u^s) \in T_{n+1} \times s^{-1}U_n s^{-1}$$

As $|s| \leq 1$, by (7.1), $T_{n+1} \times s^{-1}U_n s^{-1} \subset T_{n+1} \times U_{n+1}$.

Now, if $h = (1, v) \in U_0$, then $gh = (t, uv) \subset T_n \times U_n$. ■

Now, let $F_n = T_n \times U_{2n}$ and $F'_n = T_{2n} \times U_{2n}$. We claim that (F_n, F'_n) is a sequence of Følner pairs. As $F'_n \subset S^{3n+1}$ and $|F'_n| = 2|F_n|$, we just need to check that $S^n F_n \subset F'_n$. Let $g = (t, u) \in S^n \subset T_n \times U_n$ and $g' = (s, v) \in F_n$. By an immediate induction, (7.1) implies that $s^{-1}U_n s^n \subset U_{2n}$. Hence,

$$(t, u)(s, v) = (ts, u^s v) \in T_{2n} \times U_{2n}U_n = T_{2n} \times U_{2n},$$

which finishes the proof. ■

7.2 Random walks

We keep the notation of the introduction.

Theorem 7.3. *Let G be a unimodular elementary solvable group with exponential growth. Then for every symmetric quasi-G-transitive random walk (X, P), we have*

$$\sup_{x \in X} p_x^n(x) \approx e^{-n^{1/3}}.$$

Proof: Let (X, μ) be a quasi-G-transitive measure space. By [T4, Proposition 11.3], (X, μ) can be equipped with a G-invariant metric d on X which is proper and finite on compact sets.

As G acts properly and co-compactly on X, every orbit of G induces a coarse equivalence from G to (X, d). On the other hand, as μ is almost preserved by G, there exists a constant $C < \infty$ such that for every $r > 0$,

$$C^{-1}|S| \leq \mu(B(x, r)) \leq C|S|,$$

where $|S|$ denotes the Haar measure of the compact generating subset S of G. Hence X and G are large-scale equivalent. By Theorem 4.21, their large-scale isoperimetric profiles are asymptotically equivalent. Therefore $j_{X,2} \approx \log t$. Theorem 4.14 then implies that the probability of return of any reversible random walk at scale large enough, decreases like $e^{-n^{1/3}}$. To apply this to our random walk P, we just need to check that for k large enough, P^k defines a viewpoint at a scale h for arbitrarily large h. By the way, as d is G-invariant and finite on compacts, it can be chosen so that for every $x \in X$, $gSK \subset B(x,1)$ for some $g \in G$, which makes (X,d) a coarsely 1-geodesic space. Now, with such a metric, P is a viewpoint at scale 1. But, as (X,d) is coarsely 1-geodesic, one can check that P^k defines a viewpoint at scale $k/2$, so we are done. ■

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