GEOMETRIC PRESENTATIONS OF LIE GROUPS AND THEIR DEHN FUNCTIONS

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Abstract. We study the Dehn function of connected Lie groups. We show that this function is always exponential or polynomially bounded, according to the geometry of weights and of the 2-cohomology of these groups. Our work, which also addresses algebraic groups over local fields, uses and extends Abels' theory of multiamalgams of graded Lie algebras, in order to provide workable presentations of these groups.

Contents

Date: October 19, 2013.

2010 *Mathematics Subject Classification.* Primary 17B70, 20F05, Secondary 08B25, 13N05, 17B56, 17B65, 19C09, 20F14, 20F69, 22D05, 22E15, 22E25, 57M07.

1. Introduction

1.1. **Dehn function of Lie groups.** The object of study in this paper is the Dehn function of connected Lie groups. For a simply connected Lie group G endowed with a left-invariant Riemannian metric, this can be defined as follows: the area of a loop γ is the infimum of areas of filling discs, and the Dehn function

$\delta_G(r)$ is the supremum of areas of loops of length at most r. The asymptotic behaviour of δ_G (when $r \to +\infty$) actually does not depend on the choice of a left-invariant Riemannian metric. If G is an arbitrary connected Lie group and K a compact subgroup such that G/K is simply connected (e.g. K is a maximal compact subgroup, in which case G/K is diffeomorphic to a Euclidean space), we can endow G/K with a G-invariant Riemannian metric and thus define the Dehn function $\delta_{G/K}(r)$ in the same way; its asymptotic behaviour depends only on G, neither on K nor on the choice of the invariant Riemannian metric, and is called the Dehn function of G. For instance, if for some maximal compact subgroup K, the space G/K has a negatively curved G-invariant Riemannian metric, then the Dehn function of G has exactly linear growth. Otherwise, G is not Gromov-hyperbolic, and by a very general argument due to Bowditch (not specific to Lie groups), the Dehn function is known to be *at least* quadratic. On the other hand, the Dehn function is *at most* quadratic whenever G/K can be endowed with a non-positively curved invariant Riemannian metric, notably when G is reductive. It is worth emphasizing that many simply connected Lie groups G fail to have a non-positively curved homogeneous space G/K and nevertheless have a quadratic Dehn function. Characterizing Lie groups with a quadratic Dehn function is a very challenging problem, even in the setting of nilpotent Lie groups. Indeed, although connected nilpotent Lie groups have an at most polynomial Dehn function, there are examples with Dehn function of polynomial growth with arbitrary integer degree. In a sense, our results actually show that for connected Lie groups, polynomial Dehn functions of large polynomial growth are always related to "large" nilpotent quotients. Finally, let us observe that the Dehn function of a connected Lie group is at most exponential, the prototypical example of a Lie group with an exponential Dehn function being the three-dimensional SOL group.

A main consequence of the results we describe below is the following theorem.

Theorem A. *Let G be a connected Lie group. Then the Dehn function of G is either exponential or polynomially bounded.*

Let us mention that "polynomially bounded" cannot be improved to "of polynomial growth", since S. Wenger [We11] has exhibited some simply connected nilpotent Lie groups with a Dehn function satisfying $n^2 \ll \delta(n) \preceq n^2 \log n$.

Our results are more precise than Theorem A: we characterize *algebraically* which ones have a polynomially bounded or exponential Dehn function. To do so, we describe below two "obstructions" implying exponential Dehn function; the first being related to SOL, and the second to homology in degree 2. We prove that if none of these obstructions is fulfilled, then the group has an at most polynomial Dehn function, proving in a large number of cases that the Dehn function is at most quadratic or cubic.

These results can appear as unexpected. Indeed, it was suggested by Gromov [Gro93, 5.A_9] that the only obstruction should be related to SOL. This has been proved in several important cases [Gro93, Dru04, LP04] but turns out to be false in general.

Using the well-known fact that polycyclic groups are virtually cocompact lattices in connected Lie groups, we deduce

Corollary B. *The Dehn function of a polycyclic group is either exponential or polynomially bounded.*

The remainder of this introduction is organized as follows: in §1.2, we define a combinatorial Dehn function for compactly presented locally compact groups, and use it to state a version of Theorem A for algebraic p-adic groups. Then §1.3 is dedicated to our main results. The most difficult part of the main theorem is the fact that in the absence of the two obstructions, the Dehn function is polynomially bounded. We sketch the main ideas behind its proof in Section 1.6. In §1.4, we provide a useful characterization of these obstructions in terms of *graded Lie algebras*. We also introduce a sufficient condition for the Dehn function to be quadratic. Finally we apply these results to various concrete examples in §1.5.

1.2. **Riemannian versus Combinatorial Dehn function of Lie groups.** The previous approaches consisted in either working with groups admitting a cocompact lattice and use combinatorial methods, or use the Riemannian definition. Our method, initiated in [CT10] is largely inspired by Abel's work on p-adic algebraic groups [Ab87]. It consists in extending the combinatorial methods to general locally compact compactly generated groups. In particular, Lie groups are treated as combinatorial objects, i.e. groups endowed with a compact generating set and the corresponding Cayley graph. The object of study is the combinatorial Dehn function, usually defined for discrete groups, which turns out to be asymptotically equivalent to its Riemannian counterpart. The power of this approach relies on the dynamical structure arising from the action of G on itself by conjugation. A crucial role is played by some naturally defined subgroups that are contracted by suitable elements. The presence of these subgroups is obviously a non-discrete feature, which is invisible in any cocompact lattice (when such lattices exist). In addition, this unifying approach allows to treat p-adic algebraic groups and connected Lie groups on the same footing.

We now give the combinatorial definition of Dehn function (rechristening the above definition of Dehn function as **Riemannian Dehn function**). Let G be a locally compact group, generated by a compact subset S. Let F_S be the free group over the (abstract) set S and $F_S \to G$ the natural epimorphism, and K its kernel (its elements are called **relations**). We say that G is **compactly presented** if for some ℓ, K is generated, as a normal subgroup of F_S, by the set K_ℓ of elements with length at most ℓ with respect to S, or equivalently if K is generated, as a group, by the union $C(K_\ell)$ of conjugates of K_ℓ; this does not depend on the choice of S; the subset K_ℓ is called a set of **relators**. Assuming this, if $x \in K$, the **area** of x is by definition the number area(x) defined as its length with respect to $C(K_\ell)$. Finally, the Dehn function of G is defined as

$$\delta(n) = \sup\{\text{area}(x) : \ x \in K, \ |x| \leq n\}.$$

In the discrete setting (S finite), this function takes finite values, and this remains true in the locally compact setting. If G is not compactly presented, a good convention is to set $\delta(n) = +\infty$ for all n. The Dehn function of a compactly presented group G depends on the choices of S and ℓ, but its asymptotic behavior does not.

With this definition at hand, we can now state a version of Theorem A in a non-Archimedean setting.

Theorem C. *Let G be an algebraic group over some p-adic field. Then the Dehn function of G is at most cubic, or G is not compactly presented.*

Before providing more detailed statements, let us compare Theorems A and C. It is helpful to have in mind a certain analogy between Archimedean and non-Archimedean groups, where exponential Dehn function corresponds to not compactly presented. On the other hand, a striking difference between these two theorems is the absence for p-adic groups of polynomial Dehn functions of arbitrary degree. The explanation of this fact can be summarized as follows. In the connected Lie group setting, Dehn functions of "high polynomial degree" witness to the presence of simply connected non-abelian nilpotent quotients, see Theorem 4.M.1 for a precise statement. By way of contrast, any totally disconnected, compactly generated locally compact nilpotent group is compact-by-discrete and the group of $\mathbf{Q}_p$-points of any p-adic algebraic nilpotent group is compact-by-abelian.

1.3. **Main results.** We now turn to more comprehensive statements. Let us first introduce the two main classes of groups we will be considering in the sequel.

Definition 1.1. A **real triangulable** group is a Lie group isomorphic to a closed connected group of real triangular matrices. Equivalently, it is a simply connected solvable group in which for every g, the adjoint operator $\text{Ad}(g)$ has only real eigenvalues.

It can be shown that every connected Lie group G is quasi-isometric to a real triangulable Lie group. Namely, there exists a sequence of maps

$$G \leftarrow G_1 \rightarrow G_2 \leftarrow G_3,$$

where each arrow is a proper continuous homomorphism with cocompact image and thus is a quasi-isometry, see Lemma 3.A.1.

Let A be an abelian group and consider a representation of A on a $\mathbf{K}$-vector space V, where $\mathbf{K}$ is a finite product of complete normed fields. Let V_0 be the largest A-equivariant quotient of V on which A acts with only eigenvalues of modulus one.

Definition 1.2. A locally compact group is a **standard solvable group** if it is topologically isomorphic to a semidirect product $U \rtimes A$ so that

(1) A is a compactly generated locally compact abelian group
(2) U decomposes as a finite direct product $\prod U_i$, where each U_i is normalized by the action of A and can be written as $U_i = \mathbb{U}_i(\mathbf{K}_i)$, where $\mathbb{U}_i$ is a unipotent group over some nondiscrete locally compact field of characteristic zero $\mathbf{K}_i$;
(3) $(U/[U,U])_0 = \{0\}$.

For a group G satisfying (1) and (2), condition (3) implies that G is compactly generated, and conversely if U is totally disconnected, the failure of condition (3) implies that G is not compactly generated. If G is a compactly generated p-adic group as in Theorem C, then it has a Zariski closed cocompact subgroup which is a standard solvable group (with a single i and $\mathbf{K}_i = \mathbf{Q}_p$). Many real Lie groups have a closed cocompact standard solvable group; however, for instance, a simply connected nilpotent Lie group is not standard solvable unless it is abelian. We now introduce a very special but important class of standard solvable groups.

Definition 1.3. A **group of SOL type** is group $U \rtimes A$, where $U = \mathbf{K}_1 \times \mathbf{K}_2$, where $\mathbf{K}_1$, $\mathbf{K}_2$ are nondiscrete locally compact fields of characteristic zero, and $A \subset \mathbf{K}_1^* \times \mathbf{K}_2^*$ is a closed subgroup of $\mathbf{K}_1^* \times \mathbf{K}_2^*$ containing, as a cocompact subgroup, the cyclic group generated by some element (t_1, t_2) with $|t_1| > 1 > |t_2|$. Note that this is a standard solvable group. We call it a **non-Archimedean group of SOL type** if *both* $\mathbf{K}_1$ and $\mathbf{K}_2$ are non-Archimedean.

Example 1.4. If $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{K}$ and A is the set of pairs (t, t^{-1}), then G is the usual group $\mathrm{SOL}(\mathbf{K})$. More generally, A is the set of pairs $(t^k, t^{-\ell})$ where (k, ℓ) is a fixed pair of positive integers, then this provides another group, which is unimodular if and only $k = \ell$. Another example is $(\mathbf{R} \times \mathbf{Q}_p) \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts as the cyclic subgroup generated by (p, p) (note that $|p|_{\mathbf{R}} > 1 > |p|_{\mathbf{Q}_p}$); the latter contains the Baumslag-Solitar group $\mathbf{Z}[1/p] \rtimes \mathbf{Z}$ as a cocompact lattice.

Also, define, for $\lambda > 0$, the group SOL_λ as the semidirect product $\mathbf{R}^2 \rtimes \mathbf{R}$ where $\mathbf{R}$ is identified with the subgroup $\{(t, t^\lambda) : t > 0\}$ of $(\mathbf{R}^*)^2$. Note that SOL_1 has index 2 in $\mathrm{SOL}(\mathbf{R})$; there are obvious isomorphisms $\mathrm{SOL}_\lambda \simeq \mathrm{SOL}_{\lambda^{-1}}$, and the SOL_λ, for $\lambda \geq 1$, are pairwise non-isomorphic. These are the only real triangulable groups of SOL type.

Definition 1.5. (SOL obstruction) A locally compact group has the SOL obstruction (resp. non-Archimedean SOL obstruction) if it admits a homomorphism with dense image to a group of SOL type (resp. non-Archimedean SOL type).

Let us now define the 2-homological obstruction. For this, we need to recall a fundamental notion introduced and studied by Guivarc'h [Gui80] and later rediscovered by Osin [Os02]. Let G be a real triangulable group. Its **exponential radical** G^∞ is defined as the intersection of its descending central series and actually consists of the exponentially distorted elements in G. Let $\mathfrak{g}^\infty$ be its Lie algebra. In the case of a standard solvable group, the role of exponential radical is played by U itself (it can be checked to be equal to the derived subgroup of G, so is a characteristic subgroup).

Definition 1.6. (**2-homological obstruction**)

- The real triangulable group G is said to satisfy the 2-homological obstruction if $H_2(\mathfrak{g}^\infty)_0 \neq \{0\}$, or equivalently if the A action on $H_2(\mathfrak{g}^\infty)$ has no nonzero invariant vector.
- The standard solvable group $G = U \rtimes A$ is said to satisfy the 2-homological obstruction if $H_2(\mathfrak{u})_0 \neq \{0\}$, that is to say, $H_2(\mathfrak{u}_j)_0 \neq \{0\}$ for some j. If moreover j can be chosen so that $\mathbf{K}_j$ is non-Archimedean, we call it the **non-Archimedean 2-homological obstruction**.

In most cases, including standard solvable groups, the 2-homological obstructions can be characterized by the existence of suitable central extensions. For instance, if a real triangulable group G has a central extension $\tilde{G}$, also real triangulable, with nontrivial kernel Z such that $Z \subset (\tilde{G})^\infty$ then it satisfies the 2-homological obstruction.

The converse is true when G admits a semidirect decomposition $G^\infty \rtimes N$, but nevertheless does not in general, see §7.E.

We are now able to state our main theorem, which immediately entails Theorems A and C.

Theorem D. *Let G be a real triangulable group, or a standard solvable group.*

- *if G satisfies one of the two non-Archimedean (SOL or 2-homological) obstructions, then G is not compactly presented;*
- *otherwise G is compactly presented and has an at most exponential Dehn function. Moreover, in this case*
  - *if G satisfies one of the two (SOL or 2-homological) obstructions, then G has an exponential Dehn function;*
  - *if G satisfies none of the obstructions, then it has a polynomially bounded Dehn function; in the case of a standard solvable group, the Dehn function is at most cubic.*

This result can be seen as both a generalization and a strengthening of the following seminal result of Abels [Ab87].

Theorem (Abels)**.** *Let G be a standard solvable group over a p-adic field. Then G is compactly presented if and only if it satisfies none of the non-Archimedean obstructions.*

Let us split Theorem D into several independent statements. The first two provide lower bounds and the last two provide upper bounds on the Dehn function.

Theorem D.1. *Let G be a standard solvable or real triangulable group. If G satisfies the SOL (resp. non-Archimedean SOL) obstruction, then G has an at least exponential Dehn function (resp. is not compactly presented).*

We provide a unified proof of these two statements in Section 8. Note that the non-Archimedean case is essentially contained in the "only if" (easier) part of Abels' theorem above, itself inspired by previous work of Bieri-Strebel, notably [BiS78, Theorem A]. Part of the proof consists in estimating the size of loops in the groups of SOL type, where our proof is inspired by the original case of the real SOL, due to Thurston [ECHLPT92], which uses integration of a well-chosen differential form. Our method in Section 8 is based on a discretization of this argument, leading to both a simplification and a generalization of the argument.

Theorem D.2. *Let G be a standard solvable or real triangulable group. If G satisfies the 2-homological (resp. non-Archimedean 2-homological) obstruction, then G has an at least exponential Dehn function (resp. is not compactly presented).*

The case of standard solvable groups reduces, after a minor reduction, to a simple and classical central extension argument, see §7.B. The case of real triangulable groups is considerably more difficult; in the absence of splitting of the exponential radical, we construct an "exponentially distorted hypercentral extension". This is done in Section 7.

Theorem D.3. *Let G be a real triangulable group. Then G has an at most exponential Dehn function.*

Let $G = U \rtimes A$ be a standard solvable group and U° the identity component in U. If G/U° is compactly presented, then G is compactly presented with an at most exponential Dehn function.

Since compact presentability is stable under taking extensions [Ab72], for an arbitrary locally compact group G with a closed connected normal subgroup C, it is true that G is compactly presented if and only G/C is compactly presented. We do not know if this can be generalized to the statement that if G/C has Dehn function $\preccurlyeq f(n)$, then G has Dehn function $\preccurlyeq \max(f(n), \exp(n))$. Theorem D.3, which follows from see Theorem 3.B.1 and Corollary 3.B.5, contains two particular instances where the latter assertion holds. The first instance, namely that every connected Lie group has an at most exponential Dehn function, was asserted by Gromov, with a sketch of proof [Gro93, Corollary 3.F'_5]. The method uses an "exponentially Lipschitz" retraction and has similar consequences for higher-dimensional isoperimetry problems.

Example 1.7. Fix $n \in \mathbf{Z}$ with $|n| \geq 2$. Consider the group $G_n = (\mathbf{R} \times \mathbf{Q}_n) \rtimes_n \mathbf{Z}$, where $\mathbf{Q}_n$ is the product of $\mathbf{Q}_p$ where p ranges over distinct primes divisors of n. Here $C \simeq \mathbf{R}$ and $H \simeq \mathbf{Q}_n \rtimes_n \mathbf{Z}$, which, as a hyperbolic group (it is an HNN ascending extension of the compact group $\mathbf{Z}_n$), has a linear Dehn function. So, by Theorem D.3, G_n has an at most exponential Dehn function. Since $|n| \geq 2$, it admits a prime factor p, so G_n admits the group of SOL type $(\mathbf{R} \times \mathbf{Q}_p) \rtimes_n \mathbf{Z}$ as a quotient, and therefore G_n satisfies the SOL obstruction and thus has an at least exponential Dehn function by Theorem D.1. We conclude that G_n has an exponential Dehn function. This provides a new proof that its lattice, the Baumslag-Solitar group

$$\mathrm{BS}(1, n) = \langle t, x \mid txt^{-1} = x^n \rangle,$$

has an exponential Dehn function. This is actually true for arbitrary Baumslag-Solitar groups $\mathrm{BS}(m, n)$, $|m| \neq |n|$, for which the exponential upper bound was first established in [ECHLPT92, Theorem 7.3.4 and Example 7.4.1] and independently in [BGSS92], and the exponential lower bound, attributed to Thurston, was obtained in [ECHLPT92, Example 7.4.1].

Let us provide a useful corollary of Theorems D.1 and D.3.

Corollary D.3.a. *Let $G = U \rtimes A$ be a standard solvable group in which A has rank 1 (i.e., has a closed infinite cyclic cocompact subgroup). Then exactly one of the following occurs*

- *G satisfies the non-Archimedean SOL obstruction and thus is not compactly presented;*
- *G satisfies the SOL obstruction but not the non-Archimedean one; it is compactly presented with an exponential Dehn function;*
- *G does not satisfy the SOL obstruction; it has a linear Dehn function and is Gromov-hyperbolic.*

It is indeed an observation that if A has rank 1 and G does not satisfy the SOL obstruction, then some element of A acts on G as a "compacting automorphism" and it follows from [CCMT12] that G is Gromov-hyperbolic, or equivalently has a linear Dehn function. In this special case, the 2-homological obstruction, which may hold or not hold, implies the SOL obstruction and is accordingly unnecessary to consider; see also Theorem F.

Turning back to Theorem D, the fourth and most involved of all the steps is the following.

Theorem D.4. *Let G be a standard solvable (resp. real triangulable) group not satisfying neither the SOL nor the 2-homological obstructions. Then G has an at most cubic (resp. at most polynomial) Dehn function.*

The proof of Theorem D.4 for standard solvable groups is done in Section 4, relying on algebraic preliminaries, occupying Sections 5 and 6. We actually obtain, with relatively little additional work, a similar statement for "generalized standard solvable groups", where A is replaced by some nilpotent compactly generated group N (see Theorem 4.M.1). The case of real triangulable groups requires an additional step, namely a reduction to the case where the exponential radical is split, in which case the group is generalized standard solvable. This reduction is performed in §3.C, and relies on results from [C11].

Finally let us mention that Theorem D.4 for standard solvable groups follows from the more precise Theorem 4.L.1, which provides, in many cases, a quadratic Dehn function.

1.4. **The obstructions as "computable" invariants of the Lie algebra.** The obstructions were introduced above in a convenient way for expository reasons, but the natural framework to deal with them uses the language of graded Lie algebras, which we now describe.

Let G be either a standard solvable group $U \rtimes A$ or a real triangulable group with exponential radical also denoted, for convenience, by U. Let $\mathfrak{u}$ be the Lie algebra of U; it is a Lie algebra (over a finite product of nondiscrete locally compact fields of characteristic zero). It is, in a natural way, a graded Lie algebra. In both cases, the grading takes values into a finite-dimensional real vector space, namely $\mathrm{Hom}(G/U, \mathbf{R})$. It is introduced in §4.B.2, based on Proposition 2.E.2 (see §1.5 for a representative particular case.) In this setting, there are useful restatements (see Propositions 2.E.11 and 2.F.7) of the obstructions. We say that α is a **weight** of G (or of U, when $\mathfrak{u}$ is endowed with the grading) if $\mathfrak{u}_\alpha \neq 0$ and is a **principal weight** of G is α is a weight of $U/[U, U]$. The definitions imply that 0 is not a principal weight (although it can be a weight). We say that two nonzero weights α, β are *quasi-opposite* if $0 \in [\alpha, \beta]$, i.e., $\beta = -t\alpha$ for some $t > 0$. We write $U = U_{\mathrm{a}} \times U_{\mathrm{na}}$ as the product of its Archimedean and non-Archimedean parts. Then we have the following restatements:

- G satisfies the SOL obstruction $\Leftrightarrow$ U admits two quasi-opposite principal weights (Propositions 2.E.11 and 2.F.7);
- G satisfies the non-Archimedean SOL obstruction $\Leftrightarrow$ U_{na} admits two quasi-opposite principal weights (Proposition 2.E.11 applied to G/G_0);
- G satisfies the 2-homological obstruction $\Leftrightarrow$ $H_2(\mathfrak{u})_0 \neq \{0\}$;
- G satisfies the non-Archimedean 2-homological obstruction $\Leftrightarrow$ $H_2(\mathfrak{u}_{\mathrm{na}})_0 \neq \{0\}$.

Here, $H_2(\mathfrak{u})$ denotes the homology of the Lie algebra $\mathfrak{u}$; the grading on $\mathfrak{u}$ in the real vector space $\mathrm{Hom}(G/U, \mathbf{R})$ canonically induces a grading of $H_2(\mathfrak{u})$ in the same space (see §5.A), and $H_2(\mathfrak{u})_0$ is its component in degree zero.

Another important module associated to $\mathfrak{u}$ is $\mathrm{Kill}(\mathfrak{u})$, the quotient of the second symmetric power $\mathfrak{u} \circledcirc \mathfrak{u}$ by the submodule generated by elements of the form $[x, y] \circledcirc z - x \circledcirc [y, z]$ (thus, in case of a single field, the invariant quadratic forms on $\mathfrak{u}$ are elements in the dual of $\mathrm{Kill}(\mathfrak{u})$).

Theorem E. *Let $G = U \rtimes A$ be a standard solvable group not satisfying any of the SOL or 2-homological Dehn function. Suppose in addition that* $\mathrm{Kill}(\mathfrak{u})_0 = \{0\}$. *Then G has an at most quadratic Dehn function (thus exactly quadratic if A has dimension at least two).*

Theorem E is proved along with Theorem D.4 and involves the same difficulty, except the study of welding relations. The condition $\mathrm{Kill}(\mathfrak{u})_0 \neq \{0\}$ corresponds to the existence of certain central extensions of G as a discrete group. Using asymptotic cones, it will be shown in a subsequent paper that it implies, in many cases, that the Dehn function grows strictly faster than a quadratic function.

1.5. **Examples.** Let us give a few examples. All are standard solvable connected Lie groups $G = U \rtimes A$ so that the action of A on the Lie algebra $\mathfrak{u}$ of U is $\mathbf{R}$-diagonalizable). In this context, we call $\mathrm{Hom}(A, \mathbf{R})$ the weight space. The grading of the Lie algebra $\mathfrak{u}$ in $\mathrm{Hom}(A, \mathbf{R})$ is given by

$$\mathfrak{u}_\alpha = \{u \in \mathfrak{u} \mid \forall v \in A,\ v^{-1}uv = e^{\alpha(v)}u\}.$$

When we write the set of weights, we use boldface for the set of principal weights. We underline the zero weight (or just denote **.** to mark zero if zero is not a weight).

1.5.1. *Groups of SOL type.* For a group of SOL type, the weight space is a line and the weights lie on both sides apart zero

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By definition it satisfies the SOL obstruction. On the other hand, it satisfies the 2-homological obstruction only in a few special cases. For instance, $(\mathbf{R} \times \mathbf{Q}_p) \rtimes_p \mathbf{Z}$ does not satisfy the 2-homological obstruction, and the real group SOL_α satisfies the 2-homological obstruction only for $\alpha = 1$.

1.5.2. *Gromov's higher SOL groups.* For the group $\mathbf{R}^3 \rtimes \mathbf{R}^2$ where $\mathbf{R}^2$ acts on $\mathbf{R}^3$ as the group of diagonal matrices with positive diagonal entries and determinant one (often called higher-dimensional SOL group, but not of SOL type nor even satisfying the SOL obstruction according to our conventions), the weight space is a plane in which the weights form a triangle whose center of gravity is zero

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Since there are no opposite weights, we have $(\mathfrak{u} \otimes \mathfrak{u})_0 = 0$ and therefore $H_2(\mathfrak{u})_0$ and $\mathrm{Kill}(\mathfrak{u})_0$ (which are subquotients of $(\mathfrak{u} \otimes \mathfrak{u})_0$) are also zero. It was stated with a sketch of proof by Gromov that this group has a quadratic Dehn function [Gro93, 5.A_9]. Drutu obtained in [Dru98, Corollary 4.18] that it has a Dehn function $\preceq n^{3+\varepsilon}$, and then obtained a quadratic upper bound in [Dru04, Theorem 1.1], a result also obtained by Leuzinger and Pittet in [LP04]. The quadratic upper bound can also be viewed as an illustration of Theorem F. (The assumption that 0 is the center of gravity is unessential: the important fact is that 0 belongs to the convex hull of the three weights but does not lie in the segment joining any two weights.)

1.5.3. *Abels' first group.* The group $G = A_4(\mathbf{K})$ consists of matrices of the form

$$\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & s_2 & u_{23} & u_{24} \\ 0 & 0 & s_3 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad s_i \in \mathbf{K}^\times,\ u_{ij} \in \mathbf{K}.$$

Its weight configuration is given by

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This example is interesting because it does not satisfy the SOL obstruction but admits opposite weights. A computation shows that $H_2(\mathfrak{u})_0 = 0$ and $\mathrm{Kill}(\mathfrak{u})_0 = 0$ (see Abels [Ab87, Example 5.7.1]).

If $\bar{G} = \bar{U} \rtimes A$ is the quotient of G by its one-dimensional center, then G does not satisfy the SOL obstruction but satisfies $H_2(\bar{\mathfrak{u}})_0 \neq 0$, i.e. satisfies the 2-homological obstruction.

This example was studied specifically by the authors in the paper [CT12], where it is proved that it has a quadratic Dehn function, which also follows from Theorem E.

1.5.4. *Abels' second group.* This group was introduced in [Ab87, Example 5.7.4]. Consider the group $U \rtimes A$, where $A \simeq \mathbf{R}^2$ and U is the group corresponding to the quotient of the free 3-nilpotent Lie algebra generated by the 3-dimensional $\mathbf{K}$-vector space of basis (X_1, X_2, X_3) by the ideal generated by $[X_i, [X_i, Y_j]]$ for all $i, j \in \{1, 2, 3\}$. Its weight structure is as follows

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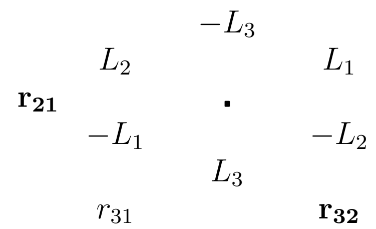
(The sign $\underline{**}$ indicates that the degree 0 subspace $\mathfrak{u}_0$ is 2-dimensional.) A computation (see Remark 5.D.3) shows that $H_2(\mathfrak{u})_0 = 0$ and $\mathrm{Kill}(\mathfrak{u})_0$ is 1-dimensional.

This example was introduced by Abels as typically difficult because although $H_2(\mathfrak{u})_0$ vanishes, U is not the multiamalgam of its tame subgroups (as we show here, this is reflected in the fact that $\mathrm{Kill}(\mathfrak{u})_0$ is nonzero). This results in a significant additional difficulty in order to estimate the Dehn function, which is at most cubic by Theorem D.4.

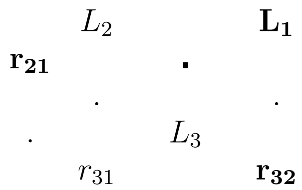
1.5.5. *Semidirect products with* SL_3. Here we consider the groups $V(\mathbf{K}) \rtimes \mathrm{SL}_3(\mathbf{K})$, where V are the following three irreducible modules: $V = V_{10}$, the standard 3-dimensional module; $V_{20} = \mathrm{Sym}^2(V_{10})$, the 6-dimensional second power of V_{10} and

V_{11}, the 8-dimensional adjoint representation. (The notation is borrowed from [FuH, Lecture 13], writing V_{ij} instead of Γ_{ij}.) These groups are not solvable but have a cocompact $\mathbf{K}$-triangulable subgroup, namely $V(\mathbf{K}) \rtimes T_3(\mathbf{K})$, where $T_3(\mathbf{K})$ is the group of lower triangular matrices. It is a simple verification that for an arbitrary nontrivial irreducible representation, the group $V(\mathbf{K}) \rtimes T_3(\mathbf{K})$ admits exactly three principal weights, namely the two principal negative roots r_{21}, r_{32} of SL_3 itself, and the highest weight of the representation V_{ab}, which is of the form $aL_1 - bL_3$, where



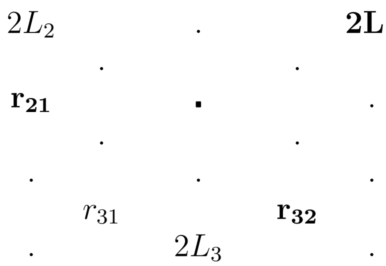
The three principal weights always form a triangle with zero contained in its interior. In particular, $V(\mathbf{K}) \rtimes T_3(\mathbf{K})$ does not satisfy the SOL obstruction. Let us write the weight diagram for each of the three examples (we mark some other points in the weight lattice as $\cdot$ for the sake of readability).

More specifically, for V_{10}, the weights configuration looks like



We see that there are no quasi-opposite weights at all. This is accordingly a case for which Theorem F below applies directly. Thus $\mathbf{K}^3 \rtimes \mathrm{SL}_3(\mathbf{K})$ has a quadratic Dehn function (it can be checked to also hold for $\mathbf{K}^d \rtimes \mathrm{SL}_d(\mathbf{K})$ for $d \geq 3$).

For V_{20}, the weights are as follows



Thus there are quasi-opposite weights but no opposite weights. Theorem 4.L.1 implies that $V_{20}(\mathbf{K}) \rtimes \mathrm{SL}_3(\mathbf{K})$ has a quadratic Dehn function.

For V_{11}, writing $L_{ij} = L_i - L_j$, the weights are as follows

	L_{23}		$\mathbf{L_{13}}$	
.		.		.
	.		.	
$\mathbf{r_{21}}, L_{21}$		$\underline{**}$		L_{12}
	.		.	
.		.		.
	r_{31}, L_{31}		$\mathbf{r_{32}}, L_{32}$	

In this case, there are opposite weights, there is an invariant quadratic form in degree zero (akin to the Killing form), defined by $\phi(r_{ji}, L_{ij}) = 1$ for all $i < j$ and all other products being zero, so $\mathrm{Kill}(\mathfrak{u})_0 \neq 0$. However, a simple computation shows that $H_2(\mathfrak{u})_0 = 0$. So Theorem D.4 implies that $\mathfrak{sl}_3(\mathbf{K}) \rtimes \mathrm{SL}_3(\mathbf{K})$ has an at most cubic Dehn function.

1.6. **Comparison to previous results and outline of the proof.** Here we discuss the most substantial part of Theorem D, namely the polynomial upper bound on the Dehn function (Theorem D.4). An essential and now classical feature we use is *Gromov's trick*. Let X be a simply connected geodesic space, and let $\mathcal{F}$ be a family of quasi-geodesic paths joining all pairs of points in X. To show a (superlinear) upper bound on the Dehn function, Gromov's observation is that it is enough to consider special loops obtained by concatenating a large but a priori bounded number of paths in $\mathcal{F}$.

In our (combinatorial) setting, we can summarize Gromov's trick by saying that in order to prove upper bounds on the Dehn function of a standard solvable group $G = U \rtimes A$, it is enough to estimate, for some fixed c, the area of words of the form

$$\prod_{i=1}^{c} g_i s_i g_i^{-1},$$

where s_i are bounded elements of U and g_i are words inside A. In [CT10] we use it for metabelian groups. The following result generalizes [Gro93, 5.A_9], [Dru04, Theorem 1.1 (2)], [LP04] and [CT10].

Theorem F. *Let $G = U \rtimes A$ be a standard solvable group. Suppose that every closed subgroup of G containing A (thus of the form $V \rtimes A$ with V a closed A-invariant subgroup of U) does not satisfy the SOL obstruction. Then G has an at most quadratic Dehn function.*

The main result of [CT10] is essentially the case when U is abelian (but on the other hand works in arbitrary characteristic). Theorem F is a particular instance of the much more general Theorem E, but is considerably easier: the material is the length estimates of the beginning of Section 4 and Gromov's trick. A direct proof of Theorem F is given in §4.D.

In [Gro93, 5.B_4'], quoth Gromov, *"We conclude our discussion on lower and upper bounds for filling area by a somewhat pessimistic note. The present methods lead to satisfactory results only in a few special cases even in the friendly geometric surroundings of solvable and nilpotent groups."*

Indeed, for standard solvable groups without the SOL obstruction, it seems that Theorem F is the best result that can be gotten without bringing forward new ideas. The first example of a standard solvable group without the SOL obstruction but not covered by Theorem F is Abels's group $A_4(\mathbf{K})$ (see the previous subsection). In this particular example, the authors obtain a quadratic upper bound for the Dehn function in [CT12]. In this case, the group is tractable enough to work with explicit matrices, but such a pedestrian approach becomes hopelessly intricate in an arbitrary group as in Theorem D.4.

Let us now describe the main ideas that underly the proof of Theorem D.4.

General picture: the multiamalgam. A central idea is to use non-positively curved subgroups (called *tame subgroups* in the sequel). This vaguely stated, it is also essential in Gromov's approach. It was previously used in Abels' work on compact presentability of p-adic groups [Ab87]. Abels considers a certain abstract group, obtained by amalgamating the tame subgroups over their intersections. Our combinatorial approach of the Dehn function allows us to take advantage of the consideration of this "multiamalgam" $\hat{G}$. One similarly defines a multiamalgam of the tame Lie algebras, denoted by $\hat{\mathfrak{g}}$.

The strategy. To simplify the discussion, let us assume that G is standard solvable over a single nondiscrete locally compact field of characteristic zero $\mathbf{K}$. Roughly speaking, the strategy is as follows. First, one would like to prove that when none of the obstructions are fulfilled, we have $\hat{G} = G$ (unfortunately, this is not exactly true as we will see below). Second, we need to be able to decompose any combinatorial loop into *boundedly many* loops corresponding to relations in the tame subgroups. It turns out that both steps are quite challenging. While Abels' work provides substantial material to tackle the first step, we had to introduce completely new ideas to solve the second one.

The first step: giving a compact presentation for G. Under the assumption that the group G does not satisfy the SOL obstruction, it follows from a theorem of Abels that the multiamalgam is a central extension of G. More precisely, in the standard solvable case, we have $\hat{G} = \hat{U} \rtimes A$, where $\hat{U}$ is a central extension in degree 0 of U. At first sight, it seems that the condition $H_2(\mathfrak{u})_0 = 0$ should be enough to ensure that $\hat{G} = G$. However, it turns out that in general, $\hat{U}$ is a "wild" central extension, in the sense that it does not carry any locally compact topology such that the projection onto U is continuous. This strange phenomenon is easier to describe at the level of the Lie algebras. There, we have that $\hat{\mathfrak{g}} = \hat{\mathfrak{u}} \rtimes \mathfrak{a}$, where $\hat{\mathfrak{u}}$ is a central extension in degree 0 of $\mathfrak{u}$, seen as *Lie algebras over* $\mathbf{Q}$. Now, if in the last statement, we could replace Lie algebras over $\mathbf{Q}$ by Lie algebras over $\mathbf{K}$, then clearly $H_2(\mathfrak{u})_0 = 0$ would imply that $\hat{\mathfrak{u}} = \mathfrak{u}$. In Section 5, we prove that this happens if and only if the natural morphism $H_2^{\mathbf{Q}}(\mathfrak{u})_0 \to H_2(\mathfrak{u})_0$ is an isomorphism, if and only if the module $\mathrm{Kill}(\mathfrak{u})_0$ (see Subsection 1.4) vanishes. In fact, there are relatively simple examples, already pointed out by Abels where $\mathrm{Kill}(\mathfrak{u})_0$ *does not* vanish (see the previous subsection). As a consequence, even when none of the obstructions hold, we need to complete the presentation of G with a family of so-called *welding relations*. At the Lie algebra level, these relations encodes $\mathbf{K}$-bilinearity of the Lie bracket.

The second step: reduction to special relations. The second main step is to reduce the estimation of area of arbitrary relations to that of relations of a special form (e.g., relations inside a tame subgroup). This idea amounts to Gromov and is instrumental in Young's approach for nilpotent groups [Y06] and for $\mathrm{SL}_{d\ge 5}(\mathbf{Z})$ [Y13]; it is also used in [CT10, CT12]. The main difference in this paper is that we have to perform such an approach without going into explicit calculations (which would be extremely complicated for an arbitrary standard solvable group, since the unipotent group U is essentially arbitrary). Our trick to avoid calculations is to use a presentation of U that is stable under "extensions of scalars". Let us be more explicit, and write $U = \mathbb{U}(\mathbf{K})$, so that $\mathbb{U}(\mathsf{A})$ makes sense for any commutative $\mathbf{K}$-algebra A. We actually provide a presentation, based on Abels' multiamalgam and welding relations, of $\mathbb{U}(\mathsf{A})$ for any $\mathbf{K}$-algebra A. When applying it to a suitable algebra A of functions of at most polynomial growth, we obtain area estimates. This is the core of our argument; it is performed in §4.I (in a particular but representative case) and in §4.K. The presentation itself is established in Sections 5 and 6.

Last step: computation of the area of special relations. For a standard solvable group not satisfying the SOL and 2-homological obstructions, these relations are of two types: those that are contained (as loops) in a tame subgroup and thus have an at most quadratic area; and the more mysterious welding relations. We show that welding relations have an at most cubic area. When the welding relations are superfluous, namely when $\mathrm{Kill}(\mathfrak{u})_0$ vanishes, Theorem 4.L.1 asserts that G then has an at most quadratic Dehn function. A study based on the asymptotic cone, in a paper in preparation by the authors, will show that conversely, in some cases where $\mathrm{Kill}(\mathfrak{u})_0$ does not vanish, the Dehn function of G grows strictly faster than quadratic. We mention this to enhance the important role played by the welding relations in the geometry of these groups. We suspect that they might be relevant as well in the study of the Dehn function of nilpotent Lie groups.

1.7. **Introduction to the Lie algebra chapters.** Although Sections 5 and 6 can be viewed as technical sections when primarily interested in the results about Dehn functions, they should also be considered as self-contained contributions to the theory of graded Lie algebras. Recall that graded Lie algebras form a very rich theory of its own interest, see for instance [Fu, Kac]. Let us therefore introduce these chapters independently. In this context, the Lie algebras are over a given commutative ring A, with no finiteness assumption. This generality is essential in our applications, since we have to consider (finite-dimensional) Lie algebras over an infinite product of fields. We actually consider Lie algebras graded in a given abelian group $\mathcal{W}$, written additively.

1.7.1. *Universal central extensions.* Recall that a Lie algebra is perfect if $g = [g, g]$. It is classical that every perfect Lie algebra admits a universal central extension. We provide a graded version of this fact. Say that a graded Lie algebra is relatively perfect in degree zero if $\mathfrak{g}_0 \subset [\mathfrak{g}, \mathfrak{g}]$ (in other words, $H_1(\mathfrak{g})_0 = \{0\}$). In §5.B, to any graded Lie algebra $\mathfrak{g}$, we canonically associate another graded Lie algebra $\tilde{\mathfrak{g}}$ along with a graded Lie algebra homomorphism $\tau : \tilde{\mathfrak{g}} \to \mathfrak{g}$, which has a central kernel, naturally isomorphic to the 0-component of the 2-homology module $H_2(\mathfrak{g})_0$.

Theorem G. *(Theorem 5.B.4) Let $\mathfrak{g}$ be a graded Lie algebra. If $\mathfrak{g}$ is relatively perfect in degree zero, then the morphism $\tilde{\mathfrak{g}} \stackrel{\tau}{\to} \mathfrak{g}$ is a graded central extension with kernel in degree zero, and is universal among such central extensions.*

An important feature of this result is that it applies to graded Lie algebras that are far from perfect: indeed, in our case, the Lie algebras are even nilpotent.

1.7.2. *Restriction of scalars.* In §5.C, we study the behavior of $H_2(\mathfrak{g})_0$ under restriction of scalars. We therefore consider a homomorphism $\mathsf{A} \to \mathsf{B}$ of commutative rings. If $\mathfrak{g}$ is a Lie algebra over B, then it is also a Lie algebra over A and therefore to avoid ambiguity we denote its 2-homology by $H_2^{\mathsf{A}}(\mathfrak{g})$ and $H_2^{\mathsf{B}}(\mathfrak{g})$ according to the choice of the ground ring. There is a canonical surjective A-module homomorphism $H_2^{\mathsf{A}}(\mathfrak{g}) \to H_2^{\mathsf{B}}(\mathfrak{g})$; we call its kernel the **welding module** and denote it by $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})$. It is a graded A-module, and $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})_0$ is also the kernel of the induced map $H_2^{\mathsf{A}}(\mathfrak{g})_0 \to H_2^{\mathsf{B}}(\mathfrak{g})_0$.

These considerations led us to introduce the *Killing module.* Let $\mathfrak{g}$ be a Lie algebra over the commutative ring B. Consider the homomorphism $\mathcal{T}$ from $\mathfrak{g}^{\otimes 3}$ to the symmetric square $\mathfrak{g} \circledcirc \mathfrak{g}$, defined by

$$\mathcal{T}(u \otimes v \otimes w) = u \circledcirc [v, w] - [u, w] \circledcirc v$$

(all tensor products are over B here). The **Killing module** is by definition the cokernel of $\mathcal{T}$. The terminology is motivated by the observation that for every B-module $\mathfrak{m}$, $\mathrm{Hom}_{\mathsf{B}}(\mathrm{Kill}^{\mathsf{B}}(\mathfrak{g}), \mathfrak{m})$ is in natural bijection with the module of invariant B-bilinear forms $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$. If $\mathfrak{g}$ is graded, then $\mathrm{Kill}(\mathfrak{g})$ is canonically graded as well.

Theorem H. *Let $Q \subset K$ be fields of characteristic zero, such that K has infinite transcendence degree over Q. Let $\mathfrak{g}$ be a finite-dimensional graded Lie algebra over K, relatively perfect in degree zero (i.e., $H_1(\mathfrak{g})_0 = \{0\}$). Then the following are equivalent:*

(i) $W_2^{Q,K}(\mathfrak{g})_0 = \{0\}$;

(ii) $W_2^{Q,\mathsf{R}}(\mathfrak{g} \otimes_K \mathsf{R})_0 = \{0\}$ *for every commutative* K*-algebra* R*;*
(iii) $\mathrm{Kill}^K(\mathfrak{g})_0 = \{0\}$.

In particular, assuming moreover that $H_2(\mathfrak{g})_0 = \{0\}$*, these are also equivalent to:*
(iv) $H_2^Q(\mathfrak{g})_0 = \{0\}$*;*
(v) $H_2^Q(\mathfrak{g} \otimes_K \mathsf{R})_0 = \{0\}$ *for every commutative* K*-algebra* R.

The interest of such a result is that (iii) appears as a checkable criterion for the vanishing of complicated and typically infinite-dimensional object (the welding module). Let us point out that this result follows, in case $\mathfrak{g}$ is defined over Q, from the results of Neeb and Wagemann [NW08]. In our application to Dehn functions (specifically, in the proof of Theorem 4.L.1), we make an essential use of the implication (iii)⇒(v), where $Q = \mathbf{Q}$, $\mathbf{K}$ is a nondiscrete locally compact field, and R is a certain algebra of functions on $\mathbf{K}$. That (iii) implies the other properties actually does not rely on the specific hypotheses (restriction to fields, finite dimension), see Corollary 5.C.8. The converse, namely that the negation of (iii) implies the negation of the other properties, follows from Theorem 5.C.13.

1.7.3. *Abels' multiamalgam.* Section 6 is devoted to the study of Abels's multiamalgam $\hat{\mathfrak{g}}$ and to its connexion with the universal central extension $\tilde{\mathfrak{g}} \to \mathfrak{g}$. Here, we again consider arbitrary Lie algebras over a commutative ring R, but we now assume that the abelian group $\mathcal{W}$ is a *real vector space.* Given a graded Lie algebra, a Lie subalgebra is called **tame** if 0 does not belong to the convex hull of its weights. Abels' multiamalgam $\hat{\mathfrak{g}}$ is the (graded) Lie algebra obtained by amalgamating all tame subalgebras of $\mathfrak{g}$ along their intersections (see 6.C for details); it comes with a natural graded Lie algebra homomorphism $\hat{\mathfrak{g}} \to \mathfrak{g}$. Abels defines a 2-tame nilpotent graded Lie algebra to be such that 0 does not belong to the convex hull of any pair of principal weights (this condition is related to the condition that the SOL-obstruction is not satisfied). A more general notion of 2-tameness, for arbitrary $\mathcal{W}$-graded Lie algebras, is introduced in 6.A. Although Abels works in a specific framework (p-adic fields, finite-dimensional nilpotent Lie algebras), his methods imply with minor changes the following result.

Theorem (essentially due to Abels, see Theorem 6.C.2)**.** *If* $\mathfrak{g}$ *is 2-tame, then* $\hat{\mathfrak{g}} \to \mathfrak{g}$ *is the universal central extension in degree 0. In other words,* $\hat{\mathfrak{g}}$ *is canonically isomorphic to* $\tilde{\mathfrak{g}}$.

This means that in this case, $\hat{\mathfrak{g}}$ is an excellent approximation of $\mathfrak{g}$, the discrepancy being encoded by the central kernel $H_2(\mathfrak{g})_0$.

We actually need the translation of this result in the group-theoretic setting, which involves significant difficulties. Assume now that the ground ring R is a commutative algebra over the field $\mathbf{Q}$ of rationals. Recall that the Baker-Campbell-Hausdorff formula defines an equivalence of categories between nilpotent Lie algebras over $\mathbf{Q}$ and uniquely divisible nilpotent groups. Then $\mathfrak{g}$ is the

Lie algebra of a certain uniquely divisible nilpotent group G, and we can define the multiamalgam $\hat{G}$ of its tame subgroups, i.e. those subgroups corresponding to tame subalgebras of $\mathfrak{g}$. Note that it does *not* follow from abstract nonsense that $\hat{G}$ is controlled in any way by $\mathfrak{g}$, because $\hat{G}$ is defined in the category of groups and not of uniquely divisible nilpotent groups. In a technical tour de force, Abels [Ab87, §4.4] managed to prove that $\hat{G}$ is nilpotent and asked whether the extension $\hat{G} \to G$ is central. The following theorem, which we need for our estimates of Dehn function, answers the latter question positively.

Theorem I (Theorem 6.D.2)**.** *Let $\mathfrak{g}$ be a 2-tame nilpotent graded Lie algebra over a commutative $\mathbf{Q}$-algebra R. If $\mathfrak{g}$ is 2-tame, then $\hat{G}$ is nilpotent and uniquely divisible, and $\hat{G} \to G$ corresponds to $\hat{\mathfrak{g}} \to \mathfrak{g}$ under the equivalence of categories. In particular, the kernel of $\hat{G} \to G$ is central and canonically isomorphic to $H_2^{\mathbf{Q}}(\mathfrak{g})_0$.*

1.8. **Guidelines.** The sections can be read independently, at the following exceptions

- The preliminary Section 2 is used at many places. More specifically, §2.A and §2.B are used throughout the paper (except in the algebraic Sections 5 and 4.E); the grading in a standard solvable group (§2.E) is used in §4 and the Cartan grading (§2.F) is used in §7.
- In Section 6, we use at many times notation introduced in Section 5.

Also, here are the logical connections between the sections:

- All sections possibly refer to Section 2;
- Section 4 makes use of the results of Section 6, which itself makes use of the results of Section 5. However, it is possible to read Section 4 taking for granted the results of the algebraic Sections 5, 6, so we chose to leave it before.
- More locally: in §7.D and §7.E, the proofs make use of facts established in 5.A and 5.B.

1.9. **Acknowledgements.** We are grateful to Christophe Pittet for several fruitful discussions. We thank Cornelia Drutu for discussions and pointing out useful references. We thank Pierre de la Harpe for many useful corrections.

2. Preliminaries

2.A. Asymptotic comparison.

Definition 2.A.1. If f, g are real functions defined on any set where ∞ makes sense (e.g., for a locally compact space, it corresponds to the filter of subsets with relatively compact complement), we say that f is **asymptotically bounded** by g and write $f \preceq g$ if there exists a constant $C \geq 1$ such that for all x close enough to ∞ we have

$$f(x) \leq Cg(x) + C;$$

if $f \preceq g \preceq f$ we write $f \simeq g$ and say that f and g have the same **asymptotic behavior**, or the same $\simeq$-asymptotic behavior.

Definition 2.A.2. If f, g are non-decreasing non-negative real functions defined on $\mathbf{R}_+$ or $\mathbf{N}$, we say that f is bi-asymptotically bounded by g and write $f \preccurlyeq g$ if there exists a constant $C \geq 1$ such that for all x

$$f(x) \leq Cg(Cx) + C;$$

if $f \preccurlyeq g \preccurlyeq f$ we write $f \approx g$, and say that f and g have the same bi-asymptotic behavior, or the same **$\approx$-asymptotic behavior**.

Besides the setting, the essential difference between $\simeq$-asymptotic and $\approx$-asymptotic behavior is the constant at the level of the source set. For instance, $2^n \approx 3^n$ but $2^n \not\simeq 3^n$.

2.B. **Definition of Dehn function.** If G is a group and S any subset, we denote by $|\cdot|_S$ the (possibly infinite) word length in G with respect to S; it takes finite values on the subgroup generated by S.

If H is a group and $R \subset H$, we define the **area** of an element $w \in H$ as the (possibly infinite) word length of w with respect to the union $\bigcup_{h \in H} hRh^{-1}$; we denote it by $\mathrm{area}_R(w)$. It takes finite values on the normal subgroup generated by R.

Now let S be an abstract set and F_S the free group over S, and $\pi : F_S \to G$ a surjective homomorphism, whose kernel K is generated by some subset R as a normal subgroup. (The elements of K are called **null-homotopic** words.) We define the Dehn function

$$\delta_{S,\pi,R}(n) = \max(\lfloor n/2 \rfloor, \sup\{\mathrm{area}_R(w) \ : \ w \in K, \ |w|_S \leq n\}).$$

(The term $n/2$ is not serious, and only avoids some pathologies. Intuitively, it corresponds to the idea that the area of a word $s^n s^{-n}$ should be at least n.) We think of this as the Dehn function of G, but in this general setting, this function as well as its asymptotic behavior can depend on the choice of S, π and R. It can also take infinite values.

Let now G be a compactly generated LC-group (LC-group means locally compact group), and S a compact generating subset. View S as an abstract set and consider the surjective homomorphism $F_S \to G$ which is the identity on S, and K its kernel. Let $R(d)$ be the intersection of K with the d-ball in $(F_S, |\cdot|_S)$. We say that G is **compactly presented** if for some d, the subset $R(d)$ generates K as a normal subgroup. This does not depend on the choice of S (but the value of d can depend). If so, the function $\delta_{S,R}$ (we omit π in the notation since it is determined by S) takes finite values, and its $\approx$-asymptotic behavior does not depend on the choice of S and R. It is then called the **Dehn function** of G. If G is not compactly presented, we say by convention that the Dehn function is infinite. For instance, when we say that a compactly generated LC-group G

has a Dehn function $\succcurlyeq f(n)$, we allow the possibility that G is not compactly presented, i.e. its Dehn function is (eventually) infinite.

The $\approx$-behavior of the Dehn function is a quasi-isometry invariant of the compactly generated locally compact group. This stems from the more general fact that for arbitrary graphs, the property that the graph can be made simply connected by adding 2-cells of with a bounded number of edges, is a quasi-isometry invariant. The argument is the same as the usual one showing that the Dehn function is a quasi-isometry invariant among finitely generated groups, see [Al91] or [BaMS93, Theorem 26].

2.C. **Dehn vs Riemannian Dehn.** If X is a Riemannian manifold, define its filling function $F(r)$ as the supremum of areas of all piecewise smooth disc fillings of Lipschitz loops of length $\le r$.

Proposition 2.C.1. *Let G be a locally compact group with a proper cocompact isometric action on a simply connected Riemannian manifold X. Then G is compactly presented and the Dehn function of G satisfies*

$$\delta(n) \approx \max(F(n), n).$$

(The $\max(\cdot, n)$ is essentially technical: unless X has dimension ≤ 1 or is compact, it can be shown that $F(r)$ grows at least linearly.)

The proof is given, for G discrete, by Bridson [Bri02, Section 5]. Here we only repeat the proof of the easier inequality $F(n) \preceq \delta(n)$, because the proof in [Bri02] makes a serious use of the assumption that G is finitely presented. For the converse inequality $\delta(n) \preceq \max(F(n), n)$, the (highly technical) proof given in [Bri02] uses general arguments of filling in Riemannian manifold and a general cellulation lemma, and the remainder of the proof carries over our more general context.

Lemma 2.C.2. *Let X be a simply connected Riemannian manifold with a cocompact isometric action of a group G. Then Riemannian area of loops of bounded length is bounded.*

Proof. By cocompactness, there exists r_0 such that for every $x \in X$, the exponential is $(1/2, 1)$-bilipschitz from the r_0-ball in T_xX to X. In particular, given a loop of length $\le r_0$, it passes through some point x; its inverse image by the exponential at x has length $\le 2r_0$ and can be filled by a disc of area $\le \pi r_0^2$ in T_xX, and its image by the exponential is a filling of area $\le \pi r_0^2$ in X.

Now fix a positive integer m_0 and $\varepsilon = 1/m_0$ with $6\varepsilon \le r_0$. Consider a compact subset Ω such that $G\Omega = X$. Consider a finite set F such that every point in Ω is ε-close to a point in X, so every point in X is ε-close to a point in $D_\varepsilon = GF$. For all $x, y \in D_\varepsilon$ with $d(x, y) \le 3\varepsilon$, fix a geodesic path $S(x, y)$ from x to y. We can suppose that there are only finitely many such segments up to G-translation.

Consider a 1-Lipschitz loop $f : [0, k] \to X$. Fix $\varepsilon > 0$ with $1/\varepsilon \in \mathbf{Z}$. If $0 \le n \le k\varepsilon^{-1}$. Let x_n be a point in D_ε that is ε-close to $f(n\varepsilon)$. Fix geodesic paths joining x_n and $f(n\varepsilon)$. So there is a homotopy from f to the concatenation of the $S(f(n\varepsilon), f((n+1)\varepsilon))$ by $k\varepsilon^{-1}$ squares of perimeter at most $6\varepsilon \le r_0$. By the above, each of these km_0 squares can be filled with area $\le \pi r_0^2$. The remaining loop is a concatenation of k segments of the form $S(x,y)$ with $d(x,y) \le 3/m_0$. For given k, there are only finitely many such loops up to translation. Since X is simply connected, each of these loops has finite area. So the remaining loop has area $\le a_k$ for some $a_k < \infty$. So we found a filling of the original loop of length $\le a_k + km_0\pi r_0^2$. □

Partial proof of Proposition 2.C.1. Fix a compact symmetric generating set S in G and by R a set of relators. Fix $x_0 \in X$. Set $r = \sup_{s\in S} d(x_0, sx_0)$. If $s \in S$, fix a r-Lipschitz map $j_s : [0,1] \to X$ mapping 0 to x_0 and 1 to sx_0. If $w = s_1 \dots s_k$, define $j_w : [0,k] \to X$ as follows: if $0 \le \ell \le k-1$ and $0 \le t \le 1$, $j_w(\ell+t) = s_1 \dots s_\ell j_{s_{\ell+1}}(t)$. It is doubly defined for an integer, but both definitions coincide. So j_w is r-Lipschitz. If w represents 1 in G, then $j_w(0) = j_w(k)$.

If w is a word in the letters in S and represents the identity, let $A(w)$ be the area of the loop j_w.

By Lemma 2.C.2, $A(w)$ is bounded when w is bounded. Also, it is clear that $A(gwg^{-1}) = A(w)$ for all group words g. This shows that there exists a constant $C > 0$, namely $C = \sup_{r\in R} A(r)$, such that $A(w) \le C\mathrm{area}(w)$ for some constant C.

For some r_0, every point in X is at distance $\le r_0$ of a point in Gx_0. Consider a loop of length k in X, given by a 1-Lipschitz function $u : [0,k] \to X$. For every n (modulo k), let $g_n x_0$ be a point in Gx_0 with $d(g_n x_0, u(n)) \le r_0$. We have $d(g_n x_0, g_{n+1} x_0) \le 2r_0 + 1$. By properness, there exists N (depending only on r_0) such that $g_n^{-1} g_{n+1} \in S^N$. If σ_n is a word of length N representing $g_n^{-1} g_{n+1}$, and $\sigma = \sigma_0 \dots \sigma_{k-1}$, then we pass from u to j_σ by a homotopy consisting of k squares of perimeter $\le 4r_0+2$. By Lemma 2.C.2, there is a bound M_0 on the Riemannian area of such squares. So the Riemannian area of u is bounded by $kM_0 + A(j_\sigma) \le kM_0 + C\mathrm{area}(\sigma) \le kM_0 + \delta_{S,R}(Nk)$. This shows that $\delta_r(k) \le kM_0 + \delta_{S,R}(Nk)$.

For the (more involved) converse inequality, we only give the following sketch: let $\rho > 0$ be such that each point in X is at distance $< \rho/8$ to Gx_0, and assume in addition that $\rho > \rho_\kappa = \frac{\pi}{2\sqrt{\kappa}}$, where κ is an upper bound on the sectional curvature of X. Let S be the set of elements in G such that $d(gx_0, x_0) \le \rho$ and R the set of words in F_S, of length at most 12 and representing 1 in G. Then the proof in [Bri02, §5.2] shows that $\langle S \mid R\rangle$ is a presentation of G with Dehn function $\delta(n)$ bounded above by $4\lambda_\kappa(F(\rho n) + \rho n + 1)$, where $\lambda_\kappa = 1/\min(4\sqrt{\kappa}/\pi, \alpha(r,\kappa))$ and $\alpha(r,\kappa)$ is the area of a disc of radius κ is the standard plane or sphere of constant curvature κ. □

2.D. **Combinatorial lemmas on the Dehn function.** This subsection contains several general lemmas about the Dehn function, which will be used at some precise parts of the paper. The reader can refer to them when necessary.

2.D.1. *Free products.*

Lemma 2.D.1. *Let f be a superadditive function. If (G_i) is a finite family of (abstract) groups, each with a presentation $\langle S_i \mid R_i\rangle$, with Dehn function $\le f$ (e.g., $f(n) = Cn^\alpha$ for $C > 0$, $\alpha \ge 1$). Then the free product H of the G_i has Dehn function $\delta \le f$ with respect to the presentation $\langle \bigsqcup S_i \mid \bigsqcup R_i\rangle$.*

Proof. Let $w = s_1 \dots s_n$ be a null-homotopic word with $n \ge 1$. Because H is a free product, there exists i and $1 \le j \le j+k-1 \le n$ such that every letter s_ℓ for $j \le \ell \le j+k-1$ is in S_i and $s_j \dots s_{j+k-1}$ represents the identity in G_i. So, with the corresponding cost, which is $\le f(k)$, we can simplify w to the null-homotopic word $s_1 \dots s_{j-1}s_k \dots s_n$. Thus $\delta(n) \le f(k) + \delta(n-k)$ (with $1 \le k \le n$). Using the property that f is superadditive, we can thus prove by induction on n that $\delta(n) \le f(n)$ for all n. (This argument is used in [GS99] for finitely generated groups.) ☐

2.D.2. *Conjugating elements.*

Lemma 2.D.2. *Let $\langle S \mid R\rangle$ be a group presentation, and r a bound on the length of the words in R. There exists C such that for every null-homotopic $w \in F_S$, with length n and area α, we can write, in F_S, $w = \prod_{i=1}^{\alpha} g_i r_i g_i^{-1}$ with $r_i \in R^{\pm 1} \cup \{1\}$ and $g_i \in F_S$, with the additional condition $|g_i|_S \le n + r\alpha$.*

Proof. We start with the following claim: consider a connected polygonal planar complex, with n vertices on the boundary (including multiplicities); suppose that the number of polygons of at most r edges is α. Fix a base-vertex. Then the distance in the one-skeleton of the base-vertex to any other vertex is $\le n + r\alpha$. Indeed, pick an injective path: it meets at most n boundary vertices. Other vertices belong to some face, but each face can be met at most r times. So the claim is proved.

Now a van Kampen diagram for a null-homotopic word of size n and area α with relators of length $\le r$ satisfies these assumptions, the distance from the identity to some vertex corresponds to the length to the conjugating element that comes into play. Thus the g_i can be chosen with $|g_i|_S \le n + r\alpha$. ☐

2.D.3. *Gromov's trick.*

Definition 2.D.3. Let F_S be the free group over an abstract set S, let G be an arbitrary group, and let $\pi : F_S \to G$ be a surjective homomorphism. We call (linear) **combing** of (G, π) (or, informally, of G if π is implicit), a subset $\mathcal{F} \subset F_S$ such that $1 \in \mathcal{F}$ and for some integer $k \ge 1$ and some constant $C > 0$, we have the property that for every $g \in G$, there exist $w_1, \dots, w_k$ in $\mathcal{F}$ with $|w_i|_S \le C|x|$ and $\pi(w_1, \dots w_k) = g$. If we need specify k, we call it a k-combing.

Remark 2.D.4. We assume neither that $\pi_{|\mathcal{F}}$ is surjective, nor injective.

Example 2.D.5. Let T be any generating subset of a finitely generated abelian group A. Suppose that $\{t_1,\dots,t_\ell\}\subset T$ is also a generating subset. Then the set of words $\{\prod_{j=1}^{\ell} t_j^{m_j}\}$ where (m_j) ranges over $\mathbf{Z}^\ell$, is a 1-combing of $F_T\to A$. If every element of T is equal or inverse to some t_i, the incurring constant C can be taken equal to 1.

The following is established in [CT10, Proposition 4.3][1].

Theorem 2.D.6 (Gromov's trick)**.** *Let S be an abstract set S, let G be an arbitrary group, and let $\pi: F_S\to G$ be a surjective homomorphism. Let $R\subset F_S$ be a subset contained in* $\mathrm{Ker}(\pi)$*. Consider a k-combing $\mathcal{F}\subset F_S$.*

Fix $c\ge 3k$. Assume that the area with respect to R of any $w\in \mathrm{Ker}(\pi)$ of the form $w=w_1\dots w_c$ of length $\le n$ with $w_i\in\mathcal{F}$ is $\le C_1n^\alpha$ for some $\alpha>1$ and $C_1>0$. Then $\langle S\mid \mathbf{R}\rangle$ is a presentation of G (i.e., R generates $\mathrm{Ker}(\pi)$ *as a normal subgroup) and for some $C_2>0$, the Dehn function of G with respect to R is $\le C_2n^\alpha$.* ☐

It will be applied in the more specific form.

Proposition 2.D.7. *Consider a group $G=U\rtimes A$, with a symmetric generating subset $S=S_U\cup T$, with $S_U\subset U$ and $T\subset A$. Assume that, for some q, there is a combing $\mathcal{F}\subset F_S$, in which every element in $\mathcal{F}$ contains at most q letters in S_U. Let $R,R'\subset F_S$ be sets of null-homotopic words; assume that $R\subset F_T$ and that the projection of R' in F_T is trivial. Assume that, for constants C_1,C_2 and $\alpha_1,\alpha_2>1$ we have*

- *$\langle T,R\rangle$ is a presentation of A with Dehn function $\le C_1n^{\alpha_1}$;*
- *for every c, there exists C_2 such that every null-homotopic of the form $\prod_{i=1}^{c} d_is_id_i^{-1}$ with $s_i\in S$, $d_i\in F_T$ and $|d_i|_T\le n$ has area $\le C_2n^{\alpha_2}$ with respect to $R\cup R'$.*

Then $\langle S\mid R\cup R'\rangle$ is a presentation of G with Dehn function $\preceq n^\alpha$, where $\alpha=$ $\max(\alpha_1,\alpha_2)$.

Proof. Consider a null-homotopic word of the form $w=w_1\dots w_c$ with $w_i\in\mathcal{F}$. So w has at most qc occurrences of letters in S_U, so it can be written as a product $\prod_{j=1}^{qc+1} s_it_i$, where s_i is a letter in S_U and t_i is a word on the letters in T (and we can assume $t_{qc+1}=1$), so that $\sum_i|t_i|_T\le n$. Write $\tau_i=t_1\dots t_{i-1}$ and $\tau=\tau_{qc+2}$. Then

$$w=\left(\prod_{i=1}^{qc+1}\tau_is_i\tau_i^{-1}\right)\tau.$$

[1][CT10, Proposition 4.3] is awkwardly stated because it purportedly considers a locally compact group without specifying a presentation and gives a conclusion on its Dehn function as a function (and not an asymptotic type of function). Actually the local compactness assumption is irrelevant and the correct statement is the one given here, the proof given in [CT10] applying without modification. The setting is just that of a group presentation; it is even not necessary to assume that the relators have bounded length.

So both $\prod_{i=1}^{qc+1} \tau_i s_i \tau_i^{-1}$ and τ represent 1 in G. Each τ_i has length $\leq qcn$, so it follows from the assumptions that w has area at most $C_1(qcn)^{\alpha_1} + C_2(qcn)^{\alpha_2} \leq Cn^\alpha$, where $C = C_1(qc)^{n_1} + C_2(qc)^{\alpha_2}$. By Theorem 2.D.6, it follows that $\langle S \mid R \cup R' \rangle$ is a presentation of G with Dehn function $\preceq n^\alpha$. $\square$

2.E. **Grading in a normed field.**

2.E.1. *Grading in a representation.*

Lemma 2.E.1. *Let $H \subset G$ be an inclusion of finite index between nilpotent groups. Then any homomorphism $f : H \to \mathbf{R}$ has a unique extension $\tilde{f} : G \to \mathbf{R}$.*

Proof. If $g \in G$ and $g^k \in H$, the element $f(g^k)/k$ does not depend on k, we define it as $\tilde{f}(g)$. We need to check that $\tilde{f}$ is multiplicative. Since this only depends on two elements, we can suppose that G and H are finitely generated. We can also suppose that they are torsion-free, as the problem is not modified if we mod out by the finite torsion subgroups. So G and H have the same Malcev closure, and every homomorphism $H \to \mathbf{R}$ extends to the Malcev closure. Necessarily, the extension is equal to $\tilde{f}$ in restriction to G, so $\tilde{f}$ is a homomorphism. (Note that $\mathbf{R}$ could be replaced in the lemma by any torsion-free divisible nilpotent group.) $\square$

Recall that for every complete normed field $\mathbf{K}$, then the norm on $\mathbf{K}$ extends to every finite extension field in a unique way [DwGS, Theorem 5.1, p. 17].

Proposition 2.E.2. *Let $\mathbf{K}$ be a non-discrete complete normed field. Let N be a topological nilpotent group and V a finite-dimensional vector space with a continuous linear N-action $\rho : N \to \mathrm{GL}(V)$. Then there is a canonical decomposition*

$$V = \bigoplus_{\alpha \in \mathrm{Hom}(N, \mathbf{R})} V_\alpha,$$

where, for $\alpha \in \mathrm{Hom}(N, \mathbf{R})$, the subspace V_α is the sum of characteristic subspaces associated to irreducible polynomials whose roots have modulus $e^{\alpha(\omega)}$ for all $\omega \in N$; moreover we have, for $\alpha \in \mathrm{Hom}(N, \mathbf{R})$

$$\begin{aligned} V_\alpha =& \{0\} \cup \left\{ v \in V \smallsetminus \{0\} : \forall \omega \in N,\ \lim_{n \to +\infty} \|\rho(\omega)^n \cdot v\|^{1/n} = e^{\alpha(\omega)} \right\} \\ =& \left\{ v \in V : \forall \omega \in N,\ \overline{\lim_{n \to +\infty}} \|\rho(\omega)^n \cdot v\|^{1/n} \leq e^{\alpha(\omega)} \right\} \end{aligned}$$

Note that we do not assume that $\rho(N)$ has a Zariski-connected closure.

Proof. Let us begin with the case when $\mathbf{K}$ is algebraically closed. Let $\mathbb{N}$ be the Zariski closure of $\rho(N)$; decompose its identity component $\mathbb{N}^0 = \mathbb{D} \times \mathbb{U}$ into diagonalizable and unipotent parts. Consider the corresponding projections d and u into $\mathbb{D}$ and $\mathbb{U}$. Define $N^0 = \rho^{-1}(\mathbb{N}^0)$; it is an open subgroup of finite index in N. Let D be the (ordinary) closure of the projection $d(\rho(N^0))$. We can decompose, with respect to D, the space V into weight subspaces: $V = \bigoplus_{\gamma\in\mathrm{Hom}(D,\mathbf{K}^*)} V_\gamma$, where $V_\gamma = \{v \in V : \forall d \in D, d \cdot v = \gamma(d)v\}$. Write $(\log|\gamma|)(v) = \log(|\gamma(v)|)$, so $\log|\gamma| \in \mathrm{Hom}(D,\mathbf{R})$. For $\delta \in \mathrm{Hom}(D,\mathbf{R})$, define $V_\delta = \bigoplus_{\{\gamma:\, \log|\gamma|=\delta\}} V_\gamma$, so that $V = \bigoplus_{\gamma\in\mathrm{Hom}(D,\mathbf{R})} V_\delta$. If $\delta \in \mathrm{Hom}(D,\mathbf{R})$, then $\delta\circ d\circ\rho \in \mathrm{Hom}(N^0,\mathbf{R})$. So by Lemma 2.E.1, it uniquely extends to a homomorphism $\hat\delta : N \to \mathbf{R}$, which is continuous because its restriction $\delta\circ d\circ\rho$ to N^0 is continuous. Note that $\delta \mapsto \hat\delta$ is obviously injective.

If $v \in V_\delta \smallsetminus \{0\}$ and $\omega \in N^0$, write it as a sum $v = \sum_{\gamma\in I} v_\gamma$ where $0 \neq v_\gamma \in V_\gamma$, where I is a non-empty finite subset of $\mathrm{Hom}(D,\mathbf{K}^*)$, actually consisting of elements γ for which $\log|\gamma| = \delta$. Then, changing the norm if necessary so that the norm is the supremum norm with respect to the norms on the V_γ (which does not affect the limits because of the exponent $1/n$), we have

$$\begin{aligned}\|\rho(\omega)^n v\|^{1/n} &= \sup_{\gamma\in I} \|u(\rho(\omega))^n d(\rho(\omega))^n v_\gamma\|^{1/n}\\ &= |\gamma(d(\rho(\omega)))| \sup_{\gamma\in I} \|u(\rho(\omega))^n v_\gamma\|^{1/n}\\ &= \exp(\hat\delta(\omega)) \sup_{\gamma\in I} \|u(\rho(\omega))^n v_\gamma\|^{1/n};\end{aligned}$$

the spectral radii of both $u(\rho(\omega))$ and its inverse being equal to 1 and $v_\gamma \neq 0$, we deduce that $\lim \|u(\rho(\omega))^n v_\gamma\|^{1/n} = 1$. It follows that $\lim \|\rho(\omega)^n v\|^{1/n} = \exp(\hat\delta(\omega))$ for all $\omega \in N^0$ and $v \in V_\delta \smallsetminus \{0\}$. If $\omega \in N$, there exists k such that $\omega^k \in N^0$. Writing $f_\omega(n) = \|\rho(\omega)^n v\|^{1/n}$, we therefore have

$$\lim_{n\to\infty} f_\omega(kn) = \lim_{n\to\infty} f_{\omega^k}(n)^{1/k} = \exp(\hat\delta(\omega^k))^{1/k} = \exp(\hat\delta(\omega));$$

on the other hand a simple verification shows that $\lim_{n\to\infty} f(n+1)/f(n) = 1$, and it follows that $\lim_{n\to\infty} f_\omega(n) = \exp(\hat\delta(\omega))$.

It follows in particular that the spectral radius of $\rho(\omega)^{\pm1}$ on V_δ is $\exp(\pm\hat\delta(\omega))$ and since the $\hat\delta$ are distinct, it follows that V_δ is the sum of common characteristic subspaces associated to eigenvalues of modulus $\exp(\hat\delta(\omega))$ for all ω.

Conversely, suppose that $v \in V$ and that there exists $\alpha \in \mathrm{Hom}(N,\mathbf{R})$ such that for all $\omega \in N$ we have $\overline{\lim}_{n\to+\infty} \|\rho(\omega)^n \cdot v\|^{1/n} \le e^{\alpha(\omega)}$, and let us check that $v \in \bigcup V_\delta$. Observe that $\overline{\lim}_{n\to+\infty} (\|\rho(\omega)^n \cdot v\|\, \|\rho(\omega^{-1})^n \cdot v\|)^{1/n} \le 1$. Write $v = \sum_{\delta\in J} v_\delta$ with $v_\delta \in V_\delta$ and suppose by contradiction that J contains two distinct elements δ_1, δ_2.

So $\hat{\delta}_1 \neq \hat{\delta}_2$ and there exist $\omega_0 \in N$ such that $\hat{\delta}_1(\omega) > \hat{\delta}_2(\omega)$. Then

$$\begin{aligned} 1 &\geq \varlimsup_{n\to+\infty} (\|\rho(\omega_0)^n \cdot v\| \, \|\rho(\omega_0^{-1})^n \cdot v\|)^{1/n} \\ &\geq \varlimsup_{n\to+\infty} \|\rho(\omega_0)^n \cdot v_{\delta_1}\|^{1/n} \|\rho(\omega_0^{-1})^n \cdot v_{\delta_2}\|^{1/n} \\ &= \exp(\hat{\delta}_1(\omega_0) - \hat{\delta}_2(\omega_0)) > 1, \end{aligned}$$

a contradiction; thus $v \in \bigcup V_\delta$. This concludes the proof in the algebraically closed case.

Now let $\mathbf{K}$ be arbitrary. The above decomposition can be done in an algebraically closure of $\mathbf{K}$, and is defined on a finite extension $\mathbf{L}$ of $\mathbf{K}$, so $V \otimes_{\mathbf{K}} \mathbf{L} = \bigoplus_{\alpha \in \mathrm{Hom}(N,\mathbf{R})} W_\alpha$, where W_α satisfies all the characterizations. Decompose $\mathbf{K}$-linearly $\mathbf{L}$ as $\mathbf{K} \oplus M$. Let V_α be the projection of W_α on V under this decomposition; clearly $\sum_\alpha V_\alpha = V$. The corresponding direct sum decomposition $V \otimes_{\mathbf{K}} \mathbf{L} = V \oplus (V \otimes_{\mathbf{K}} M)$ is preserved by the action of $\rho(N)$. It follows that elements of V_α satisfy the third property, namely for all $\omega \in N$ and $v \in V_\alpha$ we have $\varlimsup_{n\to+\infty} \|\rho(\omega)^n \cdot v\|^{1/n} \leq e^{\alpha(\omega)}$. Thus $V_\alpha \subset W_\alpha$ and hence $V = \bigoplus V_\alpha$. Elements not in $\bigcup V_\alpha$ satisfy the required properties since this is already true in $V \otimes_{\mathbf{K}} \mathbf{L}$. So the proof is complete.

Note that the proof, as a byproduct, characterizes the elements v of $\bigcup V_\alpha$ as those in V for which, for all $\omega \in N$, we have $\varlimsup_{n\to+\infty} (\|\rho(\omega)^n \cdot v\| \, \|\rho(\omega^{-1})^n \cdot v\|)^{1/n} \leq 1$ (which is actually a limit, and equal to 1, if $v \neq 0$). ☐

2.E.2. *Grading in a standard solvable group.* Let $G = U \rtimes A$ be a standard solvable group in the sense of Definition 1.2. It will be convenient to define $\mathbf{K} = \prod_{j=1}^{\tau} \mathbf{K}_j$; it is endowed with the supremum norm, and view U as $\mathbf{U}(\mathbf{K})$. We thus call G a standard solvable group over $\mathbf{K}$. In a first reading, the reader can assume there is a single field $\mathbf{K} = \mathbf{K}_1$.

Since $\mathbf{K}$ is a finite product of fields, a finite length $\mathbf{K}$-module is the same as a direct sum $V = \bigoplus V_j$, where V_j is a finite-dimensional $\mathbf{K}_j$-vector space. The length of V as a $\mathbf{K}$-module, is equal to $\sum_j \dim_{\mathbf{K}_j} V_j$.

Let $\mathfrak{u}_j$ be the Lie algebra of U_j. So $\mathfrak{u} = \prod_j \mathfrak{u}_j$ is a Lie algebra over $\mathbf{K}$ and the exponential map, which is truncated by nilpotency, is a homeomorphism $\mathfrak{u} \to U$. This conjugates the action of D on U to a linear action on $\mathfrak{u}$, preserving the Lie algebra structure; for convenience we denote it as an action by conjugation.

We endow $\mathfrak{u}_j$ with the action of A, and with the grading in $\mathrm{Hom}(A, \mathbf{R})$, as introduced in Proposition 2.E.2. Thus $\mathfrak{u}$ itself is graded by $\mathfrak{u}_\alpha = \bigoplus_j \mathfrak{u}_{j,\alpha}$. The finite-dimensional vector space $\mathcal{W} = \mathrm{Hom}(A, \mathbf{R})$ is called the **weight space**. This is a Lie algebra grading:

$$[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha+\beta}, \quad \forall \alpha, \beta.$$

Note that since A is a compactly generated locally compact abelian group, it is isomorphic to $\mathbf{R}^{d_1} \times \mathbf{Z}^{d_2} \times K$ for some integers d_1, d_2, and K a compact abelian group. In particular, if $d = d_1 + d_2$, then the weight space $\mathcal{W} = \mathrm{Hom}(A, \mathbf{R})$ is a d-dimensional real vector space.

If we split $\mathfrak{u}$ as the direct sum $\mathfrak{u} = \mathfrak{u}_{\mathrm{a}} \oplus \mathfrak{u}_{\mathrm{na}}$ of its Archimedean and non-Archimedean parts, the weights of $\mathfrak{u}_{\mathrm{a}}$ and $\mathfrak{u}_{\mathrm{na}}$, respectively, are called **Archimedean weights** and **non-Archimedean weights**.

Example 2.E.3. Assume that the action of A on $\mathfrak{u}$ is diagonalizable. If $\mathbf{K}_j = \mathbf{R}$ and the diagonal entries are positive, we have

$$(\mathfrak{u}_j)_\alpha = \{x \in \mathfrak{u}_j : \forall v \in A,\ v^{-1}xv = e^{\alpha(v)}x\}$$

and if $\mathbf{K}_j = \mathbf{Q}_p$ and the diagonal entries are powers of p, we have

$$(\mathfrak{u}_j)_\alpha = \{x \in \mathfrak{u}_j : \forall v \in A,\ v^{-1}xv = p^{-\alpha(v)/\log(p)}x\}.$$

Example 2.E.4 (Weights in groups of SOL type)**.** Let $G = (\mathbf{K}_1 \times \mathbf{K}_2) \rtimes A$ be a group of SOL type as in Definition 1.3, where A contains as a cocompact subgroup the cyclic subgroup generated by some element (t_1, t_2) with $|t_1| > 1 > |t_2|$. Then the weight space is a one-dimensional real vector space, and with a suitable normalization, the weights are $\alpha_1 = \log(|t_1|) > 0$ and $\alpha_2 = \log(|t_2|) < 0$, and $U_{\alpha_i} = \mathbf{K}_i$. It is useful to think of the weight α_i with multiplicity q_i, namely the dimension of $\mathbf{K}_i$ over the closure of $\mathbf{Q}$ in $\mathbf{K}_i$ (which is isomorphic to $\mathbf{R}$ or $\mathbf{Q}_p$ for some p). In particular, G is unimodular if and only if $q_1\alpha_1 + q_2\alpha_2 = 0$.

For instance, if $G = (\mathbf{R} \times \mathbf{Q}_p) \rtimes_p \mathbf{Z}$, then $\mathfrak{u} = \mathbf{R} \times \mathbf{Q}_p$, $\mathfrak{u}_{\log(p)} = \mathbf{R} \times \{0\}$, $\mathfrak{u}_{-\log(p)} = \{0\} \times \mathbf{Q}_p$.

For an arbitrary standard solvable group, we define the set of **weights**

$$\mathcal{W}_{\mathfrak{u}} = \{\alpha : \mathfrak{u}_\alpha \neq \{0\}\} \subset \mathrm{Hom}(A, \mathbf{R}).$$

It is finite. Weights of the abelianization $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ are called **principal weights** of $\mathfrak{u}$.

Lemma 2.E.5. *Every weight of* $\mathfrak{u}$ *is a sum of* ≥ 1 *principal weights; 0 is not a principal weight.*

Proof. The first statement is a generality about nilpotent graded Lie algebras. If P is the set of principal weights and $\mathfrak{v} = \bigoplus_{\alpha \in P} \mathfrak{u}_\alpha$, then $\mathfrak{u} = \mathfrak{v} + [\mathfrak{u}, \mathfrak{u}]$, i.e., $\mathfrak{v}$ generates $\mathfrak{u}$ modulo the derived subalgebra. A general fact about nilpotent Lie algebras (see Lemma 2.G.13 for a refinement of this) then implies that $\mathfrak{v}$ generates $\mathfrak{u}$. The first assertion follows.

The condition that 0 is not a principal weight is a restatement of Definition 1.2(3). ☐

Definition 2.E.6. If U is a locally compact group and v a topological automorphism, v is called a **compaction** if there exists a compact subset $\Omega \subset U$ that is a **vacuum subset** for v, in the sense that for every compact subset $K \subset U$ there exists $n \geq 0$ such that $v^n(K) \subset \Omega$. If every neighborhood of 1 is a vacuum subset, we say that v is a **contraction**.

Proposition 2.E.7. *Let $U \rtimes A$ be a standard solvable group. Equivalences:*
(i) some element of A acts as a compaction of U;
(ii) some element of A acts as a contraction of U;
(iii) 0 is not in the convex hull in $\mathcal{W}$ of the set of weights;
(iv) 0 is not in the convex hull in $\mathcal{W}$ of the set of principal weights.

Proof. Automorphisms of U are conjugate, through the exponential, to linear automorphisms; in particular, contractions and compactions coincide, being characterized by the condition that all eigenvalues have modulus < 1. Thus (i)$\Leftrightarrow$(ii).

By Lemma 2.E.5, weights are sums of principal weights, and thus (iii)$\Leftrightarrow$(iv).

If $v \in A$ acts as a contraction of U, then $\alpha(v) < 0$ for every weight α. Thus $\alpha \mapsto \alpha(v)$ is a linear form on $\mathcal{W}$, which is positive on all weights. Thus (ii)$\Rightarrow$(iii).

Conversely, let L be the set of linear forms ℓ of $\mathcal{W} = \mathrm{Hom}(A, \mathbf{R})$ such that $\ell(\alpha) > 0$ for every weight $\alpha \in \mathcal{W}_{\mathfrak{u}}$. Suppose that 0 is not in the convex hull of $\mathcal{W}_{\mathfrak{u}}$, or equivalently that $L \neq \emptyset$. Since L is an open convex cone, it has non-empty intersection with the image of A in the bidual $\mathcal{W}^*$, since the latter is cocompact. Thus there exists $v \in A$ such that $\alpha(v) > 0$ for every weight α. So (iii)$\Rightarrow$(ii) holds. □

Definition 2.E.8. We say that the standard solvable group $G = U \rtimes A$ (or the graded Lie algebra $\mathfrak{u}$) is

- **tame** if 0 is not in the convex hull of the set of weights;
- **2-tame** if 0 is not in the segment joining any pair of *principal* weights;
- **stably 2-tame** if 0 is not in the segment joining any pair of weights.

The definition of 2-tameness, due to Abels (in the context of p-adic groups) will be motivated on the one hand by Proposition 2.E.11, and on the other hand by Theorem 4.G.1.

Remark 2.E.9. Clearly

$$\text{tame} \;\Rightarrow\; \text{stably 2-tame} \;\Rightarrow\; \text{2-tame}\,;$$

the converse implication does not hold in general; however they hold when $\mathcal{W}$ is 1-dimensional, i.e. when A has a discrete cocompact infinite cyclic subgroup.

Also, when $\mathfrak{u}$ is abelian, then 2-tame and stably 2-tame are obviously equivalent.

The terminology is justified by the following lemma.

Lemma 2.E.10. *The standard solvable group $G = U \rtimes A$ is stably 2-tame if and only if every A-invariant $\mathbf{K}$-subalgebra of $\mathfrak{u}$ is 2-tame.*

Proof. Suppose that G is stably 2-tame. Every D-invariant $\mathbf{K}$-subalgebra of $\mathfrak{u}$ is a graded subalgebra and stable 2-tameness is clearly inherited by graded subalgebras.

Conversely, suppose that G is not stably 2-tame; let α, β be weights with $0 \in [\alpha, \beta]$. Let $\mathfrak{h}$ be the subalgebra generated by $\mathfrak{g}_\alpha + \mathfrak{g}_\beta$; it is a graded subalgebra and is not tame; all of whose weights lie on a single line; as observed in Remark 2.E.9, it follows that $\mathfrak{h}$ is not 2-tame. So $\mathfrak{u}$ does not satisfy the condition. □

The following characterization of 2-tameness will be needed to show that if G is not 2-tame then it has an at least exponential Dehn function.

Proposition 2.E.11. *Let $G = U \rtimes A$ be a standard solvable group. Then G is not 2-tame if and only if there exists a group $V \rtimes E$ of type SOL (see Definition 1.3) and a homomorphism f into $V \rtimes E$ whose image contains V and is dense.*

Proof. First recall from Example 2.E.4 that for a group of SOL type, the set of weights consists of a **quasi-opposite pair**, i.e. a pair of nonzero elements such that the segment joining them contains zero.

Consider a homomorphism $f : U \rtimes A \to V \rtimes E$ to another standard solvable group, in which E acts faithfully on V. It follows that the centralizer of any nontrivial subgroup of V is contained in V. Applying this to $f(A) \cap V$, we see that if $f(A) \cap V \neq 1$, then $f(A) \subset V$ and thus $f(U \rtimes A) \subset V$. So if we assume that $f(U \rtimes A)$ is not contained in V, we deduce that $f(A) \cap V = 1$. Since the derived subgroup of $U \rtimes A$ is U and the derived subgroup of $V \rtimes E$ is V, we have $U \subset f^{-1}(V)$, and hence $f^{-1}(V) = U$.

Consider a homomorphism $f : U \rtimes A \to V \rtimes E$ with V abelian. Assume that the image of f is dense and contains V, and thus contains a dense subgroup of E . Since $f^{-1}(V) = U$, it follows that $f(U) = V$, f is trivial on $[U, U]$ and f induces an homomorphism $A \to E$ between quotients, with dense image. This induces an inclusion of the space of weights $\mathrm{Hom}(E, \mathbf{R}) \to \mathrm{Hom}(A, \mathbf{R})$, sending the weights of $V \rtimes E$ to weights of $U/[U, U] \rtimes A$. In particular, if $V \rtimes E$ is a group of type SOL, then $U/[U, U] \rtimes D$ admits a quasi-opposite pair of weights.

Conversely, suppose that $U \rtimes A$, where $U/[U, U]$ has two nonzero weights α, β with $\beta = t\alpha$ for some $t < 0$. We say that a nonzero weight γ is *discrete* if $\gamma(A) \simeq \mathbf{Z}$, or equivalently is not dense in $\mathbf{R}$.

To construct the map f, first mod out by $[U, U]$. If α (and hence β) is discrete, we argue as follows. Write $\mathfrak{u} = \bigoplus_\gamma \mathfrak{u}_\gamma$. Modding out if necessary by all other nonzero weights, we can suppose that $\mathfrak{u} = \mathfrak{u}_\alpha \oplus \mathfrak{u}_\beta$. Then write $\mathfrak{u}_\alpha = \bigoplus_j \mathfrak{u}_{j,\alpha}$, and mod out all $\mathfrak{u}_{j,\alpha}$ except one, so that $\mathfrak{u}_\alpha$ is a Lie algebra over a single field $\mathbf{K}_j$; do the same for $\mathfrak{u}_\beta$, which is then a Lie algebra over a single field $\mathbf{K}_{j'}$. So by definition of the grading, the action of A on each of $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$ is scalar, therefore modding out by a hyperplane in each, we can suppose that each of $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$ is one-dimensional. Since α and β are proportional, they have the same kernel. We can mod out by this kernel; the resulting group has the form $(\mathbf{K}_j \times \mathbf{K}_{j'}) \rtimes \mathbf{Z}$, which is of type SOL.

If α (and hence β) are non-discrete, observe that by the definition of standard solvable groups $\mathfrak{u}_{j,\alpha} = \{0\}$ for any ultrametric $\mathbf{K}_j$. Therefore, we can argue as in the case of discrete weights, until we obtain a group of the form $\mathbf{R}^2 \rtimes \mathbf{Z}^k$, where $v \in \mathbf{Z}^k$ acts on $\mathbf{R}^2$ by $v \cdot (x, y) = (e^{\alpha(v)}x, e^{-\alpha(v)y})$, α being an homomorphism

$\mathbf{Z}^k \to \mathbf{R}$. This admits an obvious homomorphism into the real group SOL, with dense image containing the normal subgroup $\mathbf{R}^2$. □

Remark 2.E.12. The homomorphism to a group of type SOL cannot always be chosen to have a closed image. For instance, let $\mathbf{Z}^2$ acting on $\mathbf{R}^2$ by

$$(m, n) \cdot (x, y, z) = (2^{-m}3^{-n}x, 2^m 3^n y).$$

Let $G = \mathbf{R}^2 \rtimes \mathbf{Z}^2$ be the corresponding standard solvable group. Clearly, it has two opposite weights and is not 2-tame. On the other hand, since the action of $\mathbf{Z}^2$ on $\mathbf{R}^2$ is faithful, every nontrivial normal subgroup of G intersects $\mathbf{R}^2$ non-trivially. So any homomorphism with cocompact image to a group $V \rtimes E$ of type SOL, whose image contains V, is injective, and therefore induces an injective map $\mathbf{Z}^2 \to E$; in particular, $E \simeq \mathbf{R}$ and the image of $\mathbf{Z}^2$ in $\mathbf{R}$ cannot be closed.

2.F. **Cartan grading and weights.** All Lie algebras in this §2.F are finite-dimensional over a fixed field K of characteristic zero.

Definition 2.F.1. If $\mathfrak{g}$ is a Lie algebra, let $\mathfrak{g}^\infty = \bigcap_k \mathfrak{g}^k$ be the intersection of its descending central series, so that $\mathfrak{g}/\mathfrak{g}^\infty$ is the largest nilpotent quotient of $\mathfrak{g}$.

Definition 2.F.2. If G is a triangulable Lie group with Lie algebra $\mathfrak{g}$, define its **exponential radical** G^∞ as the intersection of its descending central series (so that its Lie algebra is equal to $\mathfrak{g}^\infty$).

We need to recall the notion of Cartan grading of a Lie algebra, which is used in §7.D and §7.E. Let $\mathfrak{n}$ is a nilpotent Lie algebra; denote by $\mathfrak{n}^\vee$ the space of homomorphisms from $\mathfrak{n}$ to K (that is, the linear dual of $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$).

Let $\mathfrak{v}$ be an $\mathfrak{n}$-module (finite-dimensional) with structural map $\rho : \mathfrak{n} \to \mathfrak{gl}(\mathfrak{v})$. If $\alpha \in \mathfrak{n}^\vee$, define the characteristic subspace

$$\mathfrak{v}_\alpha = \bigcup_{k \geq 1} \{v \in \mathfrak{v} : \ \forall g \in \mathfrak{n}, \ (\rho(g) - \alpha(g))^k v = 0\}.$$

The subspaces $\mathfrak{v}_\alpha$ generate their direct sum; we say that $\mathfrak{v}$ is K-**triangulable** if $\mathfrak{v} = \bigoplus_{\alpha \in \mathfrak{n}^\vee} \mathfrak{v}_\alpha$; this is automatic if K is algebraically closed. If $\mathfrak{v}$ is a K-triangulable $\mathfrak{n}$-module, the above decomposition is called the **natural grading** (in $\mathfrak{n}^\vee$) of $\mathfrak{v}$ as an $\mathfrak{n}$-module. If $\mathfrak{v} = \mathfrak{v}_0$, we call $\mathfrak{v}$ a **nilpotent** $\mathfrak{n}$-module.

Definition 2.F.3 ([Bou]). A **Cartan subalgebra** of $\mathfrak{g}$ is a nilpotent subalgebra which is equal to its normalizer.

We use the following proposition, proved in [Bou, Chap. 7,§1,2,3] (see Definition 2.F.1 for the meaning of $\mathfrak{g}^\infty$).

Proposition 2.F.4. *Every Lie algebra admits a Cartan subalgebra. If $\mathfrak{n}$ is a Cartan subalgebra, then $\mathfrak{g} = \mathfrak{n} + \mathfrak{g}^\infty$ and $\mathfrak{n}$ contains the hypercenter of $\mathfrak{g}$ (the union of the ascending central series).*

If $\mathfrak{g}$ is solvable, any two Cartan subalgebras of $\mathfrak{g}$ are conjugate by some elementary automorphism $e^{\mathfrak{ad}(x)}$ with $x \in \mathfrak{g}^\infty$ (here $\mathfrak{ad}(x)$ is a nilpotent endomorphism so its exponential makes sense). □

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ a Cartan subalgebra. Then for the adjoint representation, $\mathfrak{g}$ is an $\mathfrak{n}$-module. If $\mathfrak{g}$ is K-triangulable, as we henceforth assume, then it is K-triangulable as an $\mathfrak{n}$-module. The corresponding natural grading is then called the **Cartan grading** of $\mathfrak{g}$ (relative to the Cartan subalgebra $\mathfrak{n}$); moreover the Cartan grading determines $\mathfrak{n}$, namely $\mathfrak{n} = \mathfrak{g}_0$. We call **weights** the set of α such that $\mathfrak{g}_\alpha \neq 0$. The Cartan grading is a Lie algebra grading, i.e. satisfies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{n}^\vee$. We have $\mathfrak{n} + \mathfrak{g}^\infty = \mathfrak{g}$, and $\mathfrak{g}^\infty \subset [\mathfrak{g}, \mathfrak{g}]$, so the projection $\mathfrak{n} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is surjective, inducing an injection $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^\vee \subset \mathfrak{n}^\vee$. Actually, all weights of the Cartan grading lie inside the subspace $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^\vee$ of $\mathfrak{n}$; the advantage is that this space does not depend on $\mathfrak{n}$, allowing to refer to a weight α without reference to a the choice of a Cartan subalgebra (although the weight space $\mathfrak{g}_\alpha$ still depends on this choice). In view of Proposition 2.F.4, any two Cartan gradings of $\mathfrak{g}$ are conjugate. In particular, the set of weights, viewed as a finite subset of $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^\vee$, does not depend on the Cartan grading. Actually, the subspace linearly spanned by weights is exactly $(\mathfrak{g}/\mathfrak{r})^\vee$, where $\mathfrak{r} \supset [\mathfrak{g}, \mathfrak{g}]$ is the nilpotent radical of $\mathfrak{g}$. The **principal weights** of $\mathfrak{g}$ are by definition the nonzero weights of the Lie algebra $\mathfrak{g}/[\mathfrak{g}^\infty, \mathfrak{g}^\infty]$; note that every nonzero weight is a sum of principal weights, as a consequence of the following lemma.

Lemma 2.F.5. *$\mathfrak{g}^\infty$ is generated by $\mathfrak{g}_\triangledown$ as a Lie subalgebra.*

Proof. It is clear from the definition that $[\mathfrak{n}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ for all $\alpha \neq 0$ and therefore $\mathfrak{g}_\triangledown \subset \mathfrak{g}^\infty$. Conversely, since $\mathfrak{g}_0 = \mathfrak{n}$ is nilpotent, Lemma 5.A.5 implies that $\mathfrak{g}^\infty$ is contained in the subalgebra generated by $\mathfrak{g}_\triangledown$. □

Lemma 2.F.6. *Let $\mathfrak{g}$ be a K-triangulable Lie algebra with a Cartan grading. Then $H_2(\mathfrak{g}^\infty)^\mathfrak{g} \neq \{0\}$ if and only if $H_2(\mathfrak{g}^\infty)_0 \neq \{0\}$.*

Proof. Obviously,
$$H_2(\mathfrak{g}^\infty)^\mathfrak{g} \subset H_2(\mathfrak{g}^\infty)^\mathfrak{n} \subset H_2(\mathfrak{g}^\infty)_0;$$
this provides one implication. Conversely, suppose that $H_2(\mathfrak{g}^\infty)_0 \neq \{0\}$. Since $\mathfrak{g}^\infty$ is nilpotent, the $\mathfrak{g}^\infty$-module $\mathfrak{g}^\infty$ is nilpotent, and therefore so are the subquotients of its exterior powers; in particular, $H_2(\mathfrak{g}^\infty)_0$ is a nilpotent $\mathfrak{g}^\infty$-module. So $\mathfrak{v} = {H_2(\mathfrak{g}^\infty)_0}^{\mathfrak{g}^\infty} \neq \{0\}$. Since $\mathfrak{g}^\infty$ is an ideal, $\mathfrak{v}$ is a $\mathfrak{g}$-submodule of $H_2(\mathfrak{g}^\infty)_0$. Since it is a nonzero nilpotent $\mathfrak{n}$-module, we have $\mathfrak{v}^\mathfrak{n} \neq \{0\}$. Since the action of both $\mathfrak{g}^\infty$ and $\mathfrak{n}$ on $\mathfrak{v}^\mathfrak{n}$ is zero and $\mathfrak{g} = \mathfrak{g}^\infty + \mathfrak{n}$, we have
$$\{0\} \neq \mathfrak{v}^\mathfrak{n} = {H_2(\mathfrak{g}^\infty)_0}^\mathfrak{g} \subset H_2(\mathfrak{g}^\infty)^\mathfrak{g}$$
and the proof is complete. □

Now consider a real triangulable Lie algebra $\mathfrak{g}$, or the corresponding real triangulable Lie group. We call **weights** of $\mathfrak{g}$ the weights of graded Lie subalgebra $\mathfrak{g}^\infty$ endowed with the Cartan grading from $\mathfrak{g}$ (note that since $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}^\infty$, this is precisely the set of weights of the graded Lie algebra $\mathfrak{g}$, except possibly the 0 weight).

We define a real triangulable Lie algebra $\mathfrak{g}$, or the corresponding real triangulable Lie group G, to be **tame**, **2-tame**, or **stably 2-tame** exactly in the same fashion as in Definition 2.E.8. In the case G is also a standard solvable group $U \rtimes D$, we necessarily have $\mathfrak{u} = \mathfrak{g}^\infty$ and the its grading is also the Cartan grading; in particular whether G is tame (resp. 2-tame, stably 2-tame) does not depend on whether G is viewed as a real triangulable Lie group or a standard solvable Lie group.

For $\alpha > 0$, define SOL_α as the semidirect product $\mathbf{R}^2 \rtimes \mathbf{R}$, where the action is given by $t \cdot (x, y) = (e^t x, e^{-\alpha t} y)$. Note that apart from the obvious isomorphisms $\mathrm{SOL}_\alpha \simeq \mathrm{SOL}_{1/\alpha}$, they are pairwise non-isomorphic.

In a way analogous to Proposition 2.E.11, we have

Proposition 2.F.7. *Let G be a real triangulable group. Then G is not 2-tame if and only if it admits SOL_α as a quotient for some $\alpha > 0$.*

Proof. The "if" part can be proved in the same lines as the corresponding statement of Proposition 2.E.11 and is left to the reader.

Conversely, assume that G is not 2-tame. Clearly, G/G^∞ is not 2-tame, so we can assume that G^∞ is abelian. In the same way as in the proof of Proposition 2.E.11, we can pass to a quotient to assume that G^∞ is 2-dimensional. Let $\mathfrak{n}$ be a Cartan subalgebra. Since the weights of $\mathfrak{g}^\infty$ (in the Cartan grading of $\mathfrak{g}$) are nonzero, we have $\mathfrak{g} = \mathfrak{g}^\infty \rtimes \mathfrak{n}$; let $G^\infty \rtimes N$ be the corresponding decomposition of G. Since the action of N on G^∞ is given by two proportional weights, its kernel has codimension 1 in N; in particular this kernel is normal in G; we see that the quotient is necessarily isomorphic to SOL_α for some $\alpha > 0$. $\square$

2.G. **On nilpotent groups and Lie algebras.** We now gather some generalities concerning nilpotent groups and Lie algebras, which will be used in §6.D.

Denote by $(\!(\cdot, \cdot)\!)$ group commutators, namely

$$(\!(x, y)\!) = x^{-1} y^{-1} x y,$$

and iterated group commutators

$$(\!(x_1, \dots, x_n)\!) = (\!(x_1, (\!(x_2, \dots, x_n)\!))\!). \tag{2.G.1}$$

Define similarly iterated Lie algebra brackets.

If G is a group, its *central series* is defined by $G^1 = G$ and $G^{i+1} = (\!(G, G^i)\!)$ (the group generated by commutators $(\!(x, y)\!)$ when (x, y) ranges over $G \times G^i$). The group G is *s-nilpotent* if $G^{s+1} = \{1\}$. In particular, 0-nilpotent means trivial, 1-nilpotent means abelian, and more generally, s-nilpotent means that $(s + 1)$-iterated group commutators vanish in G. Similarly, if $\mathfrak{g}$ is a Lie algebra, its central series is defined in the same way (and does not depend on the ground commutative ring), and s-nilpotency has the same meaning.

Lemma 2.G.2. *Let N be an s-nilpotent group, and let i be an integer. Then there exists $m = m(i,s)$ such that for all $x, y \in N$,*

$$(\!(x, y^i)\!) = w_1 \dots w_m$$

where each w_j $(1 \le j \le m)$ is an iterated commutator (or its inverse) whose letters are $x^{\pm 1}$ or $y^{\pm 1}$, that is, $w_j = (\!(t_{j,1}, \dots, t_{j,k_j})\!)$ for some $k_j \ge 2$ and $t_{j,i} \in \{x^{\pm 1}, y^{\pm 1}\}$.

Example 2.G.3. In a 3-nilpotent group N, we have, for all $x, y \in N$ and $i \in \mathbf{Z}$

$$(\!(x, y^i)\!) = (\!(x, y)\!)^i (\!(y, x, y)\!)^{-i(i-1)/2}.$$

Proof of Lemma 2.G.2. We shall prove the lemma by induction on s. The statement is obvious for $s = 0$ (i.e. when N is the trivial group), so let us suppose $s \ge 1$. Applying the induction hypothesis modulo the s-th term of the descending series of N, one can write $(\!(x, y^i)\!) = wz$, where w has the form $w_1 \dots w_{m'}$ where $m' = m(i, s-1)$, and where z lies in the s-th term of the central series of the subgroup generated by x and y, which will be denoted by H. Since the word length according to $S = \{x^{\pm 1}, y^{\pm 1}\}$ of both $(\!(x, y^i)\!)$ and w is bounded by a function of i and s, this is also the case for z. Now H is generated by the set of iterated commutators $T = \{(\!(x_1, x_2 \dots, x_s)\!) \mid x_1, \dots, x_s \in S\}$. Therefore, z can be written as a word in $T^{\pm 1}$, whose length only depends on s and i. So the lemma follows. □

Recall that a group G is *divisible* (resp. *uniquely divisible*) if for every $n \ge 1$, the power map $G \to G$ mapping x to x^n, is surjective (resp. bijective).

Lemma 2.G.4. *Every torsion-free divisible nilpotent group is uniquely divisible.*

Proof. We have to check that $x^k = y^k \Rightarrow x = y$ holds in any torsion-free nilpotent group. Assume that $x^k = y^k$ and embed the finitely generated torsion-free nilpotent group $\Gamma = \langle x, y \rangle$ into the group of upper unipotent matrices over the reals. Since the latter is uniquely divisible (the k-th extraction of root being defined by some explicit polynomial), we get the result. □

Let $\mathfrak{N}_{\mathbf{Q}}$ be the category of nilpotent Lie algebras over $\mathbf{Q}$ with Lie algebras homomorphisms (with possibly infinite dimension), and $\mathcal{N}$ the category of nilpotent groups with group homomorphisms, and $\mathcal{N}_{\mathbf{Q}}$ its subcategory consisting of uniquely divisible (i.e. divisible and torsion-free, by Lemma 2.G.4) groups. If $\mathfrak{g} \in \mathfrak{N}_{\mathbf{Q}}$, consider the law $\circledast_{\mathfrak{g}}$ on $\mathfrak{g}$ defined by the Campbell-Baker-Hausdorff formula.

Theorem 2.G.5 (Malcev [Mal49a, St70])**.** *For any nilpotent Lie algebra $\mathfrak{g}$ over $\mathbf{Q}$, $(\mathfrak{g}, \circledast_{\mathfrak{g}})$ is a group and if $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then f is also a group homomorphism $(\mathfrak{g}, \circledast_{\mathfrak{g}}) \to (\mathfrak{h}, \circledast_{\mathfrak{k}})$. In other words, $\mathfrak{g} \mapsto (\mathfrak{g}, \circledast_{\mathfrak{g}})$, $f \mapsto f$ is a functor from $\mathfrak{N}_{\mathbf{Q}}$ to $\mathcal{N}$. Moreover, this functor induces an equivalence of categories $\mathfrak{N}_{\mathbf{Q}} \to \mathcal{N}_{\mathbf{Q}}$.*

The contents of the last statement is that

- any group homomorphism $(\mathfrak{g}, \circledast_{\mathfrak{g}}) \to (\mathfrak{h}, \circledast_{\mathfrak{h}})$ is a Lie algebra homomorphism;
- any uniquely divisible nilpotent group $(G, \bullet)$ has a unique $\mathbf{Q}$-Lie algebra structure $\mathfrak{g} = (G, +, [\cdot, \cdot])$ such that $\circledast_{\mathfrak{g}} = \bullet$.

Lemma 2.G.6 (see Lemma 3 in [Ho77])**.** *In a nilpotent group, divisible elements form a subgroup.*

Lemma 2.G.7 (see Theorem 14.5 in [Ba60])**.** *Let G be nilpotent and uniquely divisible, with lower central series (G^n). Then G/G^n is torsion-free (hence uniquely divisible) for all n.* □

Lemma 2.G.8. *In the category of s-nilpotent groups, any free product of uniquely divisible groups is uniquely divisible.*

Proof. First, the free groups in this category are torsion-free: to see this, it is enough to consider the case of a free group of finite rank in this category; such a group is of the form F/F^i with F free group; it is indeed is torsion free for all i: this is a result about the descending central series of a non-abelian free group and is due to Magnus [Mag35] (see also [Se, IV.6.2]).

Let G_1, G_2 be torsion-free uniquely divisible s-nilpotent groups and $G_1 *_s G_2$ their s-nilpotent free product, which is divisible by Lemma 2.G.6. Denote by (N^i) the descending central series of $G_1 * G_2$. Let g be a non-trivial element in $G_1 *_s G_2 = (G_1 * G_2)/N^{s+1}$ and let us show that g is not torsion in this group. By [Mal49b], a free product of torsion-free nilpotent groups is residually torsion-free nilpotent, and therefore there exists $t \geq s+1$ such that the image of g is not torsion in $(G_1 * G_2)/N^t$. Applying Lemma 2.G.7 to $(G_1 * G_2)/N^t$, we see that $(G_1 * G_2)/N^{s+1}$ is torsion-free, so since g is non-trivial, it is not torsion. So $G_1 *_s G_2$ is uniquely divisible by Lemma 2.G.4. □

Lemma 2.G.9. *In any uniquely divisible s-nilpotent group, if $x_i = \exp v_i$*

$$(\!(x_1, \dots, x_s)\!) = \exp[v_1, v_2, \dots, v_s]$$

Proof. Use the convenient convention to identify the group and the Lie algebra through the exponential and with this convention, the lemma simply states that

$$(\!(x_1, \dots, x_s)\!) = [x_1, x_2, \dots, x_s] \qquad \forall x_1, \dots, x_s \in G.$$

By induction, $(\!(x_2, \dots, x_n)\!) = [x_2, \dots, x_s] + O(s)$, where $O(s)$ means some combination of s-fold Lie algebra brackets.

It follows from the Baker-Campbell-Hausdorff formula that if $[x, y]$ is central then $(\!(x, y)\!) = [x, y]$. We can apply this to $x = x_1$ and $y = (\!(x_2, \dots, x_n)\!)$. This yields

$$(\!(x_1, (\!(x_2, \dots, x_n)\!))\!) = [x_1, [x_2, \dots, x_s] + O(s)] = [x_1, \dots, x_s]. \quad \square$$

Let $\mathbf{Q}F$ be the free product of two copies of $\mathbf{Q}$; we denote the two images of $1 \in \mathbf{Q}$ into $\mathbf{Q}F$ by X and Y, so that every element in $\mathbf{Q}F$ can be written as $w = \prod_{i=1}^{m} X_i^{\lambda_i}$ with $X_i \in \{X, Y\}$ and $\lambda_i \in \mathbf{Q}$.

Lemma 2.G.10. *Let x, y be elements of a uniquely divisible nilpotent group G, and n an integer. Then $(xy^{-1})^{1/n}$ is contained in the normal subgroup generated by $\{x^{1/k}y^{-1/k} : k \in \mathbf{Z}\}$.*

Proof. Let G be s-nilpotent. By [BrG11, Lemma 5.1] (see however Remark 2.G.11), there exists a sequence of rational numbers $a_1, b_1, \dots, a_k, b_k$ such that in any s-uniquely divisible s-nilpotent group H and any $u, v \in H$, we have

$$(uv^{-1})^{1/n} = u^{1/n}v^{-1/n} \prod_{i=1}^{k} u^{a_i}v^{b_i}.$$

In particular, picking $(H, u, v) = (\mathbf{R}, 1, \sqrt{2})$, we see that $\sum a_i = \sum b_i = 0$. Therefore, if d is a common denominator to $a_1, \dots, b_k$ and n, $\prod_{i=1}^{k} u^{a_i}v^{b_i}$ as well as $u^{1/n}v^{-1/n}$ belong to the normal subgroup generated by $u^{1/d}v^{-1/d}$, hence $(uv^{-1})^{1/n}$ as well. In particular, this applies to $(H, u, v) = (G, x, y)$. □

Remark 2.G.11. In the above proof, we used [BrG11, Lemma 5.1] to make short. However, this is not very natural, because the latter is proved using the Hall-Petrescu formula; the problem is that in this formula, exponents are put outside the commutators. The proof of the Hall-Petrescu formula can easily be modified to prove by induction on the degree of nilpotency a similar formula with exponents inside the commutators. In [BrG11], in order to make short (as we also do), instead of processing this induction, they work with a much simpler induction based on the Hall-Petrescu formula; this is very unnatural, because if we do not allow ourselves to use the Hall-Petrescu formula, to go through the latter is very roundabout; moreover the exponents a_k, b_k obtained in [BrG11] depend on s, while in a direct induction, we pass from the s-nilpotent case to the $(s+1)$-nilpotent case by multiplying on the right by some suitable iterated commutator of powers.

Proposition 2.G.12. *Let x, y be elements of a uniquely divisible nilpotent group G. Let N be the normal subgroup generated by the elements of the form $x^r y^{-r}$, where r ranges over $\mathbf{Q}$. Then N is divisible. Equivalently, G/N is torsion-free.*

Proof. By Lemma 2.G.10, N contains elements $(x^r y^{-r})^{\rho}$ for any $\rho \in \mathbf{Q}$. So N is generated as a normal subgroup by the divisible subgroups $N_r = \{(x^r y^{-r})^{\rho} : \rho \in \mathbf{Q}\}$, and therefore N is divisible by Lemma 2.G.6. □

The following lemma, which, unlike the previous ones, involves some topology, will be used in the proof of Proposition 4.C.2.

Lemma 2.G.13. *Fix an integer $s \geq 1$. Let $\mathbf{K} = \prod \mathbf{K}_j$ be a finite product of local field of characteristic zero (or $p > s$). Let $\mathfrak{u}$ be a s-nilpotent finite length Lie algebra over $\mathbf{K}$ and fix a norm $\|\cdot\|$ on $\mathfrak{u}$. Let U be the corresponding nilpotent group; identify U with $\mathfrak{u}$ through the exponential map. Let $(\mathfrak{u}_i)_{1\leq i\leq c}$ be $\mathbf{K}$-subalgebras of $\mathfrak{u}$ and $U_i \subset U$ the corresponding subgroups. If the $\mathfrak{u}_i$ generate $\mathfrak{u}$ modulo $[\mathfrak{u},\mathfrak{u}]$, then there exists d and a constant K such that every element $x \in U$ can be written $x_1 \dots x_d$ with $x_k \in \bigcup_i \mathfrak{u}_i$, and $\sup_k \|x_k\| \leq K\|x\|$.*

Proof. We argue by induction on s. If $s = 1$, the assumption is that $\mathfrak{u}$ is abelian and generated by the $\mathfrak{u}_i$. So there exist subspaces $\mathfrak{h}_i \subset \mathfrak{u}_i$ such that $\mathfrak{u} = \bigoplus \mathfrak{h}_i$, so if $x \in \mathfrak{u}$ and p_i is the projection to $\mathfrak{h}_i$, then $x = \sum_{i=1}^{c} p_i(x)$, and if $K_0 = \sup \|p_i\|$ (operator norm) then $\|p_i(x)\| \leq K_0\|x\|$.

Suppose the result is proved for $s-1$. We choose the norm on $\mathfrak{u}/\mathfrak{u}^{(i)}$ to be the quotient norm. We use the induction hypothesis modulo $U^{(s)}$, so that there exist d' and $K' \geq 1$ such that every $x \in U$ can be written as $y_1 \dots y_d\zeta$, with $\|y_i\| \leq K'\|x\|$ and $\zeta \in U^{(s)}$. By the Baker-Campbell-Hausdorff formula, there exists a constant $C > 0$ (depending on s and d') such that for all $x_1, \dots x_{d'+1}$ in $\mathfrak{u}$, we have $\|x_1 \cdots x_{d'+1}\| \leq C \sup \|x_i\|^s$. So $\|\zeta\| \leq C \sup(K'\|x\|)^s$. Consider the $\mathbf{K}$-multilinear map $(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])^s \to \mathfrak{u}^{(s)}$ given by the s-fold bracket. Since its image generates $\mathfrak{u}^{(s)}$ as a $\mathbf{K}$-submodule and since the $\mathfrak{u}_i$ generate $\mathfrak{u}$ modulo $[\mathfrak{u},\mathfrak{u}]$, we can find (independently of the x_i) a elements i_j and a fixed finite family $(\kappa_{jk})_{1\leq j\leq j_0, 1\leq k\leq s}$ with $\kappa_{jk} \in \mathfrak{u}_{i_j}$ such that, setting $\zeta_j = \phi(\kappa_{j1}, \dots, \kappa_{js})$ we have $\mathfrak{u}^{(s)} = \bigoplus_j \mathbf{K}\zeta_j$ (we can normalize so that $\|\zeta_j\| = 1$). Now identify the Lie algebra and the group through the exponential map. If K'' is the supremum of the norm of projections onto the submodules $\mathbf{K}\zeta_j$, then we can write $\zeta = \prod_j \lambda_j\zeta_j$ with $\lambda_j \in \mathbf{K}''$ of absolute value $|\lambda_j| \leq K''\|\zeta\| \leq CK''(K')^s \sup_i \|x_i\|^s$. We can write, in $\mathbf{K}$, $\lambda_j = \prod_{k=1}^{s} \mu_{jk}$, with $|\mu_{jk}| \leq |\lambda_j|^{1/s}\alpha_0$, where $\alpha_0 = \sup_j \inf\{|x| : |x| \in \mathbf{K}_j, |x| > 1\}$ only depends on $\mathbf{K}$. By Lemma 2.G.9, we have

$$\lambda_j\zeta_j = (\!(\mu_{j1}\kappa_{j1}, \dots, \mu_{js}\kappa_{js})\!).$$

If $C' = \sup_{j,k} \|\kappa_{jk}\|$, we have $\mu_{j1}\kappa_{j1} \leq C'(CK'')^{1/s}K' \sup_i \|x_i\|$. So

$$x = y_1 \dots y_d \prod_j (\!(\mu_{j1}\kappa_{j1}, \dots, \mu_{js}\kappa_{js})\!),$$

which is a bounded number of terms in $\bigcup U_i$, each with norm

$$\leq \max(1, C'(CK'')^{1/s})K' \sup_i \|x_i\|. \quad \square$$

Lemma 2.G.14. *Let $\mathfrak{g}$ be a Lie algebra over the commutative ring R. Let $\mathfrak{m}$ be a generating R-submodule of $\mathfrak{g}$. Define $\mathfrak{g}^{[1]} = \mathfrak{m}$, and by induction the submodule $\mathfrak{g}^{[i]} = [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i-1]}]$ for $i \geq 2$ (namely, the submodule generated by the brackets of the given form). Let $(\mathfrak{g}^i)$ be the descending central series of $\mathfrak{g}$. Then for all i we have $\mathfrak{g}^i = \sum_{m\geq i} \mathfrak{g}^{[i]}$.*

Proof. Let us check that

$$\left[\mathfrak{g}^{[i]}, \mathfrak{g}^{[j]}\right] \subset \mathfrak{g}^{[i+j]} \qquad \forall i, j \geq 1. \tag{2.G.15}$$

Note that (2.G.15) holds when either $i = 1$ or $j = 1$. We prove (2.G.15) in general by induction on $k = i + j \geq 2$, the case $k = 2$ being already settled. So suppose that $k \geq 3$ and that the result is proved for all lesser k. We argue again by induction, on $\min(i, j)$, the case $\min(i, j) = 1$ being settled. Let us suppose that $i, j \geq 2$ and $i + j = k$ and let us check that (2.G.15) holds. We can suppose that $j \leq i$. By the Jacobi identity and then the induction hypothesis

$$\begin{aligned}
[\mathfrak{g}^{[i]}, \mathfrak{g}^{[j]}] =& [\mathfrak{g}^{[i]}, [\mathfrak{g}^{[1]}, \mathfrak{g}^{[j-1]}]] \\
\subset& [\mathfrak{g}^{[1]}, [\mathfrak{g}^{[i]}, \mathfrak{g}^{[j-1]}]] + [\mathfrak{g}^{[j-1]}, [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i]}]] \\
\subset& [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i+j-1]}] + [\mathfrak{g}^{[j-1]}, \mathfrak{g}^{[1+i]}] \\
\subset& \mathfrak{g}^{[i+j]} + [\mathfrak{g}^{[j-1]}, \mathfrak{g}^{[1+i]}]
\end{aligned}$$

Since $j \leq i$, we have $\min(j-1, i+1) < \min(i, j)$, the induction hypothesis yields $[\mathfrak{g}^{[j-1]}, \mathfrak{g}^{[1+i]}] \subset \mathfrak{g}^{[i+j]}$ and (2.G.15) is proved.

It follows from (2.G.15) that, defining for all $i \geq 1$

$$\mathfrak{g}^{\{i\}} = \sum_{j \geq i} \mathfrak{g}^{[j]},$$

the graded submodule $\mathfrak{g}^{\{i\}}$ is actually a Lie subalgebra of $\mathfrak{g}$. Let us check by induction on $i \geq 1$ that $\mathfrak{g}^i = \mathfrak{g}^{\{i\}}$. Since $\mathfrak{g}^{[1]}$ generates $\mathfrak{g}$, it follows that $\mathfrak{g}^{\{1\}} = \mathfrak{g} = \mathfrak{g}^1$. Now, suppose $i \geq 2$ and the equality holds for $i - 1$. Then $\mathfrak{g}^i$ is, in our notation, the Lie subalgebra generated by

$$[\mathfrak{g}, \mathfrak{g}^{i-1}] = [\mathfrak{g}, \mathfrak{g}^{\{i-1\}}] = \left[\mathfrak{g}, \sum_{j \geq i-1} \mathfrak{g}^{[j]}\right]$$

$$= \sum_{j \geq i-1} \left[\mathfrak{g}, \mathfrak{g}^{[j]}\right] = \sum_{j \geq i-1} \mathfrak{g}^{[j+1]} = \mathfrak{g}^{\{i\}};$$

since $\mathfrak{g}^{\{i\}}$ is a Lie subalgebra we deduce that $\mathfrak{g}^i = \mathfrak{g}^{\{i\}}$. ☐

3. Metric reductions

3.A. **Reduction to triangulable groups.** The following lemma is essentially borrowed from [C08].

Lemma 3.A.1. *For every connected Lie group G, there exists a series of proper homomorphisms with cocompact images $G \leftarrow G_1 \rightarrow G_2 \leftarrow G_3$, with G_3 triangulable. In particular, G_3 is quasi-isometric to G.*

Proof. Let N be the nilradical of G. By [C08, Lemma 6.7], there exists a closed cocompact solvable subgroup G_1 of G containing N, and a cocompact embedding $G_1 \subset H_2$ with H_2 a connected solvable Lie group, such that H_2 is generated by G_1 and its center $Z(H_2)$. In particular, every normal subgroup of G_1 is normal in H_2. Let W be the largest compact normal subgroup of H_2 and define $L_2 = H_2/W$, so L_2 is a connected solvable Lie group whose derived subgroup has a simply connected closure. By [C08, Lemma 2.4], there are cocompact embeddings $L_2 \subset G_2 \supset G_3$, with G_2 and G_3 connected Lie groups, and G_3 triangulable. □

3.B. **At most exponential Dehn function.**

Theorem 3.B.1 (Gromov). *Every connected Lie group has an at most exponential Dehn function.*

As usual, Gromov gives a rough sketch of proof [Gro93, Corollary 3.F$'_5$], but we are not aware of a complete proof.

Lemma 3.B.2. *If G is a simply connected solvable Lie group with a left-invariant Riemannian metric, there is an exponentially Lipschitz strong deformation retraction of G to the trivial subgroup, i.e. a map $F : G \times [0,1] \to G$ such that for all g, $F(g,0) = g$ and $F(g,1) = 1$, and such that for some constant C, if $B(n)$ is the n-ball in G then F is* $\exp(Cn)$*-Lipschitz in restriction to $B(n) \times [0,1]$.*

Proof. We use that every simply connected solvable Lie group can be described as $(\mathbf{R}^m \times \mathbf{R}^\ell, *)$ with the law of the form

$$(u_1, v_1) * (u_2, v_2) = (u_1 + u_2, P(u_1, u_2, v_1, v_2));\ (u, v)^{-1} = (-u, -v),$$

where P is a function each component of which, if we denote by (U_i) the $2m$ coordinates of (u_1, u_2) and by (V_j) the 2ℓ coordinates of (v_1, v_2), can be described as a real-valued polynomial in the variables U_i, V_j, and $e^{\lambda_k \mathsf{U}_i}$, for some finite family of complex numbers (λ_k). For instance, the law of SOL_λ can be described as

$$(u_1, x_1, y_1) * (u_2, x_2, y_2) = (u_1 + u_2, e^{u_2} x_1 + x_2, e^{-\lambda u_2} y_1 + y_2)$$

(here $(m, \ell) = (1, 2)$).

Define, for $((u, v), t, \tau) \in G \times [0,1]^2$,

$$s((u, v), t, \tau) = (tu, \tau v).$$

Let L_g denote the left translation by g. Then

$$\begin{aligned} &\left(L^{-1}_{(t_0 u_0, \tau_0 v_0)} \circ s\right) \left(L_{(u_0,v_0)}(u, v), t, \tau\right) \\ =\ &\left(-t_0 u_0 + t u_0 + tu, P(-t_0 u_0, t(u_0 + u), -\tau_0 v_0, \tau P(u_0, u, v_0, v))\right) \end{aligned}$$

If we view this as a function of (u, v, t, τ), its differential at $(0, 0, t, \tau)$ is a number which is bounded by $c_1 e^{c_2 \|u_0\|}(1 + \|v_0\|)^{c_3}$ for some positive constants c_1, c_2, c_3 only depending on G and the choice of Riemannian metric. Therefore, the differential of s at any (u_0, v_0, t, τ), for the left-invariant Riemannian metric μ, has the same bound. If $(u_0, v_0) \in B(2n)$, then $\|u_0\| \leq n$ (up to rescaling μ) and $\|v_0\| \leq e^{c_4 n}$. So, for every $(u_0, v_0, t, \tau) \in B(2n) \times [0,1]^2$, the differential of s at (u_0, v_0, s, t) is bounded by e^{Cn}, for some constant C only depending on (G, μ). In particular, since any two points in $B(n)$ can be joined by a geodesic within $B(2n)$, we deduce that the restriction of s to $B(n) \times [0,1]^2$ is e^{Cn}-Lipschitz.

The function $(g, t) \mapsto s(g, t, t)$ is the desired retraction. (We used an extra variable τ by anticipation, in order to reuse the argument in the proof of Proposition 3.B.4.) □

Proof of Theorem 3.B.1. By Lemma 3.A.1, we can restrict to the case of a triangulable group G. Given a loop of size n in G based at the unit element, Lemma 3.B.2 provides a homotopy with exponential size to the trivial loop. □

Remark 3.B.3. Using Guivarc'h's estimates on the word length in simply connected solvable Lie groups [Gui73, Gui80], we see that there exists a constant C' such that if $B(n)$ is the n-ball in G, then $F(B(n), [0,1])$ is contained in the ball $B(C'n)$ (here F is the function constructed in the proof of Lemma 3.B.2). Thus in particular, F provides a filling of every loop of linear size, with exponential area and inside a ball of linear size. In particular, any virtually connected Lie group (and lattice therein) has a linear isodiametric Dehn function.

Let $G = (U_{\mathrm{a}} \times U_{\mathrm{na}}) \rtimes \mathbf{Z}^d$ be a standard solvable group. The group $G_1 = U_{\mathrm{a}} \rtimes \mathbf{Z}^d$ can be embedded as a closed cocompact subgroup into a virtually connected Lie group G_2 with maximal compact subgroup K. Consider a left-invariant Riemannian metric on the connected manifold G_1/K. The composite map $G_1 \to G_2/K$ is a G-equivariant quasi-isometric injective embedding; endow G_1 with the induced metric. Endow U_{na} with a word metric with respect to a compact generating set, and endow G with a metric induced by the natural quasi-isometric embedding $G \to G_1 \times G_2$.

Proposition 3.B.4. *Let G be a standard solvable group of the form $(U_{\mathrm{a}} \times U_{\mathrm{na}}) \rtimes \mathbf{Z}^d$, with the above metric. Then there is an exponentially Lipschitz homotopy between the identity map of G and its natural projection π to $U_{\mathrm{na}} \rtimes \mathbf{Z}^d$. Namely, for some constant C, there is a map*

$$\sigma : \big((U_{\mathrm{a}} \times U_{\mathrm{na}}) \rtimes \mathbf{Z}^d\big) \times [0,1] \to (U_{\mathrm{a}} \times U_{\mathrm{na}}) \rtimes \mathbf{Z}^d,$$

such that for all $g \in G$, $\sigma(g, 0) = g$ and $\sigma(g, 1) = \pi(g)$, and $\sigma(g, t) = g$ if $g \in U_{\mathrm{na}} \rtimes \mathbf{Z}^d$, and σ is (e^{Cn}, C)-Lipschitz in restriction to $B(n) \times [0,1]$.

Proof. As in the definition of standard solvable group, write $G = U \rtimes \mathbf{Z}^d$ and $U = U_{\mathrm{a}} \times U_{\mathrm{na}}$. Since U_{a} is a simply connected nilpotent Lie group, we can identify it to its Lie algebra thorough the exponential map. Define, for $(v, w, u) \in U_{\mathrm{a}} \times U_{\mathrm{na}} \rtimes \mathbf{Z}^d$ and $t \in [0,1]$, $\sigma(v, w, u, t) = (tv, w, u)$. By the computation in the proof of Lemma 3.B.2, σ is e^{Cn}-Lipschitz in restriction to $B(n) \times [0,1]$; the presence of w does not affect this computation. □

Corollary 3.B.5. *Under the assumptions of the proposition, if $G/U^0 \simeq U_{\rm na} \rtimes A$ is compactly presented with at most exponential Dehn function, then G has an at most exponential Dehn function.*

Proof. Given a loop γ of size n in G, the retraction of Proposition 3.B.4 interpolates between γ and its projection γ' to G/G^0. The interpolation has an at most exponential area because the retraction is exponentially Lipschitz; γ' has linear length and hence has at most exponential area by the assumption. So γ has an at most exponential area. □

3.C. **Reduction to split triangulable groups.** Let G be a triangulable real group and $E = G^\infty$ its exponential radical.

Proposition 3.C.1 ([C11])**.** *There exists a triangulable group $\breve{G} = E \rtimes V$ and a homeomorphism $\phi : G \to \breve{G}$, so that, denoting by d_G and $d_{\breve{G}}$ left-invariant word distances on G and $\breve{G}$*

- *ϕ restricts to the identity $E \to E$,*
- *E is the exponential radical of $\breve{G}$*
- *V is isomorphic to the simply connected nilpotent Lie group G/E,*
- *the map ϕ quasi-preserves the length: for some constant $C > 0$,*

$$C^{-1}|g| \le |\phi(g)| \le C|g|; \quad \forall g \in G$$

and is logarithmically bilipschitz

$$D(|g| + |h|)^{-1} d_G(g, h) \le d_{\breve{G}}(g, h) \le D(|g| + |h|) d_G(g, h); \quad \forall g, h \in G,$$

where $C' > 0$ is a constant and where D is an increasing function satisfying $D(n) \le C' \log(n)$ for large n.

Corollary 3.C.2. *Set $\{H, L\} = \{G, \breve{G}\}$. Suppose that the Dehn function δ_L of L satisfies $\delta_B(n) \preccurlyeq n^\alpha$.*

Then for any $\varepsilon > 0$, the Dehn function δ_H of H satisfies

$$\delta_H(n) \preccurlyeq \log(n)^{\alpha+\varepsilon} \delta_L(n \log(n)) \preccurlyeq \log(n)^{2\alpha+\varepsilon} n^\alpha.$$

Proof. Suppose, more precisely, that every loop of length n in L can be filled with area $\delta(n)$ in a ball of radius $s(n)$; note that s can be chosen to be asymptotically equal to δ (by Lemma 2.D.2).

Start with a combinatorial loop γ of length n in H. It maps (by ϕ or ϕ^{-1} to a "loop" in L, in the Cn-ball, in which every pair of consecutive vertices are at distance $\le C \log(n)$. Join those pairs by geodesic segments and fill the resulting loop γ' of length $\le Cn \log(n)$ by a disc consisting of $\delta(Cn \log(n))$ triangles of bounded radius (say, $\le C$), inside the $s(Cn \log(n))$-ball. Map this filling back to H. We obtain a "loop" γ'' consisting of $Cn \log(n)$ points, each two consecutive being at distance $\le C \log(s(Cn \log(n)))$, with a filling by $\delta(Cn \log(n))$ triangles of diameter at most$C \log(s(Cn \log(n)))$. Interpolate γ'' by geodesic segments, so as to obtain a genuine loop γ_1. So γ'' is filled by γ and $Cn\log(n)$ "small" loops of size

$$\leq C\log(s(Cn\log(n)))+1 \leq C\log(C'(Cn\log(n))^\alpha)+1 \leq C_1\log(n).$$

The loop γ'' itself is filled by $\delta(Cn\log(n))$ triangles of diameter at most $C\log(s(Cn\log(n)))$ and thus of size $\leq 3C_1\log(n)$.

We know that H has its Dehn function bounded above by C_2e^{cn}. So each of these small loops has area $\leq C_2\exp(3c(C_1\log(n))) = C'n^{c'}$. We deduce that γ can be filled by

$$(Cn\log(n)+\delta(Cn\log(n)))C'n^{c'} \preceq n^{1+c'+\max(1,\alpha)}$$

triangles of bounded diameter.

We deduce that H has a Dehn function of polynomial growth, albeit with an outrageous degree, the constants c' being out of control. Anyway, this provides a proof that H has a Dehn function $\leq C_3n^q$ for some q, and we now repeat the above argument with this additional information.

The small loops of size $C_1\log(n)$ therefore have area $\leq C_3(C_1\log(n))^q$ and the $\delta(Cn\log(n))$ triangles filling γ'' can now be filled by $\leq C_3(3C\log(s(Cn\log(n))))^q$ triangles of bounded diameter. We deduce this time that γ can be filled by at most

$$Cn\log(n)C_3(C_1\log(n))^q + 3CC_3\log(s(Cn\log(n)))^q\delta(Cn\log(n))$$
$$\approx \log(s(n\log(n)))^q\delta(n\log(n))$$

triangles of bounded diameter.

We have

$$\log(s(n\log n)) \leq \log(s(n^2)) \preceq \log(n^{2\alpha}) \preceq \log(n),$$

so we deduce that the Dehn function of γ is

$$\preccurlyeq \log(n)^q\delta(n\log(n)).$$

Since $\delta(n) \preccurlyeq n^\alpha$, the previous reasoning can be held with q of the form $\alpha+\varepsilon$ for any $\varepsilon > 0$. This proves the desired result. □

Remark 3.C.3. A variant of the proof of Corollary 3.C.2 shows that if the Dehn function of $\breve{G}$ is exponential, then the Dehn function of G is $\succcurlyeq \exp(n/\log(n)^2)$, but is not strong enough to show that the Dehn function of G is exponential, nor even $\succcurlyeq \exp(n/\log(n)^\alpha)$ for small $\alpha \geq 0$.

4. Geometric presentations

4.A. **Tame groups.** We introduce the following definition.

Definition 4.A.1. We call a **tame** group any locally compact group with a semidirect product decomposition $G = U \rtimes A$, where A is a compactly generated abelian group, such that some element of A acts as a compaction of U (in the sense of Definition 2.E.6), in which case we call $U \rtimes A$ a **tame decomposition**.

Remark 4.A.2. If $G = U \rtimes A$ is a tame decomposition, and if W is the largest compact subgroup in A, then W admits a direct factor A' in A (isomorphic to $\mathbf{R}^k \times \mathbf{Z}^\ell$ for some $k, \ell \geq 0$), so that $G = UW \rtimes A'$; if $x \in A$ acts as a compaction on U then it also acts as a compaction on UW, and so does the projection of x on A' (because the set of compactions of a given locally compact group is stable by multiplication by inner automorphisms [CCMT12, Lemma 6.16].

We need to prove that tame groups have an at most quadratic Dehn function. This is a consequence of the following very general theorem.

Theorem 4.A.3. *Let G be a tame group with a tame decomposition $U \rtimes A$. Then there exists a large-scale Lipschitz homotopy from the identity of G to a map $G \to A$. Namely, let π be the projection to A and let $v \in A$ be an element acting (by conjugation on the right) as a compaction of U and define $\gamma : G \times \mathbf{N} \to G$ by $\gamma(g, n) = gv^n$. Then for some constant C, γ is C-Lipschitz in each variable and $d(\gamma(g, n), v^n\pi(g)) \leq C$ if $n \geq C|g|$.*

Remark 4.A.4. This theorem has some similarity with a theorem of Varopoulos [Var00, Main theorem, p. 57] concerning connected Lie groups. Namely, for a simply connected Lie group of the form $U \rtimes N$ with U, N simply connected nilpotent Lie groups such that N contains an element acting as a contraction on N, he proves that there exists a "polynomially Lipschitz" homotopy from the identity of G to its projection on N.

Proof of Theorem 4.A.3. Let S be a compact symmetric generating set; we can suppose that $S = S_U \cup T$ with $S_U \subset U$, $T \subset A$. Let us assume that, in addition, $v^{-1}S_U v \subset S_U$ (so $v^{-1}Sv \subset S$). So, for $n \geq 0$, the automorphism $g \mapsto v^{-n}gv^n$ is 1-Lipschitz, and since the left multiplication by v^n is an isometry, we deduce that $\gamma(\cdot, n)$ is 1-Lipschitz. Also, assuming that $v \in T$, it is immediate that $\gamma(g, \cdot)$ is 1-Lipschitz.

It remains to prove the last statement. Let us assume that $v \in T$. There exists ℓ such that for every $w \in T$, $(vw^\ell)^{-1}S_U(vw^\ell) \subset S_U$. Consider an element in U, of size at most n. We can write it as

$$w = \prod_{i=1}^{n} s_i u_i$$

with $s_i \in T$ and $u_i \in S_U$. Defining $t_i = s_i \dots s_n$, we deduce

$$w = t_n \left(\prod_{i=1}^{n} t_i u_i t_i^{-1} \right).$$

So $t_n = \pi(w)$ and

$$v^{-\ell n} w v^{\ell n} = \pi(w) \left(\prod_{i=1}^{n} v^{-\ell n} t_i u_i t_i^{-1} v^{\ell n} \right).$$

By definition of ℓ, the element $v_i = v^{-\ell n} t_i u_i t_i^{-1} v^{\ell n}$ belongs to S_U. So

$$v^{-\ell n} w v^{\ell n} = \pi(w) \left(\prod_{i=1}^{n} v_i \right).$$

We can suppose from the beginning that S_U is a vacuum subset for the right conjugation $u \mapsto v^{-1}uv$ by v, and therefore there exists $k \geq 2$ such that $v^{-k}(S_U^2)v^k \subset S_U$. So if $j = \lceil \log_2(n) \rceil$, then $v^{-kj}(S_U^n)v^{kj} \subset S_U$. Thus we obtain

$$v^{-\ell n - k\lceil \log_2(n) \rceil} w v^{\ell n + k \lceil \log_2(n) \rceil} \in \pi(w) S_U,$$

since for all n we have $\lceil \log_2(n) \rceil \leq n$, we obtain that for every $m \geq (k+\ell)n$ we have $d(wv^m, v^m \pi(w)) \leq 1$. $\square$

Corollary 4.A.5. *If G is a tame locally compact group, then it has an at most quadratic Dehn function.*

Remark 4.A.6. If A has rank at least 2, since G has a 1-Lipschitz retraction onto A, we deduce that the Dehn function of G is exactly quadratic. On the other hand, if A has rank one, then G is hyperbolic and thus its Dehn function is linear [CCMT12].

Remark 4.A.7. Actually, the theorem also shows that each higher Dehn functions is asymptotically bounded by that of a large Euclidean space, and that all fillings preserve the Lipschitz constants (up to a fixed constant). This fact is trivial in any group quasi-isometric to a CAT(0) space, but there are instances of tame groups for which it is not known whether they are quasi-isometric to any CAT(0)-space, e.g. the semidirect product $\mathbf{R}^2 \rtimes \mathbf{R}^2$, for the action $(s,t) \cdot (u,v) = e^s(u + tv, v)$.

Remark 4.A.8. Recall that for a metric space X and nonprincipal ultrafilter ω on the set of positive integers, the asymptotic cone, denoted $\mathrm{Cone}_\omega(X)$, is obtained as follows. If (x_n), (y_n) are sequences in X, define $d_\omega((x_n),(y_n)) = \lim_\omega d(x_n,y_n)/n \in [0,\infty]$. The asymptotic cone $(\mathrm{Cone}_\omega(X), d_\omega)$ is defined as the metric space consisting of those sequences (x_n) with $d_\omega((x_n),(x_0)) < \infty$, modulo identification of (x_n) and (y_n) whenever $d_\omega((x_n),(y_n)) = 0$.

A straightforward corollary of Theorem 4.A.3 is that if G is a tame locally compact group, then all its asymptotic cones are contractible. Indeed, the large-scale Lipschitz mapping γ induces a Lipschitz map

$$\begin{array}{rcl} \tilde{\gamma} : \mathrm{Cone}_\omega(G) \times \mathbf{R}_{\geq 0} & \to & \mathrm{Cone}_\omega(G) \\ ((x_n), t) & \mapsto & \gamma(x_n, \lfloor tn \rfloor) \end{array}$$

such that $\tilde{\gamma}(x,0) = x$ and $\tilde{\gamma}(x,t) \in \mathrm{Cone}_\omega(\mathsf{A})$ if $t \geq C|x|$. Defining $h(x,t) = \tilde{\gamma}(x, t|x|)$, then h is continuous, $h(x,0) = x$ and $h(x,C) \in \mathrm{Cone}_\omega(\mathsf{A})$ for all $x \in \mathrm{Cone}_\omega(G)$. Since $\mathrm{Cone}_\omega(\mathsf{A})$ is bilipschitz homeomorphic to a Euclidean space, this shows that $\mathrm{Cone}_\omega(G)$ is contractible.

We need to make Theorem 4.A.3 more precise, so as to provide a explicit presentation in Corollary 4.A.5. Consider triples (m, S_U, T), where $m \geq 1$ is an integer, T is a compact symmetric generating subset of A with unit and S_U is a compact symmetric subset of U with unit.

Definition 4.A.9. We say that (m, S_U, T) is **adapted** to $U \rtimes A$ and v if S_U is a vacuum subset for the right conjugation $u \mapsto v^{-1}uv$ by v, and there exist integers $k, \ell \geq 1$ with $k + \ell \leq m$ such that $v^{-k}S_U^2v^k \subset S_U$ and $(v^\ell w)^{-1}S_U(v^\ell w) \subset S_U$ for all $w \in T$.

As observed in the proof of Theorem 4.A.3, there always exists such an adapted triple; more precisely, for every symmetric generating subset T of A containing $\{1, v\}$ and every symmetric vacuum subset $S_U \subset U$ of v, (m, S_U, T) is adapted for all m large enough. Also, is T is large enough, $(2, S_U, T)$ is adapted to some $v \in T$.

Theorem 4.A.10. *Let G be a tame group with a tame decomposition $U \rtimes A$, with an element v acting as a compaction of U. Let (m, S_U, T) be adapted to $U \rtimes A$ and v in the above sense. Then $S = S_U \cup T$ is a compact generating subset of G, and G being endowed with the corresponding word metric, the function*

$$\gamma : G \times \mathbf{N} \to G; \quad (g, n) \mapsto gv^n$$

is 1-Lipschitz in each variable; $\gamma(g, 0) = g$, and $d(\gamma(g, n), v^n\pi(g)) \leq 1$ whenever $g \geq mn$. □

Corollary 4.A.11. *Under the assumptions of Theorem 4.A.10, for every $x \in G$ of length n, we can write $x = \pi(x)v^{mn}sv^{-mn}$ and $s \in S_U$.*

Proof. If $m = (k + \ell)n$, define $s = (v^m\pi(x))^{-1}\gamma(x, m)$. By the theorem, $s \in S_U$, while $x = \pi(x)v^msv^{-m}$. □

The corollary allow us to introduce the following definition, which will be used in the sequel.

Definition 4.A.12. Let $G = U \rtimes A$ be tame and $v \in A$ act as a compaction on U. Let (m, S_U, T) be *adapted* to $U \rtimes A$ and v in the sense of Definition 4.A.9, and $S = S_U \cup T$. By Corollary 4.A.11, if $x \in U$ and $|x|_S = n$, the element $s = v^{-mn}xv^{mn}$ belongs to S_U. We define $\overline{x} \in F_S$ as the word $v^{mn}sv^{-mn}$, which has length $2mn + 1$ and represents x.

Corollary 4.A.13. *Under the assumptions of Theorem 4.A.10, G admits a presentation with generating set S in which the relators are the following*

(1) *All relators of the form $s_1s_2s_3$ for $s_1, s_2, s_3 \in S_U$, whenever $s_1s_2s_3 = 1$ in U;*
(2) *all relators of the form $ws_1w^{-1}s_2$ whenever $w \in T$, $s_1, s_2 \in A$ and $ws_1w^{-1}s_2 = 1$ in G;*
(3) *a finite set $\mathcal{R}_3$ of defining relators of A including all commutation relators.*

The Dehn function of this presentation is bounded above by $mn^2 + \delta_{(D,\mathcal{R}_3)}(n)$.

Proof. Start from a combinatorial loop $\gamma_0 = (u_i)_i$ of length n. It a combinatorial loop in the Cayley graph of (G, S), i.e., $d(u_i, u_{i+1}) \le 1$. Since γ is 1-Lipschitz, each $\gamma_j = (\gamma(u_i, j))_i$ is also a loop of length n. Define μ_n as the loop $\pi(u_i)v^n$ in A. We know that for $j \ge (k+1)n$, we have $d(\mu_n(i), \gamma_n(i)) \le 1$ for all i. So the "homotopy" consists in

$$\gamma_0 \rightsquigarrow \gamma_1 \rightsquigarrow \cdots \rightsquigarrow \gamma_{(k+1)n} \rightsquigarrow \mu_{(k+1)n}$$

and is finished by a homotopy from $\mu_{(k+1)n}$ to a trivial loop inside A. So we have to describe each step of this homotopy.

To go from γ_j to γ_{j+1}, we use n squares with vertices

$$(u_i v^j, u_{i+1} v^j, u_{i+1} v^{j+1}, u_i v^{j+1})$$

for $i = 1, \dots n$ (modulo n). To describe this square, we discuss whether $u_i^{-1} u_{i+1}$ belongs to S_U or T: in the first case this is a relator of the form (2) and in the second case it is a relator of the form (3).

To go from γ_j to μ_j for $j = (k+1)n$, we use n squares vith vertices

$$(u_i v^j, u_{i+1} v^j, \pi(u_{i+1}) v^j \pi(u_i) v^j).$$

We discuss again: if $u_i^{-1} u_{i+1} \in T$, this is a relator of the form (3). If $u_i^{-1} u_{i+1} \in S_U$, actually $\pi(u_i) = \pi(u_{i+1})$ and this square actually degenerates to a triangle of the form (1).

Finally, the loop can be homotoped within A. If v is part of a basis of A (as we can always assume) and T is chosen to be this basis (as well as inverses and unity), the commutators between generators are enough.

The area of the above homotopy is then bounded by $n^2(k + \ell + 1)$ (namely $n(k+\ell+1)$ steps with n squares each time), plus the area of the loop in A, which is bounded by $n^2/16$. ☐

The next proposition shows that enlarging T if necessary, we can also obtain a universal quadratic bound for the Dehn function of A.

Proposition 4.A.14. *In $\mathbf{Z}^d$, for every parallelepiped $S = \prod_{i=1}^d \{-M_i, \dots, M_i\}$ ($M_i > 0$), there exists a finite set of relators of length at most 4 with Dehn function $\le (n+3)^2/16$.*

Proof. Endow $\mathbf{R}^d$ with the ℓ^∞ norm. Identify $(\mathbf{Z}^d, S)$ with the lattice $\Lambda = \prod_{i=1}^d \frac{1}{M_i}\mathbf{Z}$, with generating subset T, which is the intersection of Λ with the unit ball of $\mathbf{R}^d$. For every $x \in \Lambda$, we have $|x|_T \ge \|x\|_\infty$ and $|x|_T \le 1$ if and only if $\|x\|_\infty \le 1$.

Let $c_0, c_1, \dots, c_{4n} = c_0$ be a combinatorial loop on the Cayley graph of (Λ, T). Then for each coordinate, the width of the projection of this loop is $\le 2n$. Therefore there exists $c \in \mathbf{Z}^d$ such that if we translate this loop by c, it is contained in $[-n, n]^d$, as we now suppose.

Define, for $k = 0, \dots, n$, $x_i^k = kc_i/n \in \mathbf{R}^d$. So we can view the $((x_i^k)_i)_k$ as a sequence of loops in $\mathbf{R}^d$ interpolating between the trivial loop for $k = 0$ and the loop (c_i) for $k = n$. It is filled by n^2 squares, each of the form $(x_i^k, x_{i+1}^k, x_{i+1}^{k+1}, x_i^{k+1})$ where i is modulo n and $k = 0, \dots, n-1$. Each edge in these square has $\|\cdot\|_\infty$-length ≤ 1, as $\|x_i^k - x_{i+1}^k\|_\infty = (k/n)\|c_i - c_{i+1}\|_\infty \le 1$ and $\|x_i^k - x_i^{k+1}\|_\infty = \|c_i/n\|_\infty \le 1$.

Let E be the map $\mathbf{R}^d \to \Lambda$, $x \mapsto (\lfloor M_1 x_1 \rfloor / M_1, \dots, \lfloor M_d x_d \rfloor / M_d)$. Although it is not 1-Lipschitz, it has the property that $\|x-y\|_\infty \le 1$ implies $|E(x)-E(y)|_T \le 1$. Moreover, it is the identity on Λ. Therefore, if we set $c_i^k = E(x_i^k)$, then $((c_i^k)_i)_k$ is a sequence of loops in Λ interpolating between the trivial loop for $k = 0$ and the loop (c_i) for $k = n$. It is filled by $2n^2$ squares, each of the form $(c_i^k, c_{i+1}^k, c_{i+1}^{k+1}, c_i^{k+1})$ where i is modulo n and $k = 0, \dots, n-1$, all of whose edges have length ≤ 1 in (Λ, T) by the aforementioned property, i.e. are edges in the Cayley graph of (Λ, T).

Thus the loop has area $\le n^2$. So $\delta(4n) \le n^2$, and thus $\delta(n) \le (n+3)^2/16$ for all n. □

In combination with Corollary 4.A.13, this yields

Corollary 4.A.15. *If $U \rtimes A$ is tame, and $A \simeq \mathbf{Z}^d$. Let S_U be a compact generating subset of U such that for some primitive element $v_1 \in \mathbf{Z}^d$ acting as a compaction of U with vacuum subset S_U, we have $v_1 \cdot S_U \subset S_U$.*

Then there exists a presentation by a compact subset $S_U \cup T$ ($T \subset D$) with relators of length ≤ 4 and Dehn function $\le 3n^2$; moreover T can be chosen to contain any prescribed finite subset T' of A and such that $(2, S_U, T)$ is adapted in the sense of Definition 4.A.9.

Proof. Complete v_1 to a basis $(v_1, w_2, \dots, w_d)$ of A. Set $v_i = v_1^k w_i$ with k large enough so that $v_i(S_U) \subset S_U$ for all i. It follows that $v = v_1 \dots v_d$ is a compaction as well. Also, (v_i) is a basis, allowing us to identify A with $\mathbf{Z}^d$.

If $\ell \ge 1$, consider the cube $T_\ell = \{-\ell, \dots, \ell\}^d$. Each element w of this cube can be written in a unique way as $v^k \prod_{i=1}^d v_i^{n_i}$ with $-\ell \le k \le \ell$ and $\inf_i n_i = 0$. It follows that $v^\ell w \cdot (S_U) \subset S_U$: indeed $v^\ell w = v^{\ell-k} \prod_{i=1}^d v_i^{n_i}$ is a product of elements, each sending S_U into itself.

If ℓ is chosen large enough, we can ensure, in addition, that $T' \subset T_\ell$ and $v^\ell(S_U S_U) \subset S_U$. □

4.B. **Tame subgroups and the grading.**

4.B.1. *Tame subgroups in a general setting.* Let G be a locally compact group with a fixed semidirect product decomposition $G = U \rtimes A$ (we shall soon specify the hypotheses).

Definition 4.B.1. A **tame subgroup** of G is a subgroup of the form $V \rtimes A$, which is tame, i.e., in which some element of A acts on V as a compaction (in the sense of Definition 2.E.6).

Note that this definition depends on the fixed semidirect product decomposition of G. As shown in Abels' seminal work [Ab87], the tame subgroups of G can provide considerable information on the geometry of G.

For $v \in A$, if we define U_v as the contraction subgroup

$$\left\{ x \in U : \lim_{n \to +\infty} v^{-n} x v^n = 1 \right\}, \tag{4.B.2}$$

then v acts as a compaction on $\overline{U_v}$ (this is [CCMT12, Prop. 6.17], but is much more elementary when further assumptions will be made on U) and therefore $\overline{U_v} \rtimes A$ is tame. We call this a **essential tame subgroup**. Finite intersections of essential tame subgroups are called **standard tame subgroups**.

4.B.2. *Tame subgroups and the grading for standard solvable groups.* In order to give a more precise description of tame subgroups, we now specify to the case of standard solvable groups (although we will momentarily return to the general setting in §4.E). We insist that the theory can be developed in a broader context, but at the price of technicalities or difficulties we chose not to encounter, our context being already very general.

So we assume that $U \rtimes A$ is a standard solvable group in the sense of Definition 1.2. The Lie algebra $\mathfrak{u}$ of U admits a grading in the weight space $\mathcal{W} = \mathrm{Hom}(A, \mathbf{R})$, introduced in §2.E.2. Define the set of weights of $\mathfrak{u}$ as the finite subset $\mathcal{W}_\mathfrak{u} = \{\alpha \in \mathcal{W} : \mathfrak{u}_\alpha \neq \{0\}\}$.

We say that a subset of $\mathcal{W}_\mathfrak{u}$ is **conic** if it is of the form $\mathcal{W}_\mathfrak{u} \cap C$, with C an open convex cone not containing 0. Let $\mathcal{C}$ be the set of conic subsets of $\mathcal{W}_\mathfrak{u}$. If $C \in \mathcal{C}$, define

$$\mathfrak{u}_C = \bigoplus_{\alpha \in C} \mathfrak{g}_\alpha;$$

this is a graded Lie subalgebra of $\mathfrak{u}$; clearly it is nilpotent. Let U_C be the closed subgroup of U corresponding to $\mathfrak{u}_C$ under the exponential map and $G_C = U_C \rtimes A$. In particular, if $v \in A$, define $H(v) = \{\alpha \in \mathcal{W}_\mathfrak{u} : \alpha(v) > 0\}$.

Lemma 4.B.3. *The $G_{H(v)}$ are the essential tame groups of G; in particular, they are finitely many. Every tame subgroup is contained in an essential tame subgroup. The G_C are the standard tame subgroups of G.*

Proof. The automorphisms of U are conjugate, through the exponential map to $\mathbf{K}$-linear automorphisms of $\mathfrak{u}$; under this identification, U_v corresponds to the sum of characteristic subspaces of the operator of conjugation by v associated to eigenvalues of modulus < 1. By definition of the grading, this is exactly $\bigoplus_{\{\alpha :\, \alpha(v)<0\}} \mathfrak{u}_\alpha = \mathfrak{u}_{H(-v)}$. Thus $U_v = U_{H(-v)}$. Thus the $G_{H(v)}$ are the essential tame subgroups.

If $V \rtimes A$ is a tame subgroup, some v acts on it as a compaction, and therefore $V \subset U_{H(-v)}$ and thus $V \rtimes A \subset G_{H(-v)}$.

For the last assertion, the finite intersection $\bigcap_i G_{H(v_i)}$ is equal to $G_{\bigcap_i H(v_i)}$ and thus each standard tame subgroup has the form G_C. Conversely, fix C and let us check that G_C is a finite intersection of essential tame subgroups. First write C as an intersection of open half-spaces $\{\ell_i > 0\}$ in $\mathcal{W}$. Since $C \cap \mathcal{W}_\mathfrak{u}$ is finite, it therefore reduces to a finite intersection, say $\{\ell_\iota > 0\} \cap \mathcal{W}_\mathfrak{u}$, where ι ranges over a finite subset of indices. Since the image of A in $\mathcal{W}$ is cocompact, we can suppose that each ℓ_i is of the form $\alpha \mapsto \alpha(v_i)$. Thus $C = \bigcap H(v_i)$, and $G_C = \bigcap G_{H(v_i)}$. □

Remark 4.B.4. Let $U \rtimes A$ be a standard solvable group. The proof of Lemma 4.B.3 shows that G is tame if and only if 0 is not in the convex hull of the set of weights, and if so, there exists $v \in A$ with $\alpha(v) < 0$ for all weights α. Note that v cannot always be found in a fixed generating set of A.

4.C. **Generating subset and length estimates.** As in §4.B.2, let $G = U \rtimes A$ be a standard solvable group over $\mathbf{K} = \prod_j \mathbf{K}_j$.

If V is any finite length $\mathbf{K}$-module, by **norm** on V we mean the supremum norm, each V_j being endowed with a $\mathbf{K}_j$-norm. As in the usual case of finite-dimensional real vector spaces, all norms are equivalent. If $U \rtimes A$ is a standard solvable group, we can fix a norm on $\mathfrak{u}$ and thus define a norm map on U through the exponential. We call this a **Lie algebra norm** on U (beware that it is generally not subadditive with respect to the group law; also note that we did not require that it is submultiplicative with respect to the Lie bracket, although this can always been assumed after multiplication by some positive scalar).

Another norm on U can be defined as follows: since $\mathrm{Aut}(U)$ is conjugate to $\mathrm{Aut}(\mathfrak{u})$ through the exponential map, the group $U \rtimes \mathrm{Aut}(U)$ can be viewed as a linear algebraic group and in particular has a linear representation of $U \rtimes A$ into some $\mathrm{GL}_q(\mathbf{K})$ (argue component by component if necessary), which is faithful in restriction to U (its kernel is reduced to the centralizer of U in A). Then, the group of matrices $\mathrm{M}_q(\mathbf{K})$ can be endowed with any submultiplicative norm, endowing on U a norm, called a **matrix norm** on U.

Definition 4.C.1. Let $G = U \rtimes A$ be a standard solvable group and (G_C) its standard tame subgroups (there are finitely many, by Lemma 4.B.3). We call **standard subset** of G a subset of the form $S_U \cup T$, where $S_U = \bigcup_C S_C$ and S_C is the exponential of the closed 1-ball of $\mathfrak{u}_C$ for some fixed norm on $\mathfrak{u}$, and T is a finite symmetric generating subset of A with unit. We will check that such a subset is necessarily generating (Proposition 4.C.2), and will then call it a **standard generating subset**.

Note that by Corollary 4.A.15, the standard generating subset can be chosen so that G_C has Dehn function $\le 3n^3$, with respect to the set of relators consisting of all relations of size ≤ 4.

Proposition 4.C.2. *Let $S_0 = S_U \cup T$ be a standard subset of the standard solvable group $G = U \rtimes A$. Let $\|\cdot\|'$ be a Lie algebra norm on U and $\|\cdot\|$ a matrix norm* *on U. Then S_0 is a compact generating subset of G. Moreover, for $u \in U$ and $v \in A$ we have*

$$|uv|_S \simeq \log(\|u\|) \vee |v|_T \simeq \log(\|u\|') \vee |v|_T,$$

where $x \vee y = \max(x, y)$ and $\simeq$ is asymptotic equivalence (Definition 2.A.1).

Proof. To avoid double subscripts, we write S instead of S_0. Let us first show $|uv|_S \succeq \log(\|u\|) \vee |v|_T$. Clearly, $|v|_S = |v|_T \le |uv|_S$. Besides, $|u|_S \le |uv|_S + |v|_T \le 2|uv|_S$. Using the above matrix representation, the norm $\|\cdot\|$ of an arbitrary element of G makes sense, so if $K = \sup\{\|x\| : x \in S^{\pm 1}\}$, we have, for all x in the n-ball, $\|x\| \le C^n$, where $C > 1$ bounds the $\|\cdot\|$-norm of any element of S. In other words, for every x we have $|x|_S \ge \log(\|x\|)/\log(C)$. So $|uv|_S \ge |u|_S/2 \ge \log(\|u\|)/(2\log(C))$.

Let us show $\log(\|u\|) \vee |v|_T \succeq \log(1 + \|u\|') \vee |v|_T$. It follows from the more precise statement that $\|u\|' \preceq 1 \vee \|u\|^s$ for $u \in U$ (s is the nilpotency length of U). Since we can choose the norm on the Lie algebra to be the restriction of $\|\cdot\|$, this means that $\|\log(u)\| \preceq 1 \vee \|u\|^s$. By the Baker-Campbell-Hausdorff formula, $\log(u)$ is a polynomial of degree s in u, so the estimate follows.

Finally, let us show that $|uv|_S \preceq \log(\|u\|') \vee |v|_T$. Since $|uv|_S \le |u|_S + |v|_T$, it is enough to show that $|u|_S \preceq \log(\|u\|') \vee 1$. By definition of S_U, it contains the elements of norm $\|\cdot\|' \le 1$ in U_C for every C.

Let us begin by the case when $\log(u) \in \mathfrak{u}_\alpha$ for some $\alpha \neq 0$. We choose (once and for all) $v_\alpha \in T$ such that $\alpha(v_\alpha) > 0$. If $c = \inf_{\alpha \neq 0} \alpha(v_\alpha) > 0$, we therefore have, for every n, $\|v_\alpha^n u v_\alpha^{-n}\|' \le e^{-cn}\|u\|'$. So if $n \ge \log(\|u\|')/c$ we have $\|v_\alpha^n u v_\alpha^{-n}\|' \le 1$, so $v_\alpha^n u v_\alpha^{-n} \in S$. If $\|u\| \ge 1$ we deduce $|u|_S \le 1 + 2(\log(\|u\|')/c + 1)$; if $\|u\| \le 1$ we have $|u\|' \le 1$.

Now let u be arbitrary. By Lemma 2.G.13, for some fixed constants k, K, we can write $u = u_1 \dots u_k$ with $\|u_i\|' \le K\|u\|$, with $u_i \in \bigcup_{\alpha \neq 0} \exp(\mathfrak{u}_\alpha)$. By the tame case, we deduce that $|u|_S \le k(3 + 2(0 \vee \log(\|u'\|))/c) \preceq 1 \vee \log(\|u'\|')$. $\square$

Now we consider a standard generating subset of G as in Definition 4.C.1. It has the form $\bigcup_C S_C \cup T$, where C ranges over standard tame subgroups. We define S as the disjoint union

$$S = \bigsqcup S_C \sqcup T$$

and will mainly estimate areas inside the free group F_S.

There exists m such that for every C, (m, S_U, T) is adapted (in the sense of Definition 4.A.9) to G_C and some compaction v_C for every C. Fixing such m and v_C, we have a well-defined map, as in Definition 4.A.12

$$\begin{array}{rcl} U_C & \to & F_S \\ x & \mapsto & \overline{x} = v_C^{mn} s v_C^{-mn} \in F_S, \end{array} \tag{4.C.3}$$

where s is the letter in S_U equal to in G_C to $v_C^{-mn} x v_C^{mn}$. Define

$$\mathcal{F}_U = \{\overline{x} : x \in U_C, C \in \mathcal{C} \subset F_S\}.$$

Let $\mathcal{F}_D \subset F_T$ be a combing of A (in the sense of Definition 2.D.3); since we assume that T contains a basis $(t_1, \dots, t_d)$ of A, we can define $\mathcal{F}$ as the set of words of the form $\prod_{i=1}^k t_i^{n_i}$. In terms of combings, the proof of Proposition 4.C.2 essentially shows the following.

Proposition 4.C.4. *Under the above assumptions, if $\mathcal{Z}$ is a 1-combing of A then $\mathcal{F} = \mathcal{F}_U \cup \mathcal{Z}$ is a combing of G.*

Proof. Fix a Lie algebra norm on U, and thus on each U_C by restriction. If $u \in U$, as in the proof of Proposition 4.C.2, write $u = u_1 \dots u_k$, with $u_i \in U_{C_i}$ and $\|u_i\| \le K\|u\|$, so $|u_i| \le K'|u|$ by Proposition 4.C.2. Then $u = \pi(\prod_{i=1}^d \overline{u_i})$. We have $|\overline{u_i}|_S \le 2m|u_i|_S + 1 \le (2m+1)|u_i|_S \le (2m+1)K'|u|_S$. It readily follows that $\mathcal{F}$ is a $(k+1)$-combing of π. □

4.D. **Conclusion in the strongly 2-tame case.**

Theorem 4.D.1. *Let $G = U \rtimes A$ be a strongly 2-tame standard solvable group. Then G has a linear or quadratic Dehn function (linear precisely when A has rank one).*

Remark 4.D.2. Theorem 4.D.1 is actually a corollary of Theorem 4.L.1, because for a strongly 2-tame group, we have $\mathrm{Kill}(\mathfrak{u})_0 = 0$ (because $\mathrm{Kill}(\mathfrak{u})_0$ is a quotient of $(\mathfrak{u} \otimes \mathfrak{u})_0$ which is itself trivial if $\mathfrak{u}$ is strongly 2-tame). The point is that Theorem 4.L.1 is considerably more difficult: it requires the remainder of this section as well as the algebraic treatment of Sections 5 and 6.

Proof of Theorem 4.D.1. If A has rank one, then G is tame and thus hyperbolic, see Remark 4.A.6.

Now let us prove the quadratic upper bound. We argue by induction on the length ℓ of $\mathfrak{u}$ as a $\mathbf{K}$-module. If $\ell \le 2$, then G is tame, hence has an at most quadratic Dehn function.

Assume now that $\ell \ge 2$ and the result is proved for lesser ℓ. We nearly use the combing from §4.C: we restrict to words of the form gsg^{-1} where $s \in \exp(\mathfrak{u}_\alpha)$ for some α. By Lemma 2.G.13, this is a combing as well (for an arbitrary standard solvable group).

Fix a weight β of $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$. Let $\mathfrak{v}$ be the kernel of the projection $\mathfrak{u} \to (\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])_\beta$; this is a graded ideal in $\mathfrak{u}$ and let V be the corresponding subgroup.

Consider a word of the form $\prod_{i=1}^c \overline{x_i}$, where $x_i \in \exp(\mathfrak{u}_{\alpha_i})$. Given one i such that $\alpha_i = \beta$, we shuffle it to the left. The cost is the appearance of elements of the form $[\overline{x_i}, \overline{x_j}]$. Fix C such that x_i and x_j belong to U_C. We can change, with at most quadratic cost $[\overline{x_i}, \overline{x_j}]$ into $\overline{[x_i, x_j]}$ (where $\overline{[x_i, x_j]}$ is defined in G_C), because the loop $[\overline{x_i}, \overline{x_j}]\overline{[x_i, x_j]}^{-1}$ entirely lies in G_C, and G_C has an at most quadratic Dehn function by Corollary 4.A.5.

We do this successively for each i such that $\alpha_i = \beta$. This, after a bounded number of operation (depending on c), each with quadratic cost, we obtain a word of the form $\prod_{i\in I}\overline{x_i}z$ where $I=\{i:\alpha_i=\beta$ and z is a product of elements $\overline{x_i}$ for $i\notin I$ and of commutators. Note that $\#(I)\le c$. Thus z represents an element of V.

Now, since all $\overline{x_i}$ are contained in a single G_C, we can change, with at most quadratic cost, $\prod\overline{x_i}$ into an element $\overline{y}$, representing an element y. Since $\overline{y}z$ represents the identity, $y\in V$. Since $\overline{y}$ has the form gsg^{-1} with g a word in A and s a conjugate of y, we deduce that the loop $\overline{y}z$ is entirely contained in V.

We finally use induction and the fact that $V\rtimes A$ is also strongly 2-tame, to argue that $V\rtimes A$ has an at most quadratic Dehn function. Thus the loop $\overline{y}z$ has an at most quadratic size, so the the original loop has an at most quadratic size, provided c is bounded. So, using Gromov's trick (Proposition 2.D.7), we deduce that G has an at most quadratic Dehn function. ☐

4.E. **Abels' multiamalgam.** Let G be any group and consider a family (G_i) of subgroups. We call multiamalgam (or colimit) an initial object in the category of groups H endowed with homomorphisms $H\to G$ and $G_i\to H$, such that all composite homomorphisms $G_i\to H\to G$ are equal to the inclusion. Such an object is defined up to unique isomorphism commuting with all homomorphisms. It can be explicitly constructed as follows: consider the free product H of all G_i, denote by $\kappa_i:G_i\to H$ the inclusion and mod out by the normal subgroup generated by the $\kappa_i(x)\kappa_j(x)^{-1}$ whenever $x\in G_i\cap G_j$. If the family (G_i) is stable under finite intersections, it is enough to mod out by the elements of the form $\kappa_i(x)\kappa_j(x)^{-1}$ whenever $x\in G_i$ and $G_i\subset G_j$. Clearly, the multiamalgam does not change if we replace the family (G_i) by a larger family (G'_j) such that each G'_j is contained in some G_i; in particular it is no restriction to assume that the family is closed under intersections. Also, the image of $\hat{G}\to G$ is obviously equal to the subgroup generated by the G_i.

Remark 4.E.1. Our (and, originally, Abels') motivation in introducing the multiamalgam is to obtain a presentation of G. Therefore, in this point of view, the ideal case is when $\hat{G}\to G$ is an isomorphism. In many interesting cases, which will be studied in the sequel, $\hat{G}\to G$ is a central extension.

At the opposite, if the G_i have pairwise trivial intersection, the multiamalgam of the G_i is barely the free product of the G_i, and this generally means that the kernel of $\hat{G}\to G$ is "large".

Example 4.E.2. Consider a group presentation $G=\langle S\mid R\rangle$ in which every relator involves at most two generators. If we consider the family of subgroups generated by 2 elements of S, the homomorphism $\hat{G}\to G$ is an isomorphism. Instances of such presentations are presentations of free abelian groups, Coxeter presentations, Artin presentations.

Lemma 4.E.3. *In general, let G be any group, (G_i) any family of subgroups. Suppose that each G_i has a presentation $\langle S_i\mid R_i\rangle$ and that $S_i\cap S_j$ generates $G_i\cap G_j$ for all i,j. Then the multiamalgam of all G_i has a presentation with generators*

$\bigsqcup S_i$, relators $\bigsqcup R_i$ and, for all (i,j) the relators of size two identifying an element of S_i and of S_j whenever they are actually equal (if (G_i) is closed under finite intersections, those (i,j) such that $G_i \subset G_j$ are enough).

Proof. This is formal. □

Following a fundamental idea of Abels, we introduce the following definition.

Definition 4.E.4. Let G be a locally compact group G with a semidirect product decomposition $G = U \rtimes A$ (as in §4.B). Consider the family $(G_C = U_C \rtimes A)_{C\in\mathcal{C}}$ of its tame subgroups $V \rtimes A$. Let $\hat{U}$ be the multiamalgam of the U_C; it admits a natural action of A, and we define $\hat{G} = \hat{U} \rtimes A$.

Remark 4.E.5. By an easy verification, $\hat{G}$ is the multiamalgam of the G_C.

4.F. **Algebraic and geometric presentations of the multiamalgam.** Let $G = U \rtimes A$ be a standard solvable group and (G_C) the family of its standard tame subgroups, $G_C = U_C \rtimes A$.

Consider the free product $H = \ast_C U_C$. There is a canonical surjective homomorphism $p : H \to \hat{U}$. Besides, there are canonical homomorphisms $i_C : U_C \to H$, so that $p \circ i_C$ is the inclusion $U_C \subset \hat{U}$.

The following lemma is an immediate consequence of the definition.

Lemma 4.F.1 (Algebraic presentation of the multiamalgam)**.** *The multiamalgam $\hat{U}$ is, through the canonical map p, the quotient of H by the normal subgroup generated by pairs $i_{C_1}(x)i_{C_2}(x)^{-1}$ where (C_1, C_2) ranges over pairs such that $C_1 \subset C_2$ and x ranges over U_{C_1}.*

Our goal is to translate this presentation into a compact presentation of $\hat{G}$. To this end, using the notation from Definition 4.C.1, recall that S denotes the disjoint union $S = \bigsqcup_C S_C \sqcup T$ and F_S the free group over S. There is a canonical surjection

$$\pi : F_S \to H \rtimes A,$$

and by composition, $p \circ \pi$ is a canonical surjection $F_S \to \hat{U}$. If ι_{C_i} is the inclusion of S_{C_i} into F_S, then $p \circ \pi \circ \iota_{C_i}$ is the inclusion of S_{C_i} into $\hat{U}$.

Let us introduce some important sets of elements in the kernel of the map $F_S \to \hat{U}$. We fix a presentation of A over T, including all commutation relators between generators.

- $R_{\text{tame}} = \bigcup_C R_{\text{tame},C}$, where $\bigcup_C R_{\text{tame},C}$ consists of all elements in $\text{Ker}(\pi) \cap F_{T\sqcup S_C}$.
- $R^1_{\text{tame}} = \bigcup_C R^1_{\text{tame},C}$, where $R^1_{\text{tame},C}$ consists of all elements in $R_{\text{tame},C}$ that are relators in the presentation of G_C given in Corollary 4.A.13, as well as the relators in $R_{\text{tame},C}$ of length two;
- R_{amalg} consists of the elements of the form $\overline{i_{C_1}(x)}\,\overline{i_{C_2}(x)}^{-1}$, where (C_1, C_2) ranges over pairs such that $C_1 \subset C_2$ and x ranges over U_{C_1}. Here, for

$z \in U_C$ for prescribed C, $\overline{z}$ is defined as in (4.C.3). We call these **amalgamation relations**;
- R^1_{amalg} consists of those tame relations for which $x \in S_{C_1}$. These are word of length at two. We call these **amalgamation relators**.

Proposition 4.F.2 (Geometric presentation of the multiamalgam)**.** *The multiamalgam $\hat{U}$ is, through the canonical map $p \circ \pi$, the quotient of F_S by the normal subgroup generated by $R^1_{\mathrm{tame}} \cup R^1_{\mathrm{amalg}}$.*

Proof. It is a general fact that if a group Γ is presented by a symmetric generating set Σ containing the unit and a set Π of relators, written as words over Σ, then the kernel of the natural projection from F_Σ to Γ is generated by Π as well as the relation of length ≤ 2, namely those saying that the unit element (which is a formal generator in F_Σ) equals 1, and those saying that any two inverse elements are actually inverse.

It therefore follows that the kernel of $F_S \to H \rtimes A$ is generated by R_{tame}; by Corollary 4.A.13 $R_{\mathrm{tame},C}$ is contained in the normal subgroup generated by R^1_{tame}, and therefore the kernel of $F_S \to H \rtimes A$ is generated by R^1_{tame}.

By Lemma 4.F.1, the kernel of the canonical map $H \rtimes A \to \hat{G}$ is generated, as a normal subgroup by elements of the form $i_{C_1}(x)i_{C_2}(x)^{-1}$ when $C_1 \subset C_2$ and $x \in U_{C_1}$. Actually, those x in S_{C_1} are enough. Indeed, if N is a normal subgroup of $H \rtimes A$, the set of $x \in U_{C_1}$ such that $i_{C_1}(x) = i_{C_2}(x)$ modulo N is a subgroup M normalized by A, so if M contains S_{C_1} then M contains U_{C_1}.

Since R^1_{amalg} contains a lift in F_S of a subset normally generating the kernel of $H \rtimes A \to \hat{G}$, and since R^1_{tame} normally generates the kernel of $F_S \to H \rtimes A$, it follows that $R^1_{\mathrm{tame}} \cup R^1_{\mathrm{amalg}}$ normally generates the kernel of $F_S \to \hat{G}$. □

Proposition 4.F.3 (Quadratic filling of tame and amalgamation relations)**.** *For the presentation*

$$\langle S \mid R^1_{\mathrm{tame}} \cup R^1_{\mathrm{amalg}} \rangle$$

of $\hat{G}$, the tame and amalgamation relations have an at most quadratic area with respect to their length.

Proof. The tame relations have an at most quadratic area by Corollary 4.A.13.

Let us consider an amalgamation relation. It has the form $w = \overline{i_{C_1}(x)}\,\overline{i_{C_2}(x)}^{-1}$ for some $x \in U_{C_1}$ and $C_1 \subset C_2$. The letters of $\overline{x}_{C_1}$ are in T except one, s, in S_{C_1}. Perform the amalgamation relator which replaces s by its identical element in S_{C_2}. This replaces, with cost one, w by a null-homotopic word w' of the same length, entirely consisting of letters in $S_{C_2} \sqcup T$. By Corollary 4.A.13, G_{C_2} has an at most quadratic Dehn function, so w' has an at most quadratic area. □

Proposition 4.F.2 is a first step towards a compact presentation of G.

4.G. **Presentation of the group in the 2-tame case.** The following two theorems, which use the notion of 2-tameness introduced in Definition 2.E.8, are established in Section 6 (Theorem 6.D.2 and Corollary 6.D.3), relying on Section 5.

Theorem 4.G.1 (Presentation of the group, weak form)**.** *Assume that $G = U \rtimes A$ is a* 2-tame *standard solvable group. Then the homomorphism $\hat{G} \to G$ has a central kernel. If moreover the degree zero component of the second homology group $H_2(\mathfrak{u})_0$ vanishes, the kernel of $\hat{G} \to G$ is generated by elements of the form*

$$\exp([\lambda x, y]) \exp([x, \lambda y])^{-1}, \qquad x, y \in \bigcup_C (U_C)_j, \ \ \lambda \in \mathbf{K}_j, \ \ j = 1 \ldots, \tau.$$

We pinpoint the striking fact that $H_2(\mathfrak{u})_0 = \{0\}$ does *not* imply that $\hat{G} \to G$ is an isomorphism. This was pointed out by Abels [Ab87, 5.7.4], and relies on the fact that the space $H_2^{\mathbf{Q}}(\mathfrak{u})_0$ of 2-homology of $\mathfrak{u}$, viewed as a (huge) Lie algebra over the rationals, can be larger than $H_2(\mathfrak{u})_0$. On the other hand, the fact that $\hat{G} \to G$ has a central kernel is a new result, even in Abels' framework ($\mathfrak{u}$ finite length Lie algebra over $\mathbf{Q}_p$); Abels however proved that $\hat{U}$ is $(s+1)$-nilpotent if U is s-nilpotent and this is a major step in the proof.

The elements $\exp([\lambda x, y]) \exp([x, \lambda y])^{-1}$ correspond to certain loops in G, which we call **welding relations**. They will be defined more precisely in §4.H.

Theorem 4.G.1 provides a nice presentation of G, whose Dehn function we bound in the remainder of this section. The basic idea is to reduce the computation of the Dehn function to area estimates of relations of a special form. However, this reduction requires a stronger statement. In short, this stronger statement asserts that the theorem holds over $\mathbf{R}$-algebras. Let us make this more precise. We can view U as the group of $\mathbf{K}$-points of $\mathbb{U}$, where $\mathbb{U}$ is an affine algebraic group over the ring $\mathbf{K} = \prod \mathbf{K}_j$. Thus, for every commutative $\mathbf{K}$-algebra A, $\mathbb{U}(\mathsf{A})$ is the group associated to the nilpotent Lie $\mathbf{Q}$-algebra $\mathfrak{u} \otimes_{\mathbf{K}} \mathsf{A}$. Similarly, $\mathbb{U}_C$ is defined so that $\mathbb{U}_C(\mathsf{A}) \subset \mathbb{U}(\mathsf{A})$ is the exponential of $\mathfrak{u}_C \otimes_{\mathbf{K}} \mathsf{A}$, and we define $\widehat{\mathbb{U}(\mathsf{A})}$ as the corresponding multiamalgam of the $\mathbb{U}_C(\mathsf{A})$.

Theorem 4.G.2 (Presentation of the group, strong (stable) form)**.** *Assume that $G = U \rtimes A$ is a 2-tame standard solvable group. Then for every $\mathbf{K}$-algebra A, the homomorphism $\widehat{\mathbb{U}(\mathsf{A})} \to \mathbb{U}(\mathsf{A})$ has a central kernel. If moreover the zero degree component of the second homology group $H_2(\mathfrak{u})_0$ vanishes, the kernel of $\widehat{\mathbb{U}(\mathsf{A})} \to \mathbb{U}(\mathsf{A})$ is generated by elements of the form*

$$\exp([\lambda x, y]) \exp([x, \lambda y])^{-1}, \qquad x, y \in \bigcup_C \mathbb{U}_{j,C}(\mathsf{A}), \ \ \lambda \in \mathsf{A}_j, \ \ j = 1 \ldots, \tau.$$

Note that Theorem 4.G.1 is equivalent to the case $\mathsf{A} = \mathbf{K}$ of Theorem 4.G.2, given the trivial observation that the kernel of $\hat{G} \to G$ and $\hat{U} \to U$ coincide.

4.H. **Welding relations and compact presentation of G.** In order translate Theorem 4.G.1 into the group-theoretic setting, we need to recall Lazard's formulas. Here we state them as follows.

Theorem 4.H.1 (Lazard [La54])**.** *For every $s \geq 1$, there exist group words $A_s, B_s \in F_2$ and positive integers q_s, q'_s, such that for every simply connected s-nilpotent Lie group G with Lie algebra $\mathfrak{g}$, and all $x, y \in G$, writing $X = \log(x) \in \mathfrak{g}$, $Y = \log(y) \in \mathfrak{g}$, we have*

$$\log(A_s(x,y)) = q_s(X+Y) \text{ and } \log(B_s(x,y)) = q'_s[X,Y]$$

Here A stands for Add, and B for Bracket.

Remark 4.H.2. We see that we have the formal equality of group words $A_s(x,1) = x^{q_s}$. Indeed, $A_s(x,1)$ has the form x^ℓ for some ℓ, and by evaluation in $\mathbf{R}$ we obtain $\ell = q_s$. Similarly, we have the formal equality $B_s(x,1) = 1$.

Corollary 4.H.3. *In every s-nilpotent group G, abbreviating $A = A_s$, $B = B_s$, $q = q_s$, we have identities $\forall x, y, z$,*

$$A(x,y) = A(y,x);\ B(x,y) = B(y,x)^{-1};$$
$$B(A(x,y),z) = A(B(x,z), B(y,z));$$
$$A(A(x,y), z^q) = A(x^q, A(y,z));$$
$$B(x^k, y) = B(x, y^k) = B(x,y)^k \quad \forall k \in \mathbf{Z}.$$

Proof. If G is a simply connected nilpotent Lie group, this follows from the corresponding identities in the Lie algebra: commutativity of addition, anti-commutativity of the bracket, distributivity, associativity of addition; in the last equality if follows from the fact that $\log(x^k) = k\log(x)$. Therefore this holds in every subgroup of such a group G, and in particular in any finitely generated free s-nilpotent group, and therefore in any nilpotent group, by substitution. □

We can now restate the second statement of Theorem 4.G.1 with no reference to the Lie algebra in the conclusion:

Theorem 4.H.4 (Compact presentation of G)**.** *Let $G = U \rtimes A$ be a 2-tame standard solvable group such that $H_2(\mathfrak{u})_0 = \{0\}$. Choose s so that U is s-nilpotent. Then the (central) kernel of $\hat{G} \to G$ is generated by elements of the form*

$$B_{s+1}(x^\lambda, y)B_{s+1}(x, y^\lambda)^{-1}, \quad x, y \in \bigcup U_{j,C}, \ \ \lambda \in \mathbf{K}_j, \ \ j = 1\dots,\tau,$$

where x^λ denotes $\exp(\lambda\log(x))$.

We can view the elements

$$B_{s+1}(x^\lambda, y)B_{s+1}(x, y^\lambda)^{-1}$$

as elements of the free product $H = \ast_C G_C$; now x, y range over the disjoint union $x, y \in \bigsqcup U_{j,C}$. We call these **welding relations** in the free product H.

By substitution, it gives rises to the set of relations in F_S

(4.H.5)
$$R_{\mathrm{weld}} = \left\{ B_{s+1}\left(\overline{x^{\lambda}}, \overline{y}\right) B_{s+1}\left(\overline{x}, \overline{y^{\lambda}}\right)^{-1} \ : x, y \in \bigsqcup U_{j,C}, \ \ \lambda \in \mathbf{K}_j, \ \ j = 1 \dots, \tau \right\}.$$

in the free group F_S. We call these **welding relations**. We define the set R^1_{weld} of **welding relators** as those welding relations for which $\|x\|', \|y\|', |\lambda| \leq 1$, where $\|\cdot\|'$ is the prescribed Lie algebra norm.

It follows from Corollary 5.C.9 that G has a presentation with relators those of $\hat{G}$ (given in Lemma 4.E.3) along with welding relators. At this point, this already reproves Abels' result.

Corollary 4.H.6 (Abels)**.** *If the standard solvable group G is 2-tame and $H_2(\mathfrak{u}_{\mathrm{na}})_0 = \{0\}$, then G is compactly presented.*

Proof. Since G is compactly presented if and only if G/G^0 is compactly presented, we can assume that G is totally disconnected, so $H_2(\mathfrak{u})_0 = \{0\}$ by assumption. The above remarks show that if π' is the natural projection $F_S \to \hat{G}$, then $\pi'(R^1_{\mathrm{weld}})$ generates normally the kernel of $\hat{G} \to G$. Therefore, by Proposition 4.F.2, G admits the compact presentation
$$\langle S \mid R^1_{\mathrm{tame}} \cup R^1_{\mathrm{amalg}} \cup R^1_{\mathrm{weld}} \rangle. \quad \square$$

Remark 4.H.7. Let us pinpoint that this does not coincide, at this point, with Abels' proof. Abels did not prove that the kernel of $\hat{G} \to G$ is generated by welding relators. Instead (assuming that G is totally disconnected), he considered the multiamalgamated product $\dot{U}$ of $\hat{U}$ and a suitable compact open subgroup Ω of U, and $\dot{G} = \dot{U} \rtimes A$. Namely, Ω is the subgroup generated by S_U; it has to be verified that it is indeed open, by a routine verification. It follows that Ω is generated by those intersections $\Omega \cap U_C$, so that $\hat{U} \to \dot{U}$ and $\hat{G} \to \dot{G}$ are surjective. It also easily follows from the definition that $\dot{G}$ is the quotient of $\hat{G}$ by the subgroup generated by generators of bounded length (this uses that Ω is actually *boundedly* generated by S_U, as we see using the Baire category theorem). There is a natural projection $\dot{G} \to G$. Since welding relators are killed by the amalgamation with Ω, we know that the natural projection $\dot{G} \to G$ is an isomorphism. Not having the presentation by welding relators, Abels used instead topological arguments [Ab87, 5.4, 5.6.1] to reach the conclusion that $\dot{G} \to G$ is indeed an isomorphism.

This approach, with the use of a compact open subgroup is, however, "unstable", in the sense that it does not yield a presentation of $\mathbb{U}(\mathsf{A}) \rtimes A$ when A is an arbitrary commutative $\mathbf{K}$-algebra, and the presentation with welding relators will be needed in the sequel in a crucial way when we obtain an upper bound on the Dehn function.

4.I. **Quadratic estimates.** In order to bound the area of welding relations, we first need to prove that certain families of loops have a quadratic filling. Denote by F_c the free group on c generators. Refer to §4.B.2 for the definition of the cones U_C.

Theorem 4.I.1. *Let $G = U \rtimes A$ be a standard solvable group. Assume that G is 2-tame and U is s-nilpotent. Then for every group word $w(x_1, \dots, x_c)$ that belongs to the $(s+2)$-th term $F_c^{(s+2)}$ of the descending central series of the free group F_c, and for any $u_1, \dots, u_c$ in $\bigsqcup_C U_C$, the relation $w(\overline{u_1}, \dots, \overline{u_c})$ in $\hat{G}$ has an at most quadratic area with respect to its total length (the constant not depending on c).*

Intuitively, the statement is that the formal relations of $(s+1)$-nilpotency, evaluated in U, have an at most quadratic area in $\hat{G}$ (and therefore in G). (Note that we consider the abstract disjoint union $\bigsqcup U_i$ whose elements are elements u in some U_C along with the datum of C; in particular $\overline{u}$ is well-defined.)

The proof of Theorem 4.I.1 uses, as an essential and new feature, the fact that $(s+1)$-nilpotency of $\hat{U}$ is a result that is "stable under passing to $\mathbf{K}$-algebras". That is, it is not enough to use Theorem 4.G.1, but we need its "stable" version, Theorem 4.G.2. Precisely, we use that $\widehat{\mathbb{U}(\mathsf{A})}$ is $(s+1)$-nilpotent. (The full result of Theorem 4.G.2 — not only the nilpotency of $\widehat{\mathbb{U}(\mathsf{A})}$ — will be used in §4.K.)

We begin by a discussion, including some notation that will be used in the proof of Theorem 4.I.1. Our goal here is to pinpoint the difficulties we have to encounter, so that the strategy developed in the proof of Theorem 4.I.1 comes up naturally.

Write $\mathcal{C} = \{C_1, \dots, C_\nu\}$, and write $U_j = U_{C_j}$ and $\mathbb{U}_j = \mathbb{U}_{C_j}$. Fix $1 \le \wp_1, \dots, \wp_c \le \nu$ and $\wp = (\wp_1, \dots, \wp_c)$. Since there are finitely many $\wp$ (for a given c), it is enough to prove that for each $\wp$, for every $(u_1, \dots, u_c) \in L^\wp = \prod_i U_{\wp_i}$, the relation $w(\overline{u_1}, \dots, \overline{u_c})$ in $\hat{G}$ has an at most quadratic area with respect to its total length.

Let us first work in the free product H of the U_i. The amalgam $\hat{U}$ is the quotient of H by a finite number of family of amalgamation relators, namely: for every pair (j', j) with $C_{j'} \subset C_j$, define $R_{j',j}$ as set of words $r'r^{-1}$, where $r' \in U_{j'}$ and r is its image in U_j by the natural inclusion. By definition R_{amalg} is the union of all $R_{j',j}$ in H; also define V as the union of all U_j in H.

By Theorem 6.D.1, $\hat{U}$ is $(s+1)$-nilpotent. Thus, for all $(x_1, \dots, x_c) \in L^\wp$, there exist integers m, μ, elements $g_{k\ell}$ in V, $1 \le k \le m$, $1 \le \ell \le \mu$, and amalgamation relators $r_k \in R$ such that, setting

$$g_k = \prod_{\ell=1}^{\mu} g_{k\ell},$$

we have, in H

$$w(x_1,\dots,x_c)=\prod_{k=1}^{m} g_k r_k g_k^{-1}.$$

A problem is that this provides m,μ depending on $(x_1,\dots,x_c)$. To obtain them uniformly, we use the fact that the hypotheses of the theorem are preserved when passing to some suitable algebras of functions. To make this statement clear, let us be more explicit.

If A is any $\mathbf{K}$-algebra, let $\mathbb{U}(\mathsf{A})$ and $\widehat{\mathbb{U}(\mathsf{A})}$ be defined as in §4.G. For any commutative $\mathbf{K}$-algebra, define $\mathbb{H}[\mathsf{A}]$ as the free product of the $\mathbb{U}_C(\mathsf{A})$ (it is a priori not representable by a group scheme). Applying this to relators, we can write, in an obvious natural way, $R_{j',j}=\mathbb{R}_{j',j}(\mathbf{K})$ and thus define $\mathbb{R}(\mathsf{A})$ as the union in $\mathbb{H}[\mathsf{A}]$ of the $\mathbb{R}_{j',j}(\mathsf{A})$, for any commutative $\mathbf{K}$-algebra A. Also define $\mathbb{V}(\mathsf{A})$ as the union of all $\mathbb{U}_j(\mathsf{A})$ in $\mathbb{H}[\mathsf{A}]$. Define $\mathbb{L}^{\wp}(\mathsf{A})=\prod_i \mathbb{U}_{\wp_i}(\mathsf{A})$, so that $L^{\wp}=\mathbb{L}^{\wp}(\mathbf{K})$.

We apply Theorem 6.D.1 to the product algebra $\mathsf{A}=\mathbf{K}^Y$, where Y is an abstract set. It states that $\widehat{\mathbb{U}(\mathsf{A})}$ is $(s+1)$-nilpotent. Observe the obvious identifications

$$\mathbb{U}_j(\mathsf{A})=U_j^Y,\quad \mathbb{L}^{\wp}(\mathsf{A})=(L^{\wp})^Y,\quad \mathbb{R}(\mathsf{A})=R^Y. \tag{4.I.2}$$

This yields, for every $(f_1,\dots,f_c)\in L^Y$ with $f_i\in\mathbb{U}_{\wp_i}(\mathsf{A})=(U_{\wp_i})^Y$ the existence of integers m,μ, elements $h_{k\ell}$ in $\mathbb{V}(\mathsf{A})$, $1\le k\le m$, $1\le\ell\le\mu$, and amalgamation relators $r_k\in\mathbb{R}(\mathsf{A})$ such that, setting

$$h_k=\prod_{\ell=1}^{\mu} h_{k\ell},$$

we have

$$w(f_1,\dots,f_c)=\prod_{k=1}^{m} h_k\rho_k h_k^{-1}. \tag{4.I.3}$$

Now pick Y to be of large enough cardinality (continuum is enough), and apply this to a single $(f_1,\dots,f_c)$ with each $(f_1,\dots,f_c)$ surjective as a function from Y to $L^{\wp}=\prod U_{\wp_i}$. By substitution, we deduce that for all $x_1,\dots,x_c$ with $x_i\in U_{\wp_i}$, there exist $g_{k\ell}$ in V, $1\le k\le m$, $1\le\ell\le\mu$, and amalgamation relators $r_k\in R$ such that, setting $g_k=\prod_{\ell=1}^{\mu} g_{k\ell}$, we have

$$w(x_1,\dots,x_c)=\prod_{k=1}^{m} g_k r_k g_k^{-1}.$$

This solves the problem of uniformity for m. However, to prove the theorem, we need to control the size of r_k in terms of the size of x. We achieve this by replacing $\mathbf{K}^Y$ by an algebra of functions with a suitable growth condition. We can now pass to the proof, properly speaking.

Proof of Theorem 4.I.1. We keep Y to be a large abstract set. Endow $\mathbf{K}$ with the supremum norm. Define the algebra $\mathcal{P}_Y$ as the set of functions $f : (Y \times \mathbf{R}_{\geq 0}) \to \mathbf{K}$ growing at most polynomially, uniformly in Y, i.e. satisfying

$$\exists \alpha > 0,\ \forall t \geq 0,\ \forall y \in Y,\ \|f(y,t)\| \leq (2+t)^{\alpha}.$$

Consider the faithful linear representation of U into GL_q of §4.C and the norm in $\mathrm{M}_q(\mathbf{K})$ defined there, so that the norm $\|\cdot\|$ of any element of U makes sense.

Define $f = (f_1, \dots, f_c) : (Y \times \mathbf{R}_{\geq 0}) \to L^{\wp} = \prod U_{\wp_i}$ in such a way that

- $\|f_i(y,t)\| \leq t$ for all $(y,t) \in Y \times \mathbf{R}_{\geq 0}$;
- for every $(x_1, \dots, x_c) \in L$, there exists $y \in Y$ such that
$$f_i(y, \max_i \|x_i\|) = x_i$$
for all i.

To construct such a f, pick injectively some $y = y(x)$ for each $x = (x_1, \dots, x_c)$, define $f(y(x), \max \|x_i\|) = x$ and define $f(y,t) = 0$ for every (y,t) not of the form $(y(x), \max \|x_i\|)$. By the first condition, $f \in \mathbb{L}^{\wp}(\mathcal{P}_Y)$.

We now use the statement of (4.I.3) with A replaced by $\mathcal{P}_Y$, applied to $(f_1, \dots, f_c)$. This shows that there exist m, μ and $\alpha > 0$ such that for every $(x_1, \dots, x_c)$ with $x_i \in U_i$, there exist $g_{k\ell}$ in V with $\|g_{k\ell}\| \leq \sup_i (2 + \|x_i\|)^{\alpha}$, $1 \leq k \leq m$, $1 \leq \ell \leq \mu$, and amalgamation relators $r_k = r'_k r''^{-1}_k \in R$ with both letters of norm $\leq \sup_i (2 + \|x_i\|)^{\alpha}$ such that, setting $g_k = \prod_{\ell=1}^{\mu} g_{k\ell}$, we have

$$w(x_1, \dots, x_c) = \prod_{k=1}^{m} g_k r_k g_k^{-1}. \tag{4.I.4}$$

Now consider $x_1, \dots, x_c$ with $x_i \in U_{C_i}$. Consider $(g_{k\ell})$ and r_k as in (4.I.4). Set $\overline{g_k} = \prod_{\ell=1}^{\mu} \overline{g_{k\ell}}$, and define $\overline{r_k}$ similarly. Formula (4.I.4) means that the word

$$\varpi = w(\overline{x_1}, \dots, \overline{x_c})^{-1} \prod_{k=1}^{m} \overline{g_k r_k g_k}^{-1} \tag{4.I.5}$$

represents the identity in the free product H. We need to estimate its length: since each of the r'_k, r''_k and $g_{k\ell}$ have norm $\leq \sup_i (2 + \|x_i\|)^{\alpha}$, by Proposition 4.C.2, $\overline{r_k}$ and $\overline{g_k}$ have length linearly bounded above in terms of $\sup_i |\overline{x_i}|$. So ϖ has length linearly bounded in terms of $\sup_i |\overline{x_i}|$.

By Lemma 2.D.1, the area of ϖ is quadratically bounded in terms of $\sup |\overline{x_i}|$. Therefore, to prove that $w(\overline{x_1}, \dots, \overline{x_c})$ has a quadratic area, it is enough to check that each $\overline{r_k}$ has a quadratic area. Recall from Corollary 4.A.5 that each $U_i \rtimes A$ has a quadratic Dehn function. The loop defined by the path $\overline{r_k}$ lies inside some $U_i \rtimes A$; therefore, using again that the length of $\overline{r_k}$ is linearly bounded in terms of $\sup_i |\overline{x_i}|$, its area is quadratically bounded. Thus the area of $w(\overline{x_1}, \dots, \overline{x_c})$ has a quadratic area in terms of its total length (which is $\geq \sup_i |\overline{x_i}|$, since we can assume that w is not the trivial word). ☐

4.J. **Area of welding relators.** Here we prove the following

Theorem 4.J.1. *If the standard solvable group $G = U \rtimes A$ is 2-tame, then welding relations in G have an at most cubic area.*

More precisely, there exists a constant K such that for all j, all $x, y \in \bigcup_C U_{j,C}$, and all $\lambda \in \mathbf{K}_j$, the welding relation (4.H.5) has area at most Kn^3, with $n = \log(1 + \|x\| + \|y\| + |\lambda|)$.

Let us assume that the subgroup U, in Theorem 4.J.1, is s-nilpotent. In all this subsection, we write $A = A_{s+1}$, $B = B_{s+1}$, $q = q_{s+1}$. Theorem 4.I.1 applies to the group words corresponding to equalities of Corollary 4.H.3, providing

Proposition 4.J.2. *If G is 2-tame and s-nilpotent, for all $x \in U_{C_1}$, $y \in U_{C_2}$ and $z \in U_{C_3}$, the relations*

$$A(\overline{x}, \overline{y}) = A(\overline{y}, \overline{x});\; B(\overline{x}, \overline{y}) = B(\overline{y}, \overline{x})^{-1}$$
$$B(A(\overline{x}, \overline{y}), \overline{z}) = A(B(\overline{x}, \overline{z}), B(\overline{y}, \overline{z}));$$
$$A(A(\overline{x}, \overline{y}), \overline{z}^q) = A(\overline{x}^q, A(\overline{y}, \overline{z}))$$
$$B(\overline{x}^k, \overline{y}) = B(\overline{x}, \overline{y}^k) = B(\overline{x}, \overline{y})^k$$

have an at most quadratic area in $\hat{G}$. ☐

Note that in the last case, the quadratic upper bound is of the form $c_k n^2$, where c_k may depend on k.

We now proceed to prove Theorem 4.J.1. Using (with quadratic cost) the "bilinearity" of B (or restricting scalars from the beginning), we can suppose that $\mathbf{K}_j$ is equal to $\mathbf{R}$ or $\mathbf{Q}_p$ (although the forthcoming argument can be adapted with unessential modifications to their finite extensions). If $\mathbf{K}_j = \mathbf{Q}_p$, define $\pi_j = p$. If $\mathbf{K} = \mathbf{R}$, define $\pi_j = 2$. Define $\Lambda_j^1(n) \subset \mathbf{Q}$ as

$$\Lambda_j^1(n) = \left\{ \lambda = \sum_{i=0}^{n-1} \varepsilon_i \pi_j^i :\; \varepsilon_i \in \{-\pi_j + 1, \dots, \pi_j - 1\} \right\}.$$

Lemma 4.J.3. *For every κ, there exists K such that for all n, all j, all $x, y \in \bigcup U_{j,C}$ with $\log(1 + \|x\| + \|y\|) \leq n$ and all $\lambda \in \Lambda_1^j(\kappa n)$, the welding relation*

$$B\left(\overline{x^\lambda}, \overline{y}\right) B\left(\overline{x}, \overline{y^\lambda}\right)^{-1}$$

has area $\leq Kn^3$.

Proof. We can work for a given j, so we write $\pi = \pi_j$. Write

$$\lambda = \sum_{i=0}^{\kappa n-1} \varepsilon_i \pi_j^i \quad (\varepsilon_i \in \{-\pi_j + 1, \dots, \pi_j - 1\}).$$

If we set

$$\lambda_i = \sum_{j=i}^{\kappa n-1} \varepsilon_j \pi^{j-i},$$

we have $\lambda_0 = \lambda$, $\lambda_{\kappa n} = 0$, and, for all i

$$\lambda_i = \pi\lambda_{i+1} + \varepsilon_i$$

Set $z = q^{-1}y$ and $\sigma_i = \sum_{j=1}^{i} \varepsilon_i \pi^i$ (so $\sigma_{-1} = 0$); consider the word

$$\Phi_i = A\left(B\left(\overline{\lambda_i x}, \overline{\pi^i z}\right), B\left(\overline{x}, \overline{\sigma_{i-1} z}\right)\right)$$

Here, for readability, we write $\overline{\lambda x}$ (etc.) instead of $\overline{x^\lambda}$. This is natural since x can be identified to its Lie algebra logarithm. For $i = 0$, $\sigma_{i-1} = 0$ so, since $\overline{1} = 1$ and, formally, $B(x, 1) = 1$ and $A(x, 1) = x^q$ (see Remark 4.H.2), we have

$$\Phi_0 = B\left(\overline{\lambda x}, \overline{z}\right)^q.$$

For $i = \kappa n$, $\lambda_n = 0$ and $\sigma_{i-1} = \lambda$ so this is (using that formally $B(1, y) = 1$ and $A(1, y) = y^q$)

$$\Phi_{\kappa n} = B\left(\overline{x}, \overline{\lambda z}\right)^q.$$

Let us show that we can pass from Φ_i to Φ_{i+1} with quadratic cost. In the following computation, each $\rightsquigarrow$ means one operation with quadratic cost, i.e., with cost $\le K_0 n^2$ for some constant K_0 only depending on the group presentation. The tag on the right explains why this quadratic operation is valid, namely:

- (1) means both the homotopy between loops lies in one tame subgroup.
- (2) means the operation follows from Proposition 4.J.2; to be specific:
  - $(2)_{\text{distr left}}$ for
    $$B(A(x, y), z) = A(B(x, z), B(y, z))$$
    and similarly $(2)_{\text{distr right}}$ on the right
  - $(2)_Q$ for an equality of the type
    $$B(x, y^k) = B(x^k, y) = B(x, y)^k,$$
    with k an integer satisfying $|k| \le \max(q, \pi)$.
  - $(2)_{\text{assoc}}$ for the identity $A(A(x, y), z^q) = A(x^q, A(y, z))$.
- Or a tag referring to a previous computation, written in brackets.

(4.J.4)
$$\begin{aligned}
\overline{\lambda_i x} = &\ \overline{q(\pi\lambda_{i+1} q^{-1}x + \varepsilon_i q^{-1}x)} & \\
\rightsquigarrow &\ \left(\overline{\pi\lambda_{i+1} q^{-1}x + \varepsilon_i q^{-1}x}\right)^q & (1)\\
\rightsquigarrow &\ \left(\overline{\pi\lambda_{i+1} q^{-1}x} + \overline{\varepsilon_i q^{-1}x}\right)^q & (1)\\
\rightsquigarrow &\ A\left(\overline{\pi\lambda_{i+1} q^{-1}x},\ \overline{\varepsilon_i q^{-1}x}\right) & (1)
\end{aligned}$$

So by substitution we obtain

(4.J.5)

$$\begin{aligned} B\left(\overline{\lambda_i x}, \overline{\pi^i z}\right) &\rightsquigarrow B\left(A\left(\overline{\pi\lambda_{i+1}q^{-1}x},\ \overline{\varepsilon_i q^{-1}x}\right), \overline{\pi^i z}\right) && [4.J.4] \\ &\rightsquigarrow A\left(\ B\left(\overline{\pi\lambda_{i+1}q^{-1}x}, \overline{\pi^i z}\right)\ , B\left(\overline{\varepsilon_i q^{-1}x}, \overline{\pi^i z}\right)\ \right) && (2)_{\text{distr left}} \\ &\rightsquigarrow A\left(\ B\left(\overline{\pi\lambda_{i+1}q^{-1}x}, \overline{\pi^i z}\right)\ , B\left(\overline{q^{-1}x}, \overline{\varepsilon_i\pi^i z}\right)\ \right) && (2)_Q \end{aligned}$$

Independently we have

(4.J.6)

$$\begin{aligned} B\left(\overline{x}, \overline{\sigma_{i-1}z}\right) &\rightsquigarrow B\left(\overline{q^{-1}x}^{q}, \overline{\sigma_{i-1}z}\right) && (1) \\ &\rightsquigarrow B\left(\overline{q^{-1}x}, \overline{\sigma_{i-1}z}\right)^{q} && (2)_Q \end{aligned}$$

Again by substitution, this yields

(4.J.7)

$$\begin{aligned} \Phi_i =& A\left(B\left(\overline{\lambda_i x}, \overline{\pi^i z}\right), B\left(\overline{x}, \overline{\sigma_{i-1}z}\right)\right) \\ \rightsquigarrow& A\left(A\left(\ B\left(\overline{\pi\lambda_{i+1}q^{-1}x}, \overline{\pi^i z}\right)\ , B\left(\overline{q^{-1}x}, \overline{\varepsilon_i\pi^i z}\right)\ \right), B\left(\overline{q^{-1}x}, \overline{\sigma_{i-1}z}\right)^{q}\right) && [4.J.5, 4.J.6] \\ \rightsquigarrow& A\left(B\left(\overline{\pi\lambda_{i+1}q^{-1}x}, \overline{\pi^i z}\right)^{q}, A\left(B\left(\overline{q^{-1}x}, \overline{\varepsilon_i\pi^i z}\right), B\left(\overline{q^{-1}x}, \overline{\sigma_{i-1}z}\right)\right)\right) && (2)_{\text{assoc}} \end{aligned}$$

(4.J.8)

$$\rightsquigarrow A\left(B\left(\overline{\pi\lambda_{i+1}q^{-1}x}, \overline{\pi^i z}\right)^{q}, B\left(\overline{q^{-1}x}, A\left(\overline{\varepsilon_i\pi^i z}, \overline{\sigma_{i-1}z}\right)\right)\right) \qquad (2)_{\text{distr right}}$$

By similar arguments

$$B\left(\overline{\pi\lambda_{i+1}q^{-1}x}, \overline{\pi^i z}\right)^{q} \overset{(2)_Q}{\rightsquigarrow} B\left(\overline{\lambda_{i+1}x}, \overline{\pi^{i+1}z}\right),$$

and

$$\begin{aligned} B\left(\overline{q^{-1}x}, A\left(\overline{\varepsilon_i\pi^i z}, \overline{\sigma_{i-1}z}\right)\right) &\rightsquigarrow B\left(\overline{q^{-1}x}, \overline{q(\varepsilon_i\pi^i z + \sigma_{i-1}z)}\right) && (1) \\ &\rightsquigarrow B\left(\overline{x}, \overline{\varepsilon_i\pi^i z + \sigma_{i-1}z}\right) && (2)_Q \\ &= B\left(\overline{x}, \overline{\sigma_i z}\right) \end{aligned}$$

so substituting from (4.J.8) we get

$$\Phi_i \rightsquigarrow A\left(B\left(\overline{\lambda_{i+1}x}, \overline{\pi^{i+1}z}\right), B\left(\overline{x}, \overline{\sigma_i z}\right)\right) \quad = \Phi_{i+1}$$

in quadratic cost, say $\le K_1 n^2$ (noting that each Φ_i has length $\le K_2 n$ for some fixed constant K_2). Note that the constant K_1 only depends on the group presentation, because the above estimates use a quadratic filling only finitely many times, each among finitely many types (note that we used $(2)_Q$ only for k in a bounded interval, only depending on $\mathbf{K}$ and s).

It follows that we can pass from Φ_0 to $\Phi_{\kappa n}$ with cost $\le K_1 \kappa n^3$. On the other hand, by substitution of type $(2)_Q$, we can pass with quadratic cost from $B(\overline{\lambda x}, \overline{y})$ to $B(\overline{\lambda x}, \overline{q^{-1}y})^q = \Phi_0$ and from $\Phi_{\kappa n} = B(\overline{x}, \overline{\lambda q^{-1} y})^q$ to $B(\overline{x}, \overline{\lambda y})$. So the proof of the lemma is complete. ☐

Conclusion of the proof of Theorem 4.J.1. Let now $\Lambda_j^2(n)$ be the set of quotients λ'/λ'' with $\lambda', \lambda'' \in \Lambda_j^1(n)$, $\lambda'' \neq 0$. Lemma 4.J.3 immediately extends to the case when $\lambda \in \Lambda_j^2$. It follows from the definition that $\Lambda_j^2(\kappa n)$ contains all elements of the form $\lambda = \sum_{i=-\kappa n}^{\kappa n} \varepsilon_i \pi_j^i$, with $\varepsilon_i \in \{-\pi_j, \dots, \pi_j\}$.

If $\mathbf{K}_j = \mathbf{R}$, $\pi_j^{\kappa n} \Lambda_j^2(\kappa n)$ contains all integers between $-\pi_j^{-2\kappa n}$ and $\pi_j^{2\kappa n}$. Thus $\Lambda_j^2(\kappa n)$ contains a set which is $|\pi^{-\kappa n}|$-dense in the ball of radius $|\pi^{\kappa n}|$. If $\mathbf{K}_j = \mathbf{Q}_p$, $\pi^{\kappa n} \Lambda_j^2(\kappa n)$ contains a $|\pi^{2\kappa n}|$-dense subset of $\mathbf{Z}_p$. Thus $\Lambda_j^2(\kappa n)$ contains a $|\pi^{\kappa n}|$-dense subset of the ball of radius $|\pi^{-\kappa n}|$. In both cases, defining $\varrho_j = \max(|\pi_j|, |\pi_j|^{-1})$, we obtain that $\Lambda_j^2(\kappa n)$ contains a $\varrho_j^{-\kappa n}$-dense subset of the ball of radius $\varrho_j^{\kappa n}$ in $\mathbf{K}_j$.

We now fix j and write $\varrho = \varrho_j$, $\pi = \pi_j$. We pick $\kappa = 2/\log(\varrho)$, so that $\varrho^{\kappa n} = e^{2n}$. We assume that $n \ge \log(1/|q|)$, where $|q|$ is the norm of q in $\mathbf{K}_j$ (if $\mathbf{K}_j = \mathbf{R}$ this is an empty condition). It follows that $\varrho^{-\kappa n} |q|^{-1} e^n \le 1$.

Now fix $\lambda \in \mathbf{K}_j$ with $|\lambda| \le e^n$. We need to prove that we can pass from $B(\overline{\lambda x}, \overline{y})$ to $B(\overline{x}, \overline{\lambda y})$ with cubic cost; clearly it is enough to pass from $B(\overline{\lambda x}, \overline{y})^q$ to $B(\overline{x}, \overline{\lambda y})^q$ with cubic cost.

Since $|\lambda| \le e^n \le \varrho^{\kappa n}$, we can write $\lambda = \mu + q\varepsilon$ with $\mu \in \Lambda^2(\kappa n)$ and $|q\varepsilon| \le \varrho^{-\kappa n}$. So $|\varepsilon| \le \varrho^{-\kappa n} |q|^{-1} \le e^{-n}$.

Also, assume that $n \ge \log(\varrho)$. So we can find an integer k with $n/\log(\varrho) \le k \le 2n/\log(\varrho)$. Thus, if we define $\eta = \pi^{\pm k}$, with the choice of sign so that $|\eta| > 1$; we have $e^n \le |\eta| \le e^{2n} \le e^n |\varepsilon|^{-1}$.

Using a computation as in (4.J.4), we obtain, we quadratic cost

$$\begin{aligned} B(\overline{\lambda x}, \overline{y}) =& B\left(\overline{(\mu + q\varepsilon)x}, \overline{y}\right) \\ \rightsquigarrow & B\left(A(\overline{\mu q^{-1} x}, \overline{\varepsilon x}), \overline{y}\right) && [4.J.4] \\ \rightsquigarrow & A\left(B\left(\overline{\mu q^{-1} x}, \overline{y}\right), B(\overline{\varepsilon x}, \overline{y})\right) && (2)_{\text{distr left}} \end{aligned}$$

and similarly, with quadratic cost.

$$B\left(\overline{x}, \overline{\lambda y}\right) \rightsquigarrow A\left(B(\overline{x}, \overline{\mu q^{-1} y}), B(\overline{x}, \overline{\varepsilon y})\right)$$

By the previous case, with cubic cost we have

$$B\left(\overline{\mu q^{-1}x}, \overline{y}\right) \rightsquigarrow B\left(\overline{x}, \overline{\mu q^{-1}y}\right)$$

So it remains to check that with cubic cost we have

$$B(\overline{\varepsilon x}, \overline{y}) \rightsquigarrow B(\overline{x}, \overline{\varepsilon y}). \tag{4.J.9}$$

If η is the element introduced above, observe that $\eta \in \Lambda^2_j(\kappa n)$ and $e^n \le |\eta| \le e^n|\varepsilon|^{-1}$. We have, with cubic cost

$$B(\overline{\varepsilon x}, \overline{y}) \rightsquigarrow B(\overline{\eta\varepsilon x}, \overline{\eta^{-1}y}); \quad B(\overline{\eta^{-1}x}, \overline{\eta\varepsilon y}) \rightsquigarrow B(\overline{x}, \overline{\varepsilon y}). \tag{4.J.10}$$

Since $\max(|\eta\varepsilon|, |\eta|^{-1}) \le e^{-n}$, it follows that all four elements $\eta\varepsilon x$, $\eta^{-1}y$, $\eta^{-1}x$, εy have norm at most one, and it follows that we can perform

$$B\left(\overline{\eta\varepsilon x}, \overline{\eta^{-1}y}\right) \rightsquigarrow B\left(\overline{\eta^{-1}x}, \overline{\eta\varepsilon y}\right) \tag{4.J.11}$$

by application of a single welding relator. So (4.J.9) follows from (4.J.10) and (4.J.11). ☐

4.K. **Area of words of bounded combinatorial length.** To estimate the area of arbitrary words, we need a generalization of Theorem 4.I.1. Because of the incurring formalism, let us give a self-contained treatment. Therefore, let us forget all the previously introduced notation, although we can have it in mind (so as to apply it to the previous setting in §4.L).

By $\mathbf{K}$ we mean a finite product of local fields (in a first reading, we can assume it is a single local field). Let A be a fixed discrete group. Let $\mathbb{U}_1, \dots, \mathbb{U}_\nu$ be affine $\mathbf{K}$-group schemes of finite type, each with an action of A; write $U_i = \mathbb{U}_i(\mathbf{K})$. Assume that for each i, we have a compact presentation

$$\langle S_i \mid \Pi_i \rangle$$

of $U_i \rtimes A$. Given fixed $\mathbf{K}$-embeddings of $\mathbb{U}_i$ in SL_q, the norm of an element of U_i makes sense. We assume that the length $|g|$ of any $g \in U_i$ with respect to S_i is $\simeq S_i$. For each $x \in U_i$, we assume that the word length of x with respect to S_i is $\simeq \log(1 + \|x\|)$ (if $U_i \rtimes A$ is a standard solvable group, this assumption is fulfilled by Proposition 4.C.2). Then, for $x \in U_i$, fix a representing word $\overline{x}$ in S_i, of size $\simeq \log(1 + \|x\|)$; we assume that we can do so.

We introduce some objects of the form $\mathbb{X}[\mathsf{A}]$, we use brackets rather than parentheses to emphasize that these objects are possibly not representable by a scheme.

Fix $1 \le i_1, \dots, i_k \le \nu$. Consider finitely many closed subschemes $\mathbb{R}_\ell \subset \prod_{\ell=1}^k \mathbb{U}_{i_\ell}$, globally invariant by the action of A (thought of as algebraically parameterized words, doomed to be relators). For convenience, write $\mathbb{R}[\mathsf{A}] = \bigcup_\ell \mathbb{R}_\ell(\mathsf{A})$ for any $\mathbf{K}$-algebra A (it would be representable if $\mathbf{K}$ were a field; anyway this is not an issue).

Let $\mathbb{H}[\mathsf{A}]$ be the free product $\ast_{i=1}^{\nu}\mathbb{U}_i(\mathsf{A})$. There is an obvious product map $\pi_{\mathsf{A}}:\prod_{\ell=1}^{k}\mathbb{U}_{i_\ell}(\mathsf{A})\to\mathbb{H}[\mathsf{A}]$. Define $\mathbb{Q}[\mathsf{A}]$ as the quotient of $\mathbb{H}[\mathsf{A}]$ by the normal subgroup generated by $\pi_{\mathsf{A}}(\mathbb{R}[\mathsf{A}])$. Informally, $\mathbb{Q}$ is a group generated by algebraic generators and algebraic sets of relators. A priori, $\mathbb{Q}$ is not representable by a group scheme over $\mathbf{K}$ (e.g., if $\mathbb{R}$ is empty, $\mathbb{Q}(\mathbf{K})$ is the free product $\mathbb{H}(\mathbf{K})$).

Now fix an integer $c\ge 0$ and $1\le\wp_1,\dots,\wp_c\le\nu$; if $w\in F_c$, define

$$\mathbb{L}_w^{\wp}[\mathsf{A}]=\left\{(f_1,\dots,f_c)\in\prod_{i=1}^{c}\mathbb{U}_{\wp_i}:\ w(f_1,\dots,f_c)=1\text{ in }\mathbb{Q}(\mathsf{A})\right\}.$$

In particular, $\mathbb{L}_1^{\wp}[\mathsf{A}]=\prod_{i=1}^{c}\mathbb{U}_{\wp_i}(\mathsf{A})$, so we write as $\mathbb{L}_1^{\wp}(\mathsf{A})$. In general, for $w\neq 1$, we cannot a priori represent $\mathbb{L}_w^{\wp}$ is by a scheme.

Theorem 4.K.1. *Fix c, $\wp$, $w\in F_c$, and define $\mathbb{L}_w^{\wp}$ as above. Assume that $\mathbb{L}_w^{\wp}$ is representable by a $\mathbf{K}$-scheme, i.e. there exists a closed subscheme $\mathbb{M}\subset\prod_{i=1}^{c}\mathbb{U}_{\wp_i}$ such that for every (reduced) commutative $\mathbf{K}$-algebra A we have $\mathbb{M}(\mathsf{A})=\mathbb{L}_w^{\wp}[\mathsf{A}]$ (equality as subsets of $\mathbb{L}_1^{\wp}(\mathsf{A})$). Assume in addition the following*

- *all presentations $\langle S_i\mid\Pi_i\rangle$ have Dehn function bounded above by some superadditive function δ_1;*
- *there is a compact subset $R\subset\mathbb{R}(\mathbf{K})$ such that for every $r\in\mathbb{R}(\mathbf{K})$, the area of $\overline{r}$ with respect to $\langle\bigcup S_i\mid\bigcup\Pi_i\cup R\rangle$ is finite and $\le\delta_2(|\overline{r}|)$.*

Then for every $(x_1,\dots,x_c)\in\mathbb{L}_w^{\wp}[\mathbf{K}]$, the area of $w(\overline{x_1},\dots,\overline{x_c})$ is $\preceq(\delta_1+\delta_2)(\sup|\overline{x_i}|)$ (where the constant may depend on c).

Here, if $r=(\rho_1,\dots,\rho_k)\in\mathbb{R}(\mathbf{K})$, we write $\overline{r}=\overline{\rho_1}\dots\overline{\rho_k}$, and $|\cdot|$ denotes the word length in the free group F_S, where $S=\bigsqcup S_i$.

Remark 4.K.2. The language of schemes is essentially here for convenience. For the reader not comfortable with it, we can avoid its use at the cost of introducing some further subscripts. Write $\mathbf{K}=\bigoplus\mathbf{K}_j$ with $\mathbf{K}_j$ a field, decompose $\mathbb{U}_i=\prod_j\mathbb{U}_{j,i}$, and define $\mathbb{L}_{j,w}^{\wp}\subset\prod_{i=1}^{c}\mathbb{U}_{j,\wp_i}$ accordingly. The representability assumption of Theorem 4.K.1 can be restated as follows: for all j there is a $\mathbf{K}_j$-closed subvariety $\mathbb{M}_j$ of $\prod_{i=1}^{c}\mathbb{U}$ such that for all (reduced) commutative $\mathbf{K}_j$-algebra A we have $\mathbb{L}_{j,w}^{\wp}[\mathsf{A}]=\mathbb{M}_j(\mathsf{A})$.

Proof. The proof follows the steps of that of Theorem 4.I.1.

If Y is an abstract set, let us first observe that there is an obvious inclusion $\mathbb{L}_w^{\wp}\left[\mathbf{K}^Y\right]\subset\mathbb{L}_w^{\wp}[\mathbf{K}]^Y$, but if Y is infinite, it does not follow from the definition that it is an equality. However, $\mathbb{M}$ being a affine scheme, it is obvious that $\mathbb{M}(\mathbf{K})^Y=\mathbb{M}(\mathbf{K}^Y)$. Moreover, consider the algebra $\mathcal{P}_Y=\mathcal{P}_Y(\mathbf{K})$ of functions $Y\times\mathbf{R}_{\ge 0}$ of at most polynomial growth, uniformly in Y. For any affine $\mathbf{K}$-scheme X of finite type (with a given embedding in an affine space), $X(\mathbf{K})$ inherits of a norm and the set $\mathcal{P}_Y(X(\mathbf{K}))$ is well-defined.

There is an obvious inclusion $\mathbb{L}_w^{\wp}[\mathcal{P}_Y(\mathbf{K})]\subset\mathcal{P}_Y[\mathbb{L}_w^{\wp}(\mathbf{K})]$. For the same reason as previously, it is not a priori an equality. Nevertheless, the equality $\mathbb{M}(\mathcal{P}_Y(\mathbf{K}))=\mathcal{P}_Y(\mathbb{M}(\mathbf{K}))$ is clear, so $\mathbb{L}_w^{\wp}[\mathcal{P}_Y(\mathbf{K})]=\mathcal{P}_Y[\mathbb{L}_w^{\wp}(\mathbf{K})]$.

Therefore the argument of the proof of Theorem 4.I.1 carries over. Let V be the union of all U_j in H. We obtain that there exist m, μ and $\alpha > 0$ such that for every $(x_1, \dots, x_c) \in \mathbb{L}_w^{\wp}[\mathbf{K}]$, there exist $g_{k\ell}$ in V with $\|g_{k\ell}\| \le \sup(1+\|x_i\|)^\alpha$, $1 \le k \le m$, $1 \le \ell \le \mu$, and r_k in $R = \mathbb{R}[\mathbf{K}]$, with each letter of norm $\le \sup(1 + \|x_i\|)^\alpha$, such that, setting $g_k = \prod_{\ell=1}^{\mu} g_{k\ell}$, we have, in $\mathbb{H}[\mathbf{K}]$

$$w(x_1, \dots, x_c) = \prod_{k=1}^{m} g_k r_k g_k^{-1}.$$

We continue as in the proof of Theorem 4.I.1. Since by the assumptions, for each i and $x \in U_i$ we have $|\overline{x}| \simeq \log(1 + \|x\|)$, the word

$$w(\overline{x_1}, \dots, \overline{x_c})^{-1} \prod_{k=1}^{m} \overline{g_k r_k g_k}^{-1} \tag{4.K.3}$$

has linear size, say $\le sn$ with respect to $n = \sup |\overline{x_i}|$. By Lemma 2.D.1, the area of (4.K.3) with respect to $\langle \bigsqcup S_i \mid \bigsqcup \Pi_i \rangle$ is $\le \delta_1(sn)$. The remaining contribution is that of the r_k and is $\le \sum_{k=1}^{m} \delta_2(|\overline{r_k}|) \preceq \delta_2(n)$ by assumption. ☐

4.L. Concluding step for standard solvable groups.

Theorem 4.L.1. *Let G be a standard solvable group. If G is 2-tame and $H_2(\mathfrak{u})_0 = 0$ then its Dehn function is at most cubic. If moreover* $\mathrm{Kill}(\mathfrak{u})_0 = \{0\}$ *then its Dehn function is quadratic (or linear in case A has rank one).*

Theorem 4.K.1 provides area estimates for null-homotopic words of bounded combinatorial length, i.e., of the form $w(\overline{x_1}, \dots, \overline{x_c})$ for bounded c. The remaining essential ingredient is "Gromov's trick" described in §2.D.3. It allows to reduce the study to the area of such words form some bounded c (only depending on the group).

Proof of Theorem 4.L.1. Let us begin by the second statement. If A has rank one, then 2-tame implies tame, and in that case, the Dehn function is linear (see Remark 4.A.6). Otherwise, if A has rank at least 2, the Dehn function is at least quadratic. So let us assume that G is 2-tame with $H_2(\mathfrak{u})_0 = \mathrm{Kill}(\mathfrak{u})_0 = \{0\}$ and prove that its Dehn function is at most quadratic.

Let us apply Theorem 4.K.1. The $U_i = \mathbb{U}_i(\mathbf{K})$ are the tame subgroups, with suitable compact presentations, which can be chosen to have an at most quadratic Dehn function, by Corollary 4.A.5. The $\mathbb{R}_j$ consist of the amalgamation relations. Picking a suitable compact subset of $\mathbb{R}_j(\mathbf{K})$ as in §4.F, the amalgamation relations have an at most quadratic area by Proposition 4.F.3. In the notation introduced before Theorem 4.K.1, the group $\mathbb{Q}(\mathbf{K})$ is equal to $\hat{U}$. For every commutative $\mathbf{K}$-algebra A, we have $H_2^{\mathsf{A}}(\mathfrak{u} \otimes_{\mathbf{K}} \mathsf{A})_0 = H_2^{\mathbf{K}}(\mathfrak{u})_0 \otimes_{\mathbf{K}} \mathsf{A} = \{0\}$, and similarly $\mathrm{Kill}^{\mathsf{A}}(\mathfrak{u} \otimes_{\mathbf{K}} \mathsf{A})_0 = \{0\}$. Thus by Corollary 6.D.4, $\mathbb{Q}(\mathsf{A}) \to \mathbb{U}(\mathsf{A})$ is an isomorphism. Therefore, $\mathbb{Q}$ is an affine group scheme, and thus we can apply Theorem 4.K.1. Fixing c, we apply it with $w = t_1 \dots t_c$ (where t_i are the free generators of F_c).

So for every c and $1 \le \wp_1, \dots, \wp_c \le \nu$, there exists a constant $C = C(c, \wp)$ such that for all $(x_1, \dots, x_c) \in \prod_i U_{\wp_i}$ such that $x_1 \dots x_c$ represents 1 in G (i.e., $(x_1, \dots, x_c) \in L_w^{\wp}$), the area of $\overline{x_1} \dots \overline{x_c}$ is at most $C(n^2 + 1)$, where $n = \sum |\overline{x_i}|$. Note that C can be chosen depending only on c (by considering $\sup_{\wp} C(c, \wp)$, where $\wp$ ranges over $\{1, \dots, \nu\}^c$).

Recalling that each $\overline{x_i}$ has the form dsd^{-1} with s a letter in U and d a word in A, we conclude by Proposition 2.D.7.

Now let us assume $\mathrm{Kill}(\mathfrak{u})_0$ arbitrary. The above proof can be repeated, but we have to include welding relators in the presentation. In the argument, we also have to deal, at the end, with relations of quadratic area, as well as welding relations. By Theorem 4.J.1, those have at most cubic area. This proves that for every c there exists a constant K'' such that for every null-homotopic element of the form $\overline{x_1} \cdots \overline{x_c}$ with $x_1, \dots, x_c$ in $\bigsqcup_C U_C$, has area $\le K'' n^3$. Again, we conclude by Proposition 2.D.7. □

Remark 4.L.2. If $\mathrm{Kill}(\mathfrak{u})_0 \neq \{0\}$, then the Dehn function δ of $\hat{G}$ with respect to the presentation $\langle S \mid R^1_{\mathrm{tame}} \cup R^1_{\mathrm{amalg}} \rangle$ is infinite for n large enough. Precisely, suppose $\mathrm{Kill}(\mathfrak{u})_{j,0} \neq \{0\}$. Consider the set of welding relations

$$B(\overline{\lambda x}, \overline{y}) B(\overline{x}, \overline{\lambda y})^{-1}, \quad x, y \in \bigcup_C U_{j,C},\ \|x\|', \|y\|' \le 1,\ \lambda \in \mathbf{Q} \text{ with } |\lambda|_{\mathbf{K}_j} \le 1.$$

Since $\lambda \in \mathbf{Q}$, these are indeed relations in $\hat{G}$. These are relations of length $2N$, where N is the length of the formal word $B(x, y)$. Suppose by contradiction that $\delta(2N) = k < \infty$. Then we can write, for all $x, y \in \bigcup_C U_{j,C}$ with $\|x\|' \le 1$, $\|y\|' \le 1$, $\lambda \in \mathbf{Q}$ with $|\lambda| \le 1$,

$$B(\overline{\lambda x}, \overline{y}) B(\overline{x}, \overline{\lambda y})^{-1} = \prod_{i=1}^{k} g_i r_i g_i^{-1} \qquad \text{in } F_S$$

The r_i have bounded length, and a standard argument based on van Kampen diagrams (see Lemma 2.D.2) shows that the $g_i = g_i(x, y, \lambda)$ can be chosen to have bounded length. Push this forward to $H \rtimes A$ (recall that H is the free product of all U_i). Then the tame relators and the relators of T are killed, so the remaining r_i are amalgamation relators $i_{C_1}(s) i_{C_2}(s)^{-1}$ (or their inverses) for $C_1 \subset C_2$ and $s \in S_{C_1}$. By a compactness argument, it follows that all $B(\lambda x, y) B(x, \lambda y)^{-1}$ for all λ in the closed unit ball of $\mathbf{K}_j$ and all x, y in $\bigcup_C U_{j,C}$ of norm ≤ 1, are products of amalgamation relators in H. By Corollary 5.C.9, it follows that $W_2(\mathfrak{u}_j)^{\mathbf{Q}, \mathbf{K}_j} = \{0\}$ and by Theorem 5.C.13 this contradicts $\mathrm{Kill}(\mathfrak{u})_{j,0} \neq \{0\}$.

4.M. **Generalized standard solvable groups.** We define a generalized standard solvable group as a locally compact group of the form $U \rtimes N$, where the definition is exactly as for standard solvable groups (Definition 1.2), except that N is supposed to be nilpotent instead of abelian. Such a group is *tame* if some element c of N acts on U as a compaction. Clearly split triangulable Lie groups are special cases of generalized standard solvable groups.

Theorem 4.M.1. *Let G be a generalized standard solvable group not satisfying any of the (SOL or 2-homological) obstructions. Then $\delta_G(n) \preceq n\delta_N(n)$. If moreover* $\mathrm{Kill}(\mathfrak{u})_0 = \{0\}$*, then $\delta_G(n) \preceq \delta_N(n)$*

The reduction of Theorem 4.M.1 to the case where G is tame can be transposed without any change from the standard solvable case. In particular, one obtains from the proof of Theorem 4.J.2 that δ_G is controlled by n times the maximum over Dehn functions of tame subgroups of G. If $G = U \rtimes N$ is a tame generalized standard solvable group it is tempting to believe that, in a way analogous to §4.A, there is a large-scale Lipschitz deformation retraction of G onto N. However, the proof only carries over when the element c of N acting as a compaction of N can be chosen to be central in N. Unfortunately, this can not always be assumed and we need a more complicated approach.

Theorem 4.M.2 (The tame case)**.** *Consider a generalized standard solvable group $G = U \rtimes N$ such that there exists $c \in N$ acting on U as a compaction. Then the Dehn function of G is equivalent to that of N.*

Proof. Recall that we write group commutators as $(\!(x, y)\!) = x^{-1}y^{-1}xy$; we also use the standard notation $x^y = y^{-1}xy$. Iterated commutators are defined in (2.G.1).

Since N is a Lipschitz retract of G, we have $\delta_N \preceq \delta_G$; let us prove that $\delta_G \preceq \delta_N$. Let $S = S_U \cup S_N$ be a compact generating set containing a vacuum set for c. let us define

$$\mathcal{F}_U = \{c^n s c^{-n} \in F_S,\ s \in S_U, n \in \mathbf{N}\}.$$

A straightforward adaptation of the proof of Proposition 4.C.4 implies that given a combing $\mathcal{Z}$ of N, then $\mathcal{F} = \mathcal{F}_U \cup \mathcal{Z}$ is a combing of G. Hence by Theorem 2.D.6, it is enough to control the area of relations of the form

$$s_1^{w_1} \dots s_k^{w_k},$$

where k is a fixed positive integer, $s_1, \dots, s_k \in S_U$ and $w_1, \dots, w_k$ are words of length $\le n$ in S_N. At this point one can use the fact that conjugation by c is compacting. Indeed, up to conjugating the word $s_1^{w_1} \dots s_k^{w_k}$ by c^q with $q \simeq n$, one can assume that each $s_i^{w_i}$ belongs to S_U. Since k is bounded independently of n, we deduce Theorem 4.M.2 from the following lemma. ☐

Lemma 4.M.3. *Relations of the form $s^w t$, where s and t belong to S_U and where w is a word of length $\simeq n$ in S_N have area $\preceq \delta_N(n)$.*

Proof. For the sake of readability, we first consider the (easier) case when $[N, N]$ is central in N.

Denote $j = in$, where i is an integer to be determined latter in the proof, but that will only depend on G and S (hence is to be considered as bounded).

Up to conjugating by a power of c, it is enough to evaluate the area of the relation $s^{wc^j}t^{c^j}$. Since c is a contracting element, it turns out that $t^{c^j} = u^{-1} \in S_U$. It is straightforward to check that the relation $t^{c^j}u$ has area $\preceq j$. Hence we are left to consider the relation $s^{wc^j}u$.

Denoting $y = (\!(w, c^n)\!) = w^{-1}c^{-n}wc^n$, we have

$$wc^j = c^j w(\!(w, c^j)\!) = c^j y^i. \tag{4.M.4}$$

Moreover the area of the relation $(\!(w, c^j)\!)y^{-i}$ is controlled by the Dehn function of N, so we are reduced to compute the area of

$$(s^{c^j})^{wy^i}u.$$

Denote N_a the Zariski closure of the range of N in $\mathrm{Aut}(U)$. The algebraic group N_a decomposes as DV where A (resp. V) is semi-simple (resp. unipotent). Let us write $c = c_d c_v$ and $w = w_d c_v$ according to this decomposition. Endow the Lie algebra of U with some norm. The crucial observation is that $y = (\!(w, c^n)\!) = (\!(w_v, c_v^n)\!)$. It follows that the matrix norm of y (acting on the Lie algebra of U) is at most Cn^D for some C, D depending only on G and S.

Let $K > 1$ be a constant such that the matrix norm of every subword of wy is at most K^n. Let z be a prefix of wy^i: it is of the form ry^k, where r is a subword of wy, and $k \le i$. The matrix norm of z is therefore at most Cin^DK^n.

Since c acts as a contraction, one can choose i be such that the matrix norm of c^i is less than K^{-2}. Hence the matrix norm of c^jz is less than Cin^DK^{-n} which is bounded by some function of i, C, D and K. It follows that for any prefix r of c^jwy^i has bounded matrix norm. Let a_q for $q = 1, 2, \dots$ be the sequence of letters of the word c^jwy^i, and let $z_q = a_1 \dots a_q$. It follows that the elements s^{z_q} are bounded in U. Now let t_q^{-1} be words in S_U of bounded length representing the elements s^{z_q}. We conclude by reducing successively the relations $(t_{q-1})^{a_q}t_q$ whose area are bounded. This solves the case where N is 2-steps nilpotent.

If N is not assumed to be 2-step-nilpotent, then the relation $(\!(a, b^i)\!) = (\!(a, b)\!)^i$ does not hold anymore. Therefore we cannot simply replace $(\!(w, c^j)\!)$ by $(\!(w, c^n)\!)^i$, as we did above. We shall use instead a more complicated formula, namely the one given by Lemma 2.G.2. According to that lemma, one can write $(\!(w, c^j)\!)$ as a product of m iterated commutators (or their inverses) in the letters $w^{\pm 1}$ and $c^{\pm n}$. The rest of the proof is then identical to the step 2 nilpotent case, replacing in the previous proof, the power of commutators $y^i = (\!(w, c^n)\!)^i$ by this product of (iterated) commutators. □

5. Central extensions of graded Lie algebras

This section contains results on central extensions of graded Lie algebras, which will be needed in Section 6. Let $Q \subset K$ be fields of characteristic zero (for instance, $Q = \mathbf{Q}$ and $\mathbf{K}$ is a nondiscrete locally compact field). To any graded Lie algebra, we associate a central extension in degree zero, which we call its "blow-up", whose study will be needed in Section 6. We are then led to following problem: given a Lie algebra $\mathfrak{g}$ over K, we need to compare the homologies $H_2^K(\mathfrak{g})$ and $H_2^Q(\mathfrak{g})$ of $\mathfrak{g}$ viewed as a Lie algebra over K and as a Lie algebra over Q by restriction of scalars. When $\mathfrak{g}$ is defined over Q, i.e. $\mathfrak{g} = \mathfrak{l} \otimes_Q K$, this problem has been tackled in several papers [KL82, NW08]. Here most of the work is carried out over an arbitrary commutative ring; this generality will be needed as we need to apply the results over suitable rings of functions.

5.A. **Basic conventions.** The following conventions will be used throughout this chapter. The letter R denotes an arbitrary commutative ring (associative with unit). Unless explicitly stated, modules, Lie algebras are over the ring R and are *not* assumed to be finitely generated. The reader is advised not to read this part linearly but rather refer to it when necessary.

Gradings. We fix an abelian group $\mathcal{W}$, called the weight space. By graded module, we mean an R-module V endowed with a **grading**, namely an R-module decomposition as a direct sum

$$V = \bigoplus_{\alpha \in \mathcal{W}} V_\alpha.$$

Elements of V_α are called homogeneous elements of weight α. An R-module homomorphism $f : V \to W$ between graded R-modules is **graded** if $f(V_\alpha) \subset W_\alpha$ for all α. If V is a graded module and V' is a subspace, it is a **graded submodule** if it is generated by homogeneous elements, in which case it is naturally graded and so is the quotient V/V'. By the **weights** of V we generally mean the subset $\mathcal{W}_V \subset \mathcal{W}$ consisting of $\alpha \in \mathcal{W}$ such that $V_\alpha \neq \{0\}$. We use the notation

$$V_\triangledown = \bigoplus_{\alpha \neq 0} V_\alpha.$$

By **graded Lie algebra** we mean a Lie algebra $\mathfrak{g}$ endowed with an R-module grading $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ such that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathcal{W}$.

Tensor products. If V, W are modules, the tensor product $V \otimes W = V \otimes_{\mathsf{R}} W$ is defined in the usual way. The symmetric product $V \circledcirc V$ is obtained by modding out by the R-linear span of all $v \otimes w - w \otimes v$ and the exterior product $V \wedge V$ is obtained by modding out by the R-linear span of all $v \otimes w + w \otimes v$ (or equivalently all $v \otimes v$ if 2 is invertible in R). More generally the nth exterior product $V \wedge \cdots \wedge V$ is obtained by modding out the nth tensor product $V \otimes \ldots \otimes V$ by all tensors $v_1 \otimes \cdots \otimes v_n + w_1 \otimes \cdots \otimes w_n$, whenever for some $1 \leq i \neq j \leq n$, we have $w_i = v_j$, $w_j = v_i$ and $w_k = v_k$ for all $k \neq i, j$.

If W_1, W_2 are submodules of V, we will sometimes denote by $W_1 \wedge W_2$ (resp. $W_1 \circledcirc W_2$) the image of $W_1 \otimes W_2$ in $V \wedge V$ (resp. $V \circledcirc V$). In case $W_1 = W_2 = W$, the latter map factors through a module homomorphism $W \wedge W \to V \wedge V$ (resp. $W \circledcirc W \to V \circledcirc V$), and this convention is consistent when this homomorphism is injective, for instance when W is a direct factor of V.

If V, W are graded then $V \otimes W$ is also graded by

$$(V \otimes W)_\alpha = \bigoplus_{\{(\beta,\gamma):\beta+\gamma=\alpha\}} V_\beta \otimes W_\gamma.$$

When $V = W$, we see that $V \wedge V$ and $V \circledcirc V$ are quotients of $V \otimes V$ by graded submodules and are therefore naturally graded; for instance if $\mathcal{W}$ has no 2-torsion

$$(V \wedge V)_0 = (V_0 \wedge V_0) \oplus \left(\bigoplus_{\alpha \in (\mathcal{W} - \{0\})/\pm} V_\alpha \otimes V_{-\alpha} \right).$$

Homology of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra (always over the commutative ring R). We consider the complex of R-modules

$$\cdots \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{d_4} \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{d_3} \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{d_2} \mathfrak{g} \xrightarrow{d_1} 0$$

given by

$$d_2(x_1, x_2) = -[x_1, x_2]$$
$$d_3(x_1, x_2, x_3) = x_1 \wedge [x_2, x_3] + x_2 \wedge [x_3, x_1] + x_3 \wedge [x_1, x_2]$$

and more generally the boundary map

$$d_n(x_1, \dots, x_n) = \sum_{1 \le i \le j \le n} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n;$$

and define the **second homology group** $H_2(\mathfrak{g}) = Z_2(\mathfrak{g})/B_2(\mathfrak{g})$, where $Z_2(\mathfrak{g}) = \mathrm{Ker}(d_2)$ is the set of **2-cycles** and $B_2(\mathfrak{g}) = \mathrm{Im}(d_3)$ is the set of **2-boundaries**. (We will focus on d_i for $i \le 3$ although the map d_4 will play a minor computational role in the sequel. This is of course part of the more general definition of the nth homology module $H_n(\mathfrak{g}) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$, which we will not consider.) If $A \to \mathsf{R}$ is a homomorphism of commutative rings, then $\mathfrak{g}$ is a Lie A-algebra by restriction of scalars, and its 2-homology as a Lie A-algebra is denoted by $H_2^{\mathsf{A}}(\mathfrak{g})$. If $\mathfrak{g}$ is a graded Lie algebra, then the maps d_i are graded as well, so $H_2(\mathfrak{g})$ is naturally a graded R-module.

Iterated brackets. In a Lie algebra, we define n-iterated bracket as the usual bracket for $n = 2$ and by induction for $n \ge 3$ as

$$[x_1, \dots, x_n] = [x_1, [x_2, \dots, x_n]].$$

Central series and nilpotency. Define the **descending central series** of the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$ for $i \ge 1$. We say that $\mathfrak{g}$ is s**-nilpotent** if $\mathfrak{g}^{s+1} = \{0\}$.

5.A.1. *The Hopf bracket.* Consider a central extension of Lie algebras

$$0 \to \mathfrak{z} \to \mathfrak{g} \xrightarrow{p} \mathfrak{h} \to 0.$$

Since $\mathfrak{z}$ is central, the bracket $\mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ factors through an R-module homomorphism $B : \mathfrak{h} \wedge \mathfrak{h} \to \mathfrak{g}$, called the **Hopf bracket**. It is unique for the property that $B(p(x) \wedge p(y)) = [x, y]$ for all $x, y \in \mathfrak{g}$ (uniqueness immediately follows from surjectivity of $\mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h} \wedge \mathfrak{h}$).

Lemma 5.A.1. *For all $x, y, z, t \in \mathfrak{h}$ we have $B([x, y] \wedge [z, t]) = [B(x \wedge y), B(z \wedge t)]$.*

Proof. Observe that if $\bar{x}, \bar{y}$ are lifts of x and y then $B(x \wedge y) = [\bar{x}, \bar{y}]$, and that $[\bar{x}, \bar{y}]$ is a lift of $[x, y]$. In view of this, observe that both terms are equal to $[[\bar{x}, \bar{y}], [\bar{z}, \bar{t}]]$. □

5.A.2. *1-tameness.*

Definition 5.A.2. We say that a Lie algebra $\mathfrak{g}$ is **1-tame** if it is generated by $\mathfrak{g}_{\triangledown} = \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$.

Lemma 5.A.3. *Let $\mathfrak{g}$ be a graded Lie algebra. Then $\mathfrak{g}$ is 1-tame if and only if we have $\mathfrak{g}_0 = \sum_\beta [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$, where β ranges over nonzero weights.*

Proof. One direction is trivial. Conversely, if $\mathfrak{g}$ is 1-tame, then $\mathfrak{g}_0$ is generated as an abelian group by elements of the form $x = [x_1, \dots, x_k]$ with $k \geq 2$ and x_i homogeneous of nonzero weight. So $x = [x_1, y]$ with $y = [x_2, \dots, x_k] \in \mathfrak{g}_{\triangledown}$ and $x_1 \in \mathfrak{g}_{\triangledown}$. □

Lemma 5.A.4. *Let $\mathfrak{g}$ be a graded Lie algebra. Then the ideal generated by $\mathfrak{g}_{\triangledown}$ coincides with the Lie subalgebra generated by $\mathfrak{g}_{\triangledown}$ and in particular is 1-tame.*

Proof. Let $\mathfrak{h}$ be the subalgebra generated by $\mathfrak{g}_{\triangledown}$; it is enough to check that $\mathfrak{h}$ is an ideal, and it is thus enough to check that $[\mathfrak{g}_0, \mathfrak{h}] \subset \mathfrak{h}$. Set $\mathfrak{h}_1 = \mathfrak{g}_{\triangledown}$ and $\mathfrak{h}_d = [\mathfrak{g}_{\triangledown}, \mathfrak{h}_{d-1}]$, so that $\mathfrak{h} = \sum_{d \geq 1} \mathfrak{h}_d$. It is therefore enough to check that $[\mathfrak{g}_0, \mathfrak{h}_d] \subset \mathfrak{h}_d$ for all $d \geq 1$. This is done by induction. The case $d = 1$ is clear. If $d \geq 2$, $x \in \mathfrak{g}_0$, $y \in \mathfrak{g}_{\triangledown}$, $z \in \mathfrak{h}_{d-1}$, then using the induction hypothesis

$$[x, [y, z]] = [y, [x, z]] - [[y, x], z] \in [\mathfrak{g}_{\triangledown}, \mathfrak{h}_{d-1}] \subset \mathfrak{h}_d$$

and we are done. □

We use this to obtain the following result, which will be used in Section 7.

Lemma 5.A.5. *Let $\mathfrak{g}$ be a graded Lie algebra with descending central series $(\mathfrak{g}^i)$, and assume that $\mathfrak{g}_0$ is s-nilpotent. Then $\mathfrak{g}^{s+1}$ is contained in the subalgebra generated by $\mathfrak{g}_{\triangledown}$. In particular, if $\mathfrak{g}_0$ is nilpotent and $\mathfrak{g}^\infty = \bigcap \mathfrak{g}^i$, then $\mathfrak{g}^\infty$ is contained in the subalgebra generated by $\mathfrak{g}_{\triangledown}$.*

Proof. By Lemma 5.A.4, it is enough to check that $\mathfrak{g}^{s+1}$ is contained in the *ideal* $\mathfrak{j}$ generated by $\mathfrak{g}_{\triangledown}$. It is sufficient to show that $\mathfrak{g}^{s+1} \cap \mathfrak{g}_0 \subset \mathfrak{j}$. Each element x of $\mathfrak{g}^{s+1} \cap \mathfrak{g}_0$ can be written as a sum of nonzero $(s+1)$-iterated brackets of homogeneous elements. Each of those brackets involves at least one element of nonzero degree, since otherwise all its entries would be contained in $\mathfrak{g}_0$ and it would vanish. So x belongs to the ideal generated by $\mathfrak{g}_\triangledown$. □

5.B. **The blow-up.**

Definition 5.B.1. Let $\mathfrak{g}$ be an arbitrary graded Lie algebra. Define the **blow-up** graded algebra $\tilde{\mathfrak{g}}$ as follows. As a graded vector space, $\tilde{\mathfrak{g}}_\alpha = \mathfrak{g}_\alpha$ for all $\alpha \neq 0$, and $\tilde{\mathfrak{g}}_0 = (\mathfrak{g} \wedge \mathfrak{g})_0 / d_3(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g})_0$.

Define a graded R-module homomorphism $\tau : \tilde{\mathfrak{g}} \to \mathfrak{g}$ by $\tau(x) = x$ if $x \in \tilde{\mathfrak{g}}_\triangledown$ and $\tau(x \wedge y) = [x, y]$ if $x \wedge y \in \tilde{\mathfrak{g}}_0$ (which of course factors through 2-boundaries).

Let us define the Lie algebra structure $[\cdot, \cdot]'$ on $\tilde{\mathfrak{g}}$. Suppose that $x \in \tilde{\mathfrak{g}}_\alpha, y \in \tilde{\mathfrak{g}}_\beta$.

- if $\alpha + \beta \neq 0$, define $[x, y]' = [\tau(x), \tau(y)]$;
- if $\alpha + \beta = 0$, define $[x, y]' = \tau(x) \wedge \tau(y)$.

Lemma 5.B.2. *With the above bracket, $\tilde{\mathfrak{g}}$ is a Lie algebra and τ is a Lie algebra homomorphism, whose kernel is central and naturally isomorphic to $H_2(\mathfrak{g})_0$. Its image is the ideal $\mathfrak{g}_\triangledown + [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.*

Proof. Let us first check that τ is a homomorphism (of non-associative algebras). Let $x, y \in \tilde{\mathfrak{g}}$ have nonzero weight α and β. In each case, we apply the definition of $[\cdot, \cdot]'$ and then of τ. If $\alpha + \beta \neq 0$

$$\tau([x, y]') = \tau([\tau(x), \tau(y)]) = [\tau(x), \tau(y)];$$

if $\alpha + \beta = 0$ then

$$\tau([x, y]') = \tau(\tau(x) \wedge \tau(y)) = [\tau(x), \tau(y)];$$

if x has weight $\alpha \neq 0$ and $y \wedge z$ has weight 0 then

$$\tau([x, y \wedge z]') = \tau([\tau(x), \tau(y \wedge z)]) = [\tau(x), \tau(y \wedge z)];$$

and similarly $\tau([y \wedge z, x]') = [\tau(y \wedge z), \tau(x)]$; if $x \wedge y$ and $z \wedge w$ have weight 0 then

$$\tau([x \wedge y, z \wedge w]') = \tau(\tau(x \wedge y) \wedge \tau(z \wedge w)) = [\tau(x \wedge y), \tau(z \wedge w)].$$

By linearity, we deduce that τ is a homomorphism. Since τ_α is an isomorphism for $\alpha \neq 0$ and $\tau_0 = -d_2$, the kernel of τ is *equal* by definition to $H_2(\mathfrak{g})_0$. Moreover, by definition the bracket $[x, y]'$ only depends on $\tau(x) \otimes \tau(y)$, and it immediately follows that $\mathrm{Ker}(\tau)$ is central in $\tilde{\mathfrak{g}}$.

Let us check that the bracket is a Lie algebra bracket; the antisymmetry being clear, we have to check the Jacobi identity. Take $x \in \tilde{\mathfrak{g}}_\alpha$, $y \in \tilde{\mathfrak{g}}_\beta$, $z \in \tilde{\mathfrak{g}}_\gamma$. From the definition above, we obtain (discussing on whether or not $\beta + \gamma$ is zero)

- if $\alpha + \beta + \gamma \neq 0$, $[x, [y, z]']' = [\tau(x), [\tau(y), \tau(z)]]$;
- if $\alpha + \beta + \gamma = 0$, $[x, [y, z]']' = \tau(x) \wedge [\tau(y), \tau(z)]$.

Therefore, the Jacobi identity for (x, y, z) immediately follows from that of $\mathfrak{g}$ in the first case, and from the fact we killed 2-boundaries in the second case.

We have $\tau(\tilde{\mathfrak{g}}_\triangledown) = \mathfrak{g}_\triangledown$ and $\tau(\tilde{\mathfrak{g}}_0) = [\mathfrak{g}, \mathfrak{g}]_0$. Therefore the image of τ is equal to $\mathfrak{g}_\triangledown + [\mathfrak{g}, \mathfrak{g}]$. The latter is an ideal, since it contains $[\mathfrak{g}, \mathfrak{g}]$. $\square$

Definition 5.B.3. We say that a graded Lie algebra $\mathfrak{g}$ is **relatively perfect in degree zero** if it satisfies one of the following (obviously) equivalent definitions

- 0 is not a weight of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$;
- $\mathfrak{g}_0 \subset [\mathfrak{g}, \mathfrak{g}]$;
- $\mathfrak{g} = \mathfrak{g}_\triangledown + [\mathfrak{g}, \mathfrak{g}]$;
- $\mathfrak{g}$ is generated by $\mathfrak{g}_\triangledown + [\mathfrak{g}_0, \mathfrak{g}_0]$;
- $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is surjective (in view of Lemma 5.B.2).

Note that if $\mathfrak{g}$ is 1-tame, then it is relatively perfect in degree zero, but the converse is not true, as shows the example of a nontrivial perfect Lie algebra with grading concentrated in degree zero. The interest of this notion is that it is satisfied by a wealth of graded Lie algebras that are very far from perfect (e.g., nilpotent).

Theorem 5.B.4. *Let $\mathfrak{g}$ be a graded Lie algebra. If $\mathfrak{g}$ is relatively perfect in degree zero (e.g., $\mathfrak{g}$ is 1-tame), then the blow-up $\tilde{\mathfrak{g}} \xrightarrow{\tau} \mathfrak{g}$ is a graded central extension with kernel in degree zero, and is universal among such central extensions. That is, for every surjective graded Lie algebra homomorphism $\mathfrak{h} \xrightarrow{p} \mathfrak{g}$ with central kernel $\mathfrak{z} = \mathfrak{z}_0$, there exists a unique graded Lie algebra homomorphism $\phi : \tilde{\mathfrak{g}} \to \mathfrak{h}$ so that the composite map $\tilde{\mathfrak{g}} \to \mathfrak{h} \to \mathfrak{g}$ coincides with the natural projection.*

Proof. By Lemma 5.B.2, $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is a central extension with kernel in degree zero. Denote by $B_\mathfrak{h} : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ the Hopf bracket associated to $\mathfrak{h} \to \mathfrak{g}$ (see §5.A).

Let us show uniqueness in the universal property. Clearly, ϕ is determined on $\tilde{\mathfrak{g}}_\triangledown$. So we have to check that ϕ is also determined on $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. Observe that the map $\tilde{\mathfrak{g}} \wedge \tilde{\mathfrak{g}} \to \mathfrak{h}$, $x \wedge y \mapsto \phi([x, y]) = [\phi(x), \phi(y)]$ factors through a map $w : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$, so $w(\tau(x) \wedge \tau(y)) = [\phi(x), \phi(y)]$ for all $x, y \in \tilde{\mathfrak{g}}$. Since $p \circ \phi = \tau$ and since $\mathfrak{h}$ is generated by the image of ϕ and by its central ideal $\mathrm{Ker}(p)$, we deduce that for all $x, y \in \mathfrak{h}$, we have $w(p(x) \wedge p(y)) = [x, y]$. By the uniqueness property of the Hopf bracket (see §5.A), we deduce that $w = B_\mathfrak{h}$. So for all $x, y \in \tilde{\mathfrak{g}}$, $\phi([x, y])$ is uniquely determined as $B_\mathfrak{h}(\tau(x) \wedge \tau(y))$.

Now to prove the existence, define $\phi : \tilde{\mathfrak{g}} \to \mathfrak{h}$ to be p^{-1} on $\tilde{\mathfrak{g}}_\triangledown$, and $\phi(x \wedge y) = B_\mathfrak{h}(x \wedge y)$ if $x \wedge y \in \tilde{\mathfrak{g}}_0$. It is clear that ϕ is a graded module homomorphism and that $p \circ \phi = \tau$. Let us show that ϕ is a Lie algebra homomorphism, i.e. that the graded module homomorphism $\sigma : \tilde{\mathfrak{g}} \wedge \tilde{\mathfrak{g}} \to \mathfrak{h}$, $(x \wedge y) \mapsto \phi([x, y]') - [\phi(x), \phi(y)]$ vanishes. Since $p \circ \phi$ is a homomorphism and $p_\triangledown$ is bijective, we have $(p \circ \sigma)_\triangledown = 0$, so it is enough to check that σ vanishes in degree 0. If x and y have nonzero opposite weights, noting that $p \circ \phi$ is the identity on $\mathfrak{g}_\triangledown$,

$$\phi([x, y]') = \phi(x \wedge y) = B_\mathfrak{h}(x \wedge y) = [p^{-1}(x), p^{-1}(y)] = [\phi(x), \phi(y)].$$

If $x \wedge y$ and $z \wedge t$ belong to $\tilde{\mathfrak{g}}_0 = (\mathfrak{g} \wedge \mathfrak{g})_0$, then, using Lemma 5.A.1, we have

$$\begin{aligned}\phi([x \wedge y, z \wedge t]') =& \phi([x,y] \wedge [z,t]) = B_{\mathfrak{h}}([x,y] \wedge [z,t]) \\ =& [B_{\mathfrak{h}}(x \wedge y), B_{\mathfrak{h}}(z \wedge t)] = [\phi(x \wedge y), \phi(z \wedge t)].\end{aligned}$$

By linearity, we deduce that $\sigma_0 = 0$ and therefore ϕ is a Lie algebra homomorphism. □

Corollary 5.B.5. *If $\mathfrak{g}$ is relatively perfect in degree zero then $\tilde{\tilde{\mathfrak{g}}} = \tilde{\mathfrak{g}}$.*

Proof. Observe that $\tilde{\tilde{\mathfrak{g}}} \to \mathfrak{g}$ has kernel $\mathfrak{z}$ concentrated in degree zero, so it immediately follows that $[\mathfrak{z}, \mathfrak{g}_\triangledown] = 0$. We need to show that $\mathfrak{z}$ is central in $\tilde{\tilde{\mathfrak{g}}}$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{g}_\triangledown$, it is enough to show that $[\mathfrak{z}, [\tilde{\tilde{\mathfrak{g}}}, \tilde{\tilde{\mathfrak{g}}}]] = \{0\}$. By the Jacobi identity, $[\mathfrak{z}, [\tilde{\tilde{\mathfrak{g}}}, \tilde{\tilde{\mathfrak{g}}}]] \subset [\tilde{\tilde{\mathfrak{g}}}, [\mathfrak{z}, \tilde{\tilde{\mathfrak{g}}}]]$. Now since $\tilde{\mathfrak{g}} \to \mathfrak{g}$ has a central kernel, $[\mathfrak{z}, \tilde{\tilde{\mathfrak{g}}}]$ is contained in the kernel of $\tilde{\tilde{\mathfrak{g}}} \to \tilde{\mathfrak{g}}$, which is central in $\tilde{\tilde{\mathfrak{g}}}$. So $[\tilde{\tilde{\mathfrak{g}}}, [\mathfrak{z}, \tilde{\tilde{\mathfrak{g}}}]] = \{0\}$. Thus $\mathfrak{z}$ is central in $\tilde{\tilde{\mathfrak{g}}}$. The universal property of $\tilde{\mathfrak{g}}$ then implies that $\tilde{\tilde{\mathfrak{g}}} \to \tilde{\mathfrak{g}}$ is an isomorphism. □

Lemma 5.B.6. *Let $\mathfrak{g}$ be a graded Lie algebra and $\tilde{\mathfrak{g}}$ its blow-up. If $\mathfrak{g}$ is 1-tame, then so is $\tilde{\mathfrak{g}}$.*

Proof. Suppose that $\mathfrak{g}$ is 1-tame. Then by linearity, it is enough to check that for every $x \in \mathfrak{g}_0$ and u, v of nonzero opposite weights, the element $x \wedge [u,v]$ belongs to $[\tilde{\mathfrak{g}}_\triangledown, \tilde{\mathfrak{g}}_\triangledown]$. This is the case since modulo 2-boundaries, this element is equal to $u \wedge [x,v] + v \wedge [u,x] \in (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$. □

Lemma 5.B.7. *Let $\mathfrak{g}_i$ be finitely many graded Lie algebras (all graded in the same abelian group) and $\tilde{\mathfrak{g}}_i$ their blow-up. If $\mathfrak{g} = \prod \mathfrak{g}_i$ satisfies the assumption that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ has no opposite weights, then the natural homomorphism $\widetilde{\prod \mathfrak{g}_i} \to \prod \tilde{\mathfrak{g}}_i$ is an isomorphism. Equivalently, $H_2(\prod_i \mathfrak{g}_i)_0 \to \bigoplus H_2(\mathfrak{g}_i)_0$ is an isomorphism.*

Proof. For each i, there are homomorphisms $\mathfrak{g}_i \to \prod \mathfrak{g}_j \to \mathfrak{g}_i$, whose composition is the identity, and hence $H_2(\mathfrak{g}_i)_0 \to H_2(\prod \mathfrak{g}_j)_0 \to H_2(\mathfrak{g}_i)_0$, whose composition is the identity again. So we obtain homomorphisms

$$\bigoplus H_2(\mathfrak{g}_i)_0 \to H_2\left(\prod \mathfrak{g}_i\right)_0 \to \bigoplus H_2(\mathfrak{g}_i)_0,$$

whose composition is the identity. To finish the proof, we have to check that $\bigoplus H_2(\mathfrak{g}_i)_0 \to H_2(\prod \mathfrak{g}_i)_0$ is surjective, or equivalently that in $Z_2(\prod \mathfrak{g}_i)_0$, every element x is the sum of an element in $\bigoplus_i Z_2(\mathfrak{g}_i)_0$ and a boundary. Now observe that

$$Z_2\left(\prod \mathfrak{g}_i\right)_0 = \bigoplus Z_2(\mathfrak{g}_i)_0 \oplus \bigoplus_{i<j} (\mathfrak{g}_i \wedge \mathfrak{g}_j)_0.$$

So we have to prove that for $i \neq j$, $\mathfrak{g}_i \wedge \mathfrak{g}_j$ consists of boundaries. Given an element $x \wedge y$ ($x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$ homogeneous), the assumption on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ implies that, for instance, y is a sum $\sum_k [z_k, w_k]$ of commutators. Projecting if necessary into $\mathfrak{g}_j$, we can suppose that all z_k and w_k belong to $\mathfrak{g}_j$. So

$$x \wedge y = x \wedge \sum_k [z_k, w_k] = \sum_k d_3(x \wedge z_k \wedge w_k). \quad \square$$

5.C. **Homology and restriction of scalars.** We now deal with two commutative rings A, B coming with a ring homomorphism $\mathsf{A} \to \mathsf{B}$ (we avoid using R as the previous results will be used both with $\mathsf{R} = \mathsf{A}$ and $\mathsf{R} = \mathsf{B}$). If $\mathfrak{g}$ is a graded Lie algebra over B, it can then be viewed as a graded Lie algebra over A by restriction of scalars. This affects the definition of the blow-up. There is an obvious surjective graded Lie algebra homomorphism $\tilde{\mathfrak{g}}^{\mathsf{A}} \to \tilde{\mathfrak{g}}^{\mathsf{B}}$. The purpose of this part is to describe the kernel of this homomorphism (or equivalently of the homomorphism $H_2^{\mathsf{A}}(\mathfrak{g})_0 \to H_2^{\mathsf{B}}(\mathfrak{g})_0$), and to characterize, under suitable assumptions, when it is an isomorphism.

Our main object of study is the following kernel.

Definition 5.C.1. If $\mathfrak{g}$ is a Lie algebra over B, we define the **welding module** $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})$ as the kernel of the natural homomorphism $H_2^{\mathsf{A}}(\mathfrak{g}) \to H_2^{\mathsf{B}}(\mathfrak{g})$, or equivalently of the homomorphism $(\mathfrak{g} \wedge_{\mathsf{A}} \mathfrak{g})/B_2^{\mathsf{A}}(\mathfrak{g}) \to (\mathfrak{g} \wedge_{\mathsf{B}} \mathfrak{g})/B_2^{\mathsf{B}}(\mathfrak{g})$. If $\mathfrak{g}$ is graded, it is a graded module as well, and $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})_0$ then also coincides with the kernel of $\tilde{\mathfrak{g}}^{\mathsf{A}} \to \tilde{\mathfrak{g}}^{\mathsf{B}}$.

The following module will also play an important role.

Definition 5.C.2. If $\mathfrak{g}$ is a Lie algebra over B, define its *Killing module* $\mathrm{Kill}(\mathfrak{g})$ (or $\mathrm{Kill}^{\mathsf{B}}(\mathfrak{g})$ if the base ring need be specified) as the cokernel of the homomorphism

$$\begin{array}{rcl} \mathcal{T} : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} & \to & \mathfrak{g} \circledcirc \mathfrak{g} \\ u \otimes v \otimes w & \mapsto & u \circledcirc [v, w] + v \circledcirc [u, w]. \end{array}$$

If $\mathfrak{g}$ is graded, it is graded as well.

Note that we can also write $\mathcal{T}(u \otimes w \otimes v) = [u, v] \circledcirc w - u \circledcirc [v, w]$, and thus we see that the set of B-linear homomorphisms from $\mathrm{Kill}(\mathfrak{g})$ to any B-module M is naturally identified to the set of the so-called invariant bilinear maps $\mathfrak{g} \times \mathfrak{g} \to M$.

Lemma 5.C.3. *Let $\mathfrak{v}$ be a B-module. Then the kernel of the natural surjective A-module homomorphism $\mathfrak{v} \otimes_{\mathsf{A}} \mathfrak{v} \to \mathfrak{v} \otimes_{\mathsf{B}} \mathfrak{v}$ is generated, as an abelian group, by by elements*

$$\lambda x \otimes_{\mathsf{A}} y - x \otimes_{\mathsf{A}} \lambda y \tag{5.C.4}$$

with $\lambda \in \mathsf{B}$, $x, y \in \mathfrak{v}$. The same holds with $\otimes$ replaced by $\circledcirc$ or $\wedge$.

Proof. Endow $\mathfrak{v} \otimes_{\mathsf{A}} \mathfrak{v}$ with a structure of a B-module, using the structure of B-module of the left-hand $\mathfrak{v}$, namely $\lambda(x \otimes y) = (\lambda x \otimes y)$ if $\lambda \in B$, $x, y \in \mathfrak{v}$.

Let W be the subgroup generated by elements of the form (5.C.4); it is clearly an A-submodule, and is actually a B-submodule as well. The natural A-module surjective homomorphism $\phi : (\mathfrak{v} \otimes_{\mathsf{A}} \mathfrak{v})/W \to \mathfrak{v} \otimes_{\mathsf{B}} \mathfrak{v}$ is B-linear. To show it is a bijection, we observe that by the universal property of $\mathfrak{v}\otimes_{\mathsf{B}}\mathfrak{v}$, we have a B-module homomorphism $\psi : \mathfrak{v}\otimes_{\mathsf{B}}\mathfrak{v} \to (\mathfrak{v}\otimes_{\mathsf{A}}\mathfrak{v})/W$ mapping $x\otimes_{\mathsf{B}} y$ to $x\otimes_{\mathsf{A}} y$ modulo W. Clearly, ψ and ϕ are inverse to each other.

Let us deal with $\wedge$, the case of $\circledcirc$ being similar. The group $\mathfrak{v}\wedge_{\mathsf{A}}\mathfrak{v}$ is defined as the quotient of $\mathfrak{v}\otimes_{\mathsf{A}}\mathfrak{v}$ by symmetric tensors (i.e. by the subgroup generated by elements of the form $x\otimes y+y\otimes x$), and the group $\mathfrak{v}\wedge_{\mathsf{B}}\mathfrak{v}$ is defined the quotient of $\mathfrak{v}\otimes_{\mathsf{B}}\mathfrak{v}$ by symmetric tensors. By the case of $\otimes$, this means that $\mathfrak{v}\wedge_{\mathsf{B}}\mathfrak{v}$ is the quotient of $\mathfrak{v}\otimes_{\mathsf{A}}\mathfrak{v}$ by the subgroup generated by symmetric tensors and elements (5.C.4). This implies that that $\mathfrak{v}\wedge_{\mathsf{B}}\mathfrak{v}$ is the quotient of $\mathfrak{v}\wedge_{\mathsf{A}}\mathfrak{v}$ by the subgroup generated by elements (5.C.4) (with $\otimes$ replaced by $\wedge$) ☐

Proposition 5.C.5. *For any Lie algebra $\mathfrak{g}$, the A-module homomorphism*

$$\begin{aligned}\Phi : \mathsf{B}\otimes_{\mathsf{A}}\mathfrak{g}\otimes_{\mathsf{A}}\mathfrak{g} &\to W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})\\ \lambda\otimes x\otimes y &\mapsto \lambda x\wedge y - x\wedge\lambda y\end{aligned}$$

is surjective. If $\mathfrak{g}$ is graded and is 1-tame, then $\Phi_0 : \mathsf{B}\otimes_{\mathsf{A}}(\mathfrak{g}\otimes_{\mathsf{A}}\mathfrak{g})_0 \to W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})_0$ is surjective in restriction to $\mathsf{B}\otimes(\mathfrak{g}_\triangledown\otimes\mathfrak{g}_\triangledown)_0$.

Proof. The group $(\mathfrak{g}\wedge_{\mathsf{A}}\mathfrak{g})/B_2^{\mathsf{A}}(\mathfrak{g})$ is defined as the quotient of $\mathfrak{g}\wedge_{\mathsf{A}}\mathfrak{g}$ by 2-boundaries, and $(\mathfrak{g}\wedge_{\mathsf{B}}\mathfrak{g})/B_2^{\mathsf{B}}(\mathfrak{g})$ is the quotient of $\mathfrak{g}\wedge_{\mathsf{B}}\mathfrak{g}$ by 2-boundaries, or equivalently, by Lemma 5.C.3, is the quotient of $\mathfrak{g}\wedge_{\mathsf{A}}\mathfrak{g}$ by 2-boundaries and elements of the form

$$\lambda x\wedge_{\mathsf{A}} y - x\wedge_{\mathsf{A}}\lambda y. \tag{5.C.6}$$

It follows that the kernel of $(\mathfrak{g}\wedge_{\mathsf{A}}\mathfrak{g})/B_2^{\mathsf{A}}(\mathfrak{g}) \to (\mathfrak{g}\wedge_{\mathsf{B}}\mathfrak{g})/B_2^{\mathsf{B}}(\mathfrak{g})$ is generated by elements of the form (5.C.6), proving the surjectivity of Φ.

For the additional statement, define $W' = \Phi(\mathsf{B}\otimes(\mathfrak{g}_\triangledown\otimes\mathfrak{g}_\triangledown))_0$. We have to show that any element $\lambda x\wedge_{\mathsf{A}} y - x\wedge_{\mathsf{A}}\lambda y$ as in (5.C.6) with x, y of zero weight belongs to W'. By linearity and Lemma 5.A.3, we can suppose that $y = [z, w]$ with z, w of nonzero opposite weight. So, modulo boundaries,

$$\begin{aligned}\lambda x\wedge[w,z] - x\wedge\lambda[w,z] &= -\,w\wedge[z,\lambda x] - z\wedge[\lambda x, w] - x\wedge\lambda[w,z]\\ &= -\,w\wedge\lambda[z,x] + \lambda w\wedge[z,x],\end{aligned}$$

which belongs to W'. ☐

Proposition 5.C.7. *Let $\mathfrak{g}$ be a Lie algebra over B. Then the map*

$$\Phi : \mathsf{B}\otimes_{\mathsf{A}}\mathfrak{g}\otimes_{\mathsf{A}}\mathfrak{g} \to W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})$$

of Proposition 5.C.5 factors through the natural projection

$$\mathsf{B}\otimes_{\mathsf{A}}\mathfrak{g}\otimes_{\mathsf{A}}\mathfrak{g} \to \mathsf{B}\otimes_{\mathsf{A}}\mathrm{Kill}^{\mathsf{A}}(\mathfrak{g});$$

moreover in restriction to $\mathsf{B}\otimes_{\mathsf{A}}\mathfrak{g}\otimes_{\mathsf{A}}[\mathfrak{g},\mathfrak{g}]$, it factors through $\mathsf{B}\otimes_{\mathsf{A}}\mathrm{Kill}^{\mathsf{B}}(\mathfrak{g})$. In particular, if $\mathfrak{g}$ is graded and $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ has no opposite weights, then Φ_0 factors through $\mathsf{B}\otimes_{\mathsf{A}}\mathrm{Kill}^{\mathsf{B}}(\mathfrak{g})$.

Proof. All $\otimes$, $\wedge$, $\circledcirc$ are meant over A.

Write $\Phi^\lambda(x \otimes y) = \Phi(\lambda \otimes x \otimes y)$. It is immediate that $\Phi^\lambda(y \otimes x) = \Phi^\lambda(x \otimes y)$ (even before modding out by 2-boundaries), and thus Φ^λ factors through $\mathfrak{g} \circledcirc \mathfrak{g}$. Let us now check that Φ^λ factors through $\mathrm{Kill}^{\mathsf{A}}(\mathfrak{g})$. Modulo 2-boundaries:

$$\begin{aligned}\Phi^\lambda(x \circledcirc [y,z] - y \circledcirc [z,x]) =& \lambda x \wedge [y,z] - x \wedge \lambda[y,z] \\ & - \lambda y \wedge [z,x] + y \wedge \lambda[z,x] \\ =& - z \wedge [\lambda x, y] + z \wedge [x, \lambda y] = 0.\end{aligned}$$

For the last statement, by Lemma 5.C.3, we have to show that $\Phi^\lambda(\mu x \circledcirc [y,z]) = \Phi^\lambda(x \circledcirc \mu[y,z])$ for all $x, y, z \in \mathfrak{g}$ and $\mu \in \mathsf{B}$. Indeed, using the latter vanishing we get

$$\Phi^\lambda(\mu x \circledcirc [y,z]) = \Phi^\lambda(y \circledcirc [z, \mu x]) = \Phi^\lambda(y \circledcirc [\mu z, x]) = \Phi^\lambda(x, [y, \mu z]). \qquad \square$$

In turn, we obtain, as an immediate consequence.

Corollary 5.C.8. *Let $\mathfrak{g}$ be a graded Lie algebra over B such that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ has no opposite weights. If $\mathrm{Kill}^{\mathsf{B}}(\mathfrak{g})_0 = \{0\}$ then $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})_0 = \{0\}$.*

Proof. By Propositions 5.C.5 and 5.C.7, Φ_0 induces a surjection $\mathsf{B} \otimes_{\mathsf{A}} \mathrm{Kill}(\mathfrak{g})_0^{\mathsf{B}} \to W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})_0$. $\square$

Under the additional assumptions that $\mathsf{A} = Q$ is a field of characteristic $\neq 2$ and $\mathfrak{g}$ is defined over Q, i.e. has the form $\mathfrak{l} \otimes_Q \mathsf{B}$ for some Lie algebra $\mathfrak{l}$ over Q, Corollary 5.C.8 easy follows from [NW08, Theorem 3.4]. We are essentially concerned with finite-dimensional Lie algebras $\mathfrak{g}$ over a field K of characteristic zero (K playing the role of B) and $\mathsf{A} = \mathbf{Q}$, but nevertheless in general we *cannot* assume that $\mathfrak{g}$ be defined over $\mathbf{Q}$.

We will also use the more specific application.

Corollary 5.C.9. *Assume that $\mathsf{A} = \mathbf{Q}$ and $\mathsf{B} = \mathbf{K} = \prod_{j=1}^{\tau} \mathbf{K}_j$ is a finite product of locally compact fields $\mathbf{K}_j$, each isomorphic to $\mathbf{R}$ or some $\mathbf{Q}_p$. Let $B_{\mathbf{K}_j}$ be the closed unit ball in $\mathbf{K}_j$. Assume that $\mathfrak{g}$ is finite-dimensional over $\mathbf{K}$, that is, $\mathfrak{g} = \prod_j \mathfrak{g}_j$ with $\mathfrak{g}_j$ finite-dimensional over $\mathbf{K}_j$. Suppose that $\mathfrak{g}$ is 1-tame and $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ has no opposite weights. For every j and weight α, let $V_{j,\alpha}$ be a neighbourhood of 0 in $\mathfrak{g}_{j,\alpha} = (\mathfrak{g}_j)_\alpha$. Then the welding module $W_2(\mathfrak{g})^{\mathbf{Q},\mathbf{K}} = \mathrm{Ker}\big(H_2(\mathfrak{g})_0^{\mathbf{Q}} \to H_2(\mathfrak{g})_0^{\mathbf{K}}\big)$ is generated by elements of the form $\Phi(\lambda \otimes x \otimes y)$ with $\lambda \in B_{\mathbf{K}_j}$ and $x \in V_{j,\alpha}$, $y \in V_{j,-\alpha}$, where α ranges over nonzero weights and $j = 1, \dots, \tau$.*

Proof. By Proposition 5.C.5, $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})$ is generated by elements of the form $\Phi(\lambda' \otimes x' \otimes y')$, with $\lambda' \in \mathbf{K}$ and $x', y' \in \mathfrak{g}$ are homogeneous of nonzero opposite weight. By linearity and Lemma 5.B.7, these elements for which

$$(\lambda', x', y') \in \bigcup_j \bigcup_{\alpha \neq 0} \mathbf{K}_j \times \mathfrak{g}_{j,\alpha} \times \mathfrak{g}_{j,-\alpha}$$

are enough. Given such an element $(\lambda', x', y') \in \mathbf{K}_j \times \mathfrak{g}_{j,\alpha} \times \mathfrak{g}_{j,-\alpha}$, using that $\mathbf{K}_j = \mathbf{Q} + B_{\mathbf{K}_j}$, write $\lambda' = \alpha + \lambda$ with $\alpha \in \mathbf{Q}$ and $\lambda \in B_{\mathbf{K}}$. Also, since $\mathfrak{g}_{j,\pm\alpha} = \mathbf{Q}V_{j,\pm\alpha}$, write $x' = \mu x$ and $y' = \gamma y$ with $\mu, \gamma \in \mathbf{Q}$, $x \in V_{j,\alpha}$, $y \in V_{j,-\alpha}$. Clearly, by the definition of Φ, we have $\Phi(\alpha \otimes x' \otimes y') = 0$, so

$$\Phi(\lambda' \otimes x' \otimes y') = \Phi(\lambda \otimes \mu x \otimes \gamma y) = \mu\gamma\Phi(\lambda \otimes x \otimes y),$$

and $\Phi(\lambda \otimes x \otimes y)$ is of the required form. □

5.C.1. *Construction of 2-cycles.* We now turn to a partial converse to Corollary 5.C.8.

Define $\mathrm{HC}_1^{\mathsf{A}}(\mathsf{B})$ (or $\mathrm{HC}_1(\mathsf{B})$ if A is implicit) as the quotient of $\mathsf{B} \wedge_{\mathsf{A}} \mathsf{B}$ by the A-submodule generated by elements of the form $uv \wedge w + vw \wedge u + wu \wedge v$. This is the **first cyclic homology group** of B (which is usually defined in another manner; we refer to [NW08] for the canonical isomorphism between the two).

Lemma 5.C.10. *Assume that K is a field of characteristic zero and Q a subfield. Suppose that K contains an element t that is transcendental over Q. Assume that either K has characteristic zero, or $K \subset Q(\!(t)\!)$. Then $\mathrm{HC}_1^Q(K) \neq \{0\}$. More precisely, the image of $t \wedge t^{-1}$ in $\mathrm{HC}_1^Q(K)$ is nonzero.*

Proof. We denote W by the Q-linear subspace of $K \wedge_Q K$ generated by elements of the form $uv \wedge w + vw \wedge u + wu \wedge v$.

Let us begin by the case of $Q(\!(t)\!)$. Define a Q-bilinear map $F : Q(\!(t)\!)^2 \to Q$ by

$$F\left(\sum x_i t^i, \sum y_j t^j\right) = \sum_{k \in \mathbf{Z}} k x_k y_{-k}. \tag{5.C.11}$$

Observe that the latter sum is finitely supported. (In $Q[t, t^{-1}]$, the above map appears in the definition of the defining 2-cocycle of affine Lie algebras, see [Fu].) We see that F is alternating by a straightforward computation, and that $F(t, t^{-1}) = 1$. Setting $f(x \wedge y) = F(x, y)$, if $x = \sum x_i t^i$, etc., we have

$$\begin{aligned} f(xy \wedge z) &= \sum_{k \in \mathbf{Z}} k (xy)_k z_{-k} \\ &= \sum_{k \in \mathbf{Z}} k \sum_{i+j=k} x_i y_j z_{-k} \\ &= -\sum_{i+j+k=0} k x_i y_j z_k; \end{aligned}$$

thus

$$f(xy \wedge z + yz \wedge x + zx \wedge y) = -\sum_{i+j+k=0} (k+i+j) x_i y_j z_k = 0.$$

Hence f factors through a Q-linear map from $\mathrm{HC}_1^Q(Q(\!(t)\!)) \to Q$ mapping $t \wedge t^{-1}$ to 1. The proof is thus complete if $K \subset Q(\!(t)\!)$.

Now assume that K has characteristic zero; let us show that $t \wedge t^{-1}$ has a nontrivial image in $\mathrm{HC}_1(K)$. Since K has characteristic zero, the above definition

(5.C.11) immediately extends to the field $Q(\!(t^{-\infty})\!) = \bigcup_{n>0} Q(\!(t^{1/n})\!)$ of Puiseux series and hence $t \wedge t^{-1}$ has a nontrivial image in $\mathrm{HC}_1^Q(Q(\!(t^{-\infty})\!))$ for every field Q of characteristic zero.

To prove the general result, let $(u_j)_{j\in J}$ be a transcendence basis of K over $Q(t)$. Replacing Q by $Q(t_j : j \in J)$ if necessary, we can suppose that K is algebraic over $Q(t)$. Let $\hat{Q}$ be an algebraic closure of Q. By the Newton-Puiseux Theorem, $\hat{Q}(\!(t^{-\infty})\!)$ is algebraically closed. By the Steinitz Theorem, there exists a $Q(t)$-embedding of K into $L = \hat{Q}(\!(t^{-\infty})\!)$. This induces a Q-linear homomorphism $\mathrm{HC}_1^Q(K) \to \mathrm{HC}_1^Q(L)$ mapping the class of $t\wedge t^{-1}$ in $\mathrm{HC}_1^Q(K)$ to the class of $t\wedge t^{-1}$ in $\mathrm{HC}_1^Q(L)$; the latter is mapped in turn to the class $t \wedge t^{-1}$ in $\mathrm{HC}_1^{\widehat{Q}}(L)$, which is nonzero. So the class of $t \wedge t^{-1}$ in $\mathrm{HC}_1^Q(K)$ is nonzero. □

Theorem 5.C.12. *Let $\mathfrak{g}$ be a Lie algebra over B, which is defined over A, i.e. $\mathfrak{g} \simeq \mathsf{B} \otimes_{\mathsf{A}} \mathfrak{g}_{\mathsf{A}}$ for some Lie algebra $\mathfrak{g}_{\mathsf{A}}$ over A. Consider the homomorphism*

$$\begin{array}{rcl}\varphi : (\mathfrak{g} \wedge_{\mathsf{A}} \mathfrak{g})/B_2^{\mathsf{A}}(\mathfrak{g}) & \to & M = \mathrm{HC}_1^{\mathsf{A}}(\mathsf{B}) \otimes \mathrm{Kill}^{\mathsf{A}}(\mathfrak{g}_{\mathsf{A}}) \\ (\lambda \otimes x) \wedge (\mu \otimes y) & \mapsto & (\lambda \wedge \mu) \otimes (x \circledcirc y).\end{array}$$

Then φ is well-defined and surjective, and $\varphi(W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})) = 2M$. In particular, if 2 is invertible in A, $\mathfrak{g}_{\mathsf{A}}$ is a graded Lie algebra and $M_0 = \mathrm{HC}_1(\mathsf{B}) \otimes \mathrm{Kill}(\mathfrak{g}_{\mathsf{A}})_0 \neq \{0\}$, then $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})_0 \neq \{0\}$.

The above map φ was considered in [NW08], for similar motivations. Assuming that A is a field of characteristic zero, the methods in [NW08] can provide a more precise description (as the cokernel of an explicit homomorphism) of the kernel $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g}) = \mathrm{Ker}(H_2^{\mathsf{A}}(\mathfrak{g}) \to H_2^{\mathsf{B}}(\mathfrak{g}))$. Since we do not need this description and in order not to introduce further notation, we do not include it.

Proof of Theorem 5.C.12. Let us first view ϕ as defined on $\mathfrak{g}\wedge_{\mathsf{A}}\mathfrak{g}$. The surjectivity is trivial. By Proposition 5.C.5 (with grading concentrated in degree 0), we see that $W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})$ is generated by elements of the form $\lambda x \wedge \mu y - \mu x \wedge \lambda y$ with $x, y \in \mathfrak{g}_{\mathsf{A}}$ (we omit the $\otimes$ signs, which can here be thought of as scalar multiplication); the image by φ of such an element is $2(\lambda\wedge\mu)(x\circledcirc y)$, which belongs to $2M$. Conversely, since $2M$ is generated by elements of the form $2(\lambda \wedge \mu)(x \circledcirc y)$, we deduce that $\varphi(W_2^{\mathsf{A},\mathsf{B}}(\mathfrak{g})) = 2M$.

Let us check that φ vanishes on 2-boundaries (so that it is well-defined on $\mathfrak{g} \wedge_{\mathsf{A}} \mathfrak{g}$ modulo 2-boundaries):

$$\begin{aligned}\varphi(tx \wedge [uy, vz]) &= (t \wedge uv) \otimes (x \circledcirc [y, z]);\\ \varphi(uy \wedge [vz, tx]) &= (u \wedge vt) \otimes (y \circledcirc [z, x]) = (u \wedge vt) \otimes (x \circledcirc [y, z]);\\ \varphi(vz \wedge [tx, uy]) &= (v \wedge tu) \otimes (z \circledcirc [x, y]) = (v \wedge tu) \otimes (x \circledcirc [y, z])\end{aligned}$$

and since $t \wedge uv + u \wedge vt + v \wedge tu = 0$ in $\mathrm{HC}_1(\mathsf{A})$, the sum of these three terms is zero.

The last statement clearly follows. □

5.C.2. *The characterization.* Using all results established in the preceding paragraphs, we obtain

Theorem 5.C.13. *Let $\mathfrak{g}$ be a finite-dimensional graded Lie algebra over a field K of characteristic zero, assume that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ has no opposite weights. Let Q be a subfield of K, so that K has infinite transcendence degree over Q. We have equivalences*

- $W_2^{Q,K}(\mathfrak{g})_0 = \{0\}$ *(i.e., $H_2^Q(\mathfrak{g})_0 \to H_2^K(\mathfrak{g})_0$ is an isomorphism);*
- $\mathrm{Kill}^K(\mathfrak{g})_0 = \{0\}$.

Corollary 5.C.14. *Under the same assumptions, we have equivalences*

- $H_2^Q(\mathfrak{g})_0 = \{0\}$ *(i.e. the blow-up $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism);*
- $H_2^K(\mathfrak{g})_0 = \mathrm{Kill}^K(\mathfrak{g})_0 = \{0\}$. □

The interest is that in both the theorem and the corollary, the first condition is a problem of linear algebra in infinite dimension, while the second is linear algebra in finite dimension (not involving Q) and is therefore directly computable in terms of the structure constants of $\mathfrak{g}$.

Proof of Theorem 5.C.13. Suppose that $\mathrm{Kill}^K(\mathfrak{g})_0 = 0$. By Corollary 5.C.8, the induced homomorphism $H_2^Q(\mathfrak{g})_0 \to H_2(\mathfrak{g})_0$ is bijective.

Conversely, suppose that $\mathrm{Kill}^K(\mathfrak{g})_0 \neq 0$. Since $\mathfrak{g}$ is finite-dimensional over K, there exists a subfield $L \subset K$, finitely generated over Q, such that $\mathfrak{g}$ is defined over Q, i.e. we can write $\mathfrak{g} = \mathfrak{g}_L \otimes_L K$. Obviously, $\mathrm{Kill}^K(\mathfrak{g}) = \mathrm{Kill}^L(\mathfrak{g}_L) \otimes_L K$, so $\mathrm{Kill}^L(\mathfrak{g}_L) \neq 0$. Let $(x \circledcirc_Q y)$ be the representative of a nonzero element in $\mathrm{Kill}^L(\mathfrak{g}_L)$. Let λ be an element of K, transcendental over L. By Lemma 5.C.10, the element $\lambda \wedge \lambda^{-1}$ has a nontrivial image in $\mathrm{HC}_1^L(K)$. By Theorem 5.C.12 (applied with $(\mathsf{A},\mathsf{B}) = (L,K)$), we deduce that

$$c_L = \lambda x \wedge_L \lambda^{-1} y - \lambda^{-1} x \wedge_L \lambda y$$

is not a 2-boundary, i.e. is nonzero in $H_2^L(\mathfrak{g})_0$. In particular the element c_Q (written as c_L with $\wedge_Q$ instead of $\wedge_L$) is nonzero in $H_2^K(\mathfrak{g})_0$ since its image in $H_2^L(\mathfrak{g})$ is c_L, while its image c_K in $\mathfrak{g} \wedge_K \mathfrak{g}$ and hence in $H_2^K(\mathfrak{g})_0$ is obviously zero. □

5.D. **Auxiliary descriptions of $H_2(\mathfrak{g})_0$ and $\mathrm{Kill}(\mathfrak{g})_0$.** In this subsection, all Lie algebras are over a fixed commutative ring R.

Definition 5.D.1. Let $\mathfrak{g}$ be a graded Lie algebra. We say that $\mathfrak{g}$ is **doubly 1-tame** if for every α we have $\mathfrak{g}_0 = \sum_{\beta \notin \{0,\alpha,-\alpha\}} [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$.

In view of Lemma 5.A.3, doubly 1-tame implies 1-tame, and the reader can easily find counterexamples to the converse. This definition will be motivated in Section 6, because it is a consequence of 2-tameness (Lemma 6.B.1(1)), which will be introduced therein.

The purpose of this subsection is to provide descriptions of $H_2(\mathfrak{g})_0$ and $\mathrm{Kill}(\mathfrak{g})_0$.

5.D.1. *The tame 2-homology module.* Let us begin by the trivial observation that if $\alpha+\beta+\gamma=0$ and $\alpha,\beta,\gamma\neq 0$, then $\alpha+\beta,\beta+\gamma,\gamma+\alpha\neq 0$. It follows that d_3 maps $(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)$ into $\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown$. Define the **tame 2-homology module**

$$H_2^\triangledown(\mathfrak{g})_0=(\mathrm{Ker}(d_2)\cap(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0)/d_3(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0.$$

We are going to prove the following result.

Theorem 5.D.2. *Let $\mathfrak{g}$ be a graded Lie algebra. The natural homomorphism $H_2^\triangledown(\mathfrak{g})_0\to H_2(\mathfrak{g})_0$ induced by the inclusion $(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0\to(\mathfrak{g}\wedge\mathfrak{g})_0$ is surjective if $\mathfrak{g}$ is 1-tame, and is an isomorphism if $\mathfrak{g}$ is doubly 1-tame.*

Remark 5.D.3. In Abels' second group, $(\mathfrak{g}\wedge\mathfrak{g})_0$ and $(\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g})_0$ have dimension 4 and 5, while $(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0$ and $(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0$ have dimension 3 and 2. Thus, we see that the computation of $H_2^\triangledown(\mathfrak{g})_0$ is in practice easier than the computation of $H_2(\mathfrak{g})_0$.

Lemma 5.D.4. *Let $\mathfrak{g}$ be any graded Lie algebra. If $\mathfrak{g}$ is 1-tame, then*

$$\mathfrak{g}_0\wedge\mathfrak{g}_0\subset\mathrm{Im}(d_3)+(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0; \tag{1}$$

$$\mathfrak{g}_0\wedge\mathfrak{g}_0\wedge\mathfrak{g}_0\subset\mathrm{Im}(d_4)+\mathfrak{g}_0\wedge(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0; \tag{2}$$

Proof. Observe that (1) is a restatement of Lemma 5.B.6.

The second assertion is similar: if u,v have nonzero opposite weights and $x,y\in\mathfrak{g}_0$, then, modulo $\mathrm{Im}(d_4)$, the element $x\wedge y\wedge[u,v]$ is equal to

$$y\wedge u\wedge[v,x]-[x,y]\wedge u\wedge v+x\wedge v\wedge[u,y]-y\wedge v\wedge[u,x]-x\wedge u\wedge[v,y],$$

which belongs to $\mathfrak{g}_0\wedge(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0$. ☐

Proposition 5.D.5. *Consider the following R-module homomorphism*

$$\begin{aligned}\Phi:\ \mathfrak{g}_\triangledown\otimes\mathfrak{g}_\triangledown\otimes\mathfrak{g}_\triangledown\otimes\mathfrak{g}_\triangledown &\to (\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)/d_3(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)\\ u\otimes v\otimes x\otimes y &\mapsto x\wedge[y,[u,v]]-y\wedge[x,[u,v]].\end{aligned}$$

If $\mathfrak{g}$ is doubly 2-tame, then there exists an R-module homomorphism

$$\Psi:\mathfrak{g}_0\otimes\mathfrak{g}_0\to(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0/d_3(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0$$

such that whenever α,β are non-collinear weights, we have

$$\Psi([x,y]\otimes[u,v])=\Phi(u\otimes v\otimes x\otimes y),\quad\forall x\in\mathfrak{g}_\alpha,\ y\in\mathfrak{g}_{-\alpha},\ u\in\mathfrak{g}_\beta,\ v\in\mathfrak{g}_{-\beta}.$$

Moreover, Ψ is antisymmetric, i.e. factors through $\mathfrak{g}_0\wedge\mathfrak{g}_0$.

Proof. Suppose that α,β are non-collinear weights and that $x\in\mathfrak{g}_\alpha$, $y\in\mathfrak{g}_{-\alpha}$, $u\in\mathfrak{g}_\beta$, $v\in\mathfrak{g}_{-\beta}$. We have $d_3\circ d_4(x\wedge y\wedge u\wedge v)=0$. If we write this down in $(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0/d_3(\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown\wedge\mathfrak{g}_\triangledown)_0$, four out of six terms vanish and we get

$$d_3([x,y]\wedge u\wedge v)+d_3([u,v]\wedge x\wedge y)=0,$$

which expands as

$$\Phi(u\otimes v\otimes x\otimes y)+\Phi(x\otimes y\otimes u\otimes v)=0. \tag{5.D.6}$$

Define, for $w \in \mathfrak{g}_0$, $\Psi_{x,y}(w) = x \wedge [y,w] - y \wedge [x,w]$. The mapping

$$(x,y) \mapsto \Psi_{x,y} \in \mathrm{Hom}(\mathfrak{g}_0, (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0 / d_3(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0\,)$$

is bilinear and in particular extends to a homomorphism

$$\sigma : (\mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0 \to \mathrm{Hom}(\mathfrak{g}_0, (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0 / d_3(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0\,).$$

Since $\mathfrak{g}$ is doubly 1-tame, any $w \in \mathfrak{g}_0$ can be written as $\sum [u_i, v_i]$ with $u_i \in \mathfrak{g}_{\beta_i}$, $v_i \in \mathfrak{g}_{-\beta_i}$, $\beta_i \notin \{0, \pm\alpha\}$, so, using (5.D.6)

$$\begin{aligned}\Psi_{x,y}(w) &= \sum_i \Phi(u_i \otimes v_i \otimes x \otimes y) \\ &= -\sum_i \Phi(x \otimes y \otimes u_i \otimes v_i) = \sum_i \Psi_{u_i,v_i}([x,y]).\end{aligned}$$

This shows that $\sigma(x \otimes y)$ only depends on $[x,y]$, i.e. we can write $\sigma(x \otimes y) = \sigma'([x,y])$. Define, for $z, w \in \mathfrak{g}_0$

$$\Psi(z \otimes w) = \sigma'(z)(w).$$

By construction, whenever $z = [x,y]$ and $w = [u,v]$, with $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, $u \in \mathfrak{g}_\beta$, $v \in \mathfrak{g}_{-\beta}$ and α, β are not collinear, we have

$$\Psi([x,y] \otimes [u,v]) = \Phi(u \otimes v \otimes x \otimes y);$$

from (5.D.6) we see in particular that Ψ is antisymmetric. □

Proof of Theorem 5.D.2. If $\mathfrak{g}$ is 1-tame, the surjectivity immediately follows from Lemma 5.D.4(1).

Now to show the injectivity of the map of the theorem, suppose that $c \in (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$ is a 2-boundary and let us show that c belongs to $d_3(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$. In view of Lemma 5.D.4(2), we already know that c belongs to $d_3(\mathfrak{g} \wedge \mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$, and let us work again modulo $d_3(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$, so that we can suppose that c belongs to $d_3(\mathfrak{g}_0 \wedge (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0)$, and we wish to check that $c = 0$. Since $\mathfrak{g}$ is doubly 1-tame, we can write

$$c = \sum d_3([u_i, v_i] \wedge x_i \wedge y_i)$$

with $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$, $u_i \in \mathfrak{g}_{\beta_i}$, $v_i \in \mathfrak{g}_{-\beta_i}$, α_i, β_i nonzero and $\alpha_i \neq \pm\beta_i$. Write $w_i = [u_i, v_i]$. Then

$$c = \left(\sum_i w_i \wedge [x_i, y_i]\right) + \left(\sum_i (x_i \wedge [y_i, w_i] + y_i \wedge [w_i, x_i])\right),$$

the first term belongs to $\mathfrak{g}_0 \wedge \mathfrak{g}_0$ and the second to $(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0 / d_3(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$; since c is assumed to lie in $(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$ we deduce that

$$\sum_i w_i \wedge [x_i, y_i] = 0 \tag{5.D.7}$$

in $\mathfrak{g}_0 \wedge \mathfrak{g}_0$. Therefore

$$c\sum_i x_i \wedge [y_i, w_i] + y_i \wedge [w_i, x_i] = \sum_i \Phi(u_i \otimes v_i \otimes x_i \otimes y_i).$$

Now using Proposition 5.D.5 we get

$$c = \sum_i \Psi(w_i \wedge [x_i, y_i]) = \Psi\left(\sum_i w_i \wedge [x_i, y_i]\right) = 0$$

again by (5.D.7). □

5.D.2. *The tame Killing module.*

Definition 5.D.8. Let $\mathfrak{g}$ be a graded Lie algebra over R. Consider the R-module homomorphism

$$\begin{aligned} \mathcal{T} : (\mathfrak{g} \circledcirc_{\mathsf{R}} \mathfrak{g}) \otimes_{\mathsf{R}} \mathfrak{g} \quad &\to \quad \mathfrak{g} \circledcirc_{\mathsf{R}} \mathfrak{g} \\ u \circledcirc v \otimes w \quad &\mapsto \quad u \circledcirc [v, w] + v \circledcirc [u, w]. \end{aligned}$$

By definition, $\mathrm{Kill}(\mathfrak{g})$ is the cokernel of $\mathcal{T}$. We define $\mathrm{Kill}^{\triangledown}(\mathfrak{g})_0$ as the cokernel of the restriction of $\mathcal{T}$ to

$$((\mathfrak{g}_{\triangledown} \circledcirc \mathfrak{g}_{\triangledown}) \otimes \mathfrak{g}_{\triangledown})_0 \to (\mathfrak{g}_{\triangledown} \circledcirc \mathfrak{g}_{\triangledown})_0.$$

Note that $\mathcal{T}$ satisfies the identities, for all x, y, z

$$\mathcal{T}(x \circledcirc y \otimes z) + \mathcal{T}(y \circledcirc z \otimes x) + \mathcal{T}(z \circledcirc x \otimes y) = 0.$$

There is an canonical homomorphism $\mathrm{Kill}^{\triangledown}(\mathfrak{g})_0 \to \mathrm{Kill}(\mathfrak{g})_0$.

Theorem 5.D.9. *Let $\mathfrak{g}$ be a graded Lie algebra. If $\mathfrak{g}$ is 1-tame then the homomorphism $\mathrm{Kill}^{\triangledown}(\mathfrak{g})_0 \to \mathrm{Kill}(\mathfrak{g})_0$ is surjective; if $\mathfrak{g}$ is doubly 1-tame then it is an isomorphism.*

Lemma 5.D.10. *Let $\mathfrak{g}$ be an arbitrary Lie algebra. Then we have the identity*

$$\begin{aligned} \mathcal{T}(w, x, [y, z]) = \quad & \mathcal{T}([x, z], w, y) - \mathcal{T}([x, y], w, z) \\ & - \mathcal{T}([w, y], x, z) + \mathcal{T}([w, z], x, y); \end{aligned}$$

Proof. Use the four equalities

$$\begin{aligned} \mathcal{T}([x, z], w, y) =& w \circledcirc [[x, z], y] + [x, z] \circledcirc [w, y], \\ \mathcal{T}([y, x], w, z) =& w \circledcirc [[y, x], z] + [y, x] \circledcirc [w, z], \\ \mathcal{T}([w, z], x, y) =& x \circledcirc [[w, z], y] + [w, z] \circledcirc [x, y], \\ \mathcal{T}([y, w], x, z) =& x \circledcirc [[y, w], z] + [y, w] \circledcirc [x, z]; \end{aligned}$$

the sum of the four right-hand terms is, by the Jacobi identity and cancelation of $([\cdot, \cdot] \circledcirc [\cdot, \cdot])$-terms, equal to

$$-w \circledcirc [[z, y], x] - x \circledcirc [[z, y], w] = \mathcal{T}(w, x, [z, y]). \quad \square$$

Lemma 5.D.11. *Let $\mathfrak{g}$ be an arbitrary graded Lie algebra. Let $\alpha, \alpha', \beta, \beta'$ be nonzero weights, with $\alpha+\beta, \alpha+\beta', \alpha'+\beta, \alpha'+\beta' \neq 0$, and $(x, x', y, y') \in \mathfrak{g}_\alpha \times \mathfrak{g}_{\alpha'} \times \mathfrak{g}_\beta \times \mathfrak{g}_{\beta'}$. Then, modulo $\mathcal{T}(\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)$, we have*

$$\mathcal{T}(x, x', [y, y']) = 0. \tag{1}$$

and

$$\mathcal{T}([x, x'], y, y') = \mathcal{T}([y, y'], x, x'). \tag{2}$$

Let $\alpha_0, \beta, \gamma, \gamma'$ be weights with $\beta, \gamma, \gamma' \neq 0$, $\alpha_0 \notin \{-\gamma, -\gamma'\}$. For all $w_0 \in \mathfrak{g}_{\alpha_0}, x \in \mathfrak{g}_\beta, y \in \mathfrak{g}_\gamma, y' \in \mathfrak{g}_{\gamma'}$ we have

$$\mathcal{T}(w_0, x, [y, y']) = \mathcal{T}([x, y'], w_0, y) - \mathcal{T}([x, y], w_0, y'). \tag{3}$$

Let $\alpha, \alpha', \beta, \beta'$ be nonzero weights with $\alpha+\alpha', \beta+\beta' \neq 0$. For all $x \in \mathfrak{g}_\alpha, x' \in \mathfrak{g}_{\alpha'}, y \in \mathfrak{g}_{\beta'}, y' \in \mathfrak{g}_{\beta'}$ we have

$$\mathcal{T}(x, y, [x', y']) = \mathcal{T}([x, y'], y, x') - \mathcal{T}([y, x'], x, y'). \tag{4}$$

Proof. This are immediate from the formula given by Lemma 5.D.10 applied to (x, x', y, y'), resp. (x, y, x', y'), resp. (w_0, x, y, z), resp. (x, y, x', y'). ☐

Lemma 5.D.12. *Let $\mathfrak{g}$ be a doubly 1-tame graded Lie algebra. Let α, β, γ be weights with $\alpha, \beta \neq 0$ and $\alpha+\beta+\gamma = 0$, and $(x, y, z) \in \mathfrak{g}_\alpha \times \mathfrak{g}_\beta \times \mathfrak{g}_\gamma$. Then, modulo $\mathcal{T}(\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)$, we have*

$$\mathcal{T}(x, y, z) = 0. \tag{1}$$

Let $\alpha, \alpha', \beta, \beta'$ be weights, with $\alpha, \beta \neq 0$ and $\alpha+\alpha'+\beta+\beta' = 0$, and $(x, x', y, y') \in \mathfrak{g}_\alpha \times \mathfrak{g}_{\alpha'} \times \mathfrak{g}_\beta \times \mathfrak{g}_{\beta'}$. Then, modulo $\mathcal{T}(\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)$, we have and

$$\mathcal{T}([x, y'], y, x') = \mathcal{T}([y, x'], x, y'). \tag{2}$$

Proof. Let us check the first assertion. If $\gamma \neq 0$ this is trivial, so assume $\gamma = 0$ (so $\alpha = -\beta$). Since $\mathfrak{g}$ is doubly 1-tame, we can write $z = \sum [u_i, v_i]$ with u_i, v_i of opposite weights, not equal to $\pm\alpha$. Then Lemma 5.D.11(1) applies.

From (1) and Lemma 5.D.11(4), we obtain (2) when $\alpha+\alpha', \beta+\beta' \neq 0$. Since $\alpha+\alpha'+\beta+\beta' = 0$, the only remaining case is when $\alpha+\alpha' = \beta+\beta' = 0$. If $\alpha+\beta' \neq 0$, then both sides in (2) are in $((\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown) \otimes \mathfrak{g}_\triangledown)_0$. So (2) is proved whenever $(\alpha, \beta, \alpha', \beta') \neq (\alpha, \alpha, -\alpha, -\alpha)$.

To tackle this last case, let us use this to prove first the following: if γ is a nonzero weight, $w_0 \in \mathfrak{g}_0$, $z \in \mathfrak{g}_\gamma$,$z' \in \mathfrak{g}_{-\gamma}$, then $\mathcal{T}(w_0, z, z') + \mathcal{T}(w_0, z', z) = 0$. Indeed, since $\mathfrak{g}$ is doubly 1-tame, this reduces by linearity to $w_0 = [u, u']$ with $u \in \mathfrak{g}_\delta$, $u' \in \mathfrak{g}_{-\delta}$ and $\delta \notin \{0, \pm\gamma\}$. So, using twice (2) in one of the cases already proved, we obtain

$$\begin{aligned}\mathcal{T}(w_0, z, z') =& \mathcal{T}([u, u'], z, z')\\ =& \mathcal{T}([z, z'], u, u')\\ =& -\mathcal{T}([z', z], u, u')\\ =& -\mathcal{T}([u, u'], z', z)\\ =& -\mathcal{T}(w_0, z', z)\end{aligned}$$

Now suppose that $(\alpha, \beta, \alpha', \beta') = (\alpha, \alpha, -\alpha, -\alpha)$. Then, using the antisymmetry property above and again using one last time one already known case of (2), we obtain

$$\begin{aligned}\mathcal{T}([x, y'], y, x') =& -\mathcal{T}([x, y'], x', y)\\ =& -\mathcal{T}([x', y], x, y')\\ =& \ \mathcal{T}([y, x'], x, y').\end{aligned}$$

□

Lemma 5.D.13. *Let $\mathfrak{g}$ be a 1-tame graded Lie algebra. Then*

(1) $(\mathfrak{g} \circledcirc \mathfrak{g})_0 = \mathcal{T}(\mathfrak{g} \circledcirc \mathfrak{g} \otimes \mathfrak{g})_0 + (\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown)_0$*;*
(2) $\mathcal{T}(\mathfrak{g} \circledcirc \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0 = \mathcal{T}(\mathfrak{g} \circledcirc \mathfrak{g} \otimes \mathfrak{g})_0$.

Proof. Suppose that $x, y \in \mathfrak{g}_0$. To show that $x \circledcirc y$ belongs to the right-hand term in (1), it suffices by linearity to deal with the case when $y = [u, v]$ with u, v homogeneous of nonzero opposite weight. Then

$$x \circledcirc [u, v] = \mathcal{T}(x, u, v) - u \circledcirc [x, v],$$

which is the sum of an element in $((\mathfrak{g} \circledcirc \mathfrak{g}) \otimes \mathfrak{g})_0$ and an element in $(\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown)_0$. So (1) is proved.

Let us prove (2). By linearity, it is enough to prove that any element $\mathcal{T}(x, y, [u, v])$, where x, y have weight zero and u, v have nonzero opposite weight, belongs to $\mathcal{T}(\mathfrak{g} \otimes \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0$: the formula in Lemma 5.D.10 expresses $\mathcal{T}(x, y, [u, v])$ as a sum of four terms in $\mathcal{T}(\mathfrak{g} \otimes \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0$. □

Proposition 5.D.14. *Let $\mathfrak{g}$ be a doubly 1-tame graded Lie algebra. Then*

$$\mathcal{T}(\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \cap (\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown)_0 \subset \mathcal{T}((\mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0).$$

Proof. Fix $u, v \in \mathfrak{n}_\beta, \mathfrak{n}_{-\beta}$ ($\beta \neq 0$) and consider the R-module homomorphism

$$\begin{aligned}\Phi_{u,v} : \mathfrak{g}_0 \ &\to \ M = (\mathfrak{g} \circledcirc \mathfrak{g})_0 / \mathcal{T}((\mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0)\\ w \ &\mapsto \ u \circledcirc [v, w].\end{aligned}$$

The mapping $(u, v) \mapsto \Phi_{u,v} \in \mathrm{Hom}_{\mathsf{R}}(\mathfrak{g}_0, M)$ is bilinear. Therefore it extends to a mapping $s \mapsto \hat{\Phi}_s$ defined for all $s \in (\mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0$.

If $s = \sum x_i \otimes y_i \in \mathfrak{g} \wedge \mathfrak{g}$, we write $\langle s \rangle = \sum [x_i, y_i]$. Now consider, for $s, s' \in (\mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)_0$, $\hat{\Psi}(s \otimes s') = \hat{\Phi}_s(\langle s' \rangle) \in M$. In other words,

$$\hat{\Psi}((u \otimes v) \otimes (x \otimes y)) = u \circledcirc [v, [x, y]].$$

We have[2]

$$\begin{aligned} \hat{\Psi}((u \otimes v) \otimes (x \otimes y)) =& u \circledcirc [v, [x, y]] \\ =& - \mathcal{T}([x, y], u, v) + [x, y] \circledcirc [u, v] \end{aligned}$$

and similarly

$$\hat{\Psi}((x \otimes y) \otimes (u \otimes v)) = -\mathcal{T}([u, v], x, y) + [u, v] \circledcirc [x, y],$$

so

$$\begin{aligned} &\hat{\Psi}((u \otimes v) \otimes (x \otimes y)) - \hat{\Psi}((x \otimes y) \otimes (u \otimes v)) \\ =& \mathcal{T}([u, v], x, y) - \mathcal{T}([x, y], u, v) = 0 \end{aligned}$$

by Lemma 5.D.12(2). Thus, $\hat{\Psi}$ is symmetric and we can write $\hat{\Psi}(s \otimes s') = \hat{\Psi}(s \circledcirc s')$. Note that (trivially) $\Psi(s \circledcirc s') = 0$ whenever s' is a 2-cycle (i.e. $\langle s' \rangle = 0$), so by the symmetry $\hat{\Psi}$ factors through a map $\Psi : \mathfrak{g}_0 \circledcirc \mathfrak{g}_0 \to M$ such that

$$\hat{\Psi}(s \circledcirc s') = \Psi(\langle s' \rangle \circledcirc \langle s' \rangle)$$

for all $s, s' \in (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$. In other words, we can write

$$u \circledcirc [v, [x, y]] = \Psi([u, v] \circledcirc [x, y]).$$

Now consider some element in $\mathcal{T}(\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \cap (\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown)_0$. By Lemma 5.D.13, it can be taken in $\mathcal{T}(\mathfrak{g} \otimes \mathfrak{g}_\triangledown \otimes \mathfrak{g}_\triangledown)$. We write it as

$$\tau = \sum \mathcal{T}(x_i, y_i, z_i).$$

With x_i, y_i, z_i homogenous and y_i, z_i of nonzero weight. Since we work modulo $((\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown) \otimes \mathfrak{g}_\triangledown)_0$, we can suppose x_i is of weight zero for all i, and we have to prove that $\tau = 0$. So

$$\tau = \left(\sum_i x_i \circledcirc [y_i, z_i] \right) + \left(\sum_i y_i \circledcirc [x_i, z_i] \right),$$

the first term belongs to $\mathfrak{g}_0 \wedge \mathfrak{g}_0$ and the second to the quotient $\mathrm{Kill}^\triangledown(\mathfrak{g})_0$ of $(\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown)_0$ by $((\mathfrak{g}_\triangledown \circledcirc \mathfrak{g}_\triangledown) \otimes \mathfrak{g}_\triangledown)_0$, so

$$\sum_i x_i \circledcirc [y_i, z_i] = 0. \tag{5.D.15}$$

[2]Given a map defined on a tensor product such as Ψ, we freely view it as a multilinear map when it is convenient.

Now, writing $x_i = \sum_j [u_{ij}, v_{ij}]$ with u_{ij}, v_{ij} of nonzero opposite weight

$$\begin{aligned}\tau &= \sum_i y_i \circledcirc [x_i, z_i] \\ &= \sum_{i,j} y_i \circledcirc [z_i, [u_{ij}, v_{ij}]] \\ &= \sum_{i,j} \Psi([y_i, z_i] \circledcirc [u_{ij}, v_{ij}]) \\ &= \sum_i \Psi([y_i, z_i] \circledcirc x_i) = \Psi\left(\sum_i [y_i, z_i] \circledcirc x_i\right) = 0 \quad \text{by (5.D.15).}\end{aligned}$$ □

Proof of Theorem 5.D.9. The first statement follows from Lemma 5.D.13(1) and the second from Proposition 5.D.14. □

6. Abels' multiamalgam

6.A. **2-tameness.** In this section, we deal with real-graded Lie algebras, that is, Lie algebras graded in a real vector space $\mathcal{W}$. As in Section 5, Lie algebras are, unless explicitly specified, over the ground commutative ring R.

Let $\mathfrak{g}$ be a real-graded Lie algebra. We say that $\mathcal{P} \subset \mathcal{W}$ is $\mathfrak{g}$-principal if $\mathfrak{g}$ is generated, as a Lie algebra, by $\mathfrak{g}_{\mathcal{P}} = \sum_{\alpha \in \mathcal{P}} \mathfrak{g}_\alpha$ (note that this only depends on the structure of Lie ring, not on the ground ring R). We say that $\mathcal{P}$ (or $(\mathfrak{g}, \mathcal{P})$) is *$k$-tame* if whenever $\alpha_1, \dots, \alpha_k \in \mathcal{P}$, there exists an $\mathbf{R}$-linear form ℓ on $\mathcal{W}$ such that $\ell(\alpha_i) > 0$ for all $i = 1, \dots, k$. Note that $\mathcal{P}$ is 1-tame if and only if $0 \notin \mathcal{P}$ and is 2-tame[3] if and only if for all $\alpha, \beta \in \mathcal{P}$ we have $0 \notin [\alpha, \beta]$. Note that k-tame trivially implies $(k-1)$-tame.

We say that the graded Lie algebra $\mathfrak{g}$ is *k-tame* if there exists a $\mathfrak{g}$-principal k-tame subset. Note that for $k = 1$ this is compatible with the definition in §5.A.2.

Example 6.A.1. As usual, we write $\mathcal{W}_{\mathfrak{g}} = \{\alpha : \mathfrak{g}_\alpha \neq \{0\}\}$.

- $\mathfrak{g} = \mathfrak{sl}_3$ with its standard Cartan grading, $\mathcal{W}_{\mathfrak{g}} = \{\alpha_{ij} : 1 \le i \neq j \le 3\} \cup \{0\}$ (with $\alpha_{ij} = e_i - e_j$, (e_i) denoting the canonical basis of $\mathbf{R}^3$); then $\{\alpha_{12}, \alpha_{23}, \alpha_{31}\}$ and $\{\alpha_{21}, \alpha_{13}, \alpha_{32}\}$ are $\mathfrak{g}$-principal and 2-tame.
- If $\mathcal{P}_1$ is the set of weights of the graded Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, then any $\mathfrak{g}$-principal set contains $\mathcal{P}_1$; conversely if $\mathfrak{g}$ is nilpotent then $\mathcal{P}_1$ itself is $\mathfrak{g}$-principal. Thus if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is k-tame if and only if 0 is not in the convex hull of k weights of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

[3]This paper will not deal with k-tameness for $k \ge 3$ but this notion is relevant to the study of higher-dimensional isoperimetry problems.

6.B. **Lemmas related to 2-tameness.** This subsection gathers a few technical lemmas needed in the study of the multiamalgam in §6.C and §6.D. The reader can skip it in a first reading.

The following lemma was proved by Abels under more specific hypotheses ($\mathfrak{g}$ nilpotent and finite-dimensional over a p-adic field). As usual, by $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma]$ we mean the *module* generated by such brackets.

Lemma 6.B.1. *Let $\mathfrak{g}$ be a real-graded Lie algebra and $\mathcal{P} \subset \mathcal{W}$ a $\mathfrak{g}$-principal subset. Suppose that $(\mathfrak{g}, \mathcal{P})$ is 2-tame. Then*

(1) for any $\omega \in \mathcal{W}$, we have $\mathfrak{g}_0 = \sum_\beta [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$, with β ranging over $\mathcal{W} - \mathbf{R}\omega$;

(2) if $\mathbf{R}_+\alpha \cap \mathcal{P} = \emptyset$, then $\mathfrak{g}_\alpha = \sum [\mathfrak{g}_\beta, \mathfrak{g}_\gamma]$, with (β, γ) ranging over pairs in $\mathcal{W} - \mathbf{R}\alpha$ such that $\beta + \gamma = \alpha$.

Lemma 6.B.1 is a consequence of the more technical Lemma 6.B.2 below (with $i = 1$). Actually, the proof of Lemma 6.B.1 is based on an induction which makes use of the full statement of Lemma 6.B.2. Besides, while Lemma 6.B.1 is enough for our purposes in the study of the multiamalgam of Lie algebras in §6.C, the statements in Lemma 6.B.2 involving the descending central series is needed when studying multiamalgams of nilpotent groups in §6.D.

Lemma 6.B.2. *Under the assumptions of Lemma 6.B.1, let $(\mathfrak{g}^i)$ be the descending central series of $\mathfrak{g}$ and $\mathfrak{g}^i_\alpha = \mathfrak{g}^i \cap \mathfrak{g}_\alpha$ (we avoid writing $\mathfrak{g}^i_\alpha$ because of the ambiguity if $\alpha = 0$). Then*

(1) for any $\omega \in \mathcal{W}$, we have $\mathfrak{g}^i_0 = \sum_{j+k=i} \sum_\beta [\mathfrak{g}^j_\beta, \mathfrak{g}^k_{-\beta}]$, with β ranging over $\mathcal{W} - \mathbf{R}\omega$;

(2) if $\mathbf{R}_+\alpha \cap \mathcal{P} = \emptyset$, then $\mathfrak{g}^i_\alpha = \sum_{j+k=i} \sum [\mathfrak{g}^j_\beta, \mathfrak{g}^k_\gamma]$, with (β, γ) ranging over pairs in $\mathcal{W} - \mathbf{R}\alpha$ such that $\beta + \gamma = \alpha$;

(3) if $\alpha \notin \mathcal{P}$, then $\mathfrak{g}^i_\alpha = \sum_{j+k=i} \sum [\mathfrak{g}^j_\beta, \mathfrak{g}^k_\gamma]$, with (β, γ) ranging over pairs in $\mathcal{W}$ such that $\beta + \gamma = \alpha$ and $0 \notin [\beta, \gamma]$.

Proof. Define $\mathfrak{g}^{[1]} = \sum_{\alpha \in \mathcal{P}} \mathfrak{g}_\alpha$, and by induction $\mathfrak{g}^{[i]} = [\mathfrak{g}^{[1]}, \mathfrak{g}^{[i-1]}]$ for $i \geq 2$; note that this depends on the choice of $\mathcal{P}$. Define $\mathfrak{g}^{[i]}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{g}^{[i]}$. Note that each $\mathfrak{g}^{[i]}$ is a graded submodule of $\mathfrak{g}$.

Let us prove by induction the following statement: in the three cases,

$$\mathfrak{g}^{[i]}_\alpha \subset \sum_{j,k \geq 1,\, j+k=i} \; \sum_{\beta+\gamma=\alpha\ldots} \left[\mathfrak{g}^{[j]}_\beta, \mathfrak{g}^{[k]}_\gamma\right] \tag{6.B.3}$$

where in each case (β, γ) satisfies the additional requirements of (1), (2), or (3) (we encode this in the notation $\sum_{\beta+\gamma=\alpha\ldots}$).

Since in all cases, $\alpha \notin \mathcal{P}$, the case $i = 1$ is an empty (tautological) statement. Suppose that $i \geq 2$ and the inclusions (6.B.3) are proved up to $i - 1$. Consider $x \in \mathfrak{g}^{[i]}_\alpha$. By definition, x is a sum of elements of the form $[y, z]$ with $y \in \mathfrak{g}^{[1]}_\beta$, $z \in \mathfrak{g}^{[i-1]}_\gamma$ with $\beta + \gamma = \alpha$. If β and γ are linearly independent over $\mathbf{R}$, the additional conditions are satisfied and we are done. Otherwise, since $\beta \in \mathcal{P}$, we have $\beta \neq 0$ and we can write $\gamma = r\beta$ for some $r \in \mathbf{R}$. There are three cases to consider:

- $r > 0$. Then we are in Case (3), and $0 \notin [\beta, \gamma]$, so the additional condition in (3) holds.
- $r = 0$. Then $\beta = \alpha \neq 0$ is a principal weight, which is consistent with none of Cases (1), (2) or (3).
- $r < 0$. Then since $\beta \in \mathcal{P}$, we have $\mathbf{R}_+\gamma \cap \mathcal{P} = \emptyset$, so we can apply the induction hypothesis of Case (2) to z; by linearity, this reduces to $x = [y, [u, v]]$ with $y \in \mathfrak{g}_\beta^{[1]}$, $u \in \mathfrak{g}_\delta^{[j]}$, $v \in \mathfrak{g}_\varepsilon^{[k]}$, $j + k = i - 1$, $\delta + \varepsilon = r\beta$, and δ, ε not collinear to β. By the Jacobi identity,

$$x \in \left[\mathfrak{g}_\delta^{[j]}, \mathfrak{g}_{\alpha-\delta}^{[k+1]}\right] + \left[\mathfrak{g}_\varepsilon^{[k]}, \mathfrak{g}_{\alpha-\varepsilon}^{[j+1]}\right]. \tag{6.B.4}$$

  If $\alpha \neq 0$, then we are in Case (3) or (2), and we get the additional conditions of (2) (and therefore of (3)). If $\alpha = 0$ (i.e. $r = -1$), we are in Case (1) and if $\omega \in \mathbf{R}\beta$, we see (6.B.4) satisfies the additional conditions of (1). However, the case $\omega \notin \mathbf{R}\beta$ is trivial since then the writing $x = [y, z]$ itself satisfies the additional condition of (1).

At this point, (6.B.3) is proved. Now write $\mathfrak{g}^{\{i\}} = \sum_{\ell \geq i} \mathfrak{g}^{[\ell]}$. Note that $\mathfrak{g}_\alpha^{\{i\}} = \sum_{\ell \geq i} \mathfrak{g}_\alpha^{[\ell]}$.

Obviously, for $\ell \geq i$ we have

$$\mathfrak{g}_\alpha^{[\ell]} \subset \sum_{j+k=\ell} \sum_{\beta+\gamma=\alpha\ldots} \left[\mathfrak{g}_\beta^{\{j\}}, \mathfrak{g}_\gamma^{\{k\}}\right] \subset \sum_{j+k=i} \sum_{\beta+\gamma=\alpha\ldots} \left[\mathfrak{g}_\beta^{\{j\}}, \mathfrak{g}_\gamma^{\{k\}}\right],$$

so

$$\mathfrak{g}_\alpha^{\{i\}} \subset \sum_{j+k=i} \sum_{\beta+\gamma=\alpha\ldots} \left[\mathfrak{g}_\beta^{\{j\}}, \mathfrak{g}_\gamma^{\{k\}}\right],$$

and since the inclusion $\left[\mathfrak{g}_\beta^{\{j\}}, \mathfrak{g}_\gamma^{\{k\}}\right] \subset \mathfrak{g}_\alpha^{\{i\}}$ is clear, we get the equality

$$\mathfrak{g}_\alpha^{\{i\}} = \sum_{j+k=i} \sum_{\beta+\gamma=\alpha\ldots} \left[\mathfrak{g}_\beta^{\{j\}}, \mathfrak{g}_\gamma^{\{k\}}\right].$$

By Lemma 2.G.14, for all $\ell \geq 1$, we have $\mathfrak{g}^{\{\ell\}} = \mathfrak{g}^\ell$, and therefore $\mathfrak{g}_\delta^{\{\ell\}} = \mathfrak{g}_\delta^\ell$ for all ℓ and all δ, whence the desired equalities. □

6.C. **Multiamalgams of Lie algebras.**

6.C.1. *The definition.* By *convex cone* in a real vector space, we mean any subset stable under addition and positive scalar multiplication (such a subset is necessarily convex). Let $\mathcal{C}$ be the set of convex cones of $\mathcal{W}$ not containing 0. Let $\mathfrak{g}$ be a real-graded Lie algebra over the ring R. If $C \in \mathcal{C}$, define

$$\mathfrak{g}_C = \bigoplus_{\alpha \in C} \mathfrak{g}_\alpha;$$

this is a graded Lie subalgebra of $\mathfrak{g}$ (if $\mathcal{W}_{\mathfrak{g}}$ is finite, $\mathfrak{g}_C$ is nilpotent). Denote by $x \mapsto \bar{x}$ the inclusion of $\mathfrak{g}_C$ into $\mathfrak{g}$.

Definition 6.C.1. Define $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}^{\mathsf{R}}$ as the multiamalgam (or colimit) of all $\mathfrak{g}_C$, where C ranges over $\mathcal{C}$.

This is by definition an initial object in the category of Lie algebras $\mathfrak{h}$ endowed with compatible homomorphisms $\mathfrak{g}_C \to \mathfrak{h}$.

It can be realized as the quotient of the Lie R-algebra free product of all $\mathfrak{g}_C$, by the ideal generated by elements $x - y$, where x, y range over elements in $\mathfrak{g}_C, \mathfrak{g}_D$ such that $\bar{x} = \bar{y}$ and C, D range over $\mathcal{C}$. Note that among those relators, we can restrict to homogeneous x, y as the other ones immediately follow. Since the free product as well as the ideal are graded, $\hat{\mathfrak{g}}$ is a graded Lie algebra; in particular, $\hat{\mathfrak{g}}$ is also the multiamalgam of the $\mathfrak{g}_C$ in the category of Lie algebras graded in $\mathcal{W}$. The inclusions $\mathfrak{g}_C \to \mathfrak{g}$ induce a natural homomorphism $\hat{\mathfrak{g}} \to \mathfrak{g}$.

6.C.2. *Link with the blow-up.* Let $\tilde{\mathfrak{g}}$ be the blow-up introduced in §5.B. For every $C \in \mathcal{C}$, the structural homomorphism $\tilde{\mathfrak{g}}_C \to \mathfrak{g}_C$ is an isomorphism, and therefore we obtain compatible homomorphisms $\mathfrak{g}_C \to \tilde{\mathfrak{g}}_C \subset \mathfrak{g}$, inducing, by the universal property, a natural graded homomorphism $\hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$.

Theorem 6.C.2. *If $\mathfrak{g}$ is 1-tame, then the natural Lie algebra homomorphism $\kappa : \hat{\mathfrak{g}}^{\mathsf{R}} \to \tilde{\mathfrak{g}}^{\mathsf{R}}$ is surjective, and if $\mathfrak{g}$ is 2-tame, κ is an isomorphism. In particular, if $\mathfrak{g}$ is 2-tame then the kernel of $\hat{\mathfrak{g}}^{\mathsf{R}} \to \mathfrak{g}$ is central in $\hat{\mathfrak{g}}$ and is canonically isomorphic (as an R-module) to $H_2^{\mathsf{R}}(\mathfrak{g})_0$.*

To prove the second statement, we need the following lemma.

Lemma 6.C.3 (Abels)**.** *If $\mathfrak{g}$ is 2-tame, then $\hat{\mathfrak{g}} \to \mathfrak{g}$ has central kernel, concentrated in degree zero.*

Proof of Theorem 6.C.2 from Lemma 6.C.3. If $\mathfrak{g}$ is 1-tame, then so is $\tilde{\mathfrak{g}}$ by Lemma 5.B.6, and the surjectivity of κ follows, proving the first assertion.

If $\mathfrak{g}$ is 2-tame, then by Lemma 6.C.3, $\hat{\mathfrak{g}} \to \mathfrak{g}$ has central kernel concentrated in degree zero, so by the universal property of the blow-up (Theorem 5.B.4), there is a section s of the natural map κ. By uniqueness in the universal properties, $s \circ \kappa$ and $\kappa \circ s$ are both identity and we are done.

The last statement is then an immediate consequence of Lemma 5.B.2. ☐

Proof of Lemma 6.C.3. If α is a nonzero weight of $\mathfrak{g}$, then there exists $C \in \mathcal{C}_{\mathfrak{g}}$ such that $\alpha \in C$. By the amalgamation relations, the composite graded homomorphism $\mathfrak{g}_\alpha \to \mathfrak{g}_C \to \hat{\mathfrak{g}}$ does not depend on the choice of C. We thus call it i_α and call its image $\mathfrak{m}_\alpha$.

Let us first check that whenever α, β are non-zero non-opposite weights, then we have the following inclusion in $\hat{\mathfrak{g}}$

$$[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \subset \mathfrak{m}_{\alpha+\beta}. \tag{6.C.4}$$

We begin with the observation that if γ, δ are nonzero weights with $\delta \notin -\mathbf{R}_+\gamma$, then

$$i_{\gamma+\delta}([\mathfrak{g}_\gamma, \mathfrak{g}_\delta]) = [\mathfrak{m}_\gamma, \mathfrak{m}_\delta]. \tag{6.C.5}$$

Indeed, in the above definition of i, we can choose C to contain both γ and δ. In particular if $\alpha \notin -\mathbf{R}_+\beta$, then (6.C.4) is clear. Let us prove (6.C.4) assuming that $\alpha \in \mathbf{R}_-\beta$. By 2-tameness, we can suppose that $\mathbf{R}_+\beta \cap \mathcal{P} = \emptyset$, so by Lemma 6.B.1(2), $\mathfrak{g}_\beta \subset \sum[\mathfrak{g}_\gamma, \mathfrak{g}_\delta]$, where (γ, β) ranges over the set $Q(\beta)$ of pair of weights such that $\gamma + \delta = \beta$ and γ, δ are not in $\mathbf{R}\beta (= \mathbf{R}\alpha)$. So, applying (6.C.5), we obtain $\mathfrak{m}_\beta \subset \sum_{(\gamma,\delta)\in Q(\beta)}[\mathfrak{m}_\gamma, \mathfrak{m}_\delta]$. Therefore

$$[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \subset \sum_{(\gamma,\delta)\in Q(\beta)} [\mathfrak{m}_\alpha, [\mathfrak{m}_\gamma, \mathfrak{m}_\delta]].$$

If we fix such $(\gamma, \delta) \in Q(\beta)$, we get, by the Jacobi identity and using that α, β, δ are pairwise non-collinear

$$\begin{aligned}[\mathfrak{m}_\alpha, [\mathfrak{m}_\gamma, \mathfrak{m}_\delta]] \subset & [\mathfrak{m}_\gamma, [\mathfrak{m}_\delta, \mathfrak{m}_\alpha]] + [\mathfrak{m}_\delta, [\mathfrak{m}_\alpha, \mathfrak{m}_\gamma]] \\ \subset & [\mathfrak{m}_\gamma, \mathfrak{m}_{\delta+\alpha}] + [\mathfrak{m}_\delta, \mathfrak{m}_{\alpha+\gamma}] \\ \subset & \mathfrak{m}_{\gamma+\delta+\alpha} = \mathfrak{m}_{\alpha+\beta},\end{aligned}$$

and finally $[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \subset \mathfrak{m}_{\alpha+\beta}$. So (6.C.4) is proved.

Now define $\mathfrak{w}_0$ as the submodule of $\hat{\mathfrak{g}}$

$$\mathfrak{w}_0 = \sum_{\gamma\neq 0}[\mathfrak{m}_\gamma, \mathfrak{m}_{-\gamma}]. \tag{6.C.6}$$

We are going to check that for every $\alpha \neq 0$

$$[\mathfrak{m}_\alpha, \mathfrak{w}_0] \subset \mathfrak{m}_\alpha \tag{6.C.7}$$

and

$$[\mathfrak{w}_0, \mathfrak{w}_0] \subset \mathfrak{w}_0. \tag{6.C.8}$$

Before proving (6.C.7), let us first check that for every nonzero α, if $\mathsf{R}(\alpha)$ is the set of nonzero weights not collinear to α then

$$[\mathfrak{m}_\alpha, \mathfrak{m}_{-\alpha}] \subset \sum_{\beta\in R(\alpha)} [\mathfrak{m}_\beta, \mathfrak{m}_{-\beta}]. \tag{6.C.9}$$

Indeed, by 2-tameness, we can suppose that $\mathbf{R}_+(-\gamma) \cap \mathcal{P} = \emptyset$, and apply Lemma 6.B.1(2), so

$$\mathfrak{g}_{-\alpha} \subset \sum_{(\gamma,\delta)\in Q(-\alpha)} [\mathfrak{g}_\gamma, \mathfrak{g}_\gamma];$$

by (6.C.5) we deduce

$$\mathfrak{m}_{-\alpha} \subset \sum_{(\gamma,\delta)\in Q(-\alpha)} [\mathfrak{m}_\alpha, \mathfrak{m}_\delta];$$

so

$$[\mathfrak{m}_\alpha, \mathfrak{m}_{-\alpha}] \subset \sum_{(\gamma,\delta)\in Q(-\alpha)} [\mathfrak{m}_\alpha, [\mathfrak{m}_\gamma, \mathfrak{m}_\delta]].$$

If $(\gamma, \delta) \in Q(-\alpha)$, we have, by the Jacobi identity and then (6.C.4)

$$\begin{aligned}[\mathfrak{m}_\alpha, [\mathfrak{m}_\gamma, \mathfrak{m}_\delta]] \subset & [\mathfrak{m}_\gamma, [\mathfrak{m}_\delta, \mathfrak{m}_\alpha]] + [\mathfrak{m}_\delta, [\mathfrak{m}_\alpha, \mathfrak{m}_\gamma]] \\ \subset & [\mathfrak{m}_\gamma, \mathfrak{m}_{\delta+\alpha}] + [\mathfrak{m}_\delta, \mathfrak{m}_{\alpha+\gamma}] \\ = & [\mathfrak{m}_\gamma, \mathfrak{m}_{-\gamma}] + [\mathfrak{m}_\delta, \mathfrak{m}_{-\delta}];\end{aligned}$$

we thus deduce (6.C.9).

We can now prove (6.C.7), namely if $\alpha, \gamma \neq 0$, then $[\mathfrak{m}_\alpha, [\mathfrak{m}_\gamma, \mathfrak{m}_{-\gamma}]] \subset \mathfrak{m}_\alpha$. By (6.C.9) we can assume that α and γ are not collinear, in which case (6.C.7) follows from (6.C.4) by an immediate application of Jacobi's identity.

To prove (6.C.8), we need to prove that $\mathfrak{w}_0$ is a subalgebra, or equivalently that for every $\alpha \neq 0$, we have

$$[\mathfrak{w}_0, [\mathfrak{m}_\alpha, \mathfrak{m}_{-\alpha}]] \subset \mathfrak{w}_0.$$

Indeed, using the Jacobi identity and then (6.C.7)

$$\begin{aligned}[\mathfrak{w}_0, [\mathfrak{m}_\alpha, \mathfrak{m}_{-\alpha}]] \subset & [\mathfrak{m}_\alpha, [\mathfrak{m}_{-\alpha}, \mathfrak{w}_0]] + [\mathfrak{m}_{-\alpha}, [\mathfrak{w}_0, \mathfrak{m}_\alpha]] \\ \subset & [\mathfrak{m}_\alpha, \mathfrak{m}_{-\alpha}] + [\mathfrak{m}_{-\alpha}, \mathfrak{m}_\alpha] \subset \mathfrak{w}_0.\end{aligned}$$

so (6.C.8) is proved.

We can now conclude the proof of the lemma. By the previous claims (6.C.4), (6.C.7), and (6.C.8), if $\mathfrak{w}_0$ is defined as in (6.C.6) the submodule $\left(\bigoplus_{\alpha\neq 0} \mathfrak{m}_\alpha\right) \oplus \mathfrak{w}_0$ is a Lie subalgebra of $\hat{\mathfrak{g}}$. Since the $\mathfrak{m}_\alpha$ for $\alpha \neq 0$ generate $\hat{\mathfrak{g}}$ by definition, this proves that this Lie subalgebra is all of $\hat{\mathfrak{g}}$.

Therefore, if $\phi : \hat{\mathfrak{g}} \to \mathfrak{g}$ is the natural map, we see that ϕ_α is the natural isomorphism $\mathfrak{m}_\alpha \to \mathfrak{g}_\alpha$. Thus $\mathrm{Ker}(\phi)$ is contained in $\hat{\mathfrak{g}}_0$. If $z \in \mathrm{Ker}(\phi)$ and $x \in \mathfrak{m}_\alpha$ for some $\alpha \neq 0$, then

$$\phi([z, x]) = [\phi(z), \phi(x)] = [0, \phi(x)] = 0,$$

so $[z, x] = \phi_\alpha^{-1}(\phi([z, x])) = 0$. Thus z centralizes $\mathfrak{m}_\alpha$ for all α; since these generate $\hat{\mathfrak{g}}$, we deduce that z is central. ☐

6.C.3. *On the non-2-tame case.* There is a partial converse to Theorem 6.C.2: if R is a field and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is not 2-tame, then $\hat{\mathfrak{g}}$ has a surjective homomorphism onto a free Lie algebra on two generators. In particular, if $\mathfrak{g}$ is nilpotent (as in all our applications), $\tilde{\mathfrak{g}}$ is nilpotent as well but $\hat{\mathfrak{g}}$ is not. The argument is straightforward: the assumption implies that there is a graded surjective homomorphism $\mathfrak{g} \to \mathfrak{h}$, where $\mathfrak{h}$ is the abelian 2-dimensional algebras with weights α, β with $\beta \in \mathbf{R}_{-}\alpha$, inducing a surjective homomorphism $\hat{\mathfrak{g}} \to \hat{\mathfrak{h}}$. Now it follows from the definition that $\mathfrak{h}$ is the free product of the 1-dimensional Lie algebras $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta$ and hence is not nilpotent.

6.D. **Multiamalgams of groups.** In this part, $\mathfrak{g}$ is a *nilpotent* real-graded Lie algebra over a commutative $\mathbf{Q}$-algebra R of characteristic zero (although the results would work in characteristic $p > s+1$, where s is the nilpotency length of the Lie algebra involved).

For $C \in \mathcal{C}_{\mathfrak{g}}$, let G and G_C be the groups associated to $\mathfrak{g}$ and $\mathfrak{g}_C$ by Malcev's equivalence of categories between nilpotent Lie algebras over $\mathbf{Q}$ and uniquely divisible nilpotent groups, described in Theorem 2.G.5. Let the embedding $G_C \to G$ corresponding to $\mathfrak{g}_C \to \mathfrak{g}$ be written as $g \mapsto \bar{g}$. Let $\hat{G}$ be the corresponding amalgam, namely the group generated by the free product of G_C for $C \in \mathcal{C}_{\mathfrak{g}}$, modded out by the relators xy^{-1} whenever $\bar{x} = \bar{y}$.

The following result was proved by Abels [Ab87, Cor. 4.4.14] assuming that $\mathsf{R} = \mathbf{Q}_p$ and that $\mathfrak{g}$ is finite-dimensional. This is one of the most delicate points in [Ab87]. We provide a sketch of the (highly technical) proof, in order to indicate how his proof works over our general hypotheses.

Theorem 6.D.1 (Abels)**.** *Suppose that $\mathfrak{g}$ is 2-tame and s-nilpotent. Then $\hat{G}$ is $(s+1)$-nilpotent.*

For the purpose of Section 4, we need a stronger result. Abels asked [Ab87, 4.7.3] whether $\hat{G} \to G$ is always a central extension. This is answered in the positive by the following theorem.

Theorem 6.D.2. *Under the same hypotheses, the nilpotent group $\hat{G}$ is uniquely divisible, and its Lie algebra is $\hat{\mathfrak{g}}^{\mathbf{Q}}$ in the natural way. In particular, the homomorphism $\hat{G} \to G$ has a central kernel, naturally isomorphic to $H_2^{\mathbf{Q}}(\mathfrak{g})_0$ as a $\mathbf{Q}$-linear space.*

Proof. Our proof is based on Theorem 6.D.1 and some generalities about nilpotent groups, which are gathered in §2.G.

Since by Theorem 6.D.1, $\hat{G}$ is nilpotent, and since it is generated by divisible subgroups, it is divisible (see Lemma 2.G.6). Therefore by the (standard) Lemma 2.G.4, to check that $\hat{G}$ is uniquely divisible, it is enough to check that it is torsion-free. Since $\hat{G}$ is known to be $(s+1)$-nilpotent, we see that $\hat{G}$ is the multiamalgam (=colimit) of the G_C *within the category $\mathcal{K}$ of $(s+1)$-nilpotent groups*. Therefore, $\hat{G}$ is the quotient of the free product W in $\mathcal{K}$ of the G_C by amalgamations relations. By Lemma 2.G.8, W is a uniquely divisible torsion-free nilpotent group. Since, for $x, y \in W$, whenever xy^{-1} is an amalgamation relation, $x^r y^{-r}$ is an amalgamation relation as well for all $r \in \mathbf{Q}$, Proposition 2.G.12 applies to show that the normal subgroup N generated by amalgamation relations is divisible. Therefore $\hat{G}/N$ is torsion-free, hence uniquely divisible.

It follows that $\hat{G}$ is also the multiamalgam of the G_C *in the category $\mathcal{K}_0$ of uniquely divisible $(s+1)$-nilpotent groups*. By Malcev's Theorem 2.G.5, this category is equivalent to the category of $(s+1)$-nilpotent Lie algebras over $\mathbf{Q}$. Therefore, if H is the group associated to $\hat{\mathfrak{g}}$ and $H \to G_C$ are the homomorphisms associated to $\mathfrak{g}_C \to \hat{\mathfrak{g}}$, then H and the family of homomorphisms $G_C \to H$ satisfy the universal property of multiamalgam in the category $\mathcal{K}_0$, and this gives rise to a canonical isomorphism $\hat{G} \to H$. In particular, by Theorem 6.C.2, the kernel W of $\hat{G} \to G$ is central in $\hat{G}$, and isomorphic to $H_2^{\mathbf{Q}}(\mathfrak{g})_0$. □

Corollary 6.D.3. *Under the same hypotheses, if moreover $H_2^{\mathsf{R}}(\mathfrak{g})_0 = \{0\}$ then the central kernel of $\hat{G} \to G$ is generated, as an abelian group, by elements of the form*

$$\exp([\lambda x, y]) \exp([x, \lambda y])^{-1}, \quad \lambda \in \mathsf{R}, \quad x, y \in \bigcup_C \mathfrak{g}_C.$$

If $\mathsf{R} = \prod_{j=1}^{\tau} \mathsf{R}_j$ is a finite product of $\mathbf{Q}$-algebras (so that $\mathfrak{g} = \prod_j \mathfrak{g}_j$ canonically), then those elements $\exp([\lambda x, y]) \exp([x, \lambda y])^{-1}$ with $\lambda \in \mathsf{R}_j$ and $x, y \in \mathfrak{g}_j$ are enough.

Proof. The additional assumption and Theorem 6.D.2 imply that the kernel of $\hat{G} \to G$ is naturally isomorphic, as a $\mathbf{Q}$-linear space, to the kernel $W_2^{\mathbf{Q},\mathsf{R}}(\mathfrak{g})$ of the natural map $H_2^{\mathbf{Q}}(\mathfrak{g})_0 \to H_2^{\mathsf{R}}(\mathfrak{g})_0$.

Let N be the normal subgroup of G generated by elements of the form

$$\exp([\lambda x, y]) \exp([x, \lambda y])^{-1}, \quad x, y \in \bigcup \mathfrak{g}_C, \ \lambda \in \mathsf{R};$$

clearly N is contained in the kernel W of $\hat{G} \to G$. By Proposition 2.G.12, N is divisible, i.e., N is a $\mathbf{Q}$-linear subspace of W. Therefore $\hat{G}/N$ is uniquely divisible and its Lie algebra can be identified with $\hat{\mathfrak{g}}/N$. By definition of N, in $\hat{G}/N$ we have $\exp([\lambda x, y]) = \exp([x, \lambda y])$ for all $x, y \in \bigcup \mathfrak{g}_C$. So in the Lie algebra of $\hat{G}/N$, which is equal to $\hat{\mathfrak{g}}/N$ we have $[\lambda x, y] = [x, \lambda y]$. Since by Proposition 5.C.5, the subgroup W is generated by elements of the form $\exp([\lambda x, y] - [x, \lambda y])$ when x, y range over $\bigcup G_C$ and λ ranges over R, it follows that $N = W$.

It remains to prove the last statement. By Lemma 5.B.7, $H_2(\mathfrak{g})_0$ can be identified with the product $\prod_j H_2(\mathfrak{g}_j)_0$. In particular, by Proposition 5.C.5, it is generated by elements of the form $[\lambda x, y] - [x, \lambda y]$ when $x, y \in \bigcup \mathfrak{g}_{j,C}$, $\lambda \in \mathsf{R}_j$, and $j = 1, \dots, \tau$. Then, we can conclude by a straightforward adaptation of the above proof. □

Corollary 6.D.4. *Under the same hypotheses, if moreover $H_2^{\mathsf{R}}(\mathfrak{g})_0 = \mathrm{Kill}^{\mathsf{R}}(\mathfrak{g})_0 = \{0\}$, then $\hat{G} \to G$ is an isomorphism.*

Proof. As observed in the proof of Corollary 6.D.3, since $H_2^{\mathsf{R}}(\mathfrak{g})_0 = 0$, the kernel of $\hat{G} \to G$ is isomorphic to $W_2^{\mathbf{Q},\mathsf{R}}(\mathfrak{g})_0$. Since $\mathrm{Kill}^{\mathsf{R}}(\mathfrak{g})_0 = 0$, Corollary 5.C.8 implies that $W_2^{\mathbf{Q},\mathsf{R}}(\mathfrak{g})_0 = 0$. □

Proof of Theorem 6.D.1 (sketched). We only sketch the proof; the proof in [Ab87] (with more restricted hypotheses) is 8 pages long.

Fix a 2-tame $\mathfrak{g}$-principal subset $\mathcal{P}$. Let $\mathcal{S}$ denote the set of all half-lines $\mathbf{R}_{>0}w$ ($w \neq 0$) in $\mathcal{W}$. Consider the descending central series (G^i). Let M_C^i be the image of $G^i \cap G_C$ in $\hat{G}$ and $M_C = M_C^1$. Let A^i be the normal subgroup of $\hat{G}$ generated by all M_C^i, where C ranges over $\mathcal{S}$.

Abels also introduces more complicated subgroups. Let $\mathcal{L}$ denote the set of lines of $\mathcal{W}$ (i.e. its projective space). Each line $L \in \mathcal{L}$ contains exactly two half lines:

$$L = S_1 \cup \{0\} \cup S_2.$$

Define the following subgroups of $\hat{G}$

$$M_{[L]} = M_{[L]}^1 = \langle M_{S_1} \cup M_{S_2} \rangle$$

and, by induction

$$M_{[L]}^i = \left\langle M_{S_1}^i \cup M_{S_2}^i \cup \bigcup_{j+k=i} \left(\!\left(M_{[L]}^j, M_{[L]}^k \right)\!\right) \right\rangle,$$

where $(\!(\cdot,\cdot)\!)$ denote the subgroup generated by group commutators.

Abels proves the following lemma [Ab87, 4.4.11]: if $L \in \mathcal{L}$ and C is a open cone in $\mathcal{W}$ such that $L + C \subset C$, then

$$\left(\!\left(M_C^j, M_{[L]}^k \right)\!\right) \subset M_C^{j+k}. \tag{6.D.5}$$

The interest is that $M_{[L]}^k$ is a complicated object, while the right-hand term M_C^{j+k} is a reasonable one.

Abels obtains [Ab87, Prop. 4.4.13] the following result, which can appear as a group version of Lemma 6.B.2(1): for any $L_0 \in \mathcal{L}$ and any i, the ith term of the descending central series $\hat{G}^i$ is generated, as a subgroup and modulo A^i, by the $M_{[L]}^i$ where L ranges over $\mathcal{L} - \{L_0\}$, or, in symbols,

$$\hat{G}^i = \left\langle \bigcup_{L \neq L_0} M_{[L]}^i \right\rangle A^i. \tag{6.D.6}$$

It is important to mention here that all three items of Lemma 6.B.2 are needed in the proof of (6.D.6) (encapsulated in the proof of [Ab87, Prop. 4.4.7]).

To conclude the proof, suppose that G is s-nilpotent. We wish to prove that $(\!(\hat{G}, \hat{G}^{s+1})\!) = \{1\}$. Since $\hat{G}$ is easily checked to be generated by M_S for S ranging over $\mathcal{S}$, it is enough to check, for each $S \in \mathcal{S}$

$$\left(\!\left(M_S, \hat{G}^{s+1} \right)\!\right) = \{1\}. \tag{6.D.7}$$

Now since G is s-nilpotent, $A^{s+1} = \{1\}$, so we can forget "modulo A^{s+1}" in (6.D.6) (with $i = s+1$), so (6.D.7) follows if we can prove

$$(\!(M_S, M_{[L]}^{s+1})\!) = \{1\}$$

for all $L \in \mathcal{L}$ with maybe one exception L_0; namely, we choose L_0 to be the line generated by $-S$. Then $S + L$ is an open cone, (6.D.5) applies and we have

$$(\!(M_S, M_{[L]}^{s+1})\!) \subset (\!(M_{S+L}, M_{[L]}^{s+1})\!) \subset M_{S+L}^{s+2} = \{1\}. \quad \square$$

7. Central and hypercentral extensions

In this section, unless explicitly specified, all Lie algebras are finite-dimensional over a field K of characteristic zero.

7.A. **Introduction of the section.** The purpose of this section is to prove the negative statements of the introduction in presence of 2-homological obstructions (Theorem D.2), gathered in the following theorem.

Theorem 7.A.1. *(1) If G is a standard solvable group (see Definition 1.2) satisfying the non-Archimedean 2-homological obstruction, then G is not compactly presented;*

(2) if G is a standard solvable group satisfying the 2-homological obstruction, then it has an least exponential Dehn function (possibly infinite);

(3) if G is a real triangulable group with the 2-homological obstruction, then it has an at least exponential Dehn function.

(1) and (2) are proved, by an elementary argument relying on central extensions, in §7.B. (3) is much more involved. The reason is that the exponential radical $\mathfrak{g}^\infty$ is not necessarily split in $\mathfrak{g}$ and the non-vanishing of $H_2(\mathfrak{g})_0$ does not necessarily yield a central extension of $\mathfrak{g}$ in degree zero (an explicit counterexample is given in §7.E). We then need some significant amount of work to show that it provides, anyway, a *hypercentral* extension.

7.B. **FC-Central extensions.** We use the following classical definition, which is a slight weakening of the notion of central extension.

Definition 7.B.1. Consider an extension

$$1 \to Z \xrightarrow{i} \widetilde{G} \to G \to 1. \tag{7.B.2}$$

We say that it is an **FC-central** extension if $i(Z)$ is FC-central in $\widetilde{G}$, in the sense that every compact subset of $i(Z)$ is contained in a compact subset of $\widetilde{G}$ that is invariant under conjugation.

This widely used terminology is (lamely) borrowed from the discrete case, in which it stands for "Finite Conjugacy (class)".

In the following, the reader can assume, in a first reading, that the FC-central extensions (defined below) are central. The greater generality allows to consider, for instance, the case when Z is a local field on which the action by conjugation is given by multiplication by elements of modulus 1.

Consider now an FC-central extension as in (7.B.2), and assume that $\widetilde{G}$ generated by a compact subset S (symmetric with 1); Fix k, set $W_k = Z \cap S^k$, and $\check{W}_k = \overline{\bigcup_{g\in\widetilde{G}} gW_kg^{-1}}$, which is a compact subset of Z by assumption. The following easy lemma, is partly a restatement of [BaMS93, Lemma 5] (which deals with finitely generated groups and assumes Z is central).

Lemma 7.B.3. *Let $\widetilde{\gamma}$ be any path in the Cayley graph of $\widetilde{G}$ with respect to S, joining 1 to an element z of Z. Let γ be the image of $\widetilde{\gamma}$ in the Cayley graph of G (with respect to the image of S). If $\widetilde{\gamma}$ can be filled by m ($\le k$)-gons, then $z \in \check{W}_k{}^m$.*

Proof. If $\widetilde{\gamma}$ can be filled by m ($\le k$)-gons, then z can be written (in the free group over S, hence in $\tilde{G}$) as $z = \prod_{i=1}^m h_i r_i h_i^{-1}$, where $r_i, h_i \in \widetilde{G}$, $r_i \in \mathrm{Ker}(\widetilde{G} \to G) = Z$ having length $\le k$ with respect to S, i.e. $r_i \in W_k$. Thus $h_i r_i h_i^{-1} \in \check{W}_k$; hence $z \in \check{W}_k{}^m$. ☐

The group Z may or not be compactly generated; if it is the case, let U a compact generating set of Z, and define the **distortion** of Z in G as

$$d_{G,Z}(n) = \max(n, \sup\{|g|_U : \ g \in Z, |g|_S \le n\});$$

if Z is not compactly generated set $d_{G,Z}(n) = +\infty$. Note that this function actually depends on Z and U as well, but its $\sim$-equivalence class only depends on (G, Z).

Proposition 7.B.4. *Given a FC-central extension as above, if G is compactly presented, then Z is compactly generated and its Dehn function satisfies $\delta_G(n) \succeq d_{G,Z}(n)$.*

Proof. By Lemma 7.B.3, if G is presented by S and relators of length $\le k$, then Z is generated by $\check{W}_k$, which is compact.

If U is a compact generating set for Z, then $W_k \subset U^\ell$ for some ℓ. Write $d(n) = d_{G,Z}(n)$ (relative to S and U) and $\delta(n) = \delta_G(n)$. Consider $g \in Z$ with $|g|_S \le n$ and $|g|_U = d(n)$. Taking $\widetilde{\gamma}$ to be a path of length $\le n$ in $\widetilde{G}$ joining 1 and g as in Lemma 7.B.3, we obtain that the loop γ in G has length $\le n$ and area m, and Lemma 7.B.3 implies that $d(n) = |g|_U \le m\ell$. So $\delta(n) \ge d(n)/\ell$. ☐

Proof of (1) and (2) in Theorem 7.A.1. In the setting of (1), we assume that $G = U \rtimes A$ satisfies the non-Archimedean 2-homological obstruction, so that for some j, $\mathbf{K}_j$ is non-Archimedean and the condition $Z = H_2(\mathfrak{u}_j)_0 \neq 0$ means that the action of A on U_j can be lifted to an action on a certain FC-central extension

$$1 \to Z \xrightarrow{i} \tilde{U}_j \to U_j \to 1,$$

so that $i(Z)$ is contained and FC-central in $[\tilde{U}_j, \tilde{U}_j]$. Thus it yields an FC-central extension

$$1 \to Z \xrightarrow{i} \tilde{U} \rtimes A \to U \rtimes A \to 1, \qquad (7.B.5)$$

where $\tilde{U} = \tilde{U}_j \times \prod_{j' \neq j} U_{j'}$, and $\tilde{U} \rtimes A$ is compactly generated. By Proposition 7.B.4, it follows that $U \rtimes A$ is not compactly presented.

In the setting of (2), the proof is similar, with $\mathbf{K}_j$ being Archimedean; there is a difference however: in (7.B.5), Z need not be FC-central in $\tilde{U} \rtimes A$, because of the possible real unipotent part of the A-action. Note that $i(Z)$ is central in

$\tilde{U}$. We then consider an A-irreducible quotient $Z'' = Z/Z'$ of Z and consider the FC-central extension

$$1 \to Z'' \xrightarrow{i} \tilde{U} \rtimes D \to U \rtimes D \to 1.$$

Since the real group U_j is exponentially distorted, it follows that $i(Z'')$ is exponentially distorted as well and by Proposition 7.B.4, it follows that $U \rtimes D$ has an at least exponential Dehn function. □

7.C. **Hypercentral extensions.** To prove Theorem 7.A.1(3), the natural approach seems to start with a real triangulable group G with $H_2(\mathfrak{g}^\infty)_0 \neq 0$ and find a central extension of G with exponentially distorted center. If the exponential radical of G is split, i.e. if $G = G^\infty \rtimes A$ for some nilpotent group A, the existence of such a central extension follows by a simple argument similar to that in the proof of Theorem 7.A.1(2). Unfortunately, in general such a central extension does not exist; although there are no simple counterexamples, we construct one in §7.E.

Nevertheless, in order to prove Theorem 7.A.1(3), the geometric part of the argument is the following variant of Proposition 7.B.4.

Proposition 7.C.1. *Let G be a connected triangulable Lie group and G^∞ its exponential radical (see Definition 2.F.2). Suppose that there exists an extension of connected triangulable Lie groups*

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$$

with N hypercentral in H (i.e. the ascending central series of H covers N) and with $N \cap H^\infty \neq \{1\}$. Then G has an (at least) exponential Dehn function.

Proof. We fix a compact generating set S in H, and its image S' in G. Let $h_n \in N$ be an element of linear size n in H and exponential size $\simeq e^n$ in H. Pick a path of size n joining 1 to h in H, i.e. represent h by a word $\gamma_n = x_1 \dots x_n$ in H with $x_i \in S$. Push this path forward to G to get a loop of size n in G; let a_n be its area. So in the free group over S, we have

$$\gamma_n = \prod_{j=1}^{a_n} g_{nj} r_{nj} g_{nj}^{-1}$$

where r_j is a relation of G (i.e. represents the identity in G) of bounded size. By a standard argument using van Kampen diagrams (see Lemma 2.D.2), we can choose the size of g_{nj} to be at most $\leq C(a_n + n)$, where C is a positive constant only depending on (G, S'). Push this forward to H to get

$$h_n = \prod_{j=1}^{a_n} g_{nj} r_{nj} g_{nj}^{-1},$$

where r_{nj} here is a bounded element of N, and g_{nj} has length $\leq C(a_n + n)$ in H. Since the action of H on N by conjugation is unipotent, we deduce that the size in N of $g_{nj}r_{nj}g_{nj}^{-1}$ is polynomially bounded with respect to a_n+n, say $\preceq (a_n+n)^d$ (uniformly in j). Therefore h_n has size $\preceq (a_n+n)^{d+1}$. Since (h_n) has exponential growth in N, we deduce that (a_n) also grows exponentially. ☐

Let us emphasize that at this point, the proof of Theorem 7.A.1(3) is not yet complete. Indeed, given a real triangulable group G with $H_2(\mathfrak{g}^\infty)_0 \neq 0$, we need to check that we can apply Proposition 7.C.1. This is the contents of the following theorem. If V is a G-module, V^G denotes the set of G-fixed points. Also, see Definition 2.F.1 for the definition of $\mathfrak{g}^\infty$.

Theorem 7.C.2. *Let G be a triangulable Lie group. Suppose that $H_2(\mathfrak{g}^\infty)^G \neq \{0\}$. Then there exists an extension of connected triangulable Lie groups*

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$$

with N hypercentral in H and with $N \cap G^\infty \neq \{1\}$.

The theorem will easily follow from the analogous (more general) result about solvable Lie algebras.

By **epimorphism** of Lie algebras we mean a surjective homomorphism, and we denote it by a two-headed arrow $\mathfrak{h} \twoheadrightarrow \mathfrak{g}$. Such an epimorphism is k-**hypercentral** if its kernel $\mathfrak{z}$ is contained in the kth term $\mathfrak{h}^k$ of the ascending central series of $\mathfrak{h}$; when $k=1$, that is, when $\mathfrak{z}$ is central in $\mathfrak{h}$, it is simply called a **central epimorphism**. Also, if $\mathfrak{m}$ is a $\mathfrak{g}$-module, we write $\mathfrak{m}^{\mathfrak{g}} = \{m \in \mathfrak{m} : \forall g \in \mathfrak{g}, gm = 0\}$.

Definition 7.C.3. We say that a hypercentral epimorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$ between Lie algebras has **polynomial distortion** if the induced epimorphism $\mathfrak{g}^\infty \twoheadrightarrow \mathfrak{h}^\infty$ is bijective; otherwise we say it has **non-polynomial distortion**.

Theorem 7.C.4. *Let $\mathfrak{g}$ be a K-triangulable Lie algebra. Assume that $H_2(\mathfrak{g}^\infty)^{\mathfrak{g}} \neq \{0\}$. Then there exists a hypercentral epimorphism $\mathfrak{h} \twoheadrightarrow \mathfrak{g}$ with non-polynomial distortion.*

The condition $H_2(\mathfrak{g}^\infty)^{\mathfrak{g}} \neq \{0\}$ can be interpreted as $H_2(\mathfrak{g}^\infty)_0 \neq \{0\}$, where $\mathfrak{g}$ is endowed with a Cartan grading (see §2.F, especially Lemma 2.F.6). Theorem 7.C.4 will be proved in §7.D.

Proof of Theorem 7.C.2 from Theorem 7.C.4. Since G is triangulable, $\mathfrak{exp}(\mathfrak{g}) = \mathfrak{g}^\infty$. Theorem 7.C.4 provides a hypercentral epimorphism $\mathfrak{h} \twoheadrightarrow \mathfrak{g}$, with kernel denoted by $\mathfrak{z}$, and with non-polynomial distortion, i.e. $\mathfrak{h}_\infty \cap \mathfrak{z} \neq \{0\}$. Since $\mathfrak{z}$ is hypercentral, the action of $\mathfrak{g}$ on $\mathfrak{z}$ is nilpotent, hence triangulable, and since moreover $\mathfrak{g}$ and $\mathfrak{z}$ are triangulable, we deduce that $\mathfrak{h}$ is triangulable. Let $H \to G$ be the corresponding surjective homomorphism of triangulable Lie groups and Z its kernel, which is hypercentral. Then $H^\infty \cap Z \neq \{1\}$, because the Lie algebra counterpart holds. So the theorem is proved. ☐

7.D. **Proof of Theorem 7.C.4.**

Lemma 7.D.1. *Let $\mathfrak{g} \twoheadrightarrow \mathfrak{h} \twoheadrightarrow \mathfrak{l}$ be epimorphisms of Lie algebras such that the composite homomorphism is a hypercentral epimorphism. If $\mathfrak{g} \twoheadrightarrow \mathfrak{l}$ has polynomial distortion then so does $\mathfrak{h} \twoheadrightarrow \mathfrak{l}$.*

Proof. This is trivial. □

Lemma 7.D.2. *Let $\mathfrak{g}$ be a Lie algebra and $(T^t)_{t\in K}$ be a one-parameter group of unipotent automorphisms of $\mathfrak{g}$. Let $\mathfrak{h} \twoheadrightarrow \mathfrak{g}$ be a central epimorphism. Then there exists a Lie algebra $\mathfrak{k}$ with an epimorphism $\rho : \mathfrak{k} \twoheadrightarrow \mathfrak{h}$ such that the composite homomorphism $\mathfrak{k} \twoheadrightarrow \mathfrak{g}$ is a central epimorphism, and such that the action of (T^t) lifts to a unipotent action on $\mathfrak{k}$.*

Remark 7.D.3. The conclusion of Lemma 7.D.2 cannot be simplified by the requirement that $\mathfrak{k} = \mathfrak{h}$, as we can see, for instance, by taking $\mathfrak{h}$ to be the direct product of the 3-dimensional Heisenberg algebra and a 1-dimensional algebra and a suitable 1-parameter subgroup of unipotent automorphisms of $\mathfrak{g} = \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$.

Proof of Lemma 7.D.2. Set $\mathfrak{z} = \mathrm{Ker}(\mathfrak{h} \twoheadrightarrow \mathfrak{g})$ and denote the Hopf bracket (see §5.A.1) by

$$[\cdot, \cdot]' : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}.$$

Pick a linear projection $\pi : \mathfrak{h} \to \mathfrak{z}$ and define $b(x \wedge y) = \pi([x, y]')$. This gives a linear identification of $\mathfrak{h}$ with $\mathfrak{g} \oplus \mathfrak{z}$, for which the law is given as

$$[\,\langle x_1, z_1\rangle\,,\,\langle x_2, z_2\rangle\,] = \langle\,[x_1, x_2]\,,\,b(x_1 \wedge x_2)\,\rangle$$

(in this proof, we write pairs $\langle x, z\rangle$ rather than (x, z) for the sake of readability). Observe that T^t naturally acts on $\mathfrak{g}\wedge\mathfrak{g}$, preserving $Z_2(\mathfrak{g})$ and $B_2(\mathfrak{g})$. If $c \in \mathfrak{g}\wedge\mathfrak{g}$, define the function

$$\begin{array}{rcl} \alpha_c : K & \to & \mathfrak{z} \\ u & \mapsto & b(T^u c). \end{array}$$

If d is the dimension of $\mathfrak{g}$, let W denote the space of K-polynomial mappings of degree $< 2d$ from K to $\mathfrak{z}$; the dimension of W is $2d\dim(\mathfrak{z})$. Now $t \mapsto T^t$ is a polynomial of degree $< d$ valued in the space of endomorphisms of $\mathfrak{g}$, so is also polynomial of degree $< 2d$ valued in the space of endomorphisms of $\mathfrak{g} \wedge \mathfrak{g}$. So α_c is a polynomial of degree $< 2d$, from K to $\mathfrak{z}$.

If $c \in \mathfrak{g} \wedge \mathfrak{g}$ is a boundary then $\alpha_c = 0$. So α defines a central extension $\mathfrak{k} = \mathfrak{g} \oplus W$ (as a vector space) of $\mathfrak{g}$ with kernel W, with law

$$[\langle x_1, \zeta_1\rangle, \langle x_2, \zeta_2\rangle] = \langle [x_1, x_2], \alpha_{x_1\wedge x_2}\rangle, \quad \langle x_1, \zeta_1\rangle, \langle x_2, \zeta_2\rangle \in \mathfrak{g} \oplus W.$$

From now on, since elements of W are functions, it will be convenient to write elements of $\mathfrak{k}$ as $\langle x, \zeta(u)\rangle$, where u is thought of as an indeterminate. For $t \in K$, the automorphism T^t lifts to an automorphism of $\mathfrak{k}$ given by

$$T^t(\langle x, \zeta(u)\rangle) = \langle T^t x, \zeta(u + t)\rangle.$$

This is obviously a one-parameter subgroup of linear automorphisms; let us check that these are Lie algebra automorphisms (in the computation, for readability we write the brackets as $[\cdot;\cdot]$, with semicolons instead of commas).

$$\begin{aligned}
[\,T^t(\langle x_1,\zeta_1(u)\rangle)\,;\,T^t(\langle x_2,\zeta_2(u)\rangle)\,] &= [\,\langle T^t x_1,\zeta_1(u+t)\rangle\,;\,\langle T^t x_2,\zeta_2(u+t)\rangle\,]\\
&= \langle\,[T^t x_1;T^t x_2]\,,\,\alpha_{T^t x_1\wedge T^t x_2}(u)\,\rangle\\
&= \langle\,[T^t x_1;T^t x_2]\,,\,b(T^u(T^t x_1\wedge T^t x_2))\,\rangle\\
&= \langle\,T^t[x_1;x_2]\,,\,b(T^{t+u}x_1\wedge T^{t+u}x_2)\,\rangle\\
&= \langle\,T^t[x_1;x_2]\,,\,\alpha_{x_1\wedge x_2}(t+u)\,\rangle\\
&= T^t(\langle\,[x_1;x_2]\,,\,\alpha_{x_1\wedge x_2}(u)\rangle\,)\\
&= T^t(\,[\,\langle x_1,\zeta_1(u)\rangle\,;\,\langle x_2,\zeta_2(u)\rangle\,]\,),
\end{aligned}$$

so these are Lie algebra automorphisms. Now the mapping

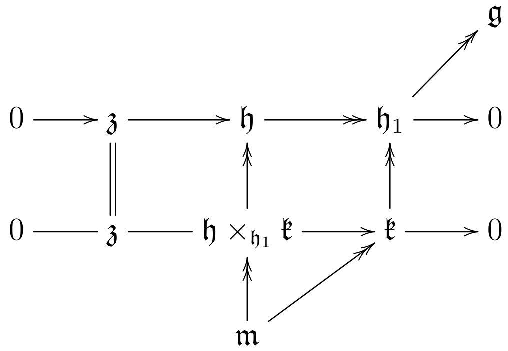
$$\begin{array}{rcl}
\mathfrak{k} & \to & \mathfrak{h}\\
\rho:\langle x,\zeta(u)\rangle & \mapsto & \langle x,\zeta(0)\rangle
\end{array}$$

is clearly a surjective linear map; it is also a Lie algebra homomorphism: indeed

$$\begin{aligned}
[\,\rho(\langle x_1,\zeta_1(u)\rangle)\,;\,\rho(\langle x_2,\zeta_2(u)\rangle)\,] &= [\,\langle x_1,\zeta_1(0)\rangle\,;\,\langle x_2,\zeta_2(0)\rangle\,]\\
&= \langle\,[x_1;x_2]\,,\,b(x_1\wedge x_2)\,\rangle\\
&= \langle\,[x_1;x_2]\,,\,\alpha_{x_1\wedge x_2}(0)\,\rangle\\
&= \rho(\langle\,[x_1;x_2]\,,\,\alpha_{x_1\wedge x_2}(u)\rangle\,)\\
&= \rho([\,\langle x_1,\zeta_1(u)\rangle\,;\,\langle x_2,\zeta_2(u)\rangle\,])
\end{aligned}$$ ☐

Lemma 7.D.4. *The statement of Lemma 7.D.2 holds true if we replace, in both the hypotheses and the conclusion, central by k-hypercentral.*

Proof. The case $k=1$ was done in Lemma 7.D.2. Decompose $\mathfrak{h}\twoheadrightarrow\mathfrak{g}$ as $\mathfrak{h}\twoheadrightarrow\mathfrak{h}_1\twoheadrightarrow\mathfrak{g}$, with $\mathfrak{h}\twoheadrightarrow\mathfrak{h}_1$ central (with kernel $\mathfrak{z}$) and $\mathfrak{h}_1\twoheadrightarrow\mathfrak{g}$ $(k-1)$-hypercentral. By induction hypothesis, there exists $\mathfrak{k}$ with $\mathfrak{k}\twoheadrightarrow\mathfrak{h}_1$ such that the composite epimorphism $\mathfrak{k}\twoheadrightarrow\mathfrak{g}$ is k-hypercentral and such that (T^t) lifts to $\mathfrak{k}$. Consider the fibered product $\mathfrak{h}\times_{\mathfrak{h}_1}\mathfrak{k}$ of the two epimorphisms $\mathfrak{h}\twoheadrightarrow\mathfrak{h}_1$ and $\mathfrak{k}\twoheadrightarrow\mathfrak{h}_1$, so that the two lines in the diagram below are central extensions and both squares commute.



Applying Lemma 7.D.2 again to $\mathfrak{h} \times_{\mathfrak{h}_1} \mathfrak{k} \twoheadrightarrow \mathfrak{k}$, we obtain $\mathfrak{m} \twoheadrightarrow \mathfrak{h} \times_{\mathfrak{h}_1} \mathfrak{l}$ so that the composite epimorphism $\mathfrak{m} \twoheadrightarrow \mathfrak{k}$ is central and so that (T^t) lifts to $\mathfrak{m}$. So the composite map $\mathfrak{m} \twoheadrightarrow \mathfrak{h}$ is the desired homomorphism. $\square$

We say that a Lie algebra $\mathfrak{g}$ is **spread** if it can be written as $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{s}$ where $\mathfrak{n}$ is nilpotent, $\mathfrak{s}$ is reductive and acts reductively on $\mathfrak{n}$. It is **spreadable** if there exists such a decomposition.

When $\mathfrak{g}$ is solvable, $\mathfrak{s}$ is abelian and a Cartan subalgebra of $\mathfrak{g}$ is given by the centralizer $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{s}) = C_{\mathfrak{n}}(\mathfrak{s}) \times \mathfrak{s}$. In particular, the $\mathfrak{s}$-characteristic decomposition of $\mathfrak{g}$ coincides with the $\mathfrak{h}$-characteristic decomposition, and the associated Cartan gradings are the same.

This remark is useful when we have to deal with a homomorphism $\mathfrak{n}_1 \rtimes \mathfrak{s} \to \mathfrak{n}_2 \rtimes \mathfrak{s}$ which is the identity on $\mathfrak{s}$: indeed such a homomorphism is graded for the Cartan gradings.

Proof of Theorem 7.C.4. We first prove the result when $\mathfrak{g}$ is spread, so $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{d}$. Let k be the dimension of $\mathfrak{u}/\mathfrak{g}^\infty$. We argue by induction on k.

Suppose that $k = 0$. We have a grading of $\mathfrak{g}^\infty$, valued in $\mathfrak{d}^\vee$. Consider the blow-up construction (Lemma 5.B.2): it gives a graded Lie algebra $\widetilde{\mathfrak{g}^\infty}$ and a graded surjective map $\widetilde{\mathfrak{g}^\infty} \twoheadrightarrow \mathfrak{g}^\infty$ with central kernel concentrated in degree zero and isomorphic to $H_2(\mathfrak{g}^\infty)_0$. This grading, valued in $\mathfrak{d}^\vee$, defines a natural action of $\mathfrak{d}$ on $\widetilde{\mathfrak{g}^\infty}$ and the epimorphism $\widetilde{\mathfrak{g}^\infty} \rtimes \mathfrak{d} \twoheadrightarrow \mathfrak{g}^\infty \rtimes \mathfrak{d}$ is central. By construction, the kernel $H_2(\mathfrak{g}^\infty)_0$ is contained in $(\widetilde{\mathfrak{g}^\infty} \rtimes \mathfrak{d})_\infty$ so this (hyper)central epimorphism has non-polynomial distortion.

Now suppose that $k \geq 1$. Let $\mathfrak{n}$ be a codimension 1 ideal of $\mathfrak{g}$ containing $\mathfrak{g}^\infty \rtimes \mathfrak{d}$. Since $\mathfrak{n}$ contains $\mathfrak{g}_\triangledown \oplus \mathfrak{d}$, the intersection of $\mathfrak{n}$ with $\mathfrak{u}_0$ is a hyperplane in $\mathfrak{u}_0$. So there exists a one-dimensional subspace $\mathfrak{l} \subset \mathfrak{u}_0$, such that $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$. Note that the grading of $\mathfrak{g}$, valued in $\mathfrak{d}^\vee$, extends that of $\mathfrak{n}$ and $\mathfrak{n}^\infty = \mathfrak{g}^\infty$ and in particular, $H_2(\mathfrak{n}^\infty)_0 \neq \{0\}$.

By induction hypothesis, there exists a hypercentral epimorphism $\mathfrak{h} \twoheadrightarrow \mathfrak{n}$, with non-polynomial distortion. By Lemma 7.D.4, there exists a hypercentral epimorphism $\mathfrak{m} \twoheadrightarrow \mathfrak{h}$ (with kernel $\mathfrak{z}$) so that the action of $e^{\mathfrak{ad}(\mathfrak{l})}$ on $\mathfrak{n}$ lifts to a unipotent action on $\mathfrak{m}$. This corresponds to a nilpotent action of $\mathfrak{l}$ on $\mathfrak{m}$. Let $\mathfrak{z}_i$ be the intersection of the ith term of the ascending central series of $\mathfrak{m}$ with $\mathfrak{z}$, so $\mathfrak{z}_\ell = \mathfrak{z}$ for some ℓ. On each $\mathfrak{z}_{i+1}/\mathfrak{z}_i$, the action of $\mathfrak{l}$ is nilpotent and the adjoint action of $\mathfrak{m}$ is trivial. So the action of $\mathfrak{m} \rtimes \mathfrak{l}$ on each $\mathfrak{z}_{i+1}/\mathfrak{z}_i$, hence on $\mathfrak{z}$, is nilpotent. That is, $\mathfrak{z}$ is hypercentral in $\mathfrak{m} \rtimes \mathfrak{l}$ (this is where the argument would fail with "hypercentral" replaced by "central"). So $\mathfrak{m} \rtimes \mathfrak{l} \twoheadrightarrow \mathfrak{g}$ is the desired hypercentral epimorphism: by Lemma 7.D.1, $\mathfrak{m} \twoheadrightarrow \mathfrak{n}$ has non-polynomial distortion and therefore so does $\mathfrak{m} \rtimes \mathfrak{l} \twoheadrightarrow \mathfrak{g}$.

Now the result is proved when $\mathfrak{g}$ is spread. In general, fix a faithful linear representation $\mathfrak{g} \to \mathfrak{gl}_n$, and let $\mathfrak{h} = \mathfrak{u} \rtimes \mathfrak{d}$ be the splittable hull of $\mathfrak{g}$ in $\mathfrak{gl}_n$ (that is, the subalgebra generated by semisimple and nilpotent parts of elements of $\mathfrak{g}$ for the additive Jordan decomposition, see [Bou, Chap. VII, §5]). If $\mathfrak{n}$ is any Cartan subalgebra of $\mathfrak{h}$, then $\mathfrak{n}' = \mathfrak{n} \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ [Bou, Chap. VII, §5, Ex. 8]. Now $\mathfrak{n} = (\mathfrak{u} \cap \mathfrak{n}) + \mathfrak{n}'$, but every $\mathfrak{n}$-weight of $\mathfrak{h}$ vanishes on $\mathfrak{u} \cap \mathfrak{n}$. So the $\mathfrak{n}$-grading of $\mathfrak{h}$ extends the $\mathfrak{n}'$-grading of $\mathfrak{g}$ (in other words, the embedding $\mathfrak{g} \subset \mathfrak{h}$ is a graded map). In particular, since $\mathfrak{g}^\infty = \mathfrak{h}^\infty$, this equality is an isomorphism of graded algebras and we deduce that $H_2(\mathfrak{h}^\infty)_0 \neq \{0\}$. So we obtain a hypercentral extension $\mathfrak{m} \twoheadrightarrow \mathfrak{h}$, with non-polynomial distortion because $\mathfrak{g}^\infty = \mathfrak{h}^\infty$. By taking the inverse image of $\mathfrak{g}$ in $\mathfrak{m}$, we obtain the desired hypercentral extension of $\mathfrak{g}$. □

7.E. **An example without central extensions.** We prove here that in the conclusion of Theorem 7.C.2, it is not always possible to replace, in the conclusion, hypercentral by central. We begin by the following useful general criterion.

Proposition 7.E.1. *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra with its Cartan grading. Then $\mathfrak{g}$ has a central extension with non-polynomial distortion if and only if the image of* $(\mathrm{Ker}(d_2) \cap (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown))_0$ *in* $H_2(\mathfrak{g})_0$ *is zero (i.e. it is contained in* $\mathrm{Im}(d_3)$*).*

Proof. Suppose that the image of the above map is nonzero. By the blow-up construction (Lemma 5.B.2), we obtain a central extension $\breve{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ with kernel $H_2(\mathfrak{g})_0$ concentrated in degree zero. By the assumption, there exist x_i, y_i in $\mathfrak{g}$, of nonzero opposite weights $\pm\alpha_i$, such that $\sum x_i \wedge y_i$ is a 2-cycle and is nonzero in $H_2(\mathfrak{g})_0$. This means that in $\breve{\mathfrak{g}}$, the element $z = \sum[x_i, y_i]$ is a nonzero element of the central kernel $H_2(\mathfrak{g})_0$. So $z \in \breve{\mathfrak{g}}^\infty$ and the central epimorphism $\breve{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ does not have polynomial distortion.

Conversely, suppose that there exists a central epimorphism $\breve{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ with non-polynomial distortion, with kernel $\mathfrak{z}$. Note that the Cartan grading lifts to $\breve{\mathfrak{g}}$, so that the kernel $\mathfrak{z}$ is concentrated in degree zero. By assumption, $\mathfrak{z} \cap \breve{\mathfrak{g}}^\infty$ contains a nonzero element z. By Lemmas 5.A.3 and 5.A.4, we can write, in $\breve{\mathfrak{g}}$, $z = \sum[x_i, y_i]$ with x_i, y_i of nonzero opposite weights. Thus in $\mathfrak{g}$, $\sum x_i \wedge y_i$ is a nonzero element of $H_2(\mathfrak{g})_0$. □

Let $\tilde{G}$ be the 15-dimensional $\mathbf{R}$-group of 6×6 upper triangular matrices of the form

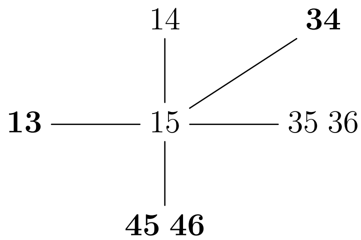
$$\begin{pmatrix} 1 & x_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ 0 & 1 & 0 & 0 & x_{25} & x_{26} \\ 0 & 0 & t_3 & u_{34} & u_{35} & u_{36} \\ 0 & 0 & 0 & t_4 & u_{45} & u_{46} \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.E.2}$$

where t_3, t_4 are nonzero. Its unipotent radical $\tilde{U}$ consists of elements of the form (7.E.2) with $t_3 = t_4 = 1$ and its exponential radical $\tilde{E}$ consists of elements in $\tilde{U}$ for which $x_{12} = x_{25} = x_{26} = x_{56} = 0$. If D denotes the (two-dimensional) diagonal subgroup in $\tilde{G}$, the quotient $\tilde{G}/\tilde{E}$ is isomorphic to the direct product of D with a 4-dimensional unipotent group (corresponding to coefficients x_{12}, x_{25}, x_{26}, x_{56}). Note that the extension $1 \to \tilde{E} \to \tilde{U} \to \tilde{U}/\tilde{E} \to 1$ is not split.

Let Z the 2-dimensional subgroup of $\tilde{U}$ consisting of matrices with all entries zero except u_{16} and x_{26}. Note that Z is hypercentral and has non-trivial intersection with the exponential radical of $\tilde{G}$.

Define $G = \tilde{G}/Z$; it is 13-dimensional. The weights of $E = \tilde{E}/Z$ are arranged as follows (the principal weights are in boldface)

(7.E.3)



and the other basis elements of weight zero in $\mathfrak{g}$ are 33, 44, 12, 25, 56.

We see that $\mathfrak{e}$ is 2-tame. Besides, $H_2(\mathfrak{e})_0 \neq 0$, as $13 \wedge 36$ is a 2-cycle in degree that is not a 2-boundary, as follows from the observation that $\mathfrak{e}$ has an obvious nontrivial central extension in degree zero, given by at the level of groups by

$$1 \to Z/Z' \to \tilde{E}/Z' \to E \to 1,$$

where Z' is the one-dimensional subgroup at position 26 (that is, the subgroup of Z consisting of matrices with $u_{16} = 0$), which is normalized by $\tilde{E}$ but not by $\tilde{U}$.

Since $H_2(\mathfrak{g}^\infty)_0 \neq \{0\}$, by Theorem 7.C.4 there exists a hypercentral epimorphism $\mathfrak{h} \to \mathfrak{g}$ with non-polynomial distortion. By contrast, every central epimorphism $\mathfrak{h} \to \mathfrak{g}$ has polynomial distortion. This follows from Proposition 7.E.1 and the following proposition.

Proposition 7.E.4. *Let $\mathfrak{g}$ be the above 13-dimensional triangulable Lie algebra. Then in $H_2(\mathfrak{g})_0$, the image of $(\mathrm{Ker}(d_2) \cap (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown))_0$ is zero.*

Proof. To streamline the notation, we denote by ij the elementary matrix usually denoted by E_{ij}, with 1 at position (i,j) and zero everywhere else (including the diagonal). By considering each pair of nonzero opposite weights in (7.E.3), we can describe the map d_2 on a basis of the 4-dimensional space $(\mathfrak{u}_\triangledown \wedge \mathfrak{u}_\triangledown)_0$.

$$13 \wedge 35 \overset{d_2}{\longmapsto} -15 \qquad\qquad 13 \wedge 36 \longmapsto 0,$$
$$14 \wedge 45 \longmapsto -15, \qquad\qquad 14 \wedge 46 \longmapsto 0;$$

accordingly a basis of $(\mathrm{Ker}(d_2) \cap (\mathfrak{g}_\triangledown \wedge \mathfrak{g}_\triangledown))_0$ is given by

$$13 \wedge 35 - 14 \wedge 45, \quad 13 \wedge 36, \quad 14 \wedge 46;$$

we have to check that these are all boundaries; let us snatch them one by one:

$$12 \wedge 25 \wedge 56 \overset{d_3}{\longmapsto} 56 \wedge 15.$$
$$13 \wedge 34 \wedge 45 \longmapsto 13 \wedge 45 - 14 \wedge 45$$
$$13 \wedge 35 \wedge 56 \longmapsto 56 \wedge 15 + 13 \wedge 36$$
$$14 \wedge 45 \wedge 56 \longmapsto 56 \wedge 15 + 14 \wedge 46.$$

□

Combining with Proposition 7.E.1, we get:

Corollary 7.E.5. *We have* $H_2(\mathfrak{g}^\infty)_0 \neq \{0\}$, *but there is no central extension of Lie groups*

$$1 \longrightarrow \mathbf{R} \overset{j}{\longrightarrow} \breve{G} \longrightarrow G \longrightarrow 1$$

with $j(\mathbf{R})$ *exponentially distorted in* $\breve{G}$. □

8. G not tame

Here we prove that any group satisfying the SOL obstruction has an at least exponential Dehn function. The method also provides the result that any group satisfying the non-Archimedean SOL obstruction is not compactly presented.

8.A. **Combinatorial Stokes formula.**

Definition 8.A.1. Let X be a set and let R be any commutative ring. We call a *closed path* a sequence $\mathbf{c} = (c_0, \dots, c_n)$ of points in X with $c_0 = c_n$ (so we can view it as indexed by $\mathbf{Z}/n\mathbf{Z}$). If α, β are functions $X \to R$, we define

$$\int_{\mathbf{c}} \beta d\alpha = \sum_{i \in \mathbf{Z}/n\mathbf{Z}} \beta(c_i)(\alpha(c_{i+1}) - \alpha(c_{i-1}))$$

Clearly, this is invariant if we shift indices. The following properties are immediate consequences of the definition.

- (Antisymmetry) We have

$$\int_{\mathbf{c}} \beta d\alpha = -\int_{\mathbf{c}} \alpha d\beta.$$

- (Concatenation) If $c_0 = c_i = c_n$ and we write $\mathbf{c}' = (c_0, \dots, c_i)$ and $\mathbf{c}'' = (c_i, \dots, c_n)$,
$$\int_{\mathbf{c}} \beta d\alpha = \int_{\mathbf{c}'} \beta d\alpha + \int_{\mathbf{c}''} \beta d\alpha.$$
- (Filiform vanishing) If $n = 2$ then the integral vanishes. More generally, the integral vanishes when $\mathbf{c}$ is filiform, i.e., n is even and $\mathbf{c}_i = \mathbf{c}_{n-i}$ for all i.

Indeed, the difference $\int_{\mathbf{c}} \beta d\alpha - \int_{\mathbf{c}'} \beta d\alpha - \int_{\mathbf{c}''} \beta d\alpha$ is equal to
$$\beta(c_0)[(\alpha(c_{i+1}) - \alpha(c_{i-1})) + (\alpha(c_1) - \alpha(c_{n-1}))$$
$$-(\alpha(c_1) - \alpha(c_{i-1})) - (\alpha(c_{i+1}) - \alpha(c_{n-1}))] = 0.$$
The filiform vanishing is immediate for $n = 2$ and follows in general by an induction based on the concatenation formula.

Now let us deal with a Cayley graph of a group G with a generating set S, and we consider paths in the graph, that is sequences of vertices linked by edges. Thus any closed path based at 1 can be encoded by a unique element of the free group F_S, which is a relation (i.e. an element of the kernel of $F_S \to G$), and conversely, if r is a relation, we denote by $[r]$ the corresponding closed path based at 1. Note that G acts by left translations on the set of closed paths. The above properties imply the following

- (Product of relations) If r, r' are relations, we have
$$\int_{[rr']} \beta d\alpha = \int_{[r]} \beta d\alpha + \int_{[r']} \beta d\alpha.$$
- (Conjugate of relations) If r is a relation and $\gamma \in F_S$,
$$\int_{[\gamma r \gamma^{-1}]} \beta d\alpha = \int_{\gamma \cdot [r]} \beta d\alpha.$$
- (Combinatorial Stokes formula) Suppose that a relation r is written as a product $r = \prod_{i=1} \gamma_i r_i \gamma_i^{-1}$ of conjugates of relations. Then
$$\int_{[r]} \beta d\alpha = \sum_{i=1}^{k} \int_{\gamma_i \cdot [r_i]} \beta d\alpha.$$

The formula for products follows from concatenation if there is no simplification in the product rr', and follows by also using the filiform vanishing otherwise. The formula for conjugates also follows using the filiform vanishing. The Stokes formula follows from the two previous by an immediate induction.

Remark 8.A.2. The above combinatorial Stokes formula is indeed analogous to the classical Stokes formula on a disc: here the left-hand term is thought of as an integral along the boundary, while the right-hand term is a discretized integral over the surface.

8.B. **Loops in groups of SOL type.** Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be two nondiscrete locally compact fields. Consider the group

$$G = (\mathbf{K}_1 \times \mathbf{K}_2) \rtimes_{(\ell_1, \ell_2^{-1})} \mathbf{Z},$$

where $|\ell_2|_{\mathbf{K}_2} \geq |\ell_1|_{\mathbf{K}_1} > 1$, with group law written so that the product depends affinely on the right term:

$$(x, y, n)(x', y', n') = (x + \ell_1^n x', y + \ell_2^{-n} y', n + n').$$

Write $|\ell_1|_{\mathbf{K}_1} = |\ell_2|_{\mathbf{K}_2}^{\mu}$ with $0 < \mu \leq 1$

Now we can also view x and y as the projections to the coordinates in the above description. In the next lemmas, we consider a normed ring $\mathbf{K}$ (whose norm is submultiplicative, not necessarily multiplicative), and functions $A : \mathbf{K}_1 \to \mathbf{K}$, and $B : \mathbf{K}_2 \to \mathbf{K}$, yielding functions $\alpha, \beta : G \to \mathbf{R}$ defined by $\alpha = A \circ x$ and $\beta = B \circ y$.

Lemma 8.B.1. *Suppose that A is 1-Lipschitz and that B satisfies the Hölder condition*

$$|B(s) - B(s')| \leq |s - s'|^{\mu}, \quad \forall s, s' \in \mathbf{K}_2.$$

Then $\int \beta d\alpha$ is bounded on triangles of bounded diameter.

Proof. Let us consider a triangle T of bounded diameter (viewed as a closed path of length three), i.e. three points (g_0, g_0h, g_0h'), where h and h' are bounded (but not g_0!). Note that as a consequence of the antisymmetry relation, $\int_T \beta d\alpha$ does not change if we add constants to both α and β. We can therefore assume that $\alpha(g_0) = \beta(g_0) = 0$. Hence

$$\int_T \beta d\alpha = \beta(g_0h)\alpha(g_0h') - \beta(g_0h')\alpha(g_0h).$$

In coordinates, suppose that $g_0 = (x_0, y_0, n_0)$, $h = (x, y, n)$ and $h' = (x', y', n')$. Then $g_0h = (x_0 + \ell_1^{n_0} x, y_0 + \ell_2^{-n_0} y, n_0 + n)$, and $g_0h' = (x_0 + \ell_1^{n_0} x', y_0 + \ell_2^{-n_0} y', n_0 + n')$. Since $A(x_0) = B(x_0) = 0$, we have

$$\int_T \beta d\alpha = B(y_0 + \ell_2^{-n_0} y) A(x_0 + \ell_1^{n_0} x') - B(y_0 + \ell_2^{-n_0} y') A(x_0 + \ell_1^{n_0} x).$$

By our assumptions on A and B, we have

$$\begin{aligned} \left| \int_T \beta d\alpha \right| &\leq |\ell_2^{-n_0} y|^{\mu} |\ell_1^{n_0} x'| + |\ell_2^{-n_0} y'|^{\mu} |\ell_1^{n_0} x| \\ &= |y|^{\mu} |x'| + |y'|^{\mu} |x|, \end{aligned}$$

which is duly bounded when h, h' are bounded. □

Fix $n \geq 1$. We consider the relation

$$\gamma_{1,n} = t^n x t^{-n} y t^n x^{-1} t^{-n} y^{-1};$$

this is a closed path of length $4n + 4$, where x, y and t denote (by abuse of notation) the elements $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of $G = \mathbf{K}_1 \times \mathbf{K}_2 \rtimes \mathbf{Z}$.

Lemma 8.B.2. *Consider A, B, α, β as introduced before Lemma 8.B.1. Suppose here that $B(0) = 1$ and B vanishes outside the open 1-ball of $\mathbf{K}_2$. Then, denoting by $|\cdot|_{\mathbf{K}}$ the norm in $\mathbf{K}$, we have*

$$\left|\int_{\gamma_{1,n}} \beta d\alpha\right|_{\mathbf{K}} = |2\ell_1^n|_{\mathbf{K}}.$$

Proof of Lemma 8.B.2. To simplify the notation, let us denote by c the closed path of length $4n+4$ defined by $\gamma_{1,n}$. In the integral $\int_{\mathbf{c}} \beta d\alpha$, only those points c_i for which "$\beta d\alpha$" is nonzero, i.e. both $\alpha(c_{i+1}) \neq \alpha(c_{i-1})$ and $\beta(c_i) \neq 0$, do contribute. In this example as well as the forthcoming ones, this will make most terms be equal to zero. The closed path c can be decomposed as

$$\begin{aligned} c_0 &= (0,0,0), (0,0,1), \dots, (0,0,n-1), (0,0,n) = c_n, \\ c_{n+1} &= (\ell_1^n, 0, n), (\ell_1^n, 0, n-1), \dots, (\ell_1^n, 0, 1), (\ell_1^n, 0, 0) = c_{2n+1}, \\ c_{2n+2} &= (\ell_1^n, 1, 0), (\ell_1^n, 1, 1), \dots, (\ell_1^n, 1, n-1), (\ell_1^n, 1, n) = c_{3n+2}, \\ c_{3n+3} &= (0,1,n), (0,1,n-1), \dots, (0,1,1), (0,1,0) = c_{4n+3}. \end{aligned}$$

We see that $x(c_{i+1}) \neq x(c_{i-1})$ only for $i = n, n+1, 3n+2, 3n+3$. Moreover, for $i = 3n+2, 3n+3$, $y(c_i) = 1$, so $B(y(c_i)) = 0$. Thus

$$\int \beta d\alpha = \sum_{i=n}^{n+1} B(y(c_i))(A(x(c_{i+1})) - A(x(c_{i-1}))) = 2B(0)(A(\ell_1^n) - A(0));$$

thus

$$\left|\int \beta d\alpha\right|_{\mathbf{K}} = |2A(\ell_1^n)|_{\mathbf{K}}.$$

□

Lemma 8.B.3. *Endow $\mathbf{K} = \mathbf{K}_1 \times \mathbf{K}_2$ with the max norm. The following functions satisfy the conditions of Lemmas 8.B.1 and 8.B.2.*

- $A_1 : \mathbf{K}_1 \to \mathbf{R}$, $x \mapsto |x|$ *and* $A_2 : \mathbf{K}_1 \to \mathbf{K}$, $x \mapsto (x, 0)$;
- $B_1 : \mathbf{K}_1 \to \mathbf{R}$, $x \mapsto \max(0, 1-|x|)$, *and, if* $\mathbf{K}_2$ *is ultrametric, the function* $\mathbf{K}_2 \to \mathbf{K}$ *mapping* x *to* $(0, 1-x)$ *if* $|x| < 1$ *and to 0 if* $|x| \geq 1$.

Proof. It is trivial that A_1 and A_2 are 1-Lipschitz. Also, note that since $\mu \leq 1$, any function B satisfying the condition $|B(s) - B(s')| \leq \max(1, |s-s'|)$ obviously satisfies the Hölder condition $|B(s) - B(s')| \leq |s-s'|^\mu$. It is clear that B_1 satisfies the former condition; if $\mathbf{K}_2$ is ultrametric then whenever $|x| < 1$ and $|x'| \geq 1$ we have $|x - x'| \geq 1$, so the former condition is also satisfied. □

To illustrate the interest of the notions developed here, note that this is enough to obtain the following result.

Proposition 8.B.4. *Under the assumptions above, if $2 \neq 0$ in $\mathbf{K}_1$, the group*

$$G = (\mathbf{K}_1 \times \mathbf{K}_2) \rtimes_{(\ell_1, \ell_2^{-1})} \mathbf{Z}$$

has at least exponential Dehn function, and if both $\mathbf{K}_1$ and $\mathbf{K}_2$ are ultrametric then G is not compactly presented.

We use the following convenient language: in a locally compact group G with a compact system of generators S, we say that a sequence of null-homotopic words (w_n) in F_S has *asymptotically infinite area* if for every R, there exists $N(R)$ such that no w_n for $n \geq N(R)$ is contained in the normal subgroup of F_S generated by null-homotopic words of length $\leq R$. By definition, the non-existence of such a sequence is equivalent to G being compactly presented.

Proof of Proposition 8.B.4. Set $I = \int_{\gamma_{1,n}} \beta d\alpha$.

We first use the functions $A = A_1$ and $B = B_1$ of Lemma 8.B.3. By Lemma 8.B.2, $|I| \geq |2A_1(\ell_1^n)|_{\mathbf{R}} = 2|\ell_1|_{\mathbf{K}_1}^n$. Suppose that $\gamma_{1,n}$ can be decomposed into j_n triangles of diameter $\leq R$. By the combinatorial Stokes formula (see §8.A) and Lemma 8.B.1, $|I| \leq Cj_n$, where the constant $C = C(R) > 0$ is provided by Lemma 8.B.1. It follows that $j_n \geq 2|\ell_1|_{\mathbf{K}_1}/C$. Hence the area of $\gamma_{1,n}$ grows at least exponentially, so the Dehn function of G grows at least exponentially.

Now assume that both $\mathbf{K}_1$ and $\mathbf{K}_2$ are ultrametric, so that $\mathbf{K}$ is ultrametric as well; we use the functions $A = A_2$ and $B = B_2$ of Lemma 8.B.3. By Lemma 8.B.2, $|I| \geq |2\ell_1^n|_{\mathbf{K}} = |2|_{\mathbf{K}_1}|\ell_1|_{\mathbf{K}_1}^n$. By Lemma 8.B.1, for each R there is a bound $C(R)$ on norm of the integral of $\beta d\alpha$ over any triangle of diameter $\leq R$. By ultrametricity and the combinatorial Stokes theorem, $C(R)$ is a bound for the norm of the integral of $\beta d\alpha$ over an arbitrary loop that can be decompose into triangles of diameter $\leq R$. Since $(|2|_{\mathbf{K}_1}|\ell_1|_{\mathbf{K}_1}^n)$ is unbounded, this shows that for every R there exists n such that $\gamma_{1,n}$ cannot be decomposed into triangles of diameter $\leq R$. Thus the sequence $(\gamma_{1,n})$ has asymptotically infinite area. This shows that G is not compactly presented. □

Remark 8.B.5. The assumption that $\mathbf{K}_1$ does not have characteristic two can be removed, but in that case we need to redefine $\int \beta d\alpha$ as $\sum_i \beta(c_i)(\alpha(c_{i+1}) - \alpha(c_i))$. The drawback of this definition is that the integral is not invariant under conjugation. However, with the help of Lemma 2.D.2, it is possible to conclude. Since we are not concerned with characteristic two here, we leave the details to the reader.

However Proposition 8.B.4 is not enough for our purposes, because we do not only wish to bound below the Dehn function of the group G, but also of various groups H mapping onto G. In general, the loop $\gamma_{1,n}$ does not lift to a loop in those groups, so we consider more complicated loops $\gamma_{k,n}$ in G, which eventually lift to the groups we have in mind. However, to estimate the area, we will go on working in G, because we know how to compute therein, and because obviously the area of a loop in H is bounded below by the area of its image in G.

Define by induction

$$\gamma_{k,n} = \gamma_{k-1,n} g_k \gamma_{k-1,n}^{-1} g_k^{-1}.$$

Here, g_k denotes the element $(0, y_k, 0)$ in the group G, where the sequence (y_i) in $\mathbf{K}_2$ satisfies the following property: $y_1 = 1$ and for any non-empty finite subset I

of integers,

$$\left|\sum_{i\in I} y_i\right| \geq 1.$$

For instance, if $\mathbf{K}_2$ is ultrametric, this is satisfied by $y_i = \ell_2^i$; if $\mathbf{K}_2 = \mathbf{R}$, the constant sequence $y_i = 1$ works. The sequence (y_i) will be fixed once and for all.

Fix n. We wish to compute, more generally, $\int_{\gamma_{k,n}} \beta d\alpha$. Write the path $\gamma_{k,n}$ as (c_i). Note that for given n, c_i does not depend on k (because $\gamma_{k,n}$ is an initial segment of $\gamma_{k+1,n}$). Write the length of $\gamma_{k,n}$ as $\lambda_{k,n}$ ($\lambda_{1,n} = 4n+4$, $\lambda_{k+1,n} = 2\lambda_{k,n}+2$).

Lemma 8.B.6. *The number n being fixed, we have*

(1) *There exists a sequence finite subsets F_i of the set of positive integers, such that for all i, we have $y(c_i) = \sum_{j\in F_i} y_j$, and satisfying in addition: for all $i < \lambda_{k,n}$ and all $k \geq 1$, we have $F_i \subset \{1,\dots,k\}$. Moreover $y(c_i) \neq 0$ (and thus $F_i \neq \emptyset$), unless either*
   - *$i \leq 2n+2$, or*
   - *$i = \lambda_{j,n}$ for some j.*

(2) *Assume that $1 \leq i \leq n-1$, or $n+2 \leq i \leq 2n+2$, or $i = \lambda_{j,n}$ for some j. Then $x(c_{i-1}) = x(c_{i+1})$.*

Proof.

(1) The sequence (F_i) is constructed for $i < \lambda_{k,n}$, by induction on k. For $k=1$, we set $F_i = \emptyset$ if $i \leq 2n-1$ and $F_i = \{1\}$ if $2n+2 \leq i \leq 4n+3 = \lambda_{1,n}-1$, and it satisfies the equality for $y(c_i)$ (see the proof of Lemma 8.B.2, where c_i is made explicit for all $i \leq \lambda_{1,n} = 4n+4$).

   Now assume that $k \geq 2$ and that F_i is constructed for $i < \lambda = \lambda_{k-1,n}$ with the required properties. We set $F_\lambda = \emptyset$; since $c_\lambda = (0,0,0)$, the condition holds for $i = \lambda$. It remains to deal with i when $\lambda < i < \lambda_{k,n}$; in this case $c_i = g_k c_{2\lambda-i}$, so $y(c_i) = y_k + y(c_i)$. Thus if we set $F_i = \{k\} \cup F_{2\lambda-i}$, remembering by induction that $F_{2\lambda-i} \subset \{1,\dots,k-1\}$, we deduce that $y(c_i) = \sum_{j\in F_i} y_j$; clearly $F_i \subset \{1,\dots,k\}$.

(2) This was already checked for $i \leq 2n+2$ (see the proof of Lemma 8.B.2). In the case $i = \lambda_{j,n}$, we have $c_{i-1} = g_j$ and $c_{i+1} = g_{j+1}$, so $x(c_{i-1}) = x(c_{i+1})$. □

Lemma 8.B.7. *Under the assumptions of Lemma 8.B.2, we have*

$$\int_{\gamma_{k,n}} \beta d\alpha = |2\ell_1|^n_{\mathbf{K}_1}.$$

Proof. By Lemma 8.B.6, if both $|y(c_i)| < 1$ and $x(c_{i-1}) \neq x(c_{i+1})$, then $i = n$ or $i = n+1$. It follows that the desired integral on $\gamma_{k,n}$ is the same as the integral on $\gamma_{1,n}$ computed in Lemma 8.B.2. □

8.C. **Groups with the SOL obstruction.**

Theorem 8.C.1. *Let G_1 be a locally compact, compactly generated group, and suppose there is a continuous surjective homomorphism*

$$G_1 \to G = (\mathbf{K}_1 \times \mathbf{K}_2) \rtimes_{(\ell_1, \ell_2^{-1})} \mathbf{Z},$$

where $0 \neq 2$ in $\mathbf{K}_1$. Suppose that G_1 has a nilpotent normal subgroup H whose image in G contains $\mathbf{K}_1 \times \mathbf{K}_2$. Then

- *the Dehn function of G is at least exponential.*
- *if both $\mathbf{K}_1$ and $\mathbf{K}_2$ are ultrametric, then G is not compactly presented.*

Proof. Let k_0 be the nilpotency length of H and fix $k \geq k_0$. Using Lemma 8.B.7 and arguing as in the proof of Proposition 8.B.4 (using Lemma 8.B.7 instead if Lemma 8.B.2), we obtain that the loops $\gamma_{k,n}$, which have linear length with respect to n, have at least exponential area, and asymptotically infinite area in case $\mathbf{K}_1$ and $\mathbf{K}_2$ are both ultrametric.

Lift x, y, and g_k to elements $\tilde{x}, \tilde{y}, \tilde{g}_k$ in H and t to an element $\tilde{t}$ in G; set $\tilde{X}_n = \tilde{t}^n \tilde{x} \tilde{t}^{-n}$; since H is normal, $\tilde{X}_n \in H$. This lifts $\gamma_{k,n}$ to a path $\widetilde{\gamma_{k,n}}$ based at 1; let $v_{k,n}$ be its value at $\lambda_{k,n}$, so $v_{1,n} = \tilde{X}_n \tilde{y} \tilde{X}_n^{-1} \tilde{y}^{-1}$ and $v_{k+1,n} = v_{k,n} \tilde{g}_k v_{k,n}^{-1} \tilde{g}_k{}^{-1}$. We see by an immediate induction that $v_{k,n}$ belongs to the $(k+1)$th term in the descending central series of H. Since $k \geq k_0$, we see that $\widetilde{\gamma_{k,n}}$ is a loop of G, of linear length with respect to n, mapping to $\gamma_{k,n}$. In particular, its area is at least the area of $\gamma_{k,n}$. So we deduce that $\widetilde{\gamma_{k,n}}$ has at least exponential area with respect to n, and has asymptotically infinite area in case $\mathbf{K}_1$ and $\mathbf{K}_2$ are both ultrametric. □

We will also need the following variant, in the real case.

Theorem 8.C.2. *Let G_1 be a locally compact, compactly generated group, and suppose there is a continuous homomorphism with dense image*

$$G_1 \to G = (\mathbf{R} \times \mathbf{R}) \rtimes \mathbf{R},$$

so that the element $t \in \mathbf{R}$ acts by the diagonal matrix (ℓ_1^t, ℓ_2^{-t}) ($\ell_2 \geq \ell_1 > 0$). Suppose that G_1 has a nilpotent normal subgroup H whose image in G contains $\mathbf{R} \times \mathbf{R}$. Then the Dehn function of G is at least exponential.

Proof. Since the homomorphism has dense image containing $\mathbf{R} \times \mathbf{R}$, there exists some element $\tilde{t}$ mapping to an element t of the form $(0, 0, \tau)$ with $\tau > 0$. Changing the parameterization of G if necessary (replacing ℓ_i by ℓ_i^τ for $i = 1, 2$), we can suppose that $\tau = 1$. Then lift x and y and pursue the proof exactly as in the proof of Theorem 8.C.1. □

Remark 8.C.3. To summarize the proof, the lower exponential bound is obtained by finding two functions α, β on G_1 such that the integral $\int \beta d\alpha$ is bounded on triangles of bounded diameter, and a sequence (γ_n) of combinatorial loops of linear diameter such that $\int_{\gamma_n} \beta d\alpha$ grows exponentially.

This approach actually also provides a lower bound on the homological Dehn function [Ger92, BaMS93, Ger99] as well. Let us recall the definition. Let G be a locally compact group with a generating set S and a subset R of the kernel of $F_S \to G$ consisting of relations of bounded length, yielding a polygonal complex structure with oriented edges and 2-faces. Let A be a commutative ring. For $i = 0, 1, 2$, let $C_i(G, A)$ be the real vector space freely spanned by the set of vertices, resp. oriented edges, resp. oriented 2-faces. Endow each $C_i(G, A)$ with the ℓ^1 norm. There are usual boundary operators

$$C_2(G, A) \stackrel{\partial_2}{\to} C_1(G, A) \stackrel{\partial_1}{\to} C_0(G, A).$$

satisfying $\partial_1 \circ \partial_2 = 0$. If $Z_1(G, A)$ is the kernel of ∂_1, then it is easy to extend, by linearity, the definition of $\int_{\mathbf{c}} \beta d\alpha$ (from §8.A) to $\mathbf{c} \in Z_1(G, A)$.

Following [Ger99], define, for $\mathbf{c} \in Z_1(G, A)$

$$\mathrm{HFill}^A_{G,S,R}(\mathbf{c}) = \inf\{\|P\|_1 : \ P \in C_2(G, A), \partial_2(P) = \mathbf{c}\}.$$

and

$$\mathrm{H}\delta^A_{G,S,R}(n) = \sup\{\mathrm{HFill}(z)^A_{G,S,R} : \ z \in Z_1(G, A), \ \|z\|_1 \leq n\}.$$

Clearly, if $\mathbf{c}$ is a basis element (so that its area makes sense)

$$\mathrm{HFill}(\mathbf{c})^{\mathbf{R}}_{G,S,R} \leq \mathrm{HFill}(\mathbf{c})^{\mathbf{Z}}_{G,S,R} \leq \mathrm{area}^{G,S,R}(\mathbf{c});$$

it follows that

$$\mathrm{H}\delta^{\mathbf{R}}_{G,S,R}(n) \leq \mathrm{H}\delta^{\mathbf{Z}}_{G,S,R}(n) \leq \delta_{G,S,R}(n).$$

The function $\mathrm{HFill}(\mathbf{c})^A_{G,S,R}$ is called the A-homological Dehn function of (G, R, S) (the function $\mathrm{HFill}(\mathbf{c})^{\mathbf{Z}}_{G,S,R}$ is called abelianized isoperimetric function in [BaMS93]). If finite, it can be shown by routine arguments that its $\approx$-asymptotic behavior only depends on G. Some Bestvina-Brady groups [BeBr97] provide examples of finitely generated groups with finite integral Dehn function but infinite Dehn function. Until recently, no example of a compactly presented group was known for which the integral (or even real) homological Dehn function is not equivalent to the integral homological Dehn function; the issue was raised, for finitely presented groups, both in [BaMS93, p. 536] and [Ger99, p. 1]; the first examples have finally been obtained by Abrams, Brady, Dani and Young in [ABDY13].

Let us turn back to G_1 (a group satisfying the hypotheses of Theorem 8.C.1 or 8.C.2): for this example, since R consists of relations of bounded length, it follows that the integral of $\beta d\alpha$ over the boundary of any polygon is bounded. Since $\int_{\gamma_n} \beta d\alpha$ grows exponentially, it readily follows that $\mathrm{HFill}^{\mathbf{R}}_{G_1}(\gamma_n)$ grows at least exponentially and hence $\mathrm{H}\delta^{\mathbf{R}}_{G_1}(n)$ (and thus $\mathrm{H}\delta^{\mathbf{Z}}_{G_1}(n)$) grows at least exponentially.

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