A NOTION OF GEOMETRIC COMPLEXITY AND ITS APPLICATION TO TOPOLOGICAL RIGIDITY

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Abstract. We introduce a geometric invariant, called finite decomposition complexity (FDC), to study topological rigidity of manifolds. We prove for instance that if the fundamental group of a compact aspherical manifold M has FDC, and if N is homotopy equivalent to M, then $M \times \mathbb{R}^n$ is homeomorphic to $N \times \mathbb{R}^n$, for n large enough. This statement is known as the stable Borel conjecture. On the other hand, we show that the class of FDC groups includes all countable subgroups of $\mathrm{GL}(n, K)$, for any field K, all elementary amenable groups, and is closed under taking subgroups, extensions, free amalgamated products, HNN extensions, and direct unions.

1. Introduction

We introduce the geometric concept of finite decomposition complexity to study questions concerning the topological rigidity of manifolds. Roughly speaking, a metric space has finite decomposition complexity when there is an algorithm to decompose the space into simpler, more manageable pieces in an asymptotic way. The precise definition is inspired by the property of finite asymptotic dimension of Gromov [G1] and is presented in Section 2.

While the property of finite decomposition complexity is flexible – the class of countable groups having finite decomposition complexity includes all linear groups (over a field with arbitrary characteristic), all hyperbolic groups and all elementary amenable groups and is closed under various operations – it is a powerful tool for studying topological rigidity – we shall see, for example, that if the fundamental group of a closed aspherical manifold has finite decomposition complexity then its universal cover is boundedly rigid, and the manifold itself is stably rigid.

The remainder of the introduction is divided into three parts. In the first we describe the applications of finite decomposition complexity to topological rigidity; in the second we outline results on the class of countable discrete groups having finite decomposition complexity; we conclude with general remarks on the organization of the paper.

Topological rigidity. A closed manifold M is *rigid* if every homotopy equivalence between M and another closed manifold is homotopic to a homeomorphism. The Borel conjecture asserts the rigidity of closed aspherical manifolds. Many important results on the Borel conjecture have been obtained by Farrell and Jones [FJ1, FJ2, FJ3, FJ4], and more recently

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Bartels and Lück [BL]. These results are proved by studying dynamical properties of actions of the fundamental group of M.

Our approach to rigidity questions is different – we shall focus not on the dynamical properties but rather on the large scale geometry of the fundamental group. As a natural byproduct, we prove the *bounded Borel conjecture*, a 'large-scale geometric' version of the Borel conjecture. Our principal result in this direction is the following theorem.

Theorem. *The bounded Borel isomorphism conjecture and the bounded Farrell-Jones L-theory isomorphism conjecture hold for a metric space with bounded geometry and finite decomposition complexity.*

We refer the reader to Section 7 for a detailed discussion and precise statements, in particular, Theorems 7.4.1 and 7.3.1. In this introduction we shall instead describe two concrete applications to questions of rigidity. Our first application concerns *bounded rigidity* of universal covers of closed aspherical manifolds.

Bounded Rigidity Theorem. *Let M be a closed aspherical manifold of dimension at least five whose fundamental group has finite decomposition complexity* (*as a metric space with a word metric*). *For every closed manifold N and homotopy equivalence $M \to N$ the corresponding bounded homotopy equivalence of universal covers is boundedly homotopic to a bounded homeomorphism.*

The universal covers of M an N as in the statement are, in particular, homeomorphic. The conclusion is actually much stronger – being *boundedly* homeomorphic means that the homeomorphism is at the same time a coarse equivalence. We defer discussion of the relevant notions concerning the *bounded category* to Section 7. See, in particular Theorem 7.3.2, of which the previous result is a special case.

Davis has given examples of aspherical manifolds whose universal covers are not homeomorphic to the Euclidean space [D]. These examples satisfy the hypothesis of the previous theorem.

A closed manifold M is *stably rigid* if there exists an n such that for every closed manifold N and every homotopy equivalence $M \to N$, the product with the identity $M \times \mathbb{R}^n \to N \times \mathbb{R}^n$ is homotopic to a homeomorphism. The *stable Borel conjecture* asserts that closed aspherical manifolds are stably rigid. The first result on the stable Borel conjecture is due to Farrell and Hsiang [FH] who proved that non positively curved Riemannian manifolds are stably rigid. Our second application is the following theorem (see Corollary 7.3.3).

Stable Rigidity Theorem. *A closed aspherical manifold whose fundamental group has finite decomposition complexity is stably rigid.*

Observe that there is no restriction on the dimension.

Groups with finite decomposition complexity. We consider countable groups equipped with a proper left-invariant metric. Recall that every countable group admits such a metric, and that any two such metrics are coarsely equivalent. As finite decomposition complexity is a coarse invariant, the statement that a countable group has finite decomposition complexity is independent of the choice of metric.

Our next results summarize the main examples of groups having finite decomposition complexity, and the stability properties of this class of groups. For the first statement, recall that a Lie group is *almost connected* if it has finitely many connected components.

Theorem. *The collection of countable groups having finite decomposition complexity contains all countable linear groups* (*over a field of arbitrary characteristic*)*, all countable subgroups of an almost connected Lie group, all hyperbolic groups and all elementary amenable groups.*

The geometry of a *discrete* subgroup of, for example, a connected semisimple Lie group such as $\mathrm{SL}(n,\mathbb{R})$ reflects the geometry of the ambient Lie group. In this case, the theorem follows from the well-known result that such groups have finite asymptotic dimension. The difficulty in the theorem concerns the case of *non-discrete or even dense* subgroups whose geometry exhibits little apparent relationship to the geometry of the ambient group. An interesting example to which our theorem applies is $\mathrm{SL}(n,\mathbb{Z}[\pi])$, which has infinite asymptotic dimension. (Here, $\pi = 3.14\ldots$.)

Theorem. *The collection of countable groups having finite decomposition complexity is closed under the formation of subgroups, products, extensions, free amalgamated products,* HNN *extensions and direct unions.*

At the moment, we know of no group *not* having finite decomposition complexity other than Gromov's random groups [G2, G3, AD]. Since finite decomposition complexity appears as a generalization of finite asymptotic dimension, we mention that in general, solvable groups, or linear groups have infinite asymptotic dimension. On the other hand, we shall prove the following result (see Theorem 5.0.1).

Theorem. *A finitely generated linear group over a field of positive characteristic has finite asymptotic dimension.*

Combined, our results greatly extend an earlier result of Ji [J] proving the stable Borel conjecture for a special class of linear groups with finite asymptotic dimension – namely, subgroups of $\mathrm{GL}(n,K)$ for a global field K, for example when $K=\mathbb{Q}$.

Organization and remarks. The paper falls into essentially two parts. In the first part, comprising Sections 2 – 6, we introduce and study finite decomposition complexity. More precisely, we introduce finite decomposition complexity in Section 2 and study its basic properties. In the subsequent section we develop the permanence characteristics of finite decomposition complexity. In Section 4 we show that a metric space having finite asymptotic dimension has finite decomposition complexity, and that one having finite decomposition complexity has Property A. As a consequence, any sequence of expanding graphs (viewed as a metric space) does not have finite decomposition complexity.

Sections 5 – 6 are devoted to examples. In these we prove that all (countable) linear groups have finite decomposition complexity and that and all (countable) elementary amenable groups have finite decomposition complexity.

The remainder of the paper, comprising Section 7 – 10 and the appendices, is devoted to applications to topological rigidity. We have split Section 7 into two parts. The first outlines two essential results, Theorems 7.1.2 and 7.1.3 – these are proven in Section 8 and 9, respectively. The remainder of Section 7 contains a self-contained exposition of our topological rigidity results in their most general and natural setting – these are deduced directly from Theorems 7.1.2 and 7.1.3.

In the first appendix, we discuss several variants of the Rips complex, the *relative* and *scaled* Rips complexes, which are important technical tools for the proofs of our results on topological rigidity. We believe this material may be of independent interest. In the second appendix, we recall the controlled Mayer-Vietoris sequences in K and L-theory, as proved in [RY1, RY2]. These are essential tools for our proofs of the bounded Borel and the Farrell-Jones L-theory isomorphism conjectures.

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2. Decomposition complexity

Our proofs of the isomorphism conjectures will be based on Mayer-Vietoris arguments – we shall apply a *large-scale* version of an appropriate Mayer-Vietoris sequence to prove that an assembly map is an isomorphism. To carry out this idea, we shall decompose a given metric space as a union of two subspaces, which are *simpler* than the original. Roughly, *simpler* is interpreted to mean that each subspace is itself a union of spaces at a pairwise distance large enough that proving the isomorphism for the subspace amounts to proving the isomorphism for these constituent pieces 'uniformly'. Further, this basic decomposition step shall be iterated a finite number of times, until we reach a bounded family. This is the idea behind finite decomposition complexity.

We shall need to formulate our notion of finite decomposition complexity not for a single metric space, but rather for a *metric family*, a (countable) family of metric spaces which we shall denote by $\mathcal{X} = \{ X \}$; throughout we view a single metric space as a family containing a single element.

In order to streamline our definitions we introduce some terminology and notation for manipulating decompositions of metric spaces and metric families. A collection of subspaces $\{ Z_i \}$ of a metric space Z is r-*disjoint* if for all $i \neq j$ we have $d(Z_i, Z_j) \geq r$. To express the idea that Z is the union of subspaces Z_i, and that the collection of these subspaces is r-disjoint we write

$$Z = \bigsqcup_{r-disjoint} Z_i.$$

A family of of metric spaces $\{ Z_i \}$ is *bounded* if there is a uniform bound on the diameter of the individual Z_i:

$$\sup \operatorname{diam}(Z_i) < \infty.$$

2.0.1. **Definition.** A metric family $\mathcal{X}$ is *r-decomposable* over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits an *r-decomposition*

$$X = X_0 \cup X_1, \quad X_i = \bigsqcup_{r-disjoint} X_{ij},$$

where each $X_{ij} \in \mathcal{Y}$. We introduce the notation $\mathcal{X} \xrightarrow{r} \mathcal{Y}$ to indicate that $\mathcal{X}$ is r-decomposable over $\mathcal{Y}$.

2.0.2. **Definition.** Let $\mathfrak{A}$ be a collection[1] of metric families. A metric family $\mathcal{X}$ is *decomposable* over $\mathfrak{A}$ if, for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{A}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection $\mathfrak{A}$ is *stable under decomposition* if every metric family which decomposes over $\mathfrak{A}$ actually belongs to $\mathfrak{A}$.

2.0.3. **Definition.** The collection $\mathfrak{D}$ of metric families with finite decomposition complexity is the minimal collection of metric families containing the bounded metric families and stable under decomposition. We abbreviate membership in $\mathfrak{D}$ by saying that a metric family in $\mathfrak{D}$ has FDC.

This notion has been inspired by the property of *finite asymptotic dimension* of Gromov [G1]. Recall that a metric space X has *finite asymptotic dimension* if there exists $d \in \mathbb{N}$ such that for every $r > 0$ the space X may be written as a union of $d+1$ subspaces, each of which may be further decomposed as an r-disjoint union:

$$X = \bigcup_{i=0}^{d} X_i, \quad X_i = \bigsqcup_{r-disjoint} X_{ij}, \tag{2.1}$$

in which the family $\{ X_{ij} \}$ (as both i and j vary) is bounded. It follows immediately from the definitions that metric families with asymptotic dimension at most one (uniformly) have finite decomposition complexity.

2.0.4. **Remark.** At the outset of this project, we defined a property weaker than FDC which is more transparently related to finite asymptotic dimension. The difference between this property – *weak finite decomposition complexity* – and the one defined here lies in the type of decomposition – we replace r-decomposability by the notion of (d, r)-decomposability.

A metric family $\mathcal{X}$ is *(d, r)-decomposable* over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits a decomposition

$$X = X_0 \cup \ldots \cup X_d, \quad X_i = \bigsqcup_{r-disjoint} X_{ij},$$

[1]While we generally prefer the term 'collection' to 'class', we do not mean to imply that a collection of metric families is a *set* of metric families. We shall not belabor the associated set-theoretic complications.

where each $X_{ij} \in \mathcal{Y}$. The metric family $\mathcal{X}$ *weakly* decomposes over the collection $\mathfrak{A}$ of metric families, if there exists a $d \in \mathbb{N}$ such that for every $r > 0$, there exists $\mathcal{Y} \in \mathfrak{A}$ and a (d, r)-decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection of metric families with *weak finite decomposition complexity* is the smallest collection containing bounded metric families, and stable under weak decomposition.

Clearly, both finite asymptotic dimension (again, uniformly in the sense of Bell and Dranishnikov [BD1]) and finite decomposition complexity imply *weak* finite decomposition complexity. While true that finite asymptotic dimension implies finite decomposition complexity, at least for proper metric spaces, this is already difficult.

2.0.5. **Question.** Are finite and weak finite decomposition complexity equivalent?

2.1. **The metric decomposition game.** The game has two players and begins with a metric family. The objective of the first player is to successfully decompose the spaces comprising the family, whereas the second player attempts to obstruct the decomposition.

Formally, let $\mathcal{X} = \mathcal{Y}_0$ be the starting family. The game begins with the first player asserting to the second they can decompose $\mathcal{Y}_0$. The second player challenges the first by asserting an integer r_1. The first turn ends with the first player exhibiting a r_1-decomposition of $\mathcal{Y}_0$ over a metric family $\mathcal{Y}_1$.

Subsequent turns are analogous: the first player asserts they can decompose the family $\mathcal{Y}_i$; the second challenges with an r_{i+1}; the first responds by exhibiting an r_{i+1}-decomposition of $\mathcal{Y}_i$ over a metric family $\mathcal{Y}_{i+1}$.

The first player has a *winning strategy* if, roughly speaking, they can produce decompositions ending in a bounded family *no matter what choices the second player makes.* While the outcome is certain, the number of turns required for eventual victory may depend on the choices made by the second player. A game in which the first player applies their winning strategy exhibits a sequence of decompositions beginning with the spaces in $\mathcal{X}$ and ending in a bounded family:

$$\mathcal{X} = \mathcal{Y}_0 \xrightarrow{r_1} \mathcal{Y}_1 \xrightarrow{r_2} \mathcal{Y}_2 \longrightarrow \dots \mathcal{Y}_{n-1} \xrightarrow{r_n} \mathcal{Y}_n, \quad \mathcal{Y}_n \text{ bounded.} \tag{2.2}$$

2.2. **Decomposition strategy.** We shall now formalize the idea of a winning strategy for the decomposition game. As we shall see in the next section, the existence of a winning strategy is equivalent to finite decomposition complexity.

A *decomposition tree* is a directed, rooted tree T satisfying the following:

(1) every non-root vertex of T is the terminal vertex of a unique edge;
(2) every non-leaf vertex of T is the initial vertex of countably many edges, which are labeled by the natural numbers;
(3) T contains no infinite ray (geodesic edge-path).

Conditions (1) and (2) are succintly expressed by saying that T is a '1-up, ∞-down' rooted tree in which the 'down' edges emanating from each vertex are in explicit bijection with $\mathbb{N}$. Figure 1 depicts a typical caret in T with labeled edges.

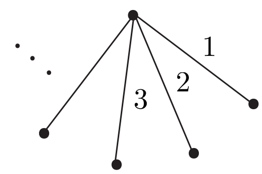

Figure 1. A caret

The vertices of a rooted tree are partially ordered by setting $v \leq w$ precisely when w lies on the (unique) geodesic edge path from v to the root vertex. Condition (3) asserts that for a decomposition tree every decreasing chain in this order eventually terminates.

Let $\mathcal{X}$ be a metric family. A *decomposition strategy* for $\mathcal{X}$ comprises a decomposition tree T, the *support tree* of the strategy, together with a labeling of the vertices of T by metric families $\mathcal{Y}$ subject to the following requirements:

(4) the root vertex of T is labeled $\mathcal{X}$;
(5) every leaf of T is labeled by a bounded family;
(6) if $\mathcal{Y}$ labels the initial vertex and $\mathcal{Z}$ the terminal vertex of an edge labeled by $r \in \mathbb{N}$ then $\mathcal{Y}$ is r-decomposable over $\mathcal{Z}$.

The intuition, of course, is that paths in T beginning at its root and ending at a leaf correspond to decompositions of the family $\mathcal{X}$ into uniformly bounded parts. Precisely, if the edges along the path are labeled $r_1, \ldots, r_n$ and the vertices are labeled $\mathcal{X}$, $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ we obtain the decomposition in (2.2).

An important feature of the decomposition game is the dependence among the r_i. Existence of a decomposition strategy implies that a partially completed decomposition may always be continued, no matter the choice of the subsequent r_i, and eventually terminates in a bounded family.

2.3. **Equivalent formulations of FDC.** We shall present two equivalent descriptions of the collection of families having finite decomposition complexity. We require the following lemma.

2.3.1. **Lemma.** *Let T be a decomposition tree. There exists a function $v \mapsto \alpha_v$ from the set of vertices of T to a set of countable ordinal numbers with the properties that $\alpha_v = 0$ if v is a leaf and*

$$\alpha_v = \sup_{w<v} \{ \alpha_w + 1 \}$$

otherwise.

Proof. Observe that, by virtue of the no-infinite-ray assumption, a decomposition tree has leaves. Define, for each countable ordinal α, a subset L_α of the vertex set of T by transfinite recursion: L_0 is the set of leaves of T; for $\alpha > 0$,

$$L_\alpha = \text{the set of leave of } T \setminus \cup_{\beta<\alpha} L_\beta,$$

if this set is nonempty, and $L_\alpha = \emptyset$ otherwise. Note that, if it is non-empty, the set $T \setminus \cup_{\beta<\alpha} L_\beta$ is again a decomposition tree, and therefore has leaves.

Let $\alpha_0 = \{\, \alpha \colon L_\alpha \neq \emptyset \,\}$ and let $\mathfrak{L} = \{L_\alpha, \alpha < \alpha_0\}$. Clearly, $\mathfrak{L}$ is a partition of the set of vertices of T, and the map $\alpha \mapsto L_\alpha : \alpha_0 \to \mathfrak{L}$ is a bijection. It follows that α_0 is countable. Finally, for every vertex v, let α_v be the unique α such that $v \in L_\alpha$. It is not difficult to see that α_v satisfies the desired properties. □

2.3.2. **Definition.** We define, for each ordinal α, a collection of metric families according to the following prescription:

(1) Let $\mathfrak{D}_0$ be the collection of bounded families:

$$\mathfrak{D}_0 = \{\, \mathcal{X} \colon \mathcal{X} \text{ is bounded} \,\}.$$

(2) If α is an ordinal greater than 0, let $\mathfrak{D}_\alpha$ be the collection of metric families decomposable over $\cup_{\beta<\alpha}\mathfrak{D}_\beta$:

$$\mathfrak{D}_\alpha = \{\, \mathcal{X} \colon \forall\, r\ \exists\, \beta < \alpha\ \exists\, \mathcal{Y} \in \mathfrak{D}_\beta \text{ such that } \mathcal{X} \xrightarrow{r} \mathcal{Y} \,\}.$$

We introduce the notation $\mathfrak{D}_{\text{fin}}$ for the union of the $\mathfrak{D}_n$, over $n \in \mathbb{N}$.

2.3.3. **Theorem.** *The following statements concerning a metric family $\mathcal{X}$ are equivalent:*

(1) *$\mathcal{X}$ has finite decomposition complexity;*
(2) *$\mathcal{X}$ admits a decomposition strategy;*
(3) *there exists a countable ordinal α such that $\mathcal{X} \in \mathfrak{D}_\alpha$.*

Proof. For purposes of the proof let $\mathfrak{D}'$ be collection of families admitting a decomposition strategy; let $\mathfrak{D}''$ be the collection of families belonging to $\mathfrak{D}_\alpha$ for some countable ordinal α. We must show $\mathfrak{D}'' = \mathfrak{D}' = \mathfrak{D}$.

A simple transfinite induction shows that $\mathfrak{D}_\alpha \subset \mathfrak{D}$ for every ordinal α. Thus, $\mathfrak{D}'' \subset \mathfrak{D}$.

Next, we show that $\mathfrak{D} \subset \mathfrak{D}'$. Since a bounded family trivially admits a decomposition strategy, it suffices to show that the collection $\mathfrak{D}'$ is closed under decomposability. Let $\mathcal{X}$ be a family decomposable over $\mathfrak{D}'$. For every $r \in \mathbb{N}$, obtain a family $\mathcal{Y}_r \in \mathfrak{D}'$ such that $\mathcal{X}$ is r-decomposable over $\mathcal{Y}_r$. A decomposition strategy for $\mathcal{X}$ is obtained by attaching strategies for the $\mathcal{Y}_r$ to the bottom of an 'infinite caret' whose root vertex is labeled $\mathcal{X}$ and whose edges are labeled by $\mathbb{N}$ as shown in Figure 2.

Finally, we show that $\mathfrak{D}' \subset \mathfrak{D}''$. Let $\mathcal{X} \in \mathfrak{D}'$. Let T be the support tree of a decomposition strategy for $\mathcal{X}$; denote the label of a vertex v by $\mathcal{Y}_v$ and let $v \mapsto \alpha_v$ be a function with the properties outlined in Lemma 2.3.1. It suffices to show that for every ordinal α we have: if $\alpha_v \leq \alpha$ then $\mathcal{Y}_v \in \mathfrak{D}_\alpha$. This follows easily by transfinite induction. □

2.3.4. **Example.** We shall require the fact, easily verified (by induction), that $\mathbb{Z}^n \in \mathfrak{D}_n$, for each natural number n.

2.3.5. **Example.** Let $G = \oplus\mathbb{Z}$ (countably infinite direct sum), equipped with a proper left-invariant metric; for concreteness use the metric determined by the requirement that the generator having a single 1 in the i^{th} position has length i. We claim that $G \in \mathfrak{D}_\omega$. Indeed,

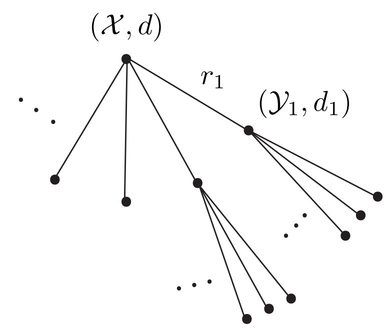

Figure 2. Concatenating strategies

let $r > 0$ be given. If n is a natural number greater than r then the decomposition of G into cosets of the subgroup $\mathbb{Z}^n$ (via the first n coordinates) is r-disjoint and the family comprised of these cosets is in $\mathfrak{D}_n$ by the previous example.

Moreover, we shall shortly see that since G does not have finite asymptotic dimension it is *not* in $\mathfrak{D}_{\mathrm{fin}}$.

2.3.6. **Example.** Let $G = \mathbb{Z} \wr \mathbb{Z}$. By considering the extension $0 \to \oplus_{n\in\mathbb{Z}}\mathbb{Z} \to G \to \mathbb{Z} \to 0$ we see that $G \in \mathfrak{D}_{\omega+1}$ (compare to Remark 3.1.6).

2.3.7. **Example.** Let $G = \oplus G_n$, where $G_n = (\dots((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z})\dots) \wr \mathbb{Z}$, the wreath product of n copies of $\mathbb{Z}$. Then $G \in \mathfrak{D}_{\omega^2}$. It is an open question whether $G \in \mathfrak{D}_\alpha$ for some $\alpha < \omega^2$.

3. Permanence of FDC

We shall study the permanence characteristics of finite decomposition complexity. While we shall focus on finite decomposition complexity, all permanence results stated in this section hold for weak finite decomposition complexity as well.

We begin by recalling some elementary concepts from coarse geometry. Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families. A *subspace* of the family $\mathcal{Y}$ is a family $\mathcal{Z}$, every element of which is a subspace of some element of $\mathcal{Y}$. A *map of families* from $\mathcal{X}$ to $\mathcal{Y}$ is a collection of functions $F = \{ f \}$, each mapping some $X \in \mathcal{X}$ to some $Y \in \mathcal{Y}$ and such that every $X \in \mathcal{X}$ is the domain of at least one $f \in F$. We use the notation $F : \mathcal{X} \to \mathcal{Y}$ and, when confusion could occur, write $f : X_f \to Y_f$ to refer to an individual function in F. The *inverse image* of the subspace $\mathcal{Z}$ is the collection

$$F^{-1}(\mathcal{Z}) = \{ f^{-1}(Z) \colon Z \in \mathcal{Z},\ f \in F \}.$$

The inverse image is a subspace of $\mathcal{X}$.

A map of families $F : \mathcal{X} \to \mathcal{Y}$ is *uniformly expansive* if there exists a non-decreasing function $\rho : [0, \infty) \to [0, \infty)$ such that for every $f \in F$ and every $x, y \in X_f$

$$d(f(x), f(y)) \le \rho(d(x, y)); \tag{3.1}$$

it is *effectively proper* if there exists a proper non-decreasing function $\delta : [0,\infty) \to [0,\infty)$ such that for every $f \in F$ and every x, $y \in X_f$

$$\delta(d(x,y)) \leq d(f(x), f(y)); \tag{3.2}$$

it is a *coarse embedding* if it is both uniformly expansive and effectively proper. (In this case, if $\mathcal{X}$ is unbounded then ρ is also proper.) Summarizing, a map of families F is a coarse embedding if the individual f are coarse embeddings *admitting a common* δ *and* ρ. Similar remarks apply to uniformly expansive and effectively proper maps.

Recall that a coarse embedding $f : X \to Y$ of metric spaces is a coarse equivalence if it admits an 'inverse' – a coarse embedding $g : Y \to X$ for which the compositions $f \circ g$ and $g \circ f$ are *close* to the identity maps on X and Y, respectively:

$$\text{there exists } C > 0 \text{ such that } d(x, gf(x)) \leq C \text{ and } d(y, gf(y)) \leq C, \tag{3.3}$$

for all $x \in X$ and $y \in Y$. So motivated, a coarse embedding $F : \mathcal{X} \to \mathcal{Y}$ of metric families is a *coarse equivalence* if each $f \in F$ is a coarse equivalence admitting an inverse g satisfying the following two conditions: first, the collection $G = \{ g \}$ is a coarse embedding $\mathcal{Y} \to \mathcal{X}$ of metric families; second, the composites $f \circ g$ and $g \circ f$ are *uniformly close* to the identity maps on the spaces comprising $\mathcal{X}$ and $\mathcal{Y}$, in the sense that the constant C in (3.3) may be chosen independently of the spaces $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Two metric families $\mathcal{X}$ and $\mathcal{Y}$ are *coarsely equivalent* if there exists a coarse equivalence $\mathcal{X} \to \mathcal{Y}$. Coarse equivalence is an equivalence relation.

3.1. **Permanence for spaces.** The primitive permanence properties for metric families are Coarse Invariance, the Fibering and Union Theorems. We shall prove these in this section.

3.1.1. **Lemma.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families and let $F : \mathcal{X} \to \mathcal{Y}$ be a uniformly expansive map. For every $r > 0$ there exists an $s > 0$ such that if $\mathcal{Z}$ and $\mathcal{Z}'$ are subspaces of $\mathcal{Y}$ and $\mathcal{Z}' \xrightarrow{s} \mathcal{Z}$ then $F^{-1}(\mathcal{Z}') \xrightarrow{r} F^{-1}(\mathcal{Z})$. Further, s depends only on r and on the non-decreasing function ρ satisfying* (3.1)*.*

Proof. Assuming F is uniformly expansive let ρ be such that (3.1) holds. Set $s = \rho(r)$ and assume $\mathcal{Z}' \xrightarrow{s} \mathcal{Z}$. An element of $F^{-1}(\mathcal{Z}')$ has the form $f^{-1}(Z)$ for some $Z \in \mathcal{Z}'$ and $f \in F$. Given such an element obtain a decomposition

$$Z = Z_0 \cup Z_1, \quad Z_i = \bigsqcup_{s-disjoint} Z_{ij},$$

in which the $Z_{ij} \in \mathcal{Z}$. We then have a decomposition

$$f^{-1}(Z) = f^{-1}(Z_0) \cup f^{-1}(Z_1), \quad f^{-1}(Z_i) = \bigcup f^{-1}(Z_{ij}),$$

in which the $f^{-1}(Z_{ij}) \in F^{-1}(\mathcal{Z})$. From the definition of s we see immediately that the union on the right is r-disjoint. □

3.1.2. **Lemma.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families and let $F : \mathcal{X} \to \mathcal{Y}$ be an effectively proper map. If $\mathcal{Z}$ is a bounded subspace of $\mathcal{Y}$ then $F^{-1}(\mathcal{Z})$ is a bounded subspace of $\mathcal{X}$.*

Proof. Assuming F is effectively proper let δ be such that (3.2) holds. Let B bound the diameter of the metric spaces in the family $\mathcal{Z}$. Using the hypothesis that δ is proper, let A be such that $\delta(A) \geq B$. Then $F^{-1}(\mathcal{Z})$ is bounded by A. $\square$

3.1.3. **Coarse Invariance.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families. If there is a coarse embedding from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{Y}$ has finite decomposition complexity, then so does $\mathcal{X}$. In particular:*

(1) *a subspace of a metric family with FDC itself has FDC;*
(2) *if $\mathcal{X}$ and $\mathcal{Y}$ are coarsely equivalent, then $\mathcal{X}$ has FDC if and only if $\mathcal{Y}$ does.*

Proof. By pruning and relabeling we can pull back a decomposition strategy for $\mathcal{Y}$ to $\mathcal{X}$. Precisely, select an increasing sequence of natural numbers $s_1, s_2, \ldots$ such that $s_i \geq i$. Prune T by removing a vertex v, together with the entire 'downward' subtree based at v and the unique upward edge incident at v, when this upward edge is labeled by an element of $\mathbb{N} \backslash \{ s_i \}$. The resulting graph T' is a subtree of T and a vertex of T' is a leaf of T' exactly when it is a leaf of T. Relabel a typical edge as shown in Figure 3. It follows from Lemmas 3.1.1 and

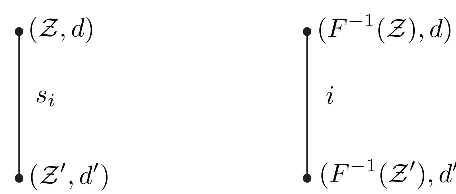

FIGURE 3. Relabeling

3.1.2 that the labeling requirements for a decomposition strategy are fulfilled. $\square$

3.1.4. **Fibering Theorem.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families and let $F : \mathcal{X} \to \mathcal{Y}$ be a uniformly expansive map. Assume $\mathcal{Y}$ has finite decomposition complexity, and that for every bounded subspace $\mathcal{Z}$ of $\mathcal{Y}$ the inverse image $F^{-1}(\mathcal{Z})$ has finite decomposition complexity. Then $\mathcal{X}$ has finite decomposition complexity.*

Proof. A decomposition strategy for $\mathcal{Y}$ pulls back, as in the previous proof, to a *partial* decomposition strategy for $\mathcal{X}$. It is partial in that the leaves of its support tree are labeled by families which are not (necessarily) bounded but rather are the inverse images of bounded subspaces of $\mathcal{Y}$. We complete the partial strategy by attaching to a leaf labeled by $F^{-1}(\mathcal{Z})$ a strategy for this family. $\square$

3.1.5. **Remark.** Directly from the definitions we see that $\mathcal{X} \in \mathfrak{D}_n$ precisely when $\mathcal{X}$ admits a decomposition strategy in which the strategy tree has *depth* not greater than n, meaning that the length of a geodesic emanating from the root vertex is at most n. In the notation of the Fibering Theorem, the previous proof shows the following: suppose that $\mathcal{Y} \in \mathfrak{D}_n$ and that there exists a natural number m such that $F^{-1}(\mathcal{Z}) \in \mathfrak{D}_m$ for every bounded subspace $\mathcal{Z}$ of $\mathcal{Y}$; then $\mathcal{X} \in \mathfrak{D}_{n+m}$.

3.1.6. **Remark.** Continuing in the spirit of the previous remark, suppose that $\mathcal{Y} \in \mathfrak{D}_{\text{fin}}$ and that $F^{-1}(\mathcal{Z}) \in \mathfrak{D}_{\text{fin}}$ for every bounded subspace $\mathcal{Z}$ of $\mathcal{Y}$. Then $\mathcal{X} \in \mathfrak{D}_{\omega+\text{fin}}$, meaning that for some natural number n we have $\mathcal{X} \in \mathfrak{D}_{\omega+n}$. The distinction between this remark and the previous is that here we assume merely that each $F^{-1}(\mathcal{Z}) \in \mathfrak{D}_m$ for some natural number m, *which may depend on* $\mathcal{Z}$.

3.1.7. **Finite Union Theorem.** *Let X be a metric space, expressed as a union of finitely many metric subspaces $X = \cup_{i=0}^n X_i$. If the metric family $\{ X_i \}$ has finite decomposition complexity so does X.*

Proof. Consider first the case $n = 2$, illustrated in Figure 4. For every $r > 0$, the metric space $X = X_1 \cup X_2$ is r-decomposable over the family $\{ X_1, X_2 \} \in \mathfrak{D}$. Thus $X \in \mathfrak{D}$. The general case follows by induction. □

3.1.8. **Union Theorem.** *Let X be a metric space, expressed as a union of metric subspaces $X = \cup_{i \in I} X_i$. Suppose that the metric family $\{ X_i \}$ has finite decomposition complexity and that for every $r > 0$ there exists a metric subspace $Y(r) \subset X$ having finite decomposition complexity and such that the subspaces $Z_i(r) = X_i \setminus Y(r)$ are pairwise r-disjoint. Then X has finite decomposition complexity.*

Proof. To conclude that X has finite decomposition complexity, it suffices to show that X is decomposable over $\mathfrak{D}$. The proof of this is illustrated in Figure 5. Formally, for every $r > 0$ let $Y(r)$ and $Z_i(r)$ be as in the statement. The decomposition

$$X = Y(r) \bigcup Z(r), \quad Z(r) = \bigsqcup_{r-disjoint} Z_i(r)$$

is a r-decomposition of X over the family $\mathcal{Y}_r = \{ Y(r) \} \cup \{ Z_i(r) : i \in I \}$. Since the $Z_i(r)$ are subspaces of the X_i and the family $\{ X_i \}$ has finite decomposition complexity, the family $\{ Z_i(r) : i \in I \}$ does as well; since $Y(r)$ has finite decomposition complexity, the family $\mathcal{Y}_r$ does as well. □

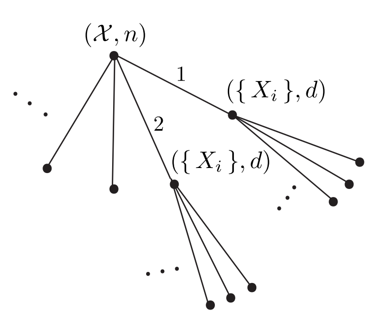

FIGURE 4. A finite union

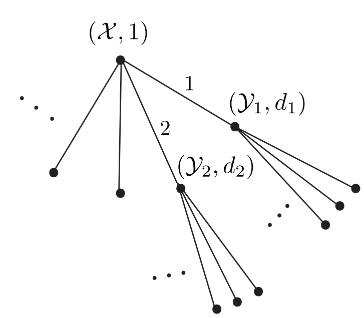

FIGURE 5. A union

3.1.9. **Remark.** While we could state union theorems in the context of metric families (instead of single metric spaces) we shall not require this level of generality.

3.2. **Permanence for groups.** Most (though not all) permanence properties for discrete groups are deduced by allowing the group to act on an appropriate metric space, and applying the permanence results for spaces detailed in the previous section.

Let G be a countable discrete group. Recall that a countable discrete group admits a proper length function ℓ and that any two metrics defined from proper length functions by the formula

$$d(s,t) = \ell(s^{-1}t)$$

are coarsely equivalent (in fact, the identity map is a coarse equivalence). As a consequence, a coarsely invariant property of metric spaces is a property of countable discrete groups – whether or not a group has the property is not an artifact of the particular metric chosen. Consequently, we say that a discrete group has finite decomposition complexity if its underlying metric space has finite decomposition complexity for some (equivalently every) metric defined as above.

3.2.1. **Proposition.** *A countable direct union of groups with finite decomposition complexity has finite decomposition complexity. Equivalently, a countable discrete group has finite decomposition complexity if and only if every finitely generated subgroup does.*

Proof. Let G be a countable discrete group, expressed as the union of a collection of subgroups each of which has finite decomposition complexity: $G = \cup G_i$. Equip G with a proper length function and associated metric. We shall show that for every $r > 0$ (the metric space) G is r-decomposable over a metric family with finite decomposition complexity; by Theorem 2.3.3 this will suffice.

Let $r > 0$. Since the ball of radius r centered at the identity in G is finite there exists $i = i(r)$ such that this ball is contained in G_i. It follows that the decomposition of G into the cosets of G_i is r-disjoint. Further, the family comprised of these cosets has finite decomposition complexity since each coset is isometric to G_i, which has finite decomposition complexity (in any proper metric so in the subspace metric) by assumption. □

Let now X be a metric space, and suppose that G acts (by isometries) on X. For $R > 0$ the *R-coarse stabilizer of x* is

$$\mathrm{Stab}(x,R) = \{\, g \in G : d(x, g\cdot x) < R \,\}.$$

In general an R-coarse stabilizer is a *subset* of G. The 0-coarse stabilizer of x is its stabilizer, a subgroup of G. The space X is *locally finite* if every ball is finite.

3.2.2. **Lemma.** *For every $x \in X$ the orbit map $g \mapsto g\cdot x : G \to X$ is uniformly expansive.* □

3.2.3. **Proposition.** *Let G be a countable discrete group acting on a metric space X with finite decomposition complexity. If there exists $x_0 \in X$ such that for every $R > 0$ the R-coarse stabilizer of x_0 has finite decomposition complexity then G has finite decomposition complexity.*

Proof. By restricting to the orbit of x_0 we may assume the action is transitive. Together with the coarse stabilizer condition, the fact that the orbit map $g \to g \cdot x_0$ is a surjective and equivariant map $G \to X$ implies that the hypothesis of the Fibering Theorem 3.1.4 are fulfilled. The proposition follows. ☐

3.2.4. **Corollary.** *Let G and X be as in the previous proposition. If X is locally finite, and if there exists $x_0 \in X$ such that the stabilizer of x_0 has finite decomposition complexity, then G has finite decomposition complexity.*

Proof. Under the stated hypotheses the Finite Union Theorem implies that the coarse stabilizers of x_0 have finite decomposition complexity. Thus, the previous proposition applies. ☐

3.2.5. **Corollary.** *The collection of countable discrete groups with finite decomposition complexity is closed under extensions.* ☐

3.2.6. **Proposition.** *If a countable discrete group acts (without inversion) on a tree, and the vertex stabilizers of the action have finite decomposition complexity, then the group itself has finite decomposition complexity.*

Proof. According to the Bass-Serre theory, a group as in the statement is built from vertex stabilizers of the action by iterated free products (with amalgam), HNN extensions and direct unions. An HNN extension, in turn, is built from free products (with amalgam), a direct union and a group extension. As we have seen that the class of (countable discrete) groups with finite decomposition complexity is closed under direct unions, subgroups and extensions the proposition follows once we show that a free product with amalgam has finite decomposition complexity if the factors do. But, this follows axiomatically from the above proven permanence results – essentially, apply fibering to the action on the Bass-Serre tree using the union theorem to conclude that the coarse stabilizers have finite decomposition complexity. For a more detailed discussion see [G] and the references therein. ☐

4. FDC, Property A and finite asymptotic dimension

In this section we shall discuss how the property of finite decomposition complexity relates to other familiar properties from coarse geometry, notably to Property A and to finite asymptotic dimension.

Above we have discussed how the definition of finite decomposition complexity is *motivated* by finite asymptotic dimension. We shall now pursue this discussion further, our goal being to prove that a metric space having finite asymptotic dimension has finite decomposition complexity as well.

Recall that a metric space is *proper* if closed and bounded sets are compact. A discrete metric space is proper precisely when it is *locally finite* in the sense that every ball is finite. It is not difficult to see that a proper metric space having finite asymptotic dimension has finite decomposition complexity. Indeed, according to a theorem of Dranishnikov-Zarichnyi a proper metric space having finite asymptotic dimension admits a coarse emebdding into the product of finitely many locally finite trees [DZ]. As trees have finite decomposition complexity, we may apply our permanence results to conclude. More generally, for metric spaces which are *not necessarily proper* we have the following theorem.

4.1. **Theorem.** *A metric space has finite asymptotic dimension if and only if it belongs to $\mathfrak{D}_{fin}$. In particular, a metric space having finite asymptotic dimension has finite decomposition complexity as well.*

We are primarily interested in the forward implication, and shall reduce the general case to the case of proper metric spaces using an ultralimit construction. Before turning to the proof, we recall the relevant background notions. Let X be a (pseudo-)metric space. The *Gromov triple product* (with respect to a base point x_0) is

$$(x|y) = \tfrac{1}{2}\left(d(x,x_0) + d(y,x_0) - d(x,y)\right).$$

The (pseudo)-metric space X is *Gromov* 0*-hyperbolic* if

$$(x|z) \geq \min\{\,(x|y),(y|z)\,\},$$

for all x, y and $z \in X$. The notion of 0-hyperbolicity is independent of the choice of base point [A, Prop. 2.2]. A Gromov 0-hyperbolic (pseudo)-metric space has asymptotic dimension at most 1. (See [R2] for a direct argument.) Hence, a Gromov 0-hyperbolic (pseudo-)metric space has finite decomposition complexity.

Proof of Theorem 4.1. A simple induction shows that a (pseudo)-metric space belonging to $\mathfrak{D}_n$ admist, for every r, a $(2^n, r)$-decomposition over a bounded family. In particular, its asymptotic dimension is at most $2^n - 1$.

For the converse, let X be a (pseudo)-metric space having finite asymptotic dimension at most n. We shall show that X has finite decomposition complexity, indeed that $X \in \mathfrak{D}_{\mathrm{fin}}$. Apply the result of Drashnikov-Zarichnyi [DZ] to the finite subsets of X – these are locally finite metric spaces and the essential observation here is that the result of Drashnikov-Zarichnyi applies *uniformly*. Precisely, there exists ρ and δ and for each finite subset $F \subset X$ a ρ-uniformly expansive and δ-effectively proper map into a product of trees:

$$F \to T_0^F \times \cdots \times T_n^F.$$

Projecting to the individual factors we lift the tree metrics back to F to obtain a family of (pseudo-)metrics $d_0^F, \ldots, d_n^F$ on F with the following two properties. First, each d_i^F is Gromov 0-hyperbolic – recall here that an $\mathbb{R}$-tree is Gromov 0-hyperbolic. Second, the identity $F \to F$ is ρ-uniformly expansive and δ-effectively proper, when the domain is equipped with the subspace metric from X and the range the sum metric $d_0^F + \cdots + d_n^F$ – explicitly, for all x, $y \in F$ we have

$$\delta(d_X(x,y)) \leq d_0^F(x,y) + \cdots + d_n^F(x,y) \leq \rho(d_X(x,y)). \tag{4.1}$$

Let now $\mathfrak{F}$ be the collection of finite subsets of X containing a fixed base point x_0, viewed as a directed set under inclusion. Let ω be an ultrafilter on the *set* $\mathfrak{F}$ with the following property: for every convergent net $(t_F)_{F\in\mathfrak{F}}$ of real numbers we have

$$\lim t_F = \omega\text{-}\lim t_F,$$

where the limit of the left is the ordinary limit of the convergent net, and the limit of the right is the limit with respect to the ultrafilter ω.

For each fixed $i = 0, \dots, n$ form the ultraproduct $X_i = \omega\text{-}\lim F_i$, where we write F_i for F equipped with the metric d_i^F. Precisely, X_i is the space of of $\mathfrak{F}$-indexed nets $\overline{x} = (x_F)$, with $x_F \in F$, for which $d_i^F(x_F, x_0)$ is bounded independent of F.[2] Define a pseudo-metric on X_i by

$$d_i(\overline{x}, \overline{y}) = \omega\text{-}\lim d_i^F(x_F, y_F),$$

where $\overline{x} = (x_F)$ and $\overline{y} = (y_F)$ are elements of X_i. Define a map $\alpha_i : X \to X_i$ by associating to x the 'constant sequence'; it follows immediately from (4.1) that

$$\alpha_i(x)_F = \begin{cases} x, & x \in F \\ x_0, & \text{else} \end{cases}$$

satisfies the boundedness condition required of elements of X_i.

Now, the individual X_i are Gromov 0-hyperbolic, essentially because the condition for 0-hyperbolicity, satisfied by the individual d_i^F, involves only finitely many points and passes to the limit intact. Thus, each X_i has finite decomposition complexity and indeed belongs to $\mathcal{D}_1$. An elementary application of permenance shows that the product $X_0 \times \cdots \times X_n$ belongs to $\mathfrak{D}_{n+1}$. See Remark 3.1.5.

The proof concludes with the observation that the product of the α_i is a coarse embedding $X \to X_0 \times \cdots \times X_n$. To verify this observe that for $x \in X$ we have $\alpha_i(x) = x$ for ω-almost every F. So, if $y \in X$ as well we have

$$\sum_{i=0}^{n} d_i(\alpha_i(x), \alpha_i(y)) = \omega\text{-}\lim \sum_{i=0}^{n} d_i^F(x, y)$$

which by (4.1) is bounded above by $\rho(d_X(x,y))$ and below by $\delta(d_X(x,y))$. $\square$

4.2. **Remark.** We are unable to find a reference for the existence of an ultrafilter as required in the previous proof; we provide instead the following simple argument. In the notation of the proof, the collection of all subsets of $\mathfrak{F}$ containing a set of the form

$$\{ F \in \mathfrak{F} : F_0 \subset F \}$$

is a filter, the *filter of tails in* $\mathfrak{F}$. An ultrafilter containing the filter of tails is as required – existence of an ultrafilter containing a given filter is a classic application of Zorn's lemma.

We turn now to a discussion of Property A, a geometric property guaranteeing coarse embeddability into Hilbert space [Y2]. We shall show that a metric space with (weak) finite decomposition complexity has Property A. As a consequence, any sequence of expanding graphs (as a metric space) does not have (weak) finite decomposition complexity since it does not admit a coarse embedding into Hilbert space.

[2]As we work with pseudo-metric spaces it is not necessary to consider equivalence classes as would be typical.

To prove the main result of this section, it is convenient to work with a characterization of Property A introduced by Dadarlat and Guentner [DG]. A metric family $\mathcal{U} = \{ U \}$ is a *cover* of a metric space X if every $U \in \mathcal{U}$ is a metric subspace of X and

$$X = \bigcup_{u \in \mathcal{U}} U.$$

A *partition of unity on X subordinate to a cover* $\mathcal{U}$ is a family of maps $\phi_U : X \to [0,1]$, one for each $U \in \mathcal{U}$, such that each ϕ_U is supported in U and such that for every $x \in X$

$$\sum_{U \in \mathcal{U}} \phi_U(x) = 1.$$

We do not require that the sum is finite for any particular $x \in X$.

4.3. **Definition.** A metric family $\mathcal{X}$ is exact if for every $R > 0$ and $\varepsilon > 0$ and for every $X \in \mathcal{X}$ there is a partition of unity $\{ \psi_U^X \}$ on X subordinate to a cover $\mathcal{U}_X$ of X such that the collection

$$\mathcal{U} = \{ U : U \in \mathcal{U}_X, \text{ some } X \}$$

is a bounded metric family and such that for every $X \in \mathcal{X}$ and every x, $y \in X$

$$d(x,y) \leq R \Longrightarrow \sum_{U \in \mathcal{U}_X} |\psi_U^X(x) - \psi_U^X(y)| \leq \varepsilon.$$

4.4. **Remark.** Our definition of exactness is equivalent to the notion of an *equi-exact family of metric spaces* introduced by Dadarlat and Guentner (compare [DG] Defs. 2.7 and 2.8). However, we have indexed our partition of unity and cover differently so our definition is not identical to the one in [DG].

For the statements of the next two results, recall that a metric space has *bounded geometry* if for every $r > 0$ there exists an $N = N(r)$ such that every ball of radius r contains at most N points.

4.5. **Theorem** ([DG] Prop. 2.10)**.** *A metric space having Property A is exact. A bounded geometry exact metric space has Property A.* ☐

4.6. **Theorem.** *A metric family having* (*weak*) *finite decomposition complexity is exact. A bounded geometry metric space having finite decomposition complexity has Property A.*

Proof. Let $\mathfrak{E}$ be the collection of exact metric families. By Theorem 2.3.3 it suffices to show that $\mathfrak{E}$ contains the bounded families and is closed under decomposability.

Clearly, $\mathfrak{E}$ contains the bounded families – for X selected from a bounded family the partition of unity comprised of the constant function at 1, subordinate to the cover $\{ X \}$, fulfills the definition.

It remains to to check that $\mathfrak{E}$ is closed under decomposability. Let $\mathcal{X}$ be a family and assume $\mathcal{X}$ is decomposable over $\mathfrak{E}$ – for every r there exists $\mathcal{Y} \in \mathfrak{E}$ such that $\mathcal{X}$ is r-decomposable over $\mathcal{Y}$. We shall apply [DG, Theorem 4.4] to show that $\mathcal{X} \in \mathfrak{E}$. Let $\delta > 0$. Select r large enough so that $r\delta \geq 2$ and obtain $\mathcal{Y}$ as above. Translating the notion of decomposability into the language of [DG] we see that $\mathcal{Y}$ is an equi-exact family with the property that that every $X \in \mathcal{X}$ admits a r-separated cover, the pieces of which belong to $\mathcal{Y}$. Thus, the hypotheses of [DG, Theorem 4.4] are satisfied and we conclude that $\mathcal{X}$ is an equi-exact family. In other words, $\mathcal{X} \in \mathfrak{E}$. $\square$

4.7. **Remark.** [DG, Theorem 4.4] is stated for a single metric space. The same argument can be used to verify that it applies to a metric family.

5. Linear groups have FDC

We devote the present section to the proof of the following result.

5.0.1. **Theorem.** *If a countable group admits a faithful, finite dimensional representation* (*as matrices over a field of arbitrary characteristic*)*, then it has finite decomposition complexity. Precisely, let G be a finitely generated subgroup of $GL(n, K)$, where K is a field. If K has characteristic zero then $G \in \mathfrak{D}_{\omega+fin}$; if K has positive characteristic then G has finite asymptotic dimension.*

5.0.2. **Example.** The wreath product $\mathbb{Z} \wr \mathbb{Z}$ can be realized as a subgroup of $\mathrm{SL}(2, \mathbb{Z}[X, X^{-1}])$ but does not have finite asymptotic dimension (it contains an infinite rank abelian subgroup). Concretely, $\mathbb{Z} \wr \mathbb{Z}$ is isomorphic to the group comprised of all matrices of the form

$$\begin{pmatrix} X^n & p(X^2) \\ 0 & X^{-n} \end{pmatrix},$$

where $n \in \mathbb{Z}$ and p is a Laurent polynomial with $\mathbb{Z}$ coefficients in the variable X^2. On the other hand $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}$, which can be similarly realized as a subgroup of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}[X, X^{-1}])$, has finite asymptotic dimension by results of Bell and Dranishnikov [BD3] and Dranishnikov and Smith [DS]. Indeed, it is an extension with both quotient $\mathbb{Z}$ and kernel $\oplus \mathbb{Z}/p\mathbb{Z}$ having finite asymptotic dimension. These examples show that the conclusion in the theorem is optimal.

In light of our permanence results, the first assertion in Theorem 5.0.1 follows from the second. For the second, let K be a field and let G be a finitely generated subgroup of $\mathrm{GL}(n, K)$. The subring of K generated by the matrix entries of a finite generating set for G is a finitely generated domain A, and we have $G \subset \mathrm{GL}(n, A)$. Thus, we are lead to consider finitely generated domains, and their fraction fields.

5.1. **Preliminaries on fields.** The proof of Theorem 5.0.1 relies on a refinement of the notion of discrete embeddability introduced earlier by Guentner, Higson and Weinberger [GHW]. A *norm*[3] on a field K is a map $\gamma : K \to [0, \infty)$ satisfying, for all x, $y \in K$

(1) $\gamma(x) = 0 \iff x = 0$
(2) $\gamma(xy) = \gamma(x)\gamma(y)$
(3) $\gamma(x + y) \leq \gamma(x) + \gamma(y)$

[3] Guentner-Higson-Weinberger use the term valuation.

A norm obtained as the restriction of the usual absolute value on $\mathbb{C}$ via a field embedding $K \to \mathbb{C}$ is *archimedean*. A norm satisfying the stronger *ultra-metric inequality*

(4) $\gamma(x+y) \leq \max\{\,\gamma(x), \gamma(y)\,\}$

in place of the triangle inequality (3) is *non-archimedean*. If in addition the range of γ on $K^\times$ is a discrete subgroup of the multiplicative group $(0, \infty)$ the norm is *discrete*. If γ is a discrete norm on a field K the subset

$$\mathcal{O} = \{\, x \in K \colon \gamma(x) \leq 1\,\}$$

is a subring of K, the *ring of integers of* γ; the subset

$$\mathfrak{m} = \{\, x \in K \colon \gamma(x) < 1\,\}$$

is a principal ideal in $\mathcal{O}$; a generator for $\mathfrak{m}$ is a *uniformizer*.

5.1.1. **Definition.** A field K is strongly discretely embeddable (for short SDE) if for every finitely generated subring A of K there exists a finite set N_A of discrete norms on K, and countable set M_A of archimedean norms on K with the following property: for every real number k there exists a finite subset $F_A(k)$ of M_A such that for every $s > 0$ the set

$$\mathcal{B}_A(k,s) = \{\, a \in A \colon \forall \gamma \in N_A \;\; \gamma(a) \leq e^k \text{ and } \forall \gamma \in F_A(k) \;\; \gamma(a) \leq s\,\}$$

is finite.

5.1.2. **Remark** (SDE versus DE). In discrete embeddability [GHW, Definition 2.1] the family of norms depends only on the subring A. In Definition 5.1.1, the subset of discrete norms depends only on A, but is required to be finite; the subset $F_A(k)$ of archimedean norms is also is required to be finite, but depends on k. One readily verifies that a strongly discretely embeddable field in the sense of Definition 5.1.1 is discretely embeddable in the sense of [GHW].

5.1.3. **Remark.** A field of positive characteristic admits no archimedean norms. In particular, a field of nonzero characteristic is strongly discretely embeddable if and only if for every finitely generated subring A there exists a finite set N_A of (discrete) norms such that for every $k \in \mathbb{N}$ the set

$$\mathcal{B}_A(k) = \{\, a \in A \colon \forall \gamma \in N_A \; \gamma(a) \leq e^k\,\}$$

is finite.

5.1.4. **Example.** Let q be a positive power of the prime p and let $\mathbb{F}_q$ be the finite field with q elements. Let $K = \mathbb{F}_q(X)$ be the rational function field. We shall show that K satisfies the definition of SDE with respect to subring of polynomials $A = \mathbb{F}_q[X] \subset K = \mathbb{F}_q(X)$. Indeed, consider the norm

$$\gamma(P/Q) = e^{deg(P) - deg(Q)}, \tag{5.1}$$

where P and Q are nonzero polynomials. For all $k \in \mathbb{N}$, we have

$$B_A(k) = \{a \in A : \gamma(a) \leq e^k\} = \mathbb{F}_q[X]_k,$$

the set of polynomials of degree at most k. As this set is already finite, it suffices to take $N_A = \{\gamma\}$.

A similar analysis applies to $A = \mathbb{F}_q[X_1, X_2, \ldots, X_n] \subset K = \mathbb{F}_q(X_1, X_2, \ldots, X_n)$. Indeed, observe that $K = K_i(X_i)$, where $K_i = \mathbb{F}_q(X_1, \ldots, \hat{X}_i, \ldots, X_n)$. Thus, in analogy with (5.1), we can define a norm reflecting the degree in the variable X_i: $\gamma_i(P/Q) = e^{deg(P)-deg(Q)}$, where $P, Q \in K_i[X_i]$. The definition is satisfied with $N_A = \{\gamma_i : 1 \leq i \leq n\}$.

5.1.5. **Example.** The case of characteristic zero is more involved, since we will have to deal with archimedean norms. Let us treat the simplest non trivial case: $A = \mathbb{Z}[X] \subset K = \mathbb{Q}(X)$. The set N_A of discrete norms will again contain the single norm γ, defined as above in (5.1). As in the previous example,

$$B_A(k) = \{\, a \in A : \gamma(a) \leq e^k \} = \mathbb{Z}[X]_k,$$

the polynomials of degree at most k. Unfortunately, since the coefficients are integers, this set is infinite – we shall need to add some archimedean norms.

Evaluation of a rational function at a transcendental element $t \in \mathbb{C}$ defines an embedding $\mathbb{Q}(X) \subset \mathbb{C}$, and hence an archimedean norm on $\mathbb{Q}(X)$. Consider the set $M_A = \{\gamma^0, \gamma^1 \ldots\}$, where the γ^i are archimedean norms constructed in this way from distinct transcendental elements, $t_0, \ldots, t_k$. We are to show that for each s the set

$$\mathcal{B}_A(k, s) = \{\, P \in \mathbb{Z}[X]_k : |P(t_i)| \leq s \text{ for all } i = 0, \ldots, k \,\}$$

is finite. This is, however, straightforward: the assignment

$$P \mapsto (P(t_0), \ldots P(t_k))$$

defines an isomorphism of complex vector spaces $\mathbb{C}[X]_k \to \mathbb{C}^{k+1}$ (with the obvious notation) and $\mathbb{Z}[X]_k \subset \mathbb{C}[X]_k$ is discrete.

The multi-variable case $\mathbb{Z}[X_1, \ldots, X_n] \subset \mathbb{Q}(X_1, \ldots, X_n)$ can be treated as in the previous example, by replacing the single discrete norm γ by the discrete norms γ_i, for $i = 1, \ldots, n$.

5.1.6. **Remark.** With these two examples in hand, the reader can omit the remainder of this section and proceed directly to Section 5.2 to complete a proof that $\mathrm{GL}(d, \mathbb{F}_q[X_1, \ldots, X_n])$ has finite asymptotic dimension, and that $\mathrm{GL}(d, \mathbb{Z}[X_1, \ldots, X_n])$ has finite decomposition complexity.

5.1.7. **Proposition.** *A finitely generated field is strongly discretely embeddable.*

This proposition follows from an adaptation either of the proof of [GHW, Theorem 2.2], or of [AS, Proposition 1.2] (which relies on Noether's normalization theorem). Below, we follow [GHW]. The proof comprises three lemmas: in the first we show that finite fields and the field of rational numbers are SDE; in the second and third we show that SDE is stable under transcendental and finite extensions, respectively.

5.1.8. **Lemma** (Finite fields and the rationals)**.** *Finite fields and the field of rational numbers are strongly discretely embeddable.*

Proof. The assertion is obvious for finite fields. Turning to the rationals, let A be a finitely generated subring of $\mathbb{Q}$. Thus, there exists a positive integer n such that $A = \mathbb{Z}[1/n]$. Let $N_A = \{ \gamma_{p_1}, \ldots, \gamma_{p_m} \}$ where, for each prime divisor p_i of n, we denote by γ_{p_i} the (discrete) p-adic norm on $\mathbb{Q}$. Let M_A consist solely of the archimedean norm coming from the inclusion $\mathbb{Q} \subset \mathbb{C}$. We leave to the reader to verify that these choices satisfy Definition 5.1.1. □

5.1.9. **Lemma** (Transcendental extensions)**.** *Strong discrete embeddability is stable under the formation of transcendental extensions.*

Proof. We shall show that the field of rational functions over a (countable) SDE field is itself SDE. To this end, let K be an SDE field and let B be a finitely generated subring of $K(X)$. There exist monic prime polynomials $Q_1, \ldots, Q_m \in K[X]$ and a finitely generated subring A of K such that $B \subset A[X][Q_1^{-1}, \ldots, Q_m^{-1}]$. According to Definition 5.1.1, applied to the subring A of K, we obtain (finitely many) discrete norms N_A, and (countably many) archimedean norms M_A.

Let N_B be the following (finite) set of discrete norms on $K(X)$:

(1) the elements of N_A extended to $K(X)$;

(At one place below we shall require the fact that if γ is a discrete norm on K then its extension $\tilde{\gamma}$ to $K(X)$ satisfies $\tilde{\gamma}(P) = \max\{ \gamma(a) \}$, where the maximum is taken over the coefficients a of the polynomial $P \in K[X]$.)

(2) the norm $\gamma_\infty(P/Q) = e^{deg(P)-deg(Q)}$;
(3) the norms $\gamma_{Q_i}(PQ_i^l) = e^{-l}$ where $\gcd(Q_i, P) = 1$ and $l \in \mathbb{Z}$
(there are m norms of this type, one for each $i = 1, \ldots, m$).

Each of the archimedean norms $\gamma \in M_A$ arises from an embedding of fields $\phi_\gamma : K \to \mathbb{C}$. Let $t_0, t_1, \ldots$ be a countable family of distinct transcendentals in $\mathbb{C}$ that are *not* in the subfield of $\mathbb{C}$ generated by the images of these embeddings – to see that this is possible, observe that since both M_A and K are countable so is the subfield generated by the images. With these choices, each embedding ϕ_γ extends to an embedding $K(X) \to \mathbb{C}$ by sending X to t_i; we denote the corresponding norm on $K(X)$ by γ_i. Let

$$M_B = \{ \gamma_i \colon \gamma \in M_A \text{ and } i = 0, 1, \ldots \},$$

a countable set of archimedean norms on $K(X)$. We record for future use that in our notation $\gamma_i(P) = |\phi_\gamma(P)(t_i)|$, for every $P \in K[X]$; here, $\phi_\gamma(P) \in \mathbb{C}[X]$ is the polynomial obtained by applying ϕ_γ to the coefficients of P.

We shall show that N_B and M_B satisfy the condition in Definition 5.1.1. For this, let $k > 0$ be given. An element of $\mathcal{B}_B(k)$ necessarily has the form

$$\frac{P}{Q} = \frac{P}{Q_1^{n_1} \ldots Q_m^{n_m}}, \tag{5.2}$$

where $n_1, \ldots, n_m$ are $\leq k$, so that also $\deg P \leq k' = k\,(1 + \sum \deg Q_i)$ – here we are using the norms in N_B of types (2) ad (3) above. In particular, the set of possible denominators

Q is finite; denote it by $\mathcal{Q}_k$. Set

$$k'' = k + \log \max\{\, \gamma(Q) \colon Q \in \mathcal{Q}_k, \gamma \in N_B \,\}$$

(actually, taking the maximum over $\gamma \in N_B$ of type (1) would suffice). Summarizing, an element of $\mathcal{B}_B(k)$ has the form (5.2) in which Q belongs to the finite set $\mathcal{Q}_k$, the degree of P is at most k' and all coefficients of P belong to $\mathcal{B}_A(k'')$ – the last assertion follows from the formula for the extension of an element of N_A to an element of N_B of type (1).

Define a finite set of archimedean norms on $K(X)$ by

$$F_B(k) = \{\, \gamma_i \in M_B \colon \gamma \in F_A(k'') \text{ and } i = 0, \dots, k' \,\}$$

Let now $s > 0$; it remains to show that $\mathcal{B}_B(k,s)$ is finite. We claim that an element of $\mathcal{B}_B(k,s)$ satisfies, in addition to the conditions outlined above for membership in $\mathcal{B}_B(k)$, the following condition: there exists an s'' such that for every norm $\gamma \in F_A(k'')$ the value of γ on each coefficient of P is at most s''; in other words, form some s'' the coefficients of P belong to $\mathcal{B}_A(k'', s'')$. If indeed this is the case, the proof is complete – $\mathcal{B}_A(k'', s'')$ is a finite set, so only finitely many polynomials P can appear in (5.2) which, combined with our remarks above concludes the proof.

It remains to prove the existence of s''. Let

$$s' = s \cdot \max\{\, \gamma(Q) \colon Q \in \mathcal{Q}_k,\, \gamma \in F_B(k) \,\}$$

so that for an element of $\mathcal{B}_B(k,s)$ written in the form (5.2) we have $\gamma_i(P) \leq s'$ for every $\gamma \in F_A(k'')$ and $i = 0, \dots, k'$. Now, the linear transformation

$$P \longmapsto (P(t_0), \dots, P(t_{k'})), \qquad \mathbb{C}[X]_{k'} \to \bigoplus_0^{k'} \mathbb{C}$$

is invertible – identifying a polynomial $P \in \mathbb{C}[X]_{k'}$ with the column vector formed by its coefficients it is given by the Vandermonde matrix corresponding to the distinct transcendentals $t_0, \dots, t_{k'}$. The condition that $\gamma_i(P) \leq s'$ for every $i = 0, \dots, k'$ and $\gamma \in F_A(k'')$ means that the coefficients of the polynomial $\phi_\gamma(P)$ lie in the subset of the domain mapping into the compact subset of the range defined by the requirement that the absolute value of each entry is at most s'. This is a compact set so that there is an s'' such that the absolute value of the coeffecients of the polynomial $\phi_\gamma(P)$ are bounded by s''; in other words, the coefficients of P are in the set $\mathcal{B}_A(k'', s'')$ as required. $\square$

5.1.10. **Lemma** (Finite extensions)**.** *Strong discrete embeddability is stable under the formation of finite extensions.*

Proof. We shall show that a finite extension of an SDE field is SDE. To this end, let L be a finite extension of an SDE field K. As a subfield of an SDE field is itself SDE we may, enlarging L as necessary, assume that L is a finite normal extension of K.

Let B be a finitely generated subring of L. Fix a basis of the K-vector space L and let A be a finitely generated subring of K containing the matrix entries of each element of B, viewed as a K-linear transformation of L. This is possible – we may take for A any subring containing the matrix entries of a finite generating set for B.

According to Definition 5.1.1 applied to the subring A of K, we obtain (finitely many) discrete norms N_A and (countably many) archimedean norms M_A. Now, every discrete norm on K admits at least one extension to a discrete norm on L; a similar statement applies to archimedean norms. See [L, Chapter 12]. Moreover, the finite group $\mathrm{Aut}_K(L)$ of K-automorphisms of L acts on the set of extensions of each individual norm on K.

Let N_B be a (finite) set of discrete norms on L comprising exactly one $\mathrm{Aut}_K(L)$-orbit of extensions of each norm in N_A; let M_B be a (countable) set of archimedean norms on L defined similarly with respect to M_A. Finally, for each k let

$$F_B(k) = \{\, \gamma \in M_B \colon \gamma \text{ extends a norm in } F_A(k') \,\};$$

here $k' = \max\{\, |f(x_0, \ldots, x_n)| \,\}$, where n is the degree of the extension and the maximum is over all elementary symmetric functions f and all tuples of real numbers $x_0, \ldots, x_n$ each of which has absolute value at most k. Each $F_B(k)$ is a finite set of archimedean norms invariant under the action of $\mathrm{Aut}_K(L)$.

Let k and $s > 0$ be given. We must show that $\mathcal{B}_B(k, s)$ is finite. We shall do this by showing that the coefficients of the characteristic polynomial of each element of $\mathcal{B}_B(k, s)$, again viewed as a K-linear transformation of L, belong to the finite set $\mathcal{B}_A(k', s')$ where s' is defined in terms of s as k' was in terms of k. Thus, every element of $\mathcal{B}_B(k, s)$ is the root of one of finitely many polynomials and $\mathcal{B}_B(k, s)$ is itself finite.

Let now $b \in \mathcal{B}_B(k, s)$. Since the extension is normal, the minimal polynomial of b (in the sense of field theory) splits in L and the group $\mathrm{Aut}_K(L)$ acts transitively on its roots. The minimal polynomials of b in the sense of field theory and as a K-linear transformation of L agree. Hence the group $\mathrm{Aut}_K(L)$ acts transitively on the roots of the characteristic polynomial Ξ_b of b.[4] It follows that all the roots of the Ξ_b belong to $\mathcal{B}_B(k, s)$. Since the coefficients of Ξ_b are symmetric functions of degree $\leq n$ of the roots, every such coefficient belongs to $\mathcal{B}_A(k', s')$ by virtue of the definitions of k' and s'. □

5.2. **The general linear group.** Let γ be a norm on a field K. Following Guentner, Higson and Weinberger define a (pseudo)-length function ℓ_γ on $\mathrm{GL}(n, K)$ as follows: if γ is non-archimedean

$$\ell_\gamma(g) = \log \max_{ij} \{\, \gamma(g_{ij}), \gamma(g^{ij}) \,\}, \tag{5.3}$$

where g_{ij} and g^{ij} are the matrix coefficients of g and g^{-1}, respectively; if γ is archimedian, arising from an embedding $K \hookrightarrow \mathbb{C}$ then

$$\ell_\gamma(g) = \log \max\{\, \|g\|, \|g^{-1}\| \,\}, \tag{5.4}$$

where $\|g\|$ is the norm of g viewed as an element of $\mathrm{GL}(n, \mathbb{C})$, and similarly for g^{-1}. The following proposition is central to our discussion of linear groups.

[4] Recall that the characteristic polynomial and the minimal polynomial of a linear transformation have the same roots (in the algebraic closure of the ground field), possibly with different multiplicities.

5.2.1. **Proposition.** *Let γ be an archimedean or a discrete norm on a field K. The group $GL(n,K)$, equipped with the (left-invariant pseudo-)metric induced by ℓ_γ, is in $\mathfrak{D}_{fin}$.*

Proof of Proposition 5.2.1 (archimedean case). The result follows immediately from the corresponding result for $\mathrm{GL}(n,\mathbb{C})$; indeed, the metric on $\mathrm{GL}(n,K)$ is the subspace metric it inherits from an embedding into $\mathrm{GL}(n,\mathbb{C})$. For $\mathrm{GL}(n,\mathbb{C})$ the result follows from standard arguments, once we observe that the length function (5.4) is continuous, hence bounded on compact sets, and proper, meaning that bounded sets are compact. In brief, $\mathrm{GL}(n,\mathbb{C})$ is coarsely equivalent to the subgroup $T(n,\mathbb{C})$ of all upper triangular matrices and a fibering argument based on Theorem 3.1.4 show thats the solvable group $T(n,\mathbb{C})$ has finite decomposition complexity. □

The discrete case is more subtle than the archimedean case, primarily because we do not assume that K is locally compact. In this case the result was proven by Matsnev [Ma]. We shall present a simplified proof, based essentially on the same ideas.

Let γ be a discrete norm on a field K and fix a uniformizer π. For the proof we shall introduce some subgroups of $\mathrm{GL}(n,K)$. Let D denote the subgroup of diagonal matrices with powers of the uniformizer on the diagonal and let U denote the unipotent upper triangular matrices. Observe that D normalizes U so that $T = DU$ is also a subgroup (namely the group upper triangular matrices). Restrict the length function ℓ_γ to each subgroup and equip each with the associated (left-invariant pseudo-)metric (which is in fact the subspace pseudo-metric from G).

5.2.2. **Lemma.** *The group U has asymptotic dimension zero. In particular, $U \in \mathfrak{D}_1$.*

Proof. The *dilation by (a nonzero) $\theta \in K$* is the function $\Theta : U \to U$ defined by

$$\Theta(u)_{ij} = \theta^{j-i} u_{ij};$$

the entries on the k^{th}-superdiagonal of n are multiplied by θ^k. (For $k = 0, \dots, n-1$ the *k^{th}-superdiagonal* of an $n \times n$ matrix consists of the positions (i,j) for which $j - i = k$.) The formula for matrix multiplication shows that Θ is an endomorphism of U. Further, it is an automorphism with inverse the dilation by θ^{-1}.

Fix $\theta \in K$ of norm greater than one – the inverse of a uniformizer will do. Let U_0 be the subgroup of U comprised of elements of length zero, and define a sequence of subgroups of U by $U_k = \Theta(U_{k-1})$. We shall show that

$$B(1, k\log\gamma(\theta)) \subset U_k \subset B(1, k(n-1)\log\gamma(\theta)). \tag{5.5}$$

The lemma follows immediately. Indeed, U is the union of the cosets of U_k and the family of these cosets is both bounded and r-disjoint, provided $k\log\gamma(\theta) > r$.

In order to verify (5.5) observe that the length function on U is given by

$$\ell_\gamma(u) = \log \max_{i<j} \{\, 1, \gamma(u_{ij}), \gamma(u^{ij}) \,\}. \tag{5.6}$$

For the first inclusion in (5.5) suppose $\ell_\gamma(u) \le k\log\gamma(\theta)$ so that in particular $\gamma(u_{ij}) \le \gamma(\theta)^k$ for all $i < j$. The non-diagonal (i,j) entry of $\Theta^{-k}(u)$ is $u_{ij}\theta^{k(i-j)}$ so that each has norm at most one. Elementary properties of the norm and (5.6) show that this implies $\Theta^{-k}(u) \in U_0$, or $u \in U_k$.

The second inclusion in (5.5) follows by induction from

$$\ell_\gamma(\Theta(u)) \leq \ell_\gamma(u) + (n-1)\log\gamma(\theta).$$

To verify this inequality, note that the non-diagonal (i,j) entry of $\Theta(u)$ is $u_{ij}\theta^{j-i}$ which has norm bounded by $\gamma(u_{ij})\gamma(\theta)^{n-1}$. Since Θ is an automorphism a similar statement applies to the entries of $\Theta(u)^{-1} = \Theta(u^{-1})$. The inequality now follows from (5.6). □

5.2.3. **Lemma.** *The group T is in $\mathfrak{D}_{n+1}$.*

Proof. Observe that $D \cong \mathbb{Z}^n$, and that the restriction of ℓ_γ to D is a proper length function – indeed it corresponds (up to a multiplicative factor) with the supremum norm on $\mathbb{Z}^n$:

$$\ell_\gamma(a) = \max|k_i| \cdot \log\gamma(\pi^{-1}),$$

where a is the diagonal matrix with entries π^{k_i}. Hence D is in $\mathfrak{D}_n$. It remains to check, as an application of fibering, that T is indeed in $\mathfrak{D}_{n+1}$.[5]

We require two observations. First, the map $T \to D$ associating to each matrix in T the matrix of its diagonal entries is a contraction. Indeed, it is a homomorphism and from the definition of ℓ_γ we see that it decreases length. Second, if $B \subset D$ is a bounded subset and $b_1 \in B$ then the subset $b_1U \subset BU$ is $\mathrm{diam}(B)$-coarsely dense. Indeed, if $bu \in BU$ then $d(bu, bub^{-1}b_1) \leq \mathrm{diam}(B)$ and, since D normalizes U,

$$bub^{-1}b_1 = b_1(b_1^{-1}b)u(b^{-1}b_1) \in b_1U.$$

We conclude by applying the Fibering Theorem 3.1.4 or, more accurately, the subsequent Remark 3.1.5, to the map $T \to D$. □

Proof of Proposition 5.2.1 (discrete case). The inclusion of T in G is isometric. Further, it is metrically onto in the sense that every element of G is at distance zero from an element of T. Indeed, let H be the subgroup of those $g \in \mathrm{GL}(n, K)$ for which the entries of g and g^{-1} are in $\mathcal{O}$. Then $G = TH$ [GHW, Lemma 4.5] and elementary calculations show that every $h \in H$ has length zero. Hence, if $g = th$ then $d(t, g) = \ell(h) = 0$. □

5.3. **Finite decomposition complexity.** We have previously reduced Theorem 5.0.1 to the case of $G = \mathrm{GL}(n, A)$, where A is a finitely generated domain. Denoting the fraction field of A by K, our strategy is to embed $\mathrm{GL}(n, A)$ into the product of several copies of $\mathrm{GL}(n, K)$ equipped with metrics associated to various norms. The proof rests on a permanence property summarized in the following lemma.

5.3.1. **Lemma.** *Let G be a countable discrete group. Suppose there exists a (pseudo-)length function ℓ' on G with the following properties:*

(1) *G is in $\mathfrak{D}_{fin}$ with respect to the associated (pseudo-)metric d'*

[5] Since $D \subset T$ isometrically, if T is in $\mathfrak{D}_\alpha$ then necessarily $\alpha \geq n$. An argument more refined than the one we present here achieves this bound: indeed $T \in \mathfrak{D}_n$.

(2) $\forall\, r > 0\ \exists\, \ell_r$, *a (pseudo-)length function on* G, *for which*
  (i) G *is in* $\mathfrak{D}_{\mathit{fin}}$ *with respect to the associated (pseudo-)metric* d_r,
  (ii) ℓ_r *is proper when restricted to* $B_{\ell'}(r)$.

Then G *has finite decomposition complexity, and indeed* $G \in \mathfrak{D}_{\omega+\mathit{fin}}$.

Condition (ii) in the lemma means precisely that $B_{\ell_r}(s) \cap B_{\ell'}(r)$ is finite for every $s > 0$.

Proof. Fix a proper length function ℓ on G, with associated metric d. By Proposition 3.2.3, applied to the action of G on the metric space (G, d'), it suffices to show that for every $r > 0$ the ball $B_{\ell'}(r)$ is in $\mathfrak{D}_{\mathrm{fin}}$ when equipped with the metric d.

Let $r > 0$. Obtain ℓ_{2r} as in the statement. The ball $B_{\ell'}(r)$ is in $\mathfrak{D}_{\mathrm{fin}}$ with respect to the metric d_{2r}. Thus, it remains to show that the metrics d and d_{2r} on $B_{\ell'}(r)$ are coarsely equivalent.

Since ℓ-balls in G are finite, we easily see that for every s there exists s' such that if $d(g, h) \leq s$ then $d_{2r}(g, h) \leq s'$; this holds for every g and $h \in G$. Conversely, for every s the set $B_{\ell'}(2r) \cap B_{\ell_{2r}}(s)$ is finite by assumption, and we obtain s' such that for every g in this set $\ell(g) \leq s'$. If now g and $h \in B_{\ell'}(r)$ are such that $d_{2r}(g, h) \leq s$ then $g^{-1}h \in B_{\ell'}(2r)$ and

$$d(g, h) = \ell(g^{-1}h) \leq s'.$$

□

Proof of Theorem 5.0.1. Let A be a finitely generated domain, K the fraction field of A and $G = \mathrm{GL}(n, A)$. (We have previously reduced the theorem to this case.) Obtain a finite family $N_A = \{\, \gamma_1, \ldots, \gamma_q \,\}$ of discrete norms on K as in the definition of strong discrete embeddability. For each norm γ_i we have the corresponding length function ℓ_{γ_i} and metric on $\mathrm{GL}(n, K)$ defined as in (5.3). Define a length function on G by

$$\ell' = \ell_{\gamma_1} + \cdots + \ell_{\gamma_q}.$$

Thus, G is metrized so that the diagonal embedding

$$G \hookrightarrow \mathrm{GL}(n, K) \times \cdots \times \mathrm{GL}(n, K)$$

is an isometry when the i^{th} factor in the product is equipped with the metric associated to the norm γ_i and the product is given the sum metric. Equipped with this metric G is in $\mathfrak{D}_{\mathrm{fin}}$ by Proposition 5.2.1, and Remark 3.1.5. To apply the lemma, we shall study the balls $B_{\ell'}(r)$ of the identity in G.

Let $r = e^k$. Obtain a family of archimedean norms $F_A(k)$ as in the definition of strong discrete embeddability. For each we have the corresponding length function and metric on $\mathrm{GL}(n, K)$ defined as in (5.4). Define a length function on G by

$$\ell_r = \sum_{\gamma \in F_A(k)} \ell_\gamma.$$

Thus, G is metrized so that the diagonal embedding

$$G \hookrightarrow \mathrm{GL}(n, K) \times \cdots \times \mathrm{GL}(n, K)$$

is an isometry when each factor in the product is equipped with the metric associated to the corresponding norm γ, and the product is given the sum metric. Equipped with this metric G is in $\mathfrak{D}_{\mathrm{fin}}$ by Proposition 5.2.1, and Remark 3.1.5. To apply the lemma, we shall study the balls $B_{\ell'}(r)$ of the identity in G.

It remains only to show that for every $s > 0$ the set $B_{\ell_r}(s) \cap B_{\ell'}(r)$ is finite. Suppose g is in this set. From the definitions of the length functions it follows that the entries of g and g^{-1} satisfy inequalities

$$\gamma(g_{ij}) \leq r, \quad \gamma(g^{ij}) \leq r,$$

for $\gamma \in N_A$, and also the inequalities

$$\gamma(g_{ij}) \leq s, \quad \gamma(g^{ij}) \leq s,$$

for $\gamma \in F_A(k)$. But, these norms were chosen according to the definition of strong discrete embeddability, so that the subset of those elements of A satisfying these inequalities is finite. In particular, the number of matrices containing only these elements as their entries is finite and the proof of the general case is complete. Further, in the case of positive characteristic, there are no archimedean norms and the above inequalities show that $B_{\ell'}(r)$ is already finite for every r. In this case, we conclude that G belongs to $\mathfrak{D}_{\mathrm{fin}}$ so that by Theorem 4.1 it has finite asymptotic dimension. $\square$

5.3.2. **Remark.** Essentially, the proofs of Lemma 5.3.1 and Theorem 5.0.1 yield the following result: if the finitely generated domain A has characteristic zero there is an action of $\mathrm{GL}(n, A)$ on a metric space in $\mathfrak{D}_{\mathrm{fin}}$ such that each coarse stabilizer is in $\mathfrak{D}_{\mathrm{fin}}$.

6. Further examples

Additional examples of groups having finite decomposition complexity are readily exhibited based on our results. In this section, we prove that all countable elementary amenable groups, all countable subgroups of almost connected Lie groups, and all countable subgroups of $\mathrm{GL}(n, R)$ for any commutative ring R with unit have finite decomposition complexity.

The class of *elementary amenable groups* is the smallest class of countable discrete groups containing all finite groups and all (countable) abelian groups, and closed under the formation of subgroups, quotients, extensions and direct unions.

6.1. **Proposition** ([C])**.** *The class of elementary amenable groups is the smallest class of countable discrete groups containing all finite groups and all* (*countable*) *abelian groups and closed under the formation of extensions and direct unions.*

Sketch of proof. Define a class of groups $\mathfrak{A}$ by transfinite recursion as follows: $\mathfrak{A}_0$ is the class of all finite and countable abelian groups; for a successor ordinal α define $\mathfrak{A}_\alpha$ to be the class of all groups obtained as a (countable) direct union or extension of groups in $\mathfrak{A}_{\alpha-1}$; for a limit ordinal α define $\mathfrak{A}_\alpha = \cup_{\beta<\alpha}\mathfrak{A}_\beta$; finally, $\mathfrak{A}$ is the collection of groups belonging to some $\mathfrak{A}_\alpha$.

From its construction $\mathfrak{A}$ is closed under extensions and (countable) direct unions, and is clearly contained in the collection of elementary amenable groups. It remains to show that $\mathfrak{A}$

is closed under subgroups and quotients. Indeed, it is readily verified by transfinite induction that each $\mathfrak{A}_\alpha$ is closed under these operations. □

6.2. **Theorem.** *Elementary amenable groups have finite decomposition complexity.*

Proof. We have observed that the class of countable discrete groups having finite decomposition complexity is closed under the formation of extensions and direct unions. Finite groups have finite decomposition complexity, as do (countable) abelian groups. Indeed, a (countable) abelian group is the direct union of its finitely generated subgroups which, according to their general structure theory, have finite decomposition complexity . □

6.3. **Question.** Does every countable amenable group have FDC? In particular, does a Grigorchuk group of intermediate growth have FDC?

6.4. **Theorem.** *A countable subgroup of an almost connected Lie group has finite decomposition complexity.* □

Proof. A group as in the statement is realized as an extension with finite quotient and with kernel a subgroup of a *connected* Lie group. A subgroup of a connected Lie group is realized as an extension with linear quotient and abelian kernel. Thus, the result follows from the stability of FDC under extensions. Compare [GHW, Thm. 6.5]. □

6.5. **Theorem.** *Let R be a commutative ring with unit. A countable subgroup of $GL(n, R)$ has finite decomposition complexity.*

The essential piece of commutative algebra we require is summarized in the following lemma.

6.6. **Lemma.** *Let R be a finitely generated commutative ring with unit and let $\mathfrak{n}$ be the nilpotent radical of R,*

$$\mathfrak{n} = \{\, r \in R : \exists n \text{ such that } r^n = 0 \,\}.$$

The quotient ring $S = R/\mathfrak{n}$ contains a finite number of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that the diagonal map

$$S \to S/\mathfrak{p}_1 \oplus \cdots \oplus S/\mathfrak{p}_n$$

embeds S into a finite direct sum of domains.

Proof. This classical fact is a consequence of the Associated Prime Theorem which states that the set of associated primes of a finitely generated module over a Noetherian ring is finite [Ei, Thm. 3.1]. Here, the module is the ring itself which is Noetherian since it is finitely generated. The mentioned theorem then says that R has finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. The conclusion follows from the fact that their intersection is $\mathfrak{n}$. □

Proof of Theorem 6.5. In views of Proposition 3.2.1, it is enough to treat the case of $\mathrm{GL}_n(R)$, where R is finitely generated. With $\mathfrak{n}$ and S as in the previous lemma, we have an exact sequence

$$1 \to I + M_n(\mathfrak{n}) \to \mathrm{GL}(n, R) \to \mathrm{GL}(n, S) \to 1,$$

in which $I + M_n(\mathfrak{n})$ is nilpotent, and therefore has finite decomposition complexity by Corollary 3.2.5. In the notation of the previous lemma, we have

$$\mathrm{GL}(n, S) \to \mathrm{GL}(n, S/\mathfrak{p}_1) \times \cdots \times \mathrm{GL}(n, S/\mathfrak{p}_n).$$

So, the quotient has finite decomposition complexity by our earlier results. □

7. Decomposition Complexity and Topological Rigidity

This section is organized into two parts. In the first part we shall state two essential results, Theorems 7.1.2 and 7.1.3, the proofs of which are defered to later sections. In the second part we shall discuss applications to topological rigidity. We shall begin by describing the bounded category, a natural framework in which to discuss bounded rigidity. We shall then state and prove our results concerning the bounded Borel and bounded Farrell-Jones L-theory isomorphism conjectures for spaces with finite decomposition complexity, Theorems 7.3.1 and 7.4.1, respectively. Finally, from these we deduce concrete applications to topological rigidity.

7.1. **Two main results.** Throughout, we shall work with a metric space Γ having *bounded geometry*: for every $r > 0$ there exists $N = N(r)$ such that every ball of radius r contains at most N elements. In several places the weaker hypothesis of *local finiteness* would suffice: every ball contains finitely many elements.

7.1.1. **Definition.** For $d \geq 0$ we define the Rips complex $P_d(\Gamma)$ to be the simplicial polyhedron with vertex set Γ, and in which a finite subset $\{\gamma_0, \ldots, \gamma_n\} \subseteq \Gamma$ spans a simplex precisely when $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$.

If Γ has bounded geometry the Rips complex is finite dimensional, with dimension bounded by $N(d) - 1$; if Γ is merely locally finite the Rips complex $P_d(\Gamma)$ is a locally finite simplicial complex.

There are in general several ways to equip the Rips complex with a metric. The *simplicial metric* is the metric induced by the (pseudo) Riemannian metric whose restriction to each n-simplex is the Riemannian metric obtained by identifying the n-simplex with the standard simplex in the Euclidean space $\mathbb{R}^n$. By convention, the distance between points in different connected components of $P_d(\Gamma)$ is infinite. Equipped with the simplicial metric the Rips complex is a *geodesic space* in the sense that every two points (at finite distance) are joined by a geodesic path.

Our first essential result is a vanishing result for the Whitehead and algebraic K-theory groups. To state the result we introduce the following notation: for a locally compact metric space X and for each $\delta \geq 0$ and $i \geq 0$ the δ-controlled locally finite Whitehead group is denoted $Wh^{\delta}_{1-i}(X)$; the δ-controlled reduced locally finite algebraic K-theory group is denoted $\widetilde{K}^{\delta}_{-i}(X)$. Both groups are defined in [RY1].[6]

[6] The group we denote $\widetilde{K}^{\delta}_0(X)$ is the group $\widetilde{K}_0(X, p_X, 0, \delta)$ defined on page 14 of [RY1], taking p_X to be the identity map $X \to X$; our $\widetilde{K}^{\delta}_{-i}(X)$ is then defined to be $\widetilde{K}^{\delta}_0(X \times \mathbb{R}^i)$. The group we denote $Wh^{\delta}(X)$

We then define, for each $i \geq 0$, the bounded locally finite Whitehead group, and bounded reduced locally finite algebraic K-theory group as follows:

$$Wh^{bdd}_{1-i}(P_d(\Gamma)) = \lim_{\delta\to\infty} Wh^{\delta}_{1-i}(P_d(\Gamma))$$
$$\tilde{K}^{bdd}_{-i}(P_d(\Gamma)) = \lim_{\delta\to\infty} \tilde{K}^{\delta}_{-i}(P_d(\Gamma)).$$

7.1.2. **Theorem.** *Let Γ be a bounded geometry metric space. Let $\widetilde{K}^{bdd}_{-i}(P_d(\Gamma))$ denote the reduced bounded locally finite algebraic K-theory group and let $Wh^{bdd}_{1-i}(P_d(\Gamma))$ denote the bounded locally finite Whitehead group of the Rips complex $P_d(\Gamma)$. If Γ has finite decomposition complexity then*

$$\begin{aligned} \lim_{d\to\infty} \widetilde{K}^{bdd}_{-i}(P_d(\Gamma)) &= 0 \\ \lim_{d\to\infty} Wh^{bdd}_{1-i}(P_d(\Gamma)) &= 0 \end{aligned}$$

for each $i \geq 0$.

Our second essential result asserts that an appropriate assembly map is an isomorphism. To state the result we introduce the following notation: $\mathbb{L}(e)$ denotes the simply connected surgery spectrum with $\pi_n(\mathbb{L}(e)) = L_n(\mathbb{Z}\{e\})$; $L^{bdd}_n(X)$ denotes the bounded, locally finite and free L-theory of the locally compact metric space X. Recall that $L^{bdd}_n(X)$ is defined using locally finite, free geometric modules and that a geometric module is locally finite if its support is locally finite. More precisely, for a locally compact metric space X and for each $\delta \geq 0$ and $n \geq 0$ the δ-controlled locally finite and free L-group in degree n is denoted $L^{\delta}_n(X)$. This group is defined in [RY2].[7] We then define the bounded locally finite L-group as follows:

$$L^{bdd}_n(P_d(\Gamma)) = \lim_{\delta\to\infty} L^{\delta}_n(P_d(\Gamma)).$$

7.1.3. **Theorem.** *Let Γ be a metric space with bounded geometry and finite decomposition complexity. The assembly map*

$$A : \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \to \lim_{d\to\infty} L^{bdd}_n(P_d(\Gamma))$$

is an isomorphism.

In the statement, the domain of assembly is the locally finite homology of the Rips complex with spectrum $\mathbb{L}(e)$, and the range is the bounded, locally finite and free L-theory of the same Rips complex.

The proofs of our essential results, Theorems 7.1.2 and 7.1.3, are accomplished using appropriate controlled Mayer-Vietoris arguments and shall be presented in Sections 8 and 9, respectively. Our proofs require the full strength of the finite decomposition complexity

is the group $Wh(X, p_X, 1, \delta)$ defined on page 22 of [RY1], where again p_X is the identity map $X \to X$; $Wh^{\delta}_{1-i}(X)$ is then defined to be $Wh^{\delta}_{1-i}(X \times \mathbb{R}^i)$.

[7] The group we denote $L^{\delta}_n(X)$ corresponds to the δ-controlled locally finite and free L-theory group $L^{\delta,\delta}_n(X; p_X, \mathbb{Z})$ in [RY2], where again p_X is the identity map $X \to X$.

hypothesis; we are unable to prove the results under the hypothesis of weak finite decomposition complexity. The remainder of the present section is devoted to a description of how the results themselves are used to deduce the rigidity statements in the introduction.

7.2. **The bounded category.** Being invariant under coarse equivalence, finite decomposition complexity is well-adapted to a topological setting where the geometry appears only 'at large scale' and the topological properties are 'uniformly' locally trivial. These ideas are formalized in the *bounded category.*

A *coarse metric manifold* is a topological manifold M equipped with a continuous (pseudo-) metric in which balls are precompact. Although Riemannian manifolds, equipped with the path length metric, are motivating examples of coarse metric manifolds, we want to make clear that our definition entails *no* assumption on the metric at 'small scale' and that the manifold M is *not* assumed to be smooth. A continuous map $f : M \to N$, between two coarse metric manifolds is *bounded* if there exists a coarse equivalence $\phi : N \to M$ and a constant $K > 0$ such that

$$d(x, \phi \circ f(x)) \leq K$$

for all $x \in M$. Coarse metric manifolds and bounded continuous maps comprise the *bounded category.*[8]

Before discussing rigidity in the bounded category, we must introduce appropriate notions of homeomorphism and homotopy. A *bounded homeomorphism* between coarse metric manifolds is a map $M \to N$ which is simultaneously a homeomorphism and a coarse equivalence. These are the isomorphisms in the bounded category.

Two bounded continuous maps $f,\ g : M \to N$ are *boundedly homotopic* if there exists a bounded homotopy between them; in other words, if there exists a continuous map $F : M \times [0,1] \to N$, for which $F(0,\cdot) = f$, $F(1,\cdot) = g$ and for which the family $(F(t,\cdot))_{t\in[0,1]}$ is bounded (uniformly in t, in the obvious sense). A bounded continuous map $f : M \to N$ is a *bounded homotopy equivalence* if there exists a bounded continuous map $g : N \to M$ such that the compositions $f \circ g$ and $g \circ f$ are boundedly homotopic to the identity.

7.2.1. **Definition.** A coarse metric manifold M is *boundedly rigid* if the following condition holds: every bounded homotopy equivalence $M \to N$ to another coarse metric manifold is boundedly homotopic to a (bounded) homeomorphism.

A coarse metric manifold M is *uniformly contractible* if for every $r > 0$, there exists $R \geq r$ such that every ball in M with radius r is contractible to a point within the larger ball of radius R and the same center. Uniform contractibility is invariant under bounded homotopy equivalence.

[8] In [CFY], the authors give an essentially equivalent definition of the bounded category in which an auxiliary metric space X is introduced. An object is a pair (M, p) where $p : M \to X$ has precompact preimages. To obtain a coarse metric manifold, one must merely pull back the metric from X to M.

A coarse metric manifold has *bounded geometry* if there exists $r > 0$ with the following property: for every $R > 0$ there exists $N > 0$ such that every ball of radius R is covered by N or fewer balls of radius r.[9]

Perhaps the most important, and motivating, example of a coarse metric manifold is the universal cover $\tilde{M}$ of a closed (topological) manifold M. To realize the structure of a coarse metric manifold on $\tilde{M}$ we need only equip it with a continuous Γ-invariant pseudo-metric in which balls are precompact, where Γ is the fundamental group of M. Let c be a continuous and compactly supported cut-off function on $\tilde{M}$ in the sense that c is a non-negative function on $\tilde{M}$ satisfying

$$\sum_{g\in\Gamma} c(gx) = 1$$

for all $x \in \tilde{M}$. We define a pseudo-metric d on $\tilde{M}$ as follows:

$$d(x, y) = \sum_{g,h\in\Gamma} c(g^{-1}x)c(h^{-1}y)d_\Gamma(g, h)$$

for all x and y in $\tilde{M}$, where d_Γ is a word metric on Γ. Equipped with this pseudo-metric, $\tilde{M}$ is coarsely equivalent to Γ. As a consequence, the coarse metric manifold structure on $\tilde{M}$ is independent of the choices made in the construction.

7.3. **Application to bounded rigidity.** The *bounded Borel isomorphism conjecture* asserts that an appropriate assembly map is an isomorphism. Precisely this conjecture asserts that for a locally finite metric space Γ the assembly map

$$A : \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \to \lim_{d\to\infty} L_n^{bdd,s}(P_d(\Gamma)) \tag{7.1}$$

is an isomorphism: as in the previous section, the domain of assembly is the locally finite homology of the Rips complex of Γ with spectrum $\mathbb{L}(e)$, the simply connected surgery spectrum with $\pi_n(\mathbb{L}(e)) = L_n^s(\mathbb{Z}\{e\}) = L_n(\mathbb{Z}\{e\})$; the range of assembly is the bounded simple L-theory of the Rips complex of Γ defined using locally finite free geometric modules.

7.3.1. **Theorem.** *The bounded Borel isomorphism conjecture is true for metric spaces with bounded geometry and finite decomposition complexity.*

Proof. By Theorem 7.1.2 and the Ranicki-Rothenberg sequence in the controlled setting [FP], we have

$$\lim_{d\to\infty} L^{bdd,s}(P_d(\Gamma)) \cong \lim_{d\to\infty} L^{bdd}(P_d(\Gamma)).$$

The result now follows from Theorem 7.1.3. □

[9] Traditionally, a Riemannian manifold is said to have bounded geometry if its curvature is bounded from below and its radius of injectivity is bounded away from zero. Such local conditions are known to imply our global condition.

As is the case for the classical Borel isomorphism conjecture, the bounded Borel isomorphism conjecture has strong topological implications. These implications, which we now describe, are to questions of topological rigidity in the bounded category of coarse metric manifolds. Our principal result in this direction is the following theorem.

7.3.2. **Theorem** (Bounded Rigidity Theorem)**.** *A uniformly contractible coarse metric manifold with bounded geometry, finite decomposition complexity, and dimension at least five is boundedly rigid.*

While we shall present the proof of this theorem at the end of this subsection let us, for the moment, apply it in the case of the universal cover of a closed aspherical manifold to deduce the following result stated in the introduction.

7.3.3. **Corollary.** *Let M be a closed aspherical manifold of dimension at least five whose fundamental group has finite decomposition complexity* (*as a metric space with a word metric*)*. For every closed manifold N and homotopy equivalence $M \to N$ the corresponding bounded homotopy equivalence of universal covers is boundedly homotopic to a homeomorphism.*

Proof. The universal cover of a closed manifold has bounded geometry as a coarse metric manifold. Further, the universal cover of a closed aspherical manifold is uniformly contractible as a coarse metric manifold. Thus, the previous theorem applies. □

Let M be a coarse metric manifold. A *net* in M is a metric subspace $\Gamma \subset M$ which is both *uniformly discrete* – the distance between distinct points of Γ is bounded uniformly away from zero – and *coarsely dense* in M – for some $C > 0$, every ball $B(x, C)$ in M intersects Γ. Clearly, the inclusion of a net into M is a coarse equivalence, so that any two nets are coarsely equivalent. If M has bounded geometry (as a coarse metric manifold) then any net in M has bounded geometry (as a discrete metric space).

7.3.4. **Proposition.** *Let M be a uniformly contractible coarse metric manifold having bounded geometry and dimension at least five. Let Γ be a net in M. The assembly map* (*7.1*) *of the bounded Borel isomorphism conjecture for Γ identifies with the assembly map for M:*

$$A : H_n(M, \mathbb{L}(e)) \to L_n^{bdd,s}(M). \tag{7.2}$$

Precisely, there are isomorphisms

$$H_n(M, \mathbb{L}(e)) \cong \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \quad \textit{and} \quad L_n^{bdd,s}(M) \cong \lim_{d\to\infty} L^{bdd,s}(P_d(\Gamma))$$

commuting with the assembly maps.

7.3.5. **Remark.** The bounded geometry condition is essential here; Dranishnikov, Ferry and Weinberger have constructed an example of a uniformly contractible manifold M for which the first asserted isomorphism fails [DFW].

Results analogous to the proposition are typically proved under the stronger assumption that M is a Riemannian manifold with bounded geometry, or at least that M is a simplicial complex equipped with a proper path metric. See, for example, [HR, Section 3]. Our proof will follow the standard arguments, based on the following lemma. For the statement define a *coarse metric CW-space* to be a CW-complex equipped with a continuous (pseudo-)metric in which balls are relatively compact, and in which the cells have uniformly bounded diameter. The latter property can always be achieved by refining the CW-structure.

7.3.6. **Lemma.** *Let X be a uniformly contractible coarse metric finite dimensional CW-space. Suppose that X admits a bounded geometry net Γ. For every sufficiently large $d > 0$ there exist continuous coarse equivalences*

$$f_d : X \to P_d(\Gamma) \quad \text{and} \quad g_d : P_d(\Gamma) \to X$$

with the following properties:

(1) *$g_d \circ f_d$ is boundedly homotopic to the identity map of X;*
(2) *$i_{dd'} \circ f_d \circ g_d$ is boundedly homotopic to the inclusion $i_{dd'} : P_d(\Gamma) \to P_{d'}(\Gamma)$, for $d' > d$ sufficiently large.*

Proof. The proof is inspired by [BR, Section 4] and [HR, Section 3]. We define f_d. Since Γ is a net, there exists a $C > 0$ such that the balls of radius C centered at the points of Γ cover X. Take a continuous partition of unity $(\phi_\gamma)_{\gamma\in\Gamma}$ in X subordinate to this cover. Define

$$f_d : X \to P_d(\Gamma), \quad f_d(x) = \sum_{\gamma\in\Gamma} \phi_\gamma(x)\gamma,$$

for all $x \in X$. Whenever $\phi_\gamma(x)$ and $\phi_{\gamma'}(x)$ are simultaneously nonzero the distance between γ and γ' is at most $2C$. Thus, f_d is properly defined provided $d > 2C$. Observe that f_d is continuous and uniformly expansive.

We define g_d recursively. On the 0-skeleton $\Gamma \subset P_d(\Gamma)$ we define g_d in the obvious way. Assuming we have defined g_d on the k-skeleton, we extend it to the $(k+1)$-skeleton using the uniform contractibility of X. Observe that g_d is continuous and uniformly expansive. Indeed, this follows from the finite dimensionality of $P_d(\Gamma)$.

Further, f_d and g_d are inverse coarse equivalences – the compositions $f_d \circ g_d$ and $g_d \circ f_d$ are close to the identity maps on X and $P_d(\Gamma)$, respectively. This is most easily verified by noting that the compositions are uniformly expansive, and close to the identity on the copies of Γ in $P_d(\Gamma)$ and X, respectively.

Then it is easy to see that for d' large enough, there exits a linear homotopy from $f_d \circ g_d$ to the identity inside $P_{d'}(\Gamma)$.

The case of $g_d \circ f_d$ is done by [BR, Lemma 4.4]. We reproduce the proof here. Consider the map $\Phi : \{0,1\} \times X \to X$ defined by $\phi(0,x) = g_d \circ f_d(x)$ and $\Phi(1,x) = x$. Note that $W = [0,1] \times X$, with the product metric, is a coarse metric CW-space, and $\{0,1\} \times X$ is a sub-complex of W. Let W_n be the union of $\{0,1\} \times X$ with the n-skeleton of W. Using the uniform contractibility of W, one can extend inductively Φ to a bounded continuous map Φ_n defined on W_n. Now, since W has finite dimension, $W = W_n$ for some n and we have constructed a bounded homotopy equivalence from $f_d \circ g_d$ to the identity. $\square$

Proof of Proposition 7.3.4. A topological manifold of dimension at least five admits the structure of a CW-complex [KS].[10] Thus a coarse metric manifold of dimension at least five is a coarse metric CW-space and the lemma applies. □

Proof of Theorem 7.3.2. Let M be as in the statement. Let N be another coarse metric manifold and suppose that N is boundedly homotopy equivalent to M. According to the bounded surgery exact sequence [FP], the bounded Borel isomorphism conjecture for M implies that N is homeomorphic to M, assuming that $\dim M \geq 5$. □

7.4. **Application to stable rigidity.** The *bounded Farrell-Jones L-theory isomorphism conjecture* asserts that a certain assembly map is an isomorphism. Precisely this conjecture asserts that for a locally finite metric space Γ the assembly map

$$A : \lim_{d\to\infty} H_n(P_d(\Gamma), \mathbb{L}(e)) \to \lim_{d\to\infty} L_n^{bdd,<-\infty>}(P_d(\Gamma))$$

is an isomorphism. Here, for a metric space X and natural number n, we define $L_n^{bdd,<-\infty>}(X)$ to be the direct limit of the bounded locally finite and free L-groups $L_n^{bdd}(X \times \mathbb{R}^k)$ with the maps given by crossing with $\mathbb{R}$. Recall that $\mathbb{L}(e)$, the (simply) connected surgery spectrum, satisfies $\pi_n(\mathbb{L}(e)) = L_n^{<-\infty>}(\mathbb{Z}\{e\})$.

7.4.1. **Theorem.** *The bounded Farrell-Jones L-theory isomorphism conjecture is true for metric spaces spaces with bounded geometry and finite decomposition complexity.*

Proof. Immediate from Theorem 7.1.3 and from the observation that if X has finite decomposition complexity, then so does $X \times \mathbb{R}^n$ for all n. □

The bounded Farrell-Jones L-theory isomorphism conjecture has implications to question of stable rigidity. Let M be a closed, aspherical manifold. By the arguments presented in the previous section, the bounded Farrell-Jones L-theory isomorphism conjectures for the universal cover of M and for the fundamental group of M are equivalent. According to the *descent principle* they imply the *integral Novikov conjecture* – a detailed argument is contained in the proof of [CP, Theorem 5.5]. For a nice exposition of the descent principle in the context of C^*-algebra K-theory see [R1].

Recall now from the introduction that a closed manifold M is *stably rigid* if there exists a natural number n with the following property: for every closed manifold N and every homotopoy equivalence $M \to N$ the map $M \times \mathbb{R}^n \to N \times \mathbb{R}^n$ is homotopic to a homeomorphism. The *stable Borel conjecture* asserts that closed aspherical manifolds are stably rigid. The fact that the integral Novikov conjecture implies the stable Borel conjecture was stated without proof in [FP]; for a detailed treatment see [J, Proposition 2.8]. From this discussion, and our previous results, we conclude:

[10] This is the only point at which we require the dimension to be ≥ 5 – the question of whether a manifold admits the structure of a CW-complex remains open in low dimensions. One could give an alternative proof of Proposition 7.3.4 using a Mayer-Vietoris argument, which would allow us to remove the dimension restriction.

7.4.2. **Theorem.** *The stable Borel conjecture holds for closed aspherical manifolds whose fundamental groups have finite decomposition complexity.* □

No restriction on the dimension is required – low dimension can compensated by increasing n. Moreover, if $\dim(M) \geq 5$, then one can take $n = 3$.

7.4.3. **Remark.** In analogy with the bounded Farrell-Jones L-theory isomorphism conjecture, we could state a bounded version of the Farrell-Jones isomorphism conjecture for algebraic K-theory. The Farrell-Jones conjecture for bounded locally finite algebraic K-theory implies vanishing of the bounded Whitehead group and the bounded reduced algebraic K-group described previously.

8. Vanishing theorem

We devote this section to the proof of Theorem 7.1.2, our vanishing result for the bounded Whitehead and bounded reduced lower algebraic K-theory groups. In view of the definitions, we obtain Theorem 7.1.2 as an immediate consequence of the following result:

8.1. **Theorem.** *Let Γ be a locally finite metric space with bounded geometry and finite decomposition complexity. The controlled locally finite Whitehead group and the controlled reduced locally finite algebraic K-theory group vanish asymptotically. Precisely, given $i \geq 0$, $\delta > 1$ and $a > 1$ there exists $b > 1$ such that, for any $Z \subset \Gamma$ the natural homomorphisms:*

$$Wh^{\delta}_{1-i}(P_a(Z)) \to Wh^{\delta}_{1-i}(P_b(Z)) \tag{8.1}$$

$$\widetilde{K}^{\delta}_{-i}(P_a(Z)) \to \widetilde{K}^{\delta}_{-i}(P_b(Z)) \tag{8.2}$$

are zero. Here $P_a(\Gamma)$ is equipped with the simplicial metric and $P_a(Z) \subset P_a(\Gamma)$ with the subspace metric (and similarly for $P_b(Z)$). The constant b depends only on i, δ, a and Γ, and not on Z.

8.2. **Remark.** To emphasize the dependence among the various constants and metric families we shall encounter we shall write, for example, $f = f(g, h)$ when f depends on g and h; if additionally $g = g(p, q)$ and $h = h(q, r)$ we write $f = f(g, h) = f(p, q, r)$.

In preparation for the proof of Theorem 8.1 we formalize the notion of a vanishing family: a collection $\mathcal{F}$ of metric subspaces of Γ is a *vanishing family* if for every $i \geq 0$, $\delta > 1$, $a > 1$, $t > 1$ and $p \geq 0$ there exists $b > 1$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the homomorphisms

$$Wh^{\delta}_{1-i}(P_a(Z) \times T^p) \to Wh^{\delta}_{1-i}(P_b(Z) \times T^p) \tag{8.3}$$

$$\widetilde{K}^{\delta}_{-i}(P_a(Z) \times T^p) \to \widetilde{K}^{\delta}_{-i}(P_b(Z) \times T^p) \tag{8.4}$$

are zero, where $N_t(X)$ is the t-neighborhood of X in Γ, i.e. $N_t(X) = \{y \in \Gamma : d(y, X) \leq t\}$. Here, T^p is the p-dimensional torus with the standard Riemannian metric of diameter one. Note that $b = b(i, p, t, a, \delta, \mathcal{F})$. We denote the collection of vanishing families by $\mathfrak{V}$.

Observe that in the definition of vanishing family we have not specified the metric to be used on $P_a(Z)$ and $P_b(Z)$. Indeed, this was intentional as we shall need to employ *two* different metrics in the proof of Theorem 8.1. The first is the *simplicial metric* on $P_a(Z)$ and the second is the *subspace metric* inherited from $P_a(\Gamma)$. Similarly we consider the simplicial and subspace metrics on $P_b(\Gamma)$.

8.3. **Proposition.** *The notion of vanishing family is independent of the choice of metric on $P_a(Z)$ and $P_b(Z)$.*

Proof. The subspace metric is always smaller than the simplicial metric. Consequently there is a hierarchy among the four (*a priori* different) definitions of vanishing family. The weakest version of vanishing states:

> For every a (8.3) and (8.4) are zero for sufficiently large b, when $P_a(Z)$ is equiped with the simplicial metric and $P_b(Z)$ with the subspace metric;

whereas the strongest version states:

> For every a (8.3) and (8.4) are zero for sufficiently large b, when $P_a(Z)$ is equiped with the subspace metric and $P_b(Z)$ with the simplicial metric.

It suffices to show that the weak version of vanishing implies the strong version. We shall focus on the Whitehead groups (the case of the K-groups being similar). Suppose that Z is a vanishing family in the weak sense. We shall show that, for sufficiently large a' depending on a and δ, there exist maps

$$Wh^{\delta}_{1-i}(P^{\mathrm{sub}}_a(Z)\times T^p)) \to Wh^{\delta}_{1-i}(P^{\mathrm{sim}}_{a'}(Z)\times T^p)); \tag{8.5}$$

here, and below, the superscript makes clear which metric is to be employed, either the subspace or the simplicial. Assuming this for the moment, the proof of the proposition is completed by considering the diagram

$$\begin{array}{ccc} Wh^{\delta}_{1-i}(P^{\mathrm{sub}}_a(Z)\times T^p) & \longrightarrow & Wh^{\delta}_{1-i}(P^{\mathrm{sim}}_b(Z)\times T^p) \\ \downarrow & & \uparrow \\ Wh^{\delta}_{1-i}(P^{\mathrm{sim}}_{a'}(Z)\times T^p) & \longrightarrow & Wh^{\delta}_{1-i}(P^{\mathrm{sub}}_{b'}(Z)\times T^p); \end{array}$$

given a we choose a' to ensure existence of the left hand vertical map as in (8.5); according to the weak version of vanishing we choose b' so that the bottom horizontal map is zero; finally, we choose b to ensure existence of the right hand vertical map as in (8.5).

It remains to verify the existence of the maps (8.5). This follows from the following two observations. First, for a' sufficiently large, the inclusion

$$P^{\mathrm{sub}}_a(Z) \to P^{\mathrm{sim}}_{a'}(Z)$$

is 1-*Lipschitz at scale* 100δ – meaning that whenever x, $y \in P^{\mathrm{sub}}_a(Z)$ satisfy $d(x,y) \le 100\delta$ then the distance between x and y in $P^{\mathrm{sim}}_{a'}(Z)$ is not greater than their distance in $P^{\mathrm{sub}}_a(Z)$. Indeed, choose $a' \ge a$ to be large enough such that any pair of points of $P^{\mathrm{sub}}_a(Z)$ at distance less than 100δ lie in a common simplex in $P_{a'}(Z)$ – this is possible because the map $P_a(\Gamma) \to \Gamma$ associating to a point some vertex of the smallest simplex containing it is uniformly expansive. Now, the first map in the composition

$$P_a^{\mathrm{sub}}(Z) \to P_{a'}^{\mathrm{sub}}(Z) \to P_{a'}^{\mathrm{sim}}(Z)$$

is contractive. The second map is isometric *for pairs of points in a simplex* – the subspace and simplicial metrics on $P_a(Z)$ coincide for pairs of points belonging to a common simplex, essentially because each simplex is a convex subspace of $P_a(\Gamma)$.

Second, the δ-controlled Whitehead groups are independent of the behavior of the metric at scales much larger than δ. More precisely, an injection $X \to Y$ which is 1-*Lipschitz at scale* 100δ induces a map $Wh_{1-i}^{\delta}(X) \to Wh_{1-i}^{\delta}(Y)$. This follows from the definitions of these groups [RY1]. □

Finally, before turning to the proof of Theorem 8.1 we pause to outline the strategy. We wish to show that a subspace of Γ (rather, a family of subspaces) is a vanishing space from the knowledge that it may be decomposed as a union of arbitrarily well-separated vanishing families. Denote the constituent families $\mathcal{C}$ and $\mathcal{D}$ – here and below we freely employ the notations of Appendix A for Rips complexes. The following diagram motivates our proof – unfortunately, it does not exist in the controlled setting and must be loosely interpreted:

$$\begin{tikzcd}
Wh(P_a(\mathcal{C})) \oplus Wh(P_a(\mathcal{D})) \arrow[r] \arrow[d] & Wh(P_a(\mathcal{C}) \cup P_a(\mathcal{D})) \arrow[r] \arrow[d] & \widetilde{K}_0(P_a(\mathcal{C}) \cap P_a(\mathcal{D})) \arrow[d, "i"] \\
Wh(P_b(\mathcal{C})) \oplus Wh(P_b(\mathcal{D})) \arrow[r] \arrow[d, "j"] & Wh(P_b(\mathcal{C}) \cup P_b(\mathcal{D})) \arrow[r] \arrow[d] & \widetilde{K}_0(P_b(\mathcal{C}) \cap P_b(\mathcal{D})) \arrow[d] \\
Wh(P_c(\mathcal{C})) \oplus Wh(P_c(\mathcal{D})) \arrow[r] & Wh(P_c(\mathcal{C}) \cup P_c(\mathcal{D})) \arrow[r] & \widetilde{K}_0(P_c(\mathcal{C}) \cap P_c(\mathcal{D})).
\end{tikzcd}$$

In this diagram $a \le b \le c$, the horizontal rows are pieces of Mayer-Vietoris sequences, and the induction hypothesis applies to the first and third columns. Thus, given a, choose b large enough so that $i = 0$; then choose c large enough so that $j = 0$; a simple diagram chase reveals that the composite of the two maps in the middle column is zero.

This heuristic does not reveal the need for the introduction of the relative Rips complex, which we have remarked is an important aspect of the proof. Roughly stated, in the diagram above upon passing from a to b we have no way to ensure that the separation of the individual spaces comprising the families $\mathcal{C}$ and $\mathcal{D}$ does not evaporate – the purpose of the relative Rips complex is to selectively rescale parts of the ambient space while maintaining the separation between them. In the proof below we shall point out where this is needed.

Proof of Theorem 8.1. Assuming that Γ has finite decomposition complexity we shall prove that the collection of vanishing families contains the bounded families and, using a controlled Mayer-Vietoris argument based on part (5) of Theorem B.1 (proved in [RY1]), is closed under decomposability. We thereby conclude that the family $\{\Gamma\}$ is a vanishing family and the theorem follows.

A uniformly bounded family of subspaces of Γ is a vanishing family, as we conclude from the following facts:

(1) If a subspace $Y \subset \Gamma$ has diameter at most b for some $b \geq 0$, then $P_b(Z)$ is Lipschitz homotopy equivalent to a point (with Lipschitz constant one); indeed the same is true for any larger b.
(2) If two metric spaces P and Q are Lipschitz homotopy equivalent (with Lipschitz constant one) then $Wh^{\delta}_{1-i}(P)$ is isomorphic to $Wh^{\delta}_{1-i}(Q)$, and similarly $\widetilde{K}^{\delta}_{-i}(P)$ is isomorphic to $\widetilde{K}^{\delta}_{-i}(Q)$.
(3) By the choice of the Riemannian metric on T^p and the assumption $\delta > 1$, $Wh^{\delta}(T^p)$ and $\tilde{K}^{\delta}_{-i}(T^p)$ vanish for each $p \geq 0$.

Now, let $\mathcal{F}$ be a family of subspaces of Γ and assume that $\mathcal{F}$ is decomposable over the collection of vanishing families. We must show that $\mathcal{F}$ is a vanishing family; precisely, there exists $b = b(i, p, t, a, \delta, \mathcal{F})$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the maps (8.3) and (8.4) are zero.

Set $r = r(t, a, \delta, \lambda)$ sufficiently large, to be specified later. Obtain an r-decomposition of $\mathcal{F}$ over a vanishing family $\mathcal{G} = \mathcal{G}(r, \mathcal{F})$. Let $X \in \mathcal{F}$. We obtain a decomposition:

$$X = A \cup B, \qquad A = \bigsqcup_r A_i, \quad B = \bigsqcup_r B_j,$$

for which all A_i and $B_j \in \mathcal{G}$. Let $Z \subset N_t(X)$; setting $C_i = Z \cap N_{t+a}(A_i)$ and $D_j = Z \cap N_{t+a}(B_j)$ we obtain an analogous decomposition:

$$Z = C \cup D, \qquad C = \bigsqcup_{r-2(t+a)} C_i, \quad D = \bigsqcup_{r-2(t+a)} D_j.$$

Denote $\mathcal{C} = \{ C_i \}$ and $\mathcal{D} = \{ D_j \}$. By the separation hypothesis we have $r - 2(t + a) > a$ so that $P_a(\mathcal{C}) = P_a(C)$ and $P_a(\mathcal{D}) = P_a(D)$. Further, $P_a(Z) = P_a(C) \cup P_a(D) = P_a(\mathcal{C} \cup \mathcal{D})$. We intend to compare the Mayer-Vietoris sequence of this pair of subspaces of $P_a(\Gamma)$ to a Mayer-Vietoris sequence for certain subspaces of an appropriate relative Rips complex. We enlarge the intersection $\mathcal{C} \cap \mathcal{D} = \{ C_i \cap D_j \}$ by setting

$$\begin{aligned} W &= N_{a\beta\lambda\delta}(C) \cap N_{a\beta\lambda\delta}(D) \cap Z \\ &= (N_{a\beta\lambda\delta}(C) \cap D) \cup (C \cap N_{a\beta\lambda\delta}(D)) \\ &= \bigsqcup_{r-2(t+a\beta\lambda\delta)} W_{ij}, \end{aligned}$$

where all the neighborhoods are in Γ and

$$W_{ij} = N_{a\beta\lambda\delta}(C_i) \cap N_{a\beta\lambda\delta}(D_j) \cap Z,$$

and where β is the constant appearing in Lemma A.3.4. Observe that $C_i \cap D_j \subset W_{ij}$, so that denoting $\mathcal{W} = \{ W_{ij} \}$ we have $\mathcal{C} \cap \mathcal{D} \subset \mathcal{W}$. Provided $a \leq b$ we have a commuting diagram

$$\begin{array}{ccc} Wh^{\delta}(P_a(\mathcal{C} \cup \mathcal{D})) & \longrightarrow & \widetilde{K}_0^{\lambda\delta}(N_{\lambda\delta}(P_a(\mathcal{C} \cap \mathcal{D}))) \\ \downarrow & & \downarrow \\ Wh^{\delta}(P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W})) & \longrightarrow & \widetilde{K}_0^{\lambda\delta}(N_{\lambda\delta}(P_b(\mathcal{W}))). \end{array} \tag{8.6}$$

The horizontal maps are boundary maps in controlled Mayer-Vietoris sequences in Appendix B: in the top row the neighborhood is taken in $P_a(\mathcal{C} \cup \mathcal{D})$, and all spaces are given the subspace metric from $P_a(\Gamma)$; in the bottom row the neighborhood is taken in $P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W})$, and all spaces are given the subspace metric from $P_{ab}(\Gamma, W)$. The vertical maps are induced from the proper contraction $P_a(\Gamma) \to P_{ab}(\Gamma, W)$. In fact, the right hand vertical map factors as the composite

$$N_{\lambda\delta}(P_a(\mathcal{C} \cap \mathcal{D})) \subset P_a(\mathcal{W}) \to P_b(\mathcal{W}) \subset N_{\lambda\delta}(P_b(\mathcal{W})); \tag{8.7}$$

in which the first two spaces are subspaces of $P_a(\mathcal{C} \cup \mathcal{D}) \subset P_a(\Gamma)$ and the last two are subspaces of $P_{ab}(\mathcal{C} \cup \mathcal{D}, \mathcal{W}) \subset P_{ab}(\Gamma, W)$. The first inclusion in (8.7) follows from

$$\begin{aligned} N_{\lambda\delta}(P_a(\mathcal{C} \cap \mathcal{D})) &= \bigcup_{i,j} N_{\lambda\delta}(P_a(C_i \cap D_j)) \\ &\subset \bigcup_{i,j} P_a(N_{a\beta\lambda\delta}(C_i) \cap N_{a\beta\lambda\delta}(D_j)) \\ &\subset \bigcup_{i,j} P_a(W_{ij}) = P_a(\mathcal{W}), \end{aligned}$$

where we have applied Lemma A.3.4 of the appendix for the first inclusion – keep in mind that the neighborhoods on the first line are taken in $P_a(\mathcal{C} \cup \mathcal{D})$.

Applying the induction hypothesis we claim that for sufficiently large b the right hand vertical map in (8.6) is zero. Indeed, the components $W_{ij} \in \mathcal{W}$ are contained in the neighborhoods $N_{t+a\beta\lambda\delta}(A_i)$ (and also of $N_{t+a\beta\lambda\delta}(B_j)$) and we can apply the hypothesis with appropriate choices of the parameters: $t' = t + a\beta\lambda\delta$, $\delta' = \lambda\delta$, $a' = a$, etc. In detail,

$$\widetilde{K}_0^{\lambda\delta}(P_a(\mathcal{W})) \xrightarrow{\cong} \prod \widetilde{K}_0^{\lambda\delta}(P_a(W_{ij})) \xrightarrow{0} \prod \widetilde{K}_0^{\lambda\delta}(P_b(W_{ij})) \longrightarrow \widetilde{K}_0^{\lambda\delta}(P_b(\mathcal{W}));$$

as the spaces $P_a(W_{ij})$ and $P_a(\mathcal{W})$ are given the subspace metric from $P_a(\Gamma)$ and the individual W_{ij} are well-separated, the first map is an isomorphism by Lemma A.3.5 (which guarantees that the various $P_a(W_{ij})$ are separated by at least $\lambda\delta$); the spaces $P_b(W_{ij})$ are given the simplicial metric and the middle map is 0 for sufficiently large b by hypothesis; the space $P_b(\mathcal{W})$ is given the subspace metric from $P_{ab}(\Gamma, \mathcal{W})$ and the last map is induced by proper contractions $P_b(W_{ij}) \subset P_b(\mathcal{W})$ onto disjoint subspaces.

Having chosen $b = b(i, p, t', a', \delta', \mathcal{G})$ we extend the diagram (8.6) to incorporate the relax-control map for the bottom sequence:
(8.8)

$$\begin{tikzcd}
 & Wh^{\delta}(P_a(\mathcal{C}\cup\mathcal{D})) \arrow[d] & \\
 & Wh^{\delta}(P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r] \arrow[d, "relax"] & \widetilde{K}_0^{\lambda\delta}(N_{\lambda\delta}(P_b(\mathcal{W}))) \\
\left.\begin{matrix} Wh^{\lambda^2\delta}(P_{ab}(\mathcal{C},\mathcal{W})\cup N_{\lambda\delta}(P_b(\mathcal{W}))) \\ \oplus \\ Wh^{\lambda^2\delta}(P_{ab}(\mathcal{D},\mathcal{W})\cup N_{\lambda\delta}(P_b(\mathcal{W}))) \end{matrix}\right\} \arrow[r] & Wh^{\lambda^2\delta}(P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) &
\end{tikzcd}$$

We conclude from the above discussion and the controlled Mayer-Vietoris sequence that the image of $Wh^{\delta}(P_a(\mathcal{C}\cup\mathcal{D}))$ under the composite of the two vertical maps is contained in the image of the bottom horizontal map. It remains to apply the induction hypothesis to $\mathcal{C}$ and $\mathcal{D}$. The case of $\mathcal{D}$ being analogous, we concentrate on $\mathcal{C}$ and shall show that for sufficiently large $c \geq b$ the composite

$$P_{ab}(\mathcal{C},\mathcal{W})\cup N_{\lambda\delta}(P_b(\mathcal{W})) \subset P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W}) \to P_b(Z) \to P_c(Z),$$

in which the arrows are induced by proper contractions $P_{ab}(\Gamma, W) \to P_b(\Gamma) \to P_c(\Gamma)$ is zero on the $\lambda^2\delta$-controlled Whitehead group. We have, as subspaces of $P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W}) \subset P_{ab}(\Gamma, W)$,

$$P_{ab}(\mathcal{C},\mathcal{W})\cup N_{\lambda\delta}(P_b(\mathcal{W})) = \bigcup_i \left(P_a(C_i) \cup \bigcup_j N_{\lambda\delta}(P_b(W_{ij}))\right), \tag{8.9}$$

in which the spaces comprising the union over i are well-separated by Lemma A.3.5 (which guarantees $\lambda^2\delta$-separation). Further, for fixed i and j we have

$$N_{\lambda\delta}(P_b(W_{ij})) \subset P_{ab}(N_{a\beta\lambda\delta}(W_{ij}), W_{ij})) \to P_b(N_{a\beta\lambda\delta}(W_{ij})) \subset P_b(N_{2a\beta\lambda\delta}(C_i)),$$

where we have applied Lemma A.3.4 for the first containment (we point out that this is one of the places where the notion of relative Rips complex is important), and the arrow represents the assertion that the space on its left maps to the space on its right under the proper contraction $P_{ab}(\Gamma, W) \to P_b(\Gamma)$. Accordingly, for each fixed i we have

$$P_a(C_i) \cup \bigcup_j P_b(N_{a\beta\lambda\delta}(W_{ij})) \to P_b(N_{2a\beta\lambda\delta}(C_i)),$$

where the arrow is interpreted as above. Now, we apply our induction hypothesis a second time, with appropriate choices of the parameters: $t'' = t + 2a\beta\lambda\delta$, $\delta'' = \lambda^2\delta$, $a'' = b$, etc, noting that $N_{2a\beta\lambda\delta}(C_i) \subset N_{t+2a\beta\lambda\delta}(A_i)$. We get $c = c(i, p, t'', a'', \delta'', \mathcal{G})$, and analyze

$$\begin{aligned}
Wh^{\lambda^2\delta}(P_{ab}(\mathcal{C}, \mathcal{W}) \cup N_{\lambda\delta}(P_b(\mathcal{W})) &\cong \prod Wh^{\lambda^2\delta}\left(P_a(C_i) \cup \bigcup_j P_b(N_{a\beta\lambda\delta}(W_{ij}))\right) \\
&\to \prod Wh^{\lambda^2\delta}(P_b(N_{2a\beta\lambda\delta}(C_i))) \\
&\to \prod Wh^{\lambda^2\delta}(P_c(N_{2a\beta\lambda\delta}(C_i))) \\
&\to Wh^{\lambda^2\delta}(P_c(Z)) \\
&\to Wh^{\delta}(P_{\lambda^2 c}(Z))
\end{aligned}$$

the $\cong$ follows from the well-separatedness in (8.9); the spaces $P_c(N_{2a\lambda\delta}(C_i))$ are given the simplicial metrics, and the second arrow is 0; the fourth arrow is induced from proper contractions onto disjoint subspaces of $P_c(Z)$. The last arrow follows from the definition of the controlled Whitehead groups. Checking the dependence of the constant c we find $c = c(i, p, t, a, \lambda, \delta, \mathcal{F})$ as required. □

9. Assembly isomorphism

We devote this section to the proof of Theorem 7.1.3, which asserts that assembly is an isomorphism for spaces having finite decomposition complexity. In view of the definitions, we obtain Theorem 7.1.3 as an immediate consequence of the following result:

9.1. **Theorem.** *Let Γ be a locally finite metric space with bounded geometry and finite decomposition complexity. Assembly for Γ is an asymptotic isomorphism. Precisely, given $n \geq 0$, $\delta > 1$ and $a > 1$ there exists $b = b(a, \delta, n) \geq a$ such that, for any $Z \subset \Gamma$,*

(1) *the kernel of $H_n(P_a(Z)) \to L_n^\delta(P_a(Z))$ is mapped to zero in $H_n(P_b(Z))$;*
(2) *the image of $L_n^\delta(P_a(Z)) \to L_n^\delta(P_b(Z))$ is contained in the image of $H_n(P_b(Z)) \to L_n^\delta(P_b(Z)$.*

We shall refer to condition (2) in the statement as *asymptotic surjectivity* and to condition (1) as *asymptotic injectivity*.

Before turning to the proof we pause to outline the strategy. The proof consists essentially of a quantitative version of the five lemma, which we shall prove using the controlled Mayer-Vietoris sequence in L-theory, precisely parts (4) and (5) of Theorem B.2. Borrowing the notation from the previous section, consider the following diagram, which again does not make sense in the controlled setting and must be loosely interpreted:

$$\begin{tikzcd}
H_n(P_a(\mathcal{C})) \oplus H_n(P_a(\mathcal{D})) \arrow[r] \arrow[d] & L_n(P_a(\mathcal{C})) \oplus L_n(P_a(\mathcal{D})) \arrow[d] \\
H_n(P_a(\mathcal{C} \cup \mathcal{D})) \arrow[r] \arrow[d] & L_n(P_a(\mathcal{C} \cup \mathcal{D})) \arrow[d] \\
H_{n-1}(P_a(\mathcal{C} \cap \mathcal{D})) \arrow[r] \arrow[d] & L_{n-1}(P_a(\mathcal{C} \cap \mathcal{D})) \arrow[d] \\
H_{n-1}(P_a(\mathcal{C})) \oplus H_{n-1}(P_a(\mathcal{D})) \arrow[r] & L_{n-1}(P_a(\mathcal{C})) \oplus L_{n-1}(P_a(\mathcal{D})).
\end{tikzcd} \tag{9.1}$$

In the diagram, the vertical exact sequences are portions of appropriate Mayer-Vietoris sequences; the horizontal maps are the assembly maps. The induction hypothesis applies to the first, third and fourth rows; we are to prove that the second horizontal map is an (asymptotic) isomorphism. In the proof below, we shall concentrate on (asymptotic) surjectivity – a simple diagram chase reveals that this follows (asymptotic) surjectivity of rows one and three and (asymptotic) injectivity of row four.

In the proof below, to help the reader follow our trajectory we shall adopt the following conventions: x, y and z will be used for elements in the bounded L-theory for unions, intersections and direct sums, respectively; x', y', z' will be used for elements in the corresponding homology groups.

As preparation for the proof we introduce the notion of an L-isomorphism family: a collection $\mathcal{F}$ of metric subspaces of Γ is an *L-isomorphism family* if for every $n \geq 0$, $\delta > 1$, $a > 1$, and $t > 1$ there exists $b = b(a, \delta, t, n) > 1$ such that for every $X \in \mathcal{F}$ and every $Z \subset N_t(X)$ the assertions (1) and (2) of the theorem are satisfied. As was the case for vanishing families the notion of an L-isomorphism family is not sensitive to the choice of metric on $P_a(Z)$ and $P_b(Z)$. Compare Proposition 8.3 – the proof in the present situation is based on the same argument.

Finally, the proof employs both the relative Rips complex, $P_{ab}(\mathcal{C}, \mathcal{W})$ and the scaled Rips complex, $P_{abm}(\mathcal{C}, \mathcal{W})$ – see Definition A.1.1 and Definition A.1.2, respectively, and also Section A.2.

Proof. The proof will be much more condensed than the proof of Theorem 8.1 which we presented in some detail; while the present proof is not technically more difficult, it is somewhat longer.

We proceed as in the proof of Theorem 8.1. Assuming Γ has finite decomposition complexity we shall show that the collection of families that are both vanishing families and L-isomorphism families contains the bounded families, and is closed under decomposability. We thereby conclude that the family $\{ \Gamma \}$ is an isomorphism family, and the theorem follows.

The case of bounded families is handled by the following facts:

(1) If a subspace $Y \subset \Gamma$ has diameter at most b for some $b \geq 0$, then $P_b(Z)$ is Lipschitz homotopy equivalent to a point (with Lipschitz constant one); indeed the same is true for any larger b.
(2) If two metric spaces P and Q are Lipschitz homotopy equivalent (with Lipschitz constant one) then $L_n^\delta(P)$ is isomorphic to $L_n^\delta(Q)$.

Now, let $\mathcal{F}$ be a family of subspaces of Γ, and assume $\mathcal{F}$ is decomposable over the collection of families that are both vanishing and L-isomorphism families. It follows from the proof of Theorem 8.1 that $\mathcal{F}$ itself is a vanishing family and we are to prove that $\mathcal{F}$ is an L-isomorphism family. We shall concentrate on proving asymptotic surjectivity; asymptotic injectivity can be proved in essentially the same manner.

Set $r = r(t, a, \delta, \lambda)$ sufficiently large, to be specified later – precisely, when a union below is called *well-separated*, this will mean for a sufficiently good choice of r, and the reader will verify that this choice depends only on the parameters t, a, δ and λ. Obtain an r-decomposition of $\mathcal{F}$ over an L-isomorphism (and vanishing) family $\mathcal{G} = \mathcal{G}(r, \mathcal{F})$.

Let $X \in \mathcal{F}$. Let Z, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{W}$ be as in the proof of Theorem 8.1. Let $x \in L_n^\delta(P_a(\mathcal{C}) \cup P_a(\mathcal{D}))$. We need to prove that x is in the image of the assembly map up to increasing a.

Step 1. Using the well-separatedness of $\mathcal{W}$, and the vanishing assumption for the family $\mathcal{W}$, we can find $b = b(a, \delta, t, n)$ such that the map

$$\widetilde{K}_0^{\lambda_n\delta}(P_a(\mathcal{W})) \to \widetilde{K}_0^{\lambda_n\delta}(P_b(\mathcal{W})) \tag{9.2}$$

is zero. This allows us to consider the boundary map

$$\partial : \ L_n^\delta(P_a(\mathcal{C} \cup \mathcal{D})) \to L_{n-1}^{\lambda_n\delta}(P_b(\mathcal{W})),$$

where ∂ is the boundary map in Theorem B.2 of Appendix B and $P_b(\mathcal{W})$ is seen as a subspace of $P_{ab}(Z, \mathcal{W})$.

Step 2. Lemma A.3.5 implies that $P_b(\mathcal{W})$ is well separated, as a subspace of $P_{ab}(Z, \mathcal{W})$. Hence

$$L_{n-1}^{\lambda_n\delta}(P_b(\mathcal{W})) \cong \prod_{i,j} L_{n-1}^{\lambda_n\delta}(P_b(W_{ij})).$$

Hence, by the surjectivity assumption for $\mathcal{W}$, there exists $c = c(a, \delta, n, t) \geq b$ and $y' \in H_{n-1}(P_c(\mathcal{W}))$ mapping to (the image of) x in $L_{n-1}^{\lambda_n\delta}(P_c(\mathcal{W}))$, which we will simply write $A(y') = \partial(x)$.

Step 3. By Theorem B.2 in Appendix B, part (5), and (9.2) (using that $c \geq b$), we have $i_* \circ \partial = 0$ in

$$L_n^\delta(P_a(\mathcal{C} \cup \mathcal{D})) \xrightarrow{\partial} L_{n-1}^{\lambda_n\delta}(P_c(\mathcal{W})) \xrightarrow{i_*} L_{n-1}^{\lambda_n\delta}(P_{ac}(\mathcal{C}, \mathcal{W})) \oplus L_{n-1}^{\lambda_n\delta}(P_{ac}(\mathcal{D}, \mathcal{W})).$$

In particular, $i_* \circ \partial(x) = i_* \circ A(y') = 0$. Considering the following commutative diagram

$$\begin{tikzcd} H_{n-1}(P_c(\mathcal{W})) \arrow[r, "i_*"] \arrow[d, "A"] & H_{n-1}(P_{ac}(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_{ac}(\mathcal{D},\mathcal{W})) \arrow[d, "A"] \\ L_{n-1}^{\lambda_n\delta}(P_c(\mathcal{W})) \arrow[r, "i_*"] & L_{n-1}^{\lambda_n\delta}(P_{ac}(\mathcal{C},\mathcal{W})) \oplus L_{n-1}^{\lambda_n\delta}(P_{ac}(\mathcal{D},\mathcal{W})), \end{tikzcd}$$

we deduce $A \circ i_*(y') = 0$.

Step 4. By the injectivity assumption for $\mathcal{W}$, there exists $d = d(a,\delta,n,t) \geq c$ such that the map

$$H_{n-1}(P_{ac}(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_{ac}(\mathcal{D},\mathcal{W})) \to H_{n-1}(P_d(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_d(\mathcal{D},\mathcal{W}))$$

sends $i_*(y')$ to 0.

Step 5. By exactness of the sequence

$$H_n(P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \xrightarrow{\partial} H_{n-1}(P_d(\mathcal{W})) \xrightarrow{i_*} H_{n-1}(P_d(\mathcal{C},\mathcal{W})) \oplus H_{n-1}(P_d(\mathcal{D},\mathcal{W})),$$

there exists $x' \in H_n(P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W}))$ such that $y' = \partial(x')$.

Step 6. If m is large enough, the metric subfamily $P_d(\mathcal{W})$ of $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})$ is well-separated by Lemma A.3.5. Hence,

$$\widetilde{K}_0^{\lambda_n^2\delta}(N_{\lambda_n^2\delta}(P_d(\mathcal{W}))) \cong \prod_{i,j} \widetilde{K}_0^{\lambda_n^2\delta}(N_{\lambda_n^2\delta}(P_d(W'_{ij}))).$$

On the other hand, by Lemma A.3.6, when m is large enough, $N_{\lambda_n^2\delta}(P_d(\mathcal{W}))$ is 2-Lipschitz homotopy equivalent to a subset of $P_d(\mathcal{W}')$ (just take the homotopy equivalence F of Lemma A.3.6, restricted to V, which in our case is $N_{\lambda_n^2\delta}(P_d(\mathcal{W}))$) where $\mathcal{W}' = N_{a\beta\lambda_n^2\delta}(\mathcal{W})$ (β is as in Lemma A.3.6) and $P_d(\mathcal{W}')$ is viewed as subspace of $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W}')$. Hence there exists $e = e(a,\delta,n,t)$ such that[11]

$$\widetilde{K}_0^{\lambda_n\delta}(N_{\lambda_n^2\delta}(P_d(\mathcal{W}))) \longrightarrow \widetilde{K}_0^{2\lambda_n\delta}(P_d(\mathcal{W}')) \xrightarrow{0} \widetilde{K}_0^{\lambda_n\delta}(P_e(\mathcal{W}')). \tag{9.3}$$

We can thus define the boundary map

$$L_n^{\lambda_n\delta}(P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \xrightarrow{\partial} L_{n-1}^{\lambda_n^2\delta}(P_e(\mathcal{W}')).$$

Step 7. Remember that $P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W})$ and $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})$ are the same topological space equipped with two different metrics. Considering the following commutative diagram,

$$\begin{tikzcd} H_n(P_d(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r, "\partial"] \arrow[d, "A"] & H_{n-1}(P_d(\mathcal{W})) \arrow[d, "A"] \\ L_n^{\lambda_n\delta}(P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r, "\partial"] & L_{n-1}^{\lambda_n^2\delta}(P_e(\mathcal{W}')), \end{tikzcd}$$

[11]as up to increasing e, one can change $2\lambda_n\delta$ to $\lambda_n\delta$ in the right-hand term.

we obtain $\partial \circ A(x') = A \circ \partial(x') = A(y') = \partial(x)$. In other words, $\partial(x - A(x'))) = 0$ in $L^{\lambda_n^2\delta}_{n-1}(P_e(\mathcal{W}'))$. Up to replacing x by $x - A(x')$, we can therefore suppose that $\partial(x) = 0$.

Step 8. Applying part (4) of Theorem B.2 with

$$\begin{tikzcd}
 & L_n^{\lambda_n\delta}(P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})) \arrow[r, "\partial"] \arrow[d] & L_{n-1}^{\lambda_n^2\delta}(\mathcal{V}) \\
L_n^{\lambda_n^3\delta}(P_{aem}(\mathcal{C},\mathcal{W}')\cup\mathcal{V}) \oplus L_n^{\lambda_n^3\delta}(P_{aem}(\mathcal{D},\mathcal{W}')\cup\mathcal{V}) \arrow[r, "j_*"] \arrow[d] & L_n^{\lambda_n^3\delta}(P_{aem}(\mathcal{C}\cup\mathcal{D},\mathcal{W}')) \arrow[d] & \\
L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{C},\mathcal{W}')) \oplus L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{D},\mathcal{W}')) \arrow[r, "j_*"] & L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{C}\cup\mathcal{D},\mathcal{W}')), &
\end{tikzcd}$$

where $\mathcal{V}$ is the $\beta\lambda_n^2\delta$-neighborhood of $P_d(\mathcal{W}')$ in $P_{adm}(\mathcal{C}\cup\mathcal{D},\mathcal{W})$. The lower part of the diagram follows from the Lipschitz-homotopy lemma (see Lemma A.3.6). Together with (9.3), we deduce the existence of z such that $x = j_*(z)$, where j_* is the map defined above.

Step 9. We have $P_{aem}(\mathcal{C},\mathcal{W}')) = \bigcup_i P_{aem}(C_i, \cup_j W'_{ij}))$, where the union over i is well-separated provided m was chosen large enough. Moreover, since $\mathcal{W}' \subset N_{2a\beta\lambda_n^2\delta}(\mathcal{C}\cap\mathcal{D})$, we have the following contractive inclusion

$$P_{aem}(C_i, \cup_j W'_{ij})) \subset P_e(N_{2a\beta\lambda_n^2}(C_i)).$$

We therefore get a map

$$L_n^{2\lambda_n^3\delta}(P_{aem}(\mathcal{C},\mathcal{W}')) \to \prod_i L_n^{2\lambda_n^3\delta}(P_e(N_{2a\beta\lambda_n^2}(C_i))).$$

The similar statement is true for $\mathcal{D}$.

Step 10. By the surjectivity assumption applied to the families $\mathcal{C}$ and $\mathcal{D}$, there exists $f = f(a,\delta,n,t)$ such that the range of

$$L_n^{2\lambda_n^3\delta}(P_e(N_{2a\beta\lambda_n^3}(C_i))) \to L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^3}(C_i)))$$

is contained in the range of

$$H_n(P_f(N_{2a\beta\lambda_n^3}(C_i))) \to L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^3}(C_i))),$$

and similarly for D_i, for all i. Hence there exists z' in

$$\prod_i \big(H_n(P_f(N_{2a\beta\lambda_n^2}(C_i))) \oplus H_n(P_f(N_{2a\beta\lambda_n^2}(D_i)))\big) \cong H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{D})))$$

such that $A(z') = z$ where z is identified with its image through the map

$$\prod_i \Big(L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(C_i))) \oplus L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(D_i))\Big) \to L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^3}(\mathcal{D}))).$$

Step 11. Finally we use the commutative diagram

$$\begin{tikzcd} H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus H_n(P_f(N_{2a\beta\lambda_n^2}(\mathcal{D}))) \arrow[r, "j_*"] \arrow[d, "A"] & H_n(P_f(Z)) \arrow[d, "A"] \\ L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(\mathcal{C}))) \oplus L_n^{2\lambda_n^3\delta}(P_f(N_{2a\beta\lambda_n^2}(\mathcal{D}))) \arrow[r, "j_*"] & L_n^{2\lambda_n^3\delta}(P_f(Z)), \end{tikzcd}$$

to get $x = j_*(z) = j_*(A(z')) = A(j_*(z'))$, viewed in $L_n^{2\lambda_n^3\delta}(P_f(Z))$. The first two equalities following from steps 8 and 10. We have therefore proved that x is in the range of

$$H_n(P_f(Z)) \to L_n^{2\lambda_n^3\delta}(P_f(Z)),$$

which is enough to conclude, as up to increasing f, we can replace $2\lambda_n^3\delta$ by δ in the right-hand term.

$\square$

10. Concluding remarks

10.1. **The Novikov conjecture for linear groups.** The methods presented here could be adapted to prove the coarse Baum-Connes conjecture for spaces with finite decomposition complexity, and therefore the Novikov conjecture for finitely decomposable groups. Of course, since finite decomposition complexity implies Property A the coarse Baum-Connes conjecture is already verified for spaces with finite decomposition complexity, as a consequence of the main result in [Y2]. The alternate proof that we suggest here would, however, not rely on infinite dimensional methods. Rather, it would be based on the more elementary approach of [Y1], where the coarse Baum-Connes conjecture is proved for groups with finite asymptotic dimension. We believe that such an alternative proof would be of some interest.

10.2. **Weak finite decomposition complexity.** A main motivation for this project was to find a weak form of finite asymptotic dimension satisfied by linear groups. As described in Remark 2.0.4, we first defined weak finite decomposition complexity. We abandoned this property when we realized we were unable to prove the bounded Borel and bounded Farrell-Jones isomorphism conjectures.

10.2.1. **Question.** Are the bounded Borel and bounded Farrell-Jones isomorphism conjectures true for bounded geometry metric spaces having weak finite decomposition complexity?

We speculate that to answer this question one would probably need to replace the controlled Mayer-Vietoris sequences used here by controlled spectral sequences in K- and L-theory.

Appendix A. Variations on the Rips complex

In this appendix, we introduce the *relative Rips complex* and the *scaled* (*relative*) *Rips complex* and prove several useful results about their geometry. These complexes, and the assorted technical results presented here, play a crucial role in the proofs of Theorems 7.1.2 and 7.1.3. The appendix is designed to be read independently and, in spite of their technical nature, we believe that the results presented may be useful in other contexts.

The appendix is organized as follows. In the first subsection, we shall introduce the *relative Rips complex* and the *scaled Rips complex*. In the second, we extend the definitions to the setting of metric families, relevant for the proofs Theorems 7.1.2 and 7.1.3. The final subsection contains a collection of lemmas, also necessary for the proofs of Theorems 7.1.2 and 7.1.3. While we shall state and prove the lemmas in the context of metric spaces they generalize immediately to the context of metric families.

Throughout, Γ is a locally finite metric space with the property that $d(x, y) \geq 1$ for each pair of distinct points x and $y \in \Gamma$. The Rips complex was defined previously (see Definition 7.1.1 and the surrounding discussion).

A.1. The relative Rips complex and the scaled Rips complex.

In this subsection, we shall introduce the *relative Rips complex* and the *scaled Rips complex*. These play important roles in the proofs of Theorems 7.1.2 and 7.1.3, respectively.

A.1.1. **Definition.** Let Σ be a subset of Γ. For $1 \leq a \leq b$ we define the relative Rips complex $P_{ab}(\Gamma, \Sigma)$ to be the simplicial polyhedron with vertex set Γ and in which a finite subset $\{ \gamma_0, \ldots, \gamma_n \}$ spans a simplex if one of the following conditions hold:

(1) $d(\gamma_i, \gamma_j) \leq a$ for all i and j;
(2) $d(\gamma_i, \gamma_j) \leq b$ for all i, j, and $\gamma_i \in \Sigma$ for all i.

The relative Rips complex is equipped with the simplicial metric.

If C is a subspace of Γ, then $P_d(C)$ is, in a natural way, a *subset* of $P_d(\Gamma)$. When $P_d(C)$ and $P_d(\Gamma)$ are equipped with the simplicial metric, the inclusion $P_d(C) \subset P_d(\Gamma)$ is contractive. Observe that $P_d(C)$ carries, in addition to the simplicial metric, a subspace metric inherited from $P_d(\Gamma)$. If $C \subset \Gamma$ and $W \subset \Sigma$ we have inclusions of sets

$$P_a(C) \subset P_{ab}(\Gamma, \Sigma), \quad P_b(W) \subset P_{ab}(\Gamma, \Sigma).$$

If $P_b(W)$ is equipped with the intrinsic metric the second inclusion is contractive; the analogous statement is generally false if $P_b(W)$ is equipped with the subspace metric inherited from $P_b(\Gamma)$. Similar remarks apply for $P_a(C)$.

A.1.2. **Definition.** Let W be a subset of the metric space Γ. For $1 \leq a \leq b$ and a sequence of positive integers $\overline{m} = m_1, \ldots, m_n, \ldots$, we define the metric space $P_{ab\overline{m}}(\Gamma; W)$ to be the polyhedron $P_b(\Gamma)$ with the metric defined as follows:

(1) each simplex K spanned by a finite subset $\{\gamma_0, \gamma_1, \cdots, \gamma_n\}$ of Γ is given by the (pseudo) Riemannian metric defined inductively on n:

(i) if K is a simplex in $P_{ab}(\Gamma; W)$, then the simplex is endowed the standard simplicial Riemannian metric;

(ii) if K is not a simplex in $P_{ab}(\Gamma; W)$ and we have inductively defined the (pseudo) Riemannian metric g_{n-1} on its $(n-1)$-skeleton $K^{(n-1)}$, then we identify K with the cone

$$([0,1] \times K^{(n-1)})/(0 \times K^{(n-1)})$$

and define a (pseudo) Riemannian metric g_n on K by:

$$g_n = m_n^2 dt^2 + t^2 g_{n-1}$$

for $t \in [0,1]$.

(2) the (pseudo) Riemannian metrics on simplices of $P_{ab}(\Gamma; W)$ can be used to define the length of any piecewise smooth path in the polyhedron. For any pair of points x and y in $P_{ab\overline{m}}(\Gamma; W)$, $d(x, y)$ is defined to be the infimum of the lengths of all piecewise smooth paths in $P_{ab\overline{m}}(\Gamma; W)$ connecting x and y.

A.1.3. **Remark.** We shall actually only use the case $\overline{m} = (m, m, \dots)$ in the proofs of Theorems 7.1.2 and 7.1.3, where we will denote $P_{ab\overline{m}}(\Gamma; W)$ by $P_{abm}(\Gamma; W)$. We however chose to introduce the more general notion since it will streamline the proofs of several results in this appendix.

A.2. **Extension of the definitions for metric families.** In this subsection, we introduce some further notations in order to deal with families of subsets of Γ instead of just one subspace at a time. In particular, we will introduce the Rips complex and the relative Rips complex for metric families. We will not treat the case of the scaled Rips complex since it is a straightforward adaptation of the case of the relative Rips complex.

For a family $\mathcal{C} = \{ C \}$ of subspaces of Γ we define

$$P_d(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} P_d(C) \subset P_d(\Gamma),$$

which we shall always equip with the subspace metric. Typically, we shall employ this notation when the family $\mathcal{C}$ is disjoint. Note that if the family $\mathcal{C}$ is d-disjoint and $\tilde{C}$ is the union of the $C \in \mathcal{C}$ then

$$P_d(\mathcal{C}) = P_d(\tilde{C}).$$

If the union of families is defined naively, and the intersection of families is *defined* to be the family of intersections $\mathcal{C} \cap \mathcal{D} = \{ C \cap D : C \in \mathcal{C}, D \in \mathcal{D} \}$ we have

$$P_d(\mathcal{C} \cup \mathcal{D}) = P_d(\mathcal{C}) \cup P_d(\mathcal{D}), \quad P_d(\mathcal{C} \cap \mathcal{D}) = P_d(\mathcal{C}) \cap P_d(\mathcal{D}).$$

Just as for the standard Rips complex, we can extend the definition of the relative Rips complex to families. For families $\mathcal{C} = \{ C \}$ and $\mathcal{W} = \{ W \}$ with each $C \subset \Gamma$ and each $W \subset \Sigma$ we define

$$P_{ab}(\mathcal{C}, \mathcal{W}) = \bigcup_{C \in \mathcal{C}} P_a(C) \cup \bigcup_{W \in \mathcal{W}} P_b(W),$$

as subspaces of $P_{ab}(\Gamma,\Sigma)$. If Σ is not explicitly specified, then Σ is understood to be the union of all W in $\mathcal{W}$. In the special case $a=b$ we have $P_{aa}(\Gamma,\Sigma)=P_a(\Gamma)$ and, more generally $P_{aa}(\mathcal{C},\mathcal{W})=P_a(\mathcal{C}\cup\mathcal{W})$. As for the standard Rips complex, we have the elementary equalities

$$P_{ab}(\mathcal{C}\cup\mathcal{D},\mathcal{W})=P_{ab}(\mathcal{C},\mathcal{W})\cup P_{ab}(\mathcal{D},\mathcal{W}),\quad P_{ab}(\mathcal{C}\cap\mathcal{D},\mathcal{W})=P_{ab}(\mathcal{C},\mathcal{W})\cap P_{ab}(\mathcal{D},\mathcal{W})$$

as subspaces of $P_{ab}(\Gamma,\Sigma)$.

A.3. **A few technical results.** In this subsection, we prove a several useful results about the geometry of the (relative) Rips and scaled Rips complex. These results are important tools in the proofs of Theorems 7.1.2 and 7.1.3.

Henceforth, we assume Γ has bounded geometry.

A.3.1. **Lemma** (Comparison lemma)**.** *Let $a\geq 1$, and let $P_a(\Gamma)$ be equipped as usual with the simplicial metric. For x and $y\in\Gamma$ we have*

$$d_\Gamma(x,y)\leq a\;\alpha\;d_{P_a(\Gamma)}(x,y),$$

for some constant α depending only on the dimension of $P_a(\Gamma)$.

Proof. As this lemma is classical, we will only sketch its proof. Let $x,y\in X$ and suppose that there exists a path γ of length l between x and y in $P_a(X)$ (since otherwise, the distance between x and y in $P_a(X)$ is infinite). Let n be the minimal integer such that γ is contained in the n-skeleton of $P_a(X)$. We will prove the inequality $d_\Gamma(x,y)\leq a\alpha l$ for some α depending only on n by induction on n. First note that if $n=1$, then the inequality $d_\Gamma(x,y)\leq al$ is trivial. Now consider a simplex Δ of dimension n whose interior intersects γ. Let $u<v$ be such that $\gamma(s)$ lies in Δ for all $u\leq s\leq v$ and the path γ intersects $\partial\Delta$ at u and v (when s is u and v), where γ is parametrized by arclength s. It is easy to see that there exists a constant C depending only on n such that the portion of γ lying between u and v can be replaced by a path in $\partial\Delta$ of length $\leq C(v-u)$. Doing this on each such n-simplex, we obtain a path in the $(n-1)$-skeleton of length $\leq Cl$ and we conclude by induction. □

A.3.2. **Lemma.** *[Comparison lemma for the scaled complex] Let $a\geq 1$, and let C be a subspace of Γ. There exists $\beta\geq 1$ depending only on the dimension of $P_a(\Gamma)$ such that for all $b\geq a$, there exists $M>0$ for which*

$$d_\Gamma(x,C)\leq a\;\beta\;d(x,P_b(C)),$$

for all $x\in\Gamma$, provided $m_k\geq M$ for all k, where the distance for the right-hand term is taken in $P_{ab\overline{m}}(\Gamma,C)$.

Proof. It is enough to show that if γ is a path of length l in $P_{ab\overline{m}}(\Gamma,C)$, parametrized by its arc length with respect to the (pseudo) riemannian metric, between $x\in\Gamma$ and $P_b(C)$, then

$$d_\Gamma(x,C)\leq a\beta l. \tag{A.1}$$

We proceed by induction on n, the minimal integer such that γ is contained in the union of $P_a(\Gamma)$ and the n-skeleton of $P_{ab\overline{m}}(\Gamma,C)$. Precisely, our induction hypothesis will be the following: for all $\beta>\alpha$, where α appears in the comparison lemma for $P_a(\Gamma)$, and every path of length l contained in the union of $P_a(\Gamma)$ and the n-skeleton, there exists M such that (A.1) holds for all $\overline{m}$ such that $m_k \geq M$ for all $1 \leq k \leq n$.

Let us start with the case $n = 1$. Note that up to replacing γ by a sub-path, we can always suppose that it does not intersect C at any $t < l$. We can also suppose that if γ meets the interior of an edge not belonging to $P_a(\Gamma)$, then this edge is completely contained in γ. Hence traveling along γ means that, either we stay in $P_a(\Gamma)$, or we jump between two points in Γ, at distance $\leq b$, through an edge of length m_1. Hence choosing $M = b$, we conclude thanks to the comparison lemma in $P_a(\Gamma)$.

Now let us suppose that $n \geq 2$. Fix some $\beta_1 > \beta_2 > \alpha$ and choose an M such that the induction hypothesis applies for $\beta = \beta_2$. We assume moreover that $M \leq m_k \leq K$ for all $1 \leq k \leq n-1$, where K is some integer. Let us assume that γ meets at least a simplex Δ of dimension n which does not belong to $P_a(\Gamma)$. Let $u < v$ be such that $\gamma(t) \in \Delta$ for $u \leq t \leq v$, and γ meets the boundary of Δ at u and v. We start with two observations. Let $\Delta = ([0,1] \times \partial\Delta)/(0 \times \partial\Delta)$ and let $\eta \in (0,1)$.

First, note that if γ meets $[0, 1-\eta] \times \partial\Delta$, then $v - u \geq \eta m_n$. But the diameter of $\partial\Delta$ is less than ρK, for some ρ depending only on n. Hence we can replace the portion of γ between u and v by a path contained in $\partial\Delta$, of length $\leq \rho K \leq \rho K(v-u)/(\eta m_n)$.

Second, if γ is contained in $[1-\eta, 1] \times \partial\Delta$, then observe that the retraction of $[1-\eta, 1] \times \partial\Delta$ onto $\partial\Delta$ is a $(1-\eta)^{-1}$-Lipschitz map, and hence, projecting γ to the boundary increases its length by at most $(1-\eta)^{-1}$. Hence there exists a path γ' completely contained in the union of $P_a(\Gamma)$ and the $(n-1)$-skeleton whose length l' satisfies

$$l' \leq (\rho K/(\eta m_n))l + (1-\eta)^{-1}l.$$

Applying the induction hypothesis to γ' yields

$$d_\Gamma(x, C) \leq a\beta_2 l' \leq a\beta_2((\rho K/(\eta m_n) + (1-\eta)^{-1})l.$$

First fix η such that

$$\beta_2(1-\eta)^{-1} < \beta_1.$$

We then take $M' \geq M$ big enough so that

$$\beta_2((\rho K/(\eta m_n) + (1-\eta)^{-1}) \leq \beta_1$$

for all $m_n \geq M'$. This gives the desired inequality

$$d_\Gamma(x, C) \leq a\beta_1 l,$$

under the assumption that $M \leq m_k \leq K$ for $1 \leq k \leq n-1$, and $m_n \geq M'$. But since increasing m_k can only increase l, this inequality remains true under the condition that $m_k \geq M'$ for all $1 \leq k \leq n$. $\square$

Next we make the following observation, from which we will immediately deduce the *neighborhood* and the *separation* lemmas below.

A.3.3. **Lemma.** *Let C be a subspace of Γ and let $\varepsilon \geq 1$ and $a \geq 1$. There exists $\beta \geq 1$ depending only on the dimension of $P_a(\Gamma)$ such that the following statements are true. Viewing $P_a(C)$ as a subspace of $P_a(\Gamma)$ we have*

$$N_\varepsilon(P_a(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C),$$

Similarly for the relative Rips complex, viewing $P_b(C)$ as a subspace of $P_{ab}(\Gamma, C)$ ($b \geq a$) we have

$$N_\varepsilon(P_b(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C).$$

Finally, for the scaled complex, viewing $P_b(C)$ as a subspace of $P_{ab\overline{m}}(\Gamma, C)$ we have

$$N_\varepsilon(P_b(C)) \cap \Gamma \subset N_{a\varepsilon\beta}(C).$$

provided that $\overline{m}$ is large enough in sense that $m_k \geq M$ for all k, where M depends only on b.

Proof. The diameter of a simplex in $P_a(\Gamma)$ is bounded by a universal constant. The first assertion therefore follows from the comparison lemmas upon approximating a given $z \in P_a(C)$ by a vertex in C.

Observe that a path in $P_{ab}(\Gamma, C)$ leaving $P_b(C)$ must pass through $P_a(C)$. Hence, moving away from $P_b(C)$ within $P_{ab}(\Gamma, C)$ is the same as moving away from $P_a(C)$ in $P_a(\Gamma)$. The second assertion therefore follows from the first one.

Finally, the third assertion follows immediately from the previous lemma. $\square$

The following lemma is an easy consequence of the previous and is left to the reader.

A.3.4. **Lemma** (Neighborhood lemma)**.** *Let $C \subset \Gamma$, $\varepsilon \geq 1$ and $a \geq 1$. Viewing $P_a(C) \subset P_a(\Gamma)$ we have*

$$N_\varepsilon(P_a(C)) \subset P_a(N_{a\varepsilon\beta}(C)),$$

for some constant β depending only on the dimension of $P_a(\Gamma)$. Similarly for the relative Rips complex, viewing $P_b(C) \subset P_{ab}(\Gamma, C)$ ($b \geq a$) we have

$$N_\varepsilon(P_b(C)) \subset P_{ab}(N_{a\varepsilon\beta}(C), C).$$

A.3.5. **Lemma** (Separation lemma)**.** *Let $\varepsilon \geq 1$ and $a \geq 1$. If the family $\mathcal{C}$ of subsets of Γ is ε-separated, then the family $P_a(\mathcal{C})$ (resp. $P_b(\mathcal{C})$) is $\varepsilon(a\beta)^{-1}$-separated in $P_a(\Gamma)$ (resp. in $P_{ab}(\Gamma, \mathcal{C})$ for $b \geq a$, and in $P_{ab\overline{m}}(\Gamma, \mathcal{C})$ for $b \geq a$ if $\overline{m}$ is large enough), where β only depends on the dimension of $P_a(\Gamma)$.*

Proof. The first two cases are direct consequences of the neighborhood lemma above. For the scaled complex, it follows from Lemma A.3.2. $\square$

Note that the neighborhood lemma does not apply to the scaled Rips complex. Instead, we have the following slightly weaker statement.

A.3.6. **Lemma** (Lipschitz homotopy lemma). *Let C and W be subspaces of the metric space Γ. Let $\varepsilon \geq 1$ and $b \geq a \geq 1$. Let V be the ε-neighborhood of $P_b(W)$ in $P_{ab\overline{m}}(\Gamma, W)$, let W' be the $a\beta\varepsilon$-neighborhood of W in Γ, where β is the constant appearing in Lemma A.3.3. Then, for all $c \geq b$, there exist $M > 0$ and a proper continuous map*

$$F : (P_{ac\overline{m}}(C, W') \cup V) \times [0,1] \to P_{ac\overline{m}}(C, W') \cup V$$

such that

1. *$F(\cdot, t)$ is 2-Lipschitz for all $t \in [0,1]$, provided that $m_k \geq M$ for all k,*
2. *for each $t \in [0,1]$, $F(\cdot, t)$ restricts to the identity map on $P_{ac\overline{m}}(C, W')$,*
3. *$F(\cdot, 0)$ is the identity map on $P_{ac\overline{m}}(C, W') \cup V$, and the image of $F(\cdot, 1)$ lies in $P_{ac\overline{m}}(C, W')$.*

Moreover, the constant M depends only on ε and the dimension of $P_c(\Gamma)$.

Proof. By Lemma A.3.3, we have $V \cap \Gamma \subset W'$. Now for each n, let X_n be the union of $P_{ac\overline{m}}(C, W')$ with the k-simplices of $P_{ab\overline{m}}(\Gamma, W)$ whose interiors meet V, for all $k \leq n$. Let $Y_n = X_n \cap (P_{ac\overline{m}}(C, W') \cup V)$. We will prove the following stronger statement by induction on n.

For all $c \geq b$, and all $\eta > 0$, if M is large enough, there exists a proper continuous map $F : Y_n \times [0,1] \to Y_n$ where for each $t \in [0,1]$, $F(\cdot, t)$ is $(1+\eta)$-Lipschitz if $m_k \geq M$ for all $1 \leq k \leq n$, and restricts to the identity map on $P_{ac\overline{m}}(C, W')$. Moreover $F(\cdot, 0)$ is the identity map on Y_n, and the image of $F(\cdot, 1)$ lies in $P_{ac\overline{m}}(C, W')$. This is obvious for $n = 0$. We therefore assume $n \geq 1$.

We suppose that this statement holds for Y_{n-1}. Note that it is enough to show that for every $\eta > 0$, there exists some proper continuous map $H : Y_n \times [0,1] \to Y_n$ such that for every $t \in [0,1]$, $H(\cdot, t)$ is $(1+\eta)$-Lipschitz as soon as m_n is large enough, H restricts to the identity map on $P_{ac\overline{m}}(C, W')$, $H(\cdot, 0)$ is the identity map on Y_n, and the image of $H(\cdot, 1)$ lies in Y_{n-1}.

Let $\Delta = ([0,1] \times \partial\Delta)/(0 \times \partial\Delta)$ be a simplex in X_n, which is not in X_{n-1}. Observe that $Y_n \cap \Delta \subset [1 - \varepsilon/m_n] \times \partial\Delta$. Consider the standard homotopy between $[1 - \varepsilon/m_n] \times \partial\Delta$ and $\partial\Delta$. We define H to be the restriction of this homotopy to $(Y_n \cap \Delta) \times [0,1]$. Clearly the Lipschitz constant is at most $1/(1 - \varepsilon/m_n)$. Hence it suffices to take m_n large enough to make this number less than $1 + \eta$. ∎

Appendix B. Mayer-Vietoris sequences in bounded K and L-theory

In this section, we recall from [RY1, RY2] the controlled Mayer-Vietoris sequences in K and L-theory. These are important tools in our proof of the bounded Borel conjecture for spaces with finite decomposition complexity.

B.1. **Theorem.** *Let X be a metric space, written as the union of closed subspaces $X = A \cup B$. There exists a universal constant $\lambda > 1$ (independent of X, A and B) such that for each $\delta > 0$,*

1. *in $Wh^\delta(A \cap B) \xrightarrow{i_*} Wh^\delta(A) \oplus Wh^\delta(B) \xrightarrow{j_*} Wh^\delta(X)$, we have $j_* i_* = 0$;*

(2) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then the relax-control image of the kernel of* j_* *in* $Wh^{\lambda^2\delta}(A\cup W)\oplus Wh^{\lambda^2\delta}(B\cup W)$ *is contained in the image of* i_* *below*

$$\begin{tikzcd} & Wh^{\delta}(A)\oplus Wh^{\delta}(B) \arrow[r, "j_*"] \arrow[d] & Wh^{\delta}(X)\ , \\ Wh^{\lambda^2\delta}(W) \arrow[r, "i_*"] & Wh^{\lambda^2\delta}(A\cup W)\oplus Wh^{\lambda^2\delta}(B\cup W) & \end{tikzcd}$$

where $N_{\lambda\delta}(A\cap B)=\{\,x\in X\colon d(x,A\cap B)\le\lambda\delta\,\}$*;*

(3) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then in*

$$Wh^{\delta}(A)\oplus Wh^{\delta}(B)\xrightarrow{j_*}Wh^{\delta}(X)\xrightarrow{\partial}\tilde K_0^{\lambda\delta}(W),$$

we have $\partial j_*=0$*;*

(4) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then the relax-control image of the kernel of* ∂ *in* $Wh^{\lambda^2\delta}(X)$ *is contained in the image of* j_* *below*

$$\begin{tikzcd} & Wh^{\delta}(X) \arrow[r, "\partial"] \arrow[d] & \tilde K_0^{\lambda\delta}(W) \\ Wh^{\lambda^2\delta}(A\cup W)\oplus Wh^{\lambda^2\delta}(B\cup W) \arrow[r, "j_*"] & Wh^{\lambda^2\delta}(X) & \end{tikzcd}$$

(5) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then in*

$$Wh^{\delta}(X)\xrightarrow{\partial}\tilde K_0^{\lambda\delta}(W)\xrightarrow{i_*}\tilde K_0^{\lambda\delta}(A\cup W)\oplus\tilde K_0^{\lambda\delta}(B\cup W),$$

we have $i_*\partial=0$*;*

(6) *if* $N_{\lambda\delta}(A\cap B)\subset W$*, then the relax-control image of the kernel of* i_* *in* $\tilde K_0^{\lambda^2\delta}(W)$ *is contained in the image of* ∂

$$\begin{tikzcd} & \tilde K_0^{\delta}(A\cap B) \arrow[r, "i_*"] \arrow[d] & \tilde K_0^{\delta}(A)\oplus\tilde K_0^{\delta}(B) \\ Wh^{\lambda\delta}(X) \arrow[r, "\partial"] & \tilde K_0^{\lambda^2\delta}(W) & \end{tikzcd}$$

The precise L-theory version we require is the following result where, for each metric space Y, each integer $n\ge 0$ and $\delta>0$, $L_n^{\delta}(Y)$ is the δ-controlled locally finite and free L-theory of Y [RY2]. This result is a consequence of Theorem 7.3 and Proposition 4.6 in [RY2]), Proposition 3.2 and Proposition 3.4 in [RY1].

B.2. **Theorem.** *Let* P *be a locally compact polyhedron and* P' *a subpolyhedron of* P*. Assume that* P *and* P' *are respectively given with metrics* d *and* d' *satisfying* $d(x,y)\le d'(x,y)$ *for all* x *and* y *in* P'*. Let* X *be a metric subspace of* P'*. Assume that* X *is written as the union of closed subspaces* $X=A\cup B$*. For every integer* $n\ge 2$ *there exists* $\lambda_n>1$*, which depends only on* n*, such that for each* $\delta>0$*,*

(1) *in* $L_n^\delta(A\cap B)\xrightarrow{i_*}L_n^\delta(A)\oplus L_n^\delta(B)\xrightarrow{j_*}L_n^\delta(X)$, *we have* $j_*i_*=0$, *where the metrics on* $A\cap B$, A, B *and* X *are inherited from the metric of* P'*;*

(2) *if* $N_{\lambda_n\delta}(A\cap B)\subseteq W\subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then the relax-control image of the kernel of* j_* *in*

$$L_n^{\lambda_n^2\delta}(A\cup W)\oplus L_n^{\lambda_n^2\delta}(B\cup W)$$

is contained in the image of i_* *below*

$$\begin{tikzcd} & L_n^\delta(A)\oplus L_n^\delta(B) \arrow[r, "j_*"] \arrow[d] & L_n^\delta(X)\ , \\ L_n^{\lambda_n\delta}(W) \arrow[r, "i_*"] & L_n^{\lambda_n^2\delta}(A\cup W)\oplus L_n^{\lambda_n^2\delta}(B\cup W) & \end{tikzcd}$$

where $N_{\lambda_n\delta}(A\cap B)=\{\,x\in X\colon d(x,A\cap B)\le\lambda_n\delta\,\}$ *is given the metric of* P', *the metrics on* A, B *and* X *are inherited from the metric of* P', *and the metrics on* W, $A\cup W$ *and* $B\cup W$ *are inherited from the metric of* P*;*

(3) *if* $N_{\lambda_n\delta}(A\cap B)\subseteq W\subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then in*

$$L_n^\delta(A)\oplus L_n^\delta(B)\xrightarrow{j_*}L_n^\delta(X)\xrightarrow{\partial}L_{n-1}^{\lambda_n\delta}(W),$$

we have $\partial j_*=0$, *where* $N_{\lambda_n\delta}(A\cap B)=\{\,x\in X\colon d(x,A\cap B)\le\lambda_n\delta\,\}$ *is given the metric of* P', *the metrics on* A, B *and* X *are inherited from the metric of* P', *and the metric on* W *is inherited from the metric of* P*;*

(4) *if* $N_{\lambda_n\delta}(A\cap B)\subseteq W\subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then the relax-control image of the kernel of* ∂ *in* $L_n^{\lambda_n^2\delta}(X)$ *is contained in the image of* j_* *below*

$$\begin{tikzcd} & L_n^\delta(X) \arrow[r, "\partial"] \arrow[d] & L_{n-1}^{\lambda_n\delta}(W) \\ L_n^{\lambda_n^2\delta}(A\cup W)\oplus L_n^{\lambda_n^2\delta}(B\cup W) \arrow[r, "j_*"] & L_n^{\lambda_n^2\delta}(X\cup W) & \end{tikzcd}$$

where $N_{\lambda_n\delta}(A\cap B)=\{\,x\in X\colon d(x,A\cap B)\le\lambda_n\delta\,\}$ *is given the metric of* P', *the metric on* X *is inherited from the metric of* P', *and the metrics on* W, $A\cup W$, $B\cup W$ *and* $X\cup W$ *are inherited from the metric of* P*;*

(5) *if* $N_{\lambda_n\delta}(A\cap B)\subseteq W\subseteq P$ *and the natural homomorphism from* $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ *to* $\tilde{K}_0^{\lambda_n\delta}(W)$ *is zero, then in*

$$L_n^\delta(X)\xrightarrow{\partial}L_{n-1}^{\lambda_n\delta}(W)\xrightarrow{i_*}L_{n-1}^{\lambda_n\delta}(A\cup W)\oplus L_{n-1}^{\lambda_n\delta}(B\cup W),$$

we have $i_*\partial = 0$, where $N_{\lambda_n\delta}(A\cap B) = \{\, x \in X : d(x, A\cap B) \le \lambda_n\delta \,\}$ is given the metric of P', the metric on X is inherited from the metric of P', and the metrics on W, $A\cup W$ and $B\cup W$ are inherited from the metric of P;

(6) *if $N_{\lambda_n\delta}(A\cap B) \subseteq W \subseteq P$ and the natural homomorphism from $\tilde{K}_0^{\lambda_n\delta}(N_{\lambda_n\delta}(A\cap B))$ to $\tilde{K}_0^{\lambda_n\delta}(W)$ is zero, then the relax-control image of the kernel of i_* in $L_{n-1}^{\lambda_n^2\delta}(W)$ is contained in the image of ∂*

$$\begin{array}{ccccc} & & L_{n-1}^{\delta}(A\cap B) & \xrightarrow{i_*} & L_{n-1}^{\delta}(A)\oplus L_{n-1}^{\delta}(B) \\ & & \downarrow & & \\ L_n^{\lambda_n\delta}(X) & \xrightarrow{\partial} & L_{n-1}^{\lambda_n^2\delta}(W) & & \end{array}$$

where $N_{\lambda_n\delta}(A\cap B) = \{\, x \in X : d(x, A\cap B) \le \lambda_n\delta \,\}$ is given the metric of P', the metrics on X, $A\cap B$, A and B are inherited from the metric of P', and the metric on W is inherited from the metric of P.

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