

# FIRST PASSAGE PERCOLATION ON A HYPERBOLIC GRAPH ADMITS BI-INFINITE GEODESICS

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ABSTRACT. Given an infinite connected graph, a way to randomly perturb its metric is to assign random i.i.d. lengths to the edges. An open question attributed to Furstenberg ([Ke86]) is whether there exists a two-sided infinite geodesic in first passage percolation on  $\mathbb{Z}^2$ , and more generally on  $\mathbb{Z}^n$  for  $n \geq 2$ . Although the answer is generally conjectured to be negative, we give a positive answer for graphs satisfying some negative curvature assumption. More precisely, we prove that if a graph with bounded degree  $X$  contains a Morse geodesic (e.g. is non-elementary Gromov hyperbolic), then almost surely, there exists a bi-infinite geodesic in first passage percolation on  $X$ .

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## 1. INTRODUCTION

First passage percolation is a model of random perturbation of a given geometry. In this paper, we shall restrict to the simplest model, where random i.i.d lengths are assigned to the edges of a fixed graph. We refer to [GK12, Ke86] for background and references.

Let us briefly recall how FPP is defined. We consider a connected non-oriented graph  $X$ , whose set of vertices (resp. edges) is denoted by  $V$  (resp.  $E$ ). For every function  $\omega : E \rightarrow (0, \infty)$ , we equip  $V$  with the weighted graph metric  $d_\omega$ , where each edge  $e$  has weight  $\omega(e)$ . In other words, for every  $v_1, v_2 \in V$ ,

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$d_\omega(v_1, v_2)$  is defined as the infimum over all path  $\gamma = (e_1, \dots, e_m)$  joining  $v_1$  to  $v_2$  of  $|\gamma|_\omega := \sum_{i=1}^m \omega(e_i)$ . Observe that the simplicial metric on  $V$  corresponds to the case where  $\omega$  is constant equal to 1, we shall simply denote it by  $d$ . We will now consider a probability measure on the set of all weight functions  $\omega$ . We let  $\nu$  be a probability measure supported on  $[0, \infty)$ . Our model consists in choosing independently at random the weights  $\omega(e)$  according to  $\nu$ . More formally, we equip the space  $\Omega = [0, \infty)^E$  with the product probability that we denote by  $P$ .

A famous open problem in percolation theory is whether with positive probability, first passage percolation on  $\mathbb{Z}^2$  admits a two-sided infinite geodesic. In his Saint-Flour course from 84', Kesten attributes this question to Furstenberg (see [Ke86]). Licea and Newman [LN96] made partial progress on this problem, which is still open and mentioned that the conjecture that there are no such geodesics arose independently in the physics community studying spin glass. Wehr and Woo [WW98] proved absence of two sided infinite geodesic in a half plane, assuming the lengths distribution is continuously distributed with a finite mean.

For Riemannian manifolds, the existence of bi-infinite geodesics is influenced by the curvature of the space. It is well-known that complete simply connected non-positively curved Riemannian manifolds (such as the euclidean space  $\mathbb{R}^n$  or the hyperbolic space  $\mathbb{H}^n$ ) admit bi-infinite geodesics. Therefore, simply connected manifolds without two-sided geodesics must have positively curved regions. It is easy to come up with examples of complete Riemannian surfaces with bubble-like structures that create short cuts avoiding larger and larger balls around some origin.

To help the reader's intuition, let us roughly describe a similar example in the graph setting. Starting with the standard Cayley graph of  $\mathbb{Z}^2$ , it is not difficult to choose edges lengths among the two possible values  $1/10$  and  $1$ , such that the resulting weighted graph has no bi-infinite geodesics. To do so, consider a sequence of squares  $C_n$  centered at the origin, whose size grows faster than any exponential sequence (e.g. like  $n^n$ ). Then attribute length  $1/10$  to the edges along  $C_n$  for all  $n$ , and  $1$  to all other edges. This creates large "bubbles" with relatively small neck in the graph (which in a sense can be interpreted as large positively curved regions). One easily checks for all  $n$ , for every pair of points at large enough distance from the origin, any geodesic between them never enters  $C_n$  (as it is more efficient to go around the shorter edges of its boundary, than traveling inside it).

The Euclidean plane being flat, its discrete counterpart  $\mathbb{Z}^2$  (and more generally  $\mathbb{Z}^d$  for  $d \geq 2$ ) is in some sense at criticality for the question of existence of bi-infinite geodesics in FPP. Therefore, we expect that in presence of negative curvature, FPP a.s. exhibits bi-infinite geodesics. For instance, this should apply to FPP on Cayley graphs of groups acting properly cocompactly by isometries on the hyperbolic space  $\mathbb{H}^d$ . This will be a consequence of our main result below.

Let us first introduce some notation. Let  $(X, \omega)$  be a weighted graph. Recall that a path  $\gamma = (e_1, \dots, e_n)$  between two vertices  $x, y$  is a sequence of consecutive edges joining  $x$  to  $y$ . Given a path  $\gamma$ , we denote  $(x = \gamma(0), \dots, \gamma(n) = y)$  the set of vertices such that for all  $0 \leq i < n$ ,  $\gamma(i)$  and  $\gamma(i+1)$  are joined by the edge  $e_{i+1}$ . For all  $i < j$ , we shall also denote by  $\gamma([i, j])$  the subpath  $(e_{i+1}, \dots, e_j)$  joining  $\gamma(i)$  to  $\gamma(j)$ . Similarly, we define infinite paths indexed by  $\mathbb{N}$  (resp. bi-infinite paths indexed by  $\mathbb{Z}$ ).

**Definition 1.1.** Let  $X$  be an infinite connected (simplicial) graph, and let  $C \geq 1$  and  $K \geq 0$ . A path  $\gamma$  of length  $n$  between two vertices  $x$  and  $y$  is called a  $(C, K)$ -quasi-geodesic finite path if for all  $0 < i < j \leq n$ ,

$$\sum_{k=i}^{j-1} d(\gamma(k), \gamma(k+1)) \leq Cd(\gamma(i), \gamma(j)) + K.$$

Similarly, we define  $(C, K)$ -quasi-geodesic infinite (or bi-infinite) paths. An infinite (or a bi-infinite) path will simply be called a quasi-geodesic if it is  $(C, K)$ -quasi-geodesic for some constants  $C$  and  $K$ .

**Definition 1.2.** A bi-infinite path  $\gamma$  in  $X$  is called a *Morse quasi-geodesic* (resp. Morse geodesic) if it is a quasi-geodesic (resp. a geodesic) and if it satisfies the so-called Morse property: for all  $C \geq 1$  and  $K > 0$ , there exists  $R$  such that every  $(C, K)$ -quasi-geodesic joining two points of  $\gamma$  remains inside the  $R$ -neighborhood of  $\gamma$ .

It is well-known and easy to deduce from its definition that a Morse geodesic always lies at a bounded distance from a bi-infinite geodesic.

{thm:Main}

**Theorem 1.3.** *Let  $X$  be an infinite connected graph with bounded degree, that contains a Morse quasi-geodesic. Assume  $\mathbb{E}\omega_e < \infty$  and  $\nu(\{0\}) = 0$ . Then for a.e.  $\omega$ ,  $X_\omega$  admits a bi-infinite geodesic.*

We briefly recall the definition of a hyperbolic graph (in the sense of Rips). A geodesic triangle in a graph  $X$  consists of a triplet of vertices  $x_0, x_1, x_2 \in V$ ,

and of geodesic paths  $\gamma_0, \gamma_1, \gamma_2$  such that  $\gamma_i$  joins  $x_{i+1}$  to  $x_{i+2}$  where  $i \in \mathbb{Z}/3\mathbb{Z}$ . Given  $\delta \geq 0$ , a geodesic triangle is called  $\delta$ -thin if for every  $i \in \mathbb{Z}/3\mathbb{Z}$ , every vertex  $v_i$  on  $\gamma_i$  lies at distance at most  $\delta$  from either  $\gamma_{i+1}$  or  $\gamma_{i+2}$ . Said informally, a geodesic triangle is  $\delta$ -thin if every side is contained in the  $\delta$ -neighborhood of the other two sides. It is well-known [Gr87] that in a hyperbolic graph, any bi-infinite quasi-geodesic is Morse. In particular, we deduce the following

**Corollary 1.4.** *Let  $X$  be a hyperbolic graph with bounded degree containing at least one bi-infinite geodesic. Assume  $\mathbb{E}\omega_e < \infty$  and  $\nu(\{0\}) = 0$ . Then for a.e.  $\omega$ ,  $X_\omega$  admits a bi-infinite geodesic.*

Note that the case where  $\nu$  is supported in an interval  $[a, b] \subset (0, \infty)$  is essentially obvious. Indeed, for all  $\omega$ , the weighed graph  $X_\omega$  is bi-Lipschitz equivalent to  $X$ . We deduce that a Morse quasi-geodesic in  $X$  remains a Morse quasi-geodesic in  $X_\omega$  (adapting the definition to weighted graphs), and therefore lies at bounded distance from an actual bi-infinite geodesic.

We finish this introduction mentioning that Theorem 1.3 applies to a wide class of Cayley graphs, including Cayley graphs of relatively hyperbolic groups, Mapping Class groups, and so on.

## 2. PRELIMINARY LEMMAS

We start with a useful characterization of Morse quasi-geodesics.

{lem:MQG} **Lemma 2.1.** [DMS08, Proposition 3.24] *Let  $X$  be an infinite connected graph with bounded degree, and assume that there exists a Morse quasi-geodesic  $\gamma_0$ . There exists an increasing function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{t \rightarrow \infty} \phi(t) = \infty$  satisfying the following property. For all  $x, y, x', y' \in X$ ,  $\gamma$  such that*

- $x, y$  belong to  $\gamma_0$ ;
- $d(x, x') = d(y, y') = R$ ;
- $\gamma$  is a path joining  $x'$  to  $y'$ , and remains outside of the  $R$ -neighborhood of  $\gamma_0$ .

Then

$$|\gamma| \geq \phi(R)d(x, y) - 4R.$$

The hypothesis  $\mathbb{E}\omega_e < \infty$  is used (only) in the following trivial lemma.

{lem:LLN}

**Lemma 2.2.** *Let  $X$  be a connected graph, and let  $\gamma$  be an injective path. Assume  $0 < b = \mathbb{E}\omega_e < \infty$ . Then for a.e.  $\omega$ , there exists  $r_0 = r_0(\omega)$  such that for all  $i \leq 0 \leq j$ ,*

$$|\gamma([i, j])|_\omega \leq 2b(j - i) + r_0.$$

*Proof.* This immediately follows from the law of large number, using that the edges length distributions are i.i.d.  $\square$

Our assumption  $\nu(\{0\}) = 0$  is used to prove the following lemmas.

**Lemma 2.3.** *Let  $X$  be an infinite connected graph with bounded degree and assume that  $\nu(\{0\}) = 0$ . There exists an increasing function  $\alpha : (0, \infty) \rightarrow (0, \infty)$  such that  $\lim_{t \rightarrow 0} \alpha(t) = 0$ , and such that for all finite path  $\gamma$  and all  $\varepsilon > 0$ ,*

$$P(|\gamma|_\omega \leq \varepsilon|\gamma|) \leq \alpha(\varepsilon)^{|\gamma|}.$$

*Proof.* The assumption implies that for all  $\lambda > 0$ , there exists  $\delta > 0$  such that  $\nu([0, 0 + \delta]) < \lambda$ . Let  $\gamma$  be a path of length  $n$ . Assume that  $|\gamma|_\omega \leq \varepsilon|\gamma|$ , and let  $N$  be the number of edges of  $\gamma$  with  $\omega$ -length  $\geq \delta n$ . It follows that

$$\delta N \leq \varepsilon n,$$

so we deduce that  $N \leq \varepsilon n / \delta$ . This imposes that at least  $(1 - \varepsilon/\delta)n$  edges of  $\gamma$  have  $\omega$ -length  $\leq \delta$ . Recall that by Stirling's formula, given some  $0 < \alpha < 1$ , the number of ways to choose  $\alpha n$  edges in a path of length  $n$  is

$$\sim \frac{n^n}{(\alpha n)^{\alpha n} ((1 - \alpha)n)^{(1 - \alpha)n}} = (1/\alpha)^{\alpha n} (1/(1 - \alpha))^{(1 - \alpha)n}.$$

Thus the probability that  $\gamma$  has  $\omega$ -length at most  $\varepsilon n$  is less than a universal constant times

$$\frac{\lambda^{(1 - \varepsilon/\delta)n}}{(\varepsilon/\delta)^{(\varepsilon/\delta)n} (1 - \varepsilon/\delta)^{(1 - \varepsilon/\delta)n}} = \left( \frac{\lambda^{1 - \varepsilon/\delta}}{(\varepsilon/\delta)^{\varepsilon/\delta} (1 - \varepsilon/\delta)^{1 - \varepsilon/\delta}} \right)^n.$$

Note that

$$\lim_{\varepsilon \rightarrow 0} \frac{\lambda^{1 - \varepsilon/\delta}}{(\varepsilon/\delta)^{\varepsilon/\delta} (1 - \varepsilon/\delta)^{1 - \varepsilon/\delta}} = \lambda,$$

and since  $\lambda$  was arbitrary, this proves the lemma.  $\square$

{lem:upperbound}

**Lemma 2.4.** *Let  $X$  be an infinite connected graph with bounded degree, and let  $o$  be some vertex of  $X$ . Assume  $\nu(\{0\}) = 0$ . Then there exists  $c > 0$  such that for a.e.  $\omega$ , there exists  $r_1 = r_1(\omega)$  such that for all finite path  $\gamma$  such that  $d(\gamma, o) \leq |\gamma|$ , one has*

$$|\gamma|_\omega \geq c|\gamma| - r_1.$$

*Proof.* The number of paths of length  $\leq n$  which are at distance at most  $n$  from  $o$  is less than  $(q+1)^{2n}$ , where  $q$  is an upper bound on the degree of  $X$ . So we deduce from the previous lemma that for  $\varepsilon$  small enough, the probability that there exists some path  $\gamma$  of length  $n$ , and at distance at most  $n$  from  $o$ , and satisfying  $|\gamma|_\omega \leq \varepsilon|\gamma|$  decays exponentially with  $n$ . Hence the lemma follows.  $\square$

## 3. PROOF OF THEOREM 1.3

We let  $\gamma_0$  be some Morse quasi-geodesic. First of all, we do not lose generality by supposing that our Morse quasi-geodesic  $\gamma_0$  is a two-sided infinite geodesic of the graph  $X$ . We let  $o \in \gamma_0$  be some vertex. We consider two sequences of vertices  $(x_n)$  and  $(y_n)$  on  $\gamma_0$  which go to infinity in opposite directions.

We let  $\Omega' \subset \Omega$  be a measurable subset of full measure such that the conclusions of Lemmas 2.2 and 2.4 hold. For all  $n$  and for all  $\omega$ , we pick measurably an  $\omega$ -geodesic  $\gamma_\omega^n$  between  $x_n$  and  $y_n$ .

If we can prove that for all  $\omega \in \Omega'$ , there exists a constant  $R_\omega > 0$  such that along a subsequence,  $d(\gamma_\omega^n, o) \leq R_\omega$ , then the conclusion of Theorem 1.3 follows by a straightforward compactness argument. So we shall assume by contradiction that for some  $\omega \in \Omega'$ , there exists a sequence  $R_n$  going to infinity such that  $\gamma_\omega^n$  avoids  $B(o, 100R_n)$ .

**Lemma 3.1.** *There exist integers  $q < p$  such that*

$$d(\gamma_\omega^n(p), \gamma_0) = d(\gamma_\omega^n(q), \gamma_0) = R_n,$$

and such that for all  $p \leq k \leq q$ ,

$$d(\gamma_\omega^n(k), \gamma_0) \geq R_n,$$

and

$$d(\gamma_\omega^n(p), \gamma_\omega^n(q)) \geq 10R_n.$$

*Proof.* (Note that this is obvious from a picture). Let  $r$  and  $l$  be integers such that

$$d(\gamma_\omega^n(r), \gamma_0) = d(\gamma_\omega^n(r), o) \geq 100R_n.$$

By triangular inequality, for all  $p < r$  and  $q > r$  such that  $d(\gamma_\omega^n(p), \gamma_\omega^n(q)) \leq 10R_n$ , we have

$$d(\gamma_\omega^n(p), \gamma_\omega^n(q)) \geq 80R_n > R_n.$$

Hence, if we let  $p$  and  $q$  be respectively the largest integer  $\leq r$  and the smallest integer  $\geq r$  such that

$$d(\gamma_\omega^n(p), \gamma_0) = d(\gamma_\omega^n(q), \gamma_0) = R_n,$$

we have that

$$d(\gamma_\omega^n(p), \gamma_\omega^n(q)) \geq 10R_n.$$

So the lemma follows. □

We now let  $i$  and  $j$  be integers such that

$$d(\gamma_\omega^n(p), \gamma_0(i)) = d(\gamma_\omega^n(q), \gamma_0(j)) = R_n.$$

Note that by triangular inequality,  $j - i = |\gamma_0([i, j])| \geq R_n$ .

By Lemmas 2.4 and 2.1, we have

$$\begin{aligned} |d(\gamma_\omega^n([p, q]))|_\omega &\geq c|d(\gamma_\omega^n([p, q]))| - r_1 \\ &\geq c\phi(R_n)|\gamma_0([i, j])| - 4cR_n - r_1 \\ &= c\phi(R_n)(j - i) - 4cR_n - r_1. \end{aligned}$$

On the other hand, since  $\gamma_\omega^n$  is a geodesic between  $x_n$  and  $y_n$ , we have

$$\begin{aligned} |d(\gamma_\omega^n([p, q]))|_\omega &\leq 2R_n + |\gamma_0([i, j])|_\omega \\ &\leq 2R_n + 2b|\gamma_0([i, j])| + r_0 \\ &= 2R_n + 2b(j - i) + r_0, \end{aligned}$$

where the second inequality follows from Lemma 2.2.

Gathering these inequalities, we obtain

$$(c\phi(R_n) - 2b)R_n \leq (c\phi(R_n) - 2b)(j - i) \leq (2 + 4c)R_n + r_1 + r_0,$$

which yields a contradiction since  $\phi(R_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . This ends the proof of Theorem 1.3.

#### 4. REMARKS AND QUESTIONS

- (Cayley graphs) We do not know a single example of an infinite Cayley graph, for which FPP a.s. admits no bi-infinite geodesics (to fix the ideas, assume the edge length distribution is supported on the interval  $[1, 2]$ ).
- (Adding dependence) Given a hyperbolic Cayley graph, rather than considering independent edges lengths it is natural to consider other group invariant distributions. Under which natural conditions (mixing?) on this distribution do bi-infinite geodesics a.s. exists?
- (Poisson Voronoi and other random models) A variant of random metric perturbation is obtained via Poisson Voronoi tiling of a measure metric space. The proof below adapts to the hyperbolic Poisson Voronoi tiling, see [BPP14]. Recently other versions of random hyperbolic triangulations were constructed, [AR13] [C14]. Since those were not constructed by perturbing an underlying hyperbolic space, our proof does not apply to this setting.

- (Variance along Morse geodesics) We *conjecture* that the variance of the random distance grows linearly along the Morse quasi-geodesic, unlike in Euclidean lattices [BKS03]. For lengths which are bounded away from zero and infinity, Morse’s property ensures that geodesics remain at uniformly bounded distance from  $\gamma$ , hence reducing the problem to “filiform graphs”, i.e. graphs quasi-isometric to  $\mathbb{Z}$ . A (very) special class of filiform graphs is dealt with in [A15].

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