CHARACTERIZING A VERTEX-TRANSITIVE GRAPH BY A LARGE BALL

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Abstract. It is well-known that a complete Riemannian manifold M which is locally isometric to a symmetric space is covered by a symmetric space. Here we prove that a discrete version of this property (called *local to global rigidity*) holds for a large class of vertex-transitive graphs, including Cayley graphs of torsion-free lattices in simple Lie groups, and Cayley graph of torsion-free virtually nilpotent groups. By contrast, we exhibit various examples of Cayley graphs of finitely presented groups (e.g. $\mathrm{PGL}(4, \mathbf{Z})$) which fail to have this property, answering a question of Benjamini and Georgakopoulos.

Answering a question of Cornulier, we also construct a continuum of non pairwise isometric *large-scale simply connected* locally finite vertex-transitive graphs. This question was motivated by the fact that large-scale simply connected Cayley graphs are precisely Cayley graphs of finitely presented groups and therefore have countably many isometric classes.

1. Introduction

Throughout this paper, we equip every connected simplicial graph X with its usual geodesic metric that assigns length 1 to each edge. To lighten the statements, we adopt the following convention: "a graph" means a connected, locally finite, simplicial graph withouth multiple edges and loops, and "$x \in X$", means that x is a vertex of X. A graph X is entirely determined by the restriction of the distance to the vertex set, because there is no multiple edges and loops. In particular the isomorphism group of the simplicial graph X coincides with the isometry group of the vertex set of X. When G is a group with a finite symmetric generating set S and associated word-length $|\cdot|_S$, the Cayley graph of G with respect to S, denoted (G, S), is the simplicial graph whose vertex set is G with distance $d(g, h) = |g^{-1}h|_S$.

Observe that given an integer $d \geq 2$, any d-regular graph X is covered by the d-regular (infinite) tree T_d. This trivial observation is a "baby case" of the phenomenon studied in this paper.

Following the terminology of [B13, G], given a graph X, we say that Y is R-locally X if for all vertex $y \in Y$ there exists $x \in X$ such that the ball $B_X(x, R)$, and $B_Y(y, R)$, equipped with their intrinsic geodesic metrics, are isometric. We now introduce the central notion studied in this paper.

Definition 1.1 (Local-Global rigidity)**.** Let X be a graph.

- **(LG-rigidity)** Let $R > 0$. X is called *local to global rigid* (for short LG-rigid) at scale R, if every graph which is R-locally X, is covered by X.
- **(SLG-rigidity)** Let $0 < r \leq R$. X is called *strongly local to global rigid* (SLG-rigid) at scales (r, R), for some $0 < r \leq R$, if the following holds. For every graph Y which is R-locally X, every isometry from a ball $B(x, R)$ in X to a ball $B(y, R)$ in Y, its restriction to $B(x, r)$ extends to a covering from X to Y.
- **(USLG-rigidity)** If in addition to the previous condition, the covering extending the partial isometry is unique, then we call X USLG-rigid at scales (r, R).

If there exists R such that X LG-rigid at scale R, then we simply call X LG-rigid. Similarly if for all large enough r there exists R such that X is SLG-rigid (resp. USLG-rigid) at scales (r, R), then X is called SLG-rigid (resp. USLG-rigid).

1.1. **Rigidity results.** Our first remark can now be reformulated as follows: T_d is SLG-rigid at scales (r, r) for all $r > 0$ (observe that it is not USLG-rigid). Let us start with a generalization to quasi-trees. Recall that a quasi-tree is a connected graph which is quasi-isometric to a tree.

Theorem A. *Let X be a quasi-tree whose group of isometries acts cocompactly. Then X is SLG-rigid.*

In particular we deduce the following

Corollary B. *Cayley graphs of virtually free finitely generated groups are SLG-rigid.*

Given a graph X, and some $k \in \mathbf{N}$, we define a polygonal 2-complex $P_k(X)$ whose 1-skeleton is X, and whose 2-cells are m-gons for $0 \leq m \leq k$, defined by simple loops $(x_0, \ldots, x_m = x_0)$ of length m in X, up to cyclic permutations.

Definition 1.2. Let us say that a graph X is simply connected at scale k (for short, k-simply connected) if $P_k(X)$ is simply connected. If there exists such a k, then we shall say that X is large-scale simply connected.

Note that k-simple connectedness automatically implies k'-simple connectedness for any $k' \geq k$. We say that a sequence of graphs Y_n is asymptotically k-simply connected if for every r, there exists $r' > r$ and $n_r \in \mathbf{N}$ such that for every $n \geq n_r$, and every $x \in Y_n$, every cycle in $B(x, r)$ is trivial in $P_k(B(x, r'))$.

As mentioned above the regular tree is LG-rigid at any positive scale. Conversely, it is easy to see that a connected d-regular graph which is LG-rigid at some scale $r < 1/2$ is necessarily simply connected, and hence isomorphic to T_d (a wedge a self-loops yields a counterexample for $r = 1/2$). The following proposition generalizes this fact to higher scales.

Proposition 1.3. *If a vertex-transitive graph is LG-rigid at scale $R \in \mathbf{N}$, then it is simply connected at scale $2R$.*

This is tight as shown by the standard Cayley X graph of $\mathbf{Z}^2$. By [BE], X is LG-rigid at scale 2. However, it is obviously not 3-simply connected as the smallest non-trivial simple loops in X have length 4.

Let G be a finitely generated group and let S be a finite symmetric generating subset. It is well-known that the Cayley graph (G, S) is large-scale simply connected if and only if G is finitely presented. More precisely (G, S) is k-simply connected if and only if G has a presentation $\langle S|R\rangle$ with relations of length at most k. By proposition 1.3, it follows that a Cayley graph of a finitely generated group that is not finitely presented is not LG-rigid.

Let us pause here, recalling that the notion of LG-rigidity was introduced by Benjamini and Georgakopoulos in [B13, G]. The main result of [G] is

Theorem. [G] *One-ended planar vertex-transitive graphs are LG-rigid.*

Examples of LG-rigid vertex-transitive graphs also include the standard Cayley graphs of $\mathbf{Z}^d$ [BE]. All these examples are now covered by the following theorem.

Theorem C. *Let X be a connected, large-scale simply connected graph with finite valency whose group of isometries Isom(X) is cocompact (e.g. X is vertex transitive). Then X is USLG-rigid if (and only if) the vertex-stabilizers of Isom(X) are finite, or equivalently if the isometry group of X is discrete.*

Note that Theorem A is not a consequence of Theorem C as the automorphism group of a tree may have infinite vertex-stabilizers. It follows from [Ba97, Theorem 3.1] that the isometry group G of a one ended planar vertex-transitif graph X embeds as a closed (hence discrete) subgroup of either $\mathrm{PSL}(2, \mathbf{R})$ or of $\mathrm{Isom}(\mathbf{R}^2)$. Hence we deduce from Theorem C that X is USLG-rigid, hence recovering Georgakopoulos' result.

Let us say that a finitely presented group is LG-rigid (resp. SLG-rigid, USLG-rigid) if all its Cayley graphs are LG-rigid (resp. SLG-rigid, USLG-rigid). Using some structural results due to Furman (for lattices) and Trofimov (for groups with polynomial growth), we obtain, as a corollary of Theorem C,

Corollary D. *Under the assumption that they are torsion-free and they are not virtually free, the following groups are USLG-rigid:*

- *lattices in simple Lie groups;*
- *groups of polynomial growth.*

In [G], the author asked the following question.

Question 1.4. Are Cayley graphs of finitely presented groups LG-rigid?

We shall see in the next section that this question has a negative answer. Before answering Question 1.4, let us give a useful characterization of LG-rigidity.

Proposition 1.5. *Let $k \in \mathbf{N}$. Let X be a k-simply connected graph with cocompact group of isometries. Then X is LG-rigid if and only there exists R such that every k-simply connected graph which is R-locally X is isometric to it.*

The same proof shows that X is SLG-rigid if and only if for all r there exists $r \leq R$ such that the restriction to a ball of radius r of every isometry from a ball of radius R in X to a ball of radius r in a R-locally X k-simply connected graph Y extends to an isometry from X to Y (and similarly for USLG-rigid).

As a corollary, we have the following result which says that for a sequence of asymptotically k-simply connected graphs to converge in the Benjamini-Schramm topology [BS] to a k-simply-connected LG-rigid graph, it is enough that the balls of a fixed radius converge. This corollary was suggested by Itai Benjamini.

Corollary 1.6. *Let X be a k-simply-connected LG-rigid graph with cocompact group of isometries. There exists R such that the following holds. If Y_n is a sequence of finite graphs such that a proportion $1 - o(1)$ of the balls of radius R in Y_n are isometric to a ball in X, and such that Y_n is asymptotically k-simply connected, then for every R', a proportion $1 - o(1)$ of the balls of radius R' in Y_n are isometric to a ball in X.*

As an almost immediate corollary of the proof of Proposition 1.5, we get

Corollary 1.7. *Let X be a Cayley graph of a finitely presented group. Then there exists $r \leq R$ such that for all Cayley graph Y which is R-locally X, every isometry from a ball of radius r in X to a ball of radius r in Y extends to a covering map from X to Y.*

In other words, Cayley graphs of finitely presented groups are SLG-rigid among Cayley graphs. We shall see later that this is not true among arbitrary graphs, not even among vertex transitive ones.

Finally we mention that in [FT15] an example of an infinite transitive graph X was given, which is isolated among all *transitive* graphs in the sense that there exists R such that X is the only transitive graph which is R-locally X.

1.2. **Flexibility in presence of a finite normal subgroup.** In [ST15] we see that the example of the building of $\mathrm{SL}(n, \mathbf{F}_p((T)))$ gives a counterexample to question 1.4 for $n \geq 3$. Below is a different class of counterexamples.

Theorem E. *Let H be a finitely presented group. Assume that H contains a finitely generated subgroup G such that $H^2(G, \mathbf{Z}/2\mathbf{Z})$ is infinite. Then there a Cayley graph of $H \times \mathbf{Z}/2\mathbf{Z}$ that is not LG-rigid.*

Requiring that G is normal and that H is a semidirect product of G by H/G, we can get a stronger form of non LG-rigidity, where the graphs negating the LG-rigidity are *transitive* graphs:

Theorem F. *Let H be a finitely presented group. Assume that H is isomorphic to a semi-direct product $G \rtimes Q$ such that G is finitely generated and $H^2(G, \mathbf{Z}/2\mathbf{Z})$*

is infinite. Assume moreover that G has an element of infinite order. Then there is a Cayley graph X of $\mathbf{Z}/2\mathbf{Z} \times H$ that is not LG-rigid. More precisely, for every $R \geq 1$, there exists a family with the cardinality of the continuum $(X_i)_{i\in I}$ of large-scale simply connected vertex-transitive graphs that are pairwise non-isometric such that for every $i \in I$,

(i) *X_i is R-locally X and 4-bilipschitz equivalent to X;*
(ii) *there is a surjective continuous proper morphism from a finite index subgroup of the isometry group of X_i onto H whose kernel is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$.*

Remark 1.8. The assumption that G has an element of infinite order is a minor technical assumption that allows to use a variant of Theorem H. Without it we would only be able to prove the Theorem for a Cayley graph of $\mathbf{Z}/N\mathbf{Z} \times H$ for some N.

An explicit example for which Theorem F applies is $H = \mathbf{F}_2 \times \mathbf{F}_2$, and G the kernel of the homomorphism $\mathbf{F}_2 \times \mathbf{F}_2 \to \mathbf{Z}$ which sends each generator of each copy of the free group $\mathbf{F}_2$ to 1. Alternatively, one could also take for H a product of two surface groups of genus at least 2. This probably well-known fact was explained to us by Jean-Claude Sikorav. We could not find a reference in the literature and instead provide a proof in Appendix A.

In particular, we deduce that Theorem E applies to any finitely presented group H containing $\mathbf{F}_2 \times \mathbf{F}_2$. For instance we deduce from Remark 1.8 that for all $n \geq 4$, $\mathrm{PSL}(n, \mathbf{Z}) \times \mathbf{Z}/2\mathbf{Z}$ admits a Cayley graph which is not LG-rigid. Since for n even, we have $\mathrm{PGL}(n, \mathbf{Z}) \simeq \mathrm{PSL}(n, \mathbf{Z}) \times \mathbf{Z}/2\mathbf{Z}$, this shows that the torsion-free assumption in Corollary D is not superfluous. We end this discussion with the following question.

Question 1.9. Among lattices in semi-simple Lie groups, which ones are LC-rigid? For instance is $\mathrm{PSL}(3, \mathbf{Z})$ SLG-rigid?

Note that since large-scale simply connected Cayley graphs are precisely Cayley graphs of finitely presented groups, there are countably many such isomorphism classes of such graphs. Cornulier asked whether there exist uncountably many isomorphism classes of large-scale simply connected vertex-transitive graphs. The previous theorem answers positively this question. It would be interesting to know whether there exist uncountably many quasi-isometry classes of large-scale simply connected vertex-transitive graphs. Observe that this is not answered by our result.

1.3. **Cayley graphs with discrete isometry group.** We conjecture that every finitely generated group has a Cayley graph (without multiple edges) with discrete isometry group. In the general case the closest to this conjecture that we can get is the following theorem.

Theorem G. *Let Γ be a finitely generated group. There is a finite cyclic group F and a Cayley graph of $\Gamma \times F$ with discrete isometry group.*

More involved is the following result, where we prove the conjecture providing the group admits an element of infinite order. A variant of this result plays a crucial role in the proof of Theorem F.

Theorem H. *Every finitely generated group Γ with an element of infinite order admits a Cayley graph (Γ, S) with discrete group of isometries. If in addition Γ is finitely presented, we deduce that (Γ, S) is USLG-rigid.*

Let us mention the following consequence, which gives a partial answer to a question by Georgakopoulos [G, Problem 1.2].

Corollary I. *Let Γ is a finitely presented group with an element of infinite order. If all the Cayley graphs X of Γ admit a sequence $(Y_n)_n$ of finite graphs which are n-locally X, then Γ is residually finite.*

1.4. **From graphs to cocompact geodesic metric spaces.** Finally, one may wonder whether Theorem C can be generalized to more general geodesic metric spaces. The following construction provides serious limitations to this hope.

Theorem J. *The exists a metric space X with the following properties.*

(i) *X is proper, geodesic, and contractible.*
(ii) *$Isom(X) \simeq \mathbf{Z}$ (in particular it has trivial points stabilizers).*
(iii) *$Isom(X)$ is cocompact. More precisely, there exists $x \in X$ such that $Isom(X) \cdot B(x, 1) = X$.*
(iv) *For every R, there exists a continuum of pairwise non isometric metric spaces Y_R which are R-locally X and satisfying (i), (ii) and (iii).*
(v) *For every R, there exists a continuum of pairwise non isometric metric spaces Y'_R which are R-locally X but have a trivial isometry group.*
(vi) *For every R, there exists a continuum of pairwise non isometric metric spaces Y''_R which are R-locally X have an uncountable isometry group (cocompact or not).*

Organization of the paper. The paper is organized as follows. Section 2 and 3 contain preliminaries on large scale simple connectedness and the proofs of Propositions 1.3 and 1.5. Section 4 and 5 contain our rigidity results for quasi-trees (Theorem A) and graphs with discrete isometry groups (Theorem C) respectively. In Section 6, we prove Corollary D. Sections 7 and 8 contain the proofs of Theorem E and F respectively, using for Theorem F the content of Section 9. Theorems G and H are proved in Section 9. Finally, the proof of Theorem J is provided in Section 10.

2. Preliminaries about k-simple connectedness

Except for the following paragraph, dealing with the quasi-isometry invariance of large-scale simple connectedness, the following material is not needed in the rest of the paper, but we include it in order to advertise the naturality of the 2-complex $P_k(X)$ for vertex-transitive graphs.

2.1. **Invariance under quasi-isometry.** Given two constants $C \geq 1$ and $K \geq 0$, a map $f : X \to Y$ between two metric spaces is a (C, K)-quasi-isometry if every $y \in Y$ lies at distance $\leq K$ from a point of $f(X)$, and if for all $x, x' \in X$,

$$C^{-1}d_X(x, x') - K \leq d_Y(f(x), f(x')) \leq Cd_X(x, x') + K.$$

Theorem 2.1. *Let $k \in \mathbf{N}^*$, $C \geq 1$, $K \geq 0$ and let X be a graph. Then there exists $k' \in \mathbf{N}^*$ such that every graph Y such that there exists a (C, K)-quasi-isometry from X to Y, is k'-simply connected.*

Proof. Since this is well-known, we only sketch its proof (which roughly follows the same lines as the proof of [CH, Proposition 6.C.4]). The strategy roughly consists in showing that simple k-connectedness is equivalent to a property that is defined in terms of the metric space X, and which will obviously be invariant under quasi-isometries (up to changing k).

In the sequel, a path γ joining two vertices x to x' in a graph X is a sequence of vertices $(x = \gamma_0, \ldots, \gamma_n = x')$ such that γ_i and γ_{i+1} are adjacent for all $0 \leq i < n$. We consider the equivalence relation $\sim_{k,x,x'}$ between such paths $\gamma = (\gamma_0, \ldots, \gamma_n)$ and $\gamma = (\gamma'_0, \ldots, \gamma'_{n'})$ generated by $\gamma \sim_{k,x,x'} \gamma'$ if they "differ by at most one 2-cell", i.e. if $n = j_1 + j_2 + j_3$, $n' = j_1 + j'_2 + j_3$ such that

- $\gamma_i = \gamma'_i$ for all $i \leq j_1$;
- $\gamma_{j_1+j_2+i} = \gamma'_{j_1+j'_2+i}$ for all $i \leq j_3$;
- $j_2 + j'_2 \leq k$.

We leave as an exercice the fact that $P_k(X)$ is simply connected if and only if for all x, x', the equivalence relation $\simeq_{k,x,x'}$ has a single equivalence class. Note that this reformulation allows to work directly in the graph X. But it still has the disadvantage that it is defined in terms of combinatorial paths in X, based on the notion of adjacent vertices (which does not behave well under quasi-isometries). In order to solve this issue, but at the cost of changing k, we now define a more flexible notion of paths in X: given a constant $C > 0$, we call a C-path in X from x to x' a sequence $x = \eta_0, \ldots, \eta_n = x'$ such that $d(\eta_i, \eta_{i+1}) \leq C$ for all $0 \leq i < n$. Given some $L > 0$, we define the equivalence relation $\sim_{C,L,x,x'}$ between C-paths joining x to x' generated by the relation $\eta \sim_{C,L,x,x'} \eta'$ if $n = j_1 + j_2 + j_3$, $n' = j_1 + j'_2 + j_3$ such that

- $\eta_i = \eta'_i$ for all $i \leq j_1$;
- $\eta_{j_1+j_2+i} = \eta'_{j_1+j'_2+i}$ for all $i \leq j_3$;
- $j_2 + j'_2 \leq L$.

It is easy to see that if X is k-simply connected, then for every C, there exists L such that for all x, x' the equivalence relation $\sim_{C,L,x,x'}$ has a single equivalence class. Conversely, if for some $C \geq 1$ and L, the equivalence relation $\sim_{C,L,x,x'}$ has a single equivalence class for all $x, x' \in X$, then X is k'-simply connected for some k' only depending on C and L. Now the latter condition is designed to be invariant under quasi-isometries, so we are done. $\square$

2.2. **Cayley-Abels graph.** Let X be a locally finite vertex-transitive graph, and let G be its full group of isometry. Recall that G is locally compact for the compact open topology. Given some vertex v_0, denote by K the stabilizer of v_0 in G: this is a compact open subgroup. Let S be the subset of G sending v_0 to its neighbors. One checks that S is a compact open symmetric generating subset of G and that S is bi-K-invariant: $S = KSK$. It follows that the Cayley graph (G, S) is invariant under the action of K by right translations, and that X naturally identifies to the quotient of (G, S) under this action. Conversely, given a totally discontinuous, compactly generated, locally compact group G, one can construct a locally finite graph on which G acts continuously, properly, and vertex-transitively. To do so, just pick a compact open subgroup K and a compact symmetric generating set T, define $S = KTK$ and consider as above the quotient of the Cayley graph (G, S) by the action of K by right translations (note that the vertex set is just G/K). This construction, known as the Cayley-Abels graph (G, K, S) of G with respect to S and K generalizes the more classical notion of Cayley graph, which corresponds to the case where $K = 1$ (and G is discrete).

2.3. **Cayley-Abels 2-complex.** We start by recalling some basic fact about group presentation and presentation complex for abstract groups (not necessarily finitely generated). Let G be a group, and let S be a symmetric generating subset of G. We consider the Cayley graph (G, S) as a graph whose edges are labelled by elements of S. Let R be a subset of the kernel of the epimorphism $\phi : F_S \to G$. Consider the polygonal 2-complex $X = X(G, S, R)$, whose 1-skeleton is the Cayley graph (G, S), and where a k-gone is attached to every k-loop labeled by an element of R. It is well-known that X is simply-connected if and only if the normal subgroup generated by R is $\ker \phi$. In this case, $\langle S; R\rangle$ defines a presentation of G, and X is called the Cayley 2-complex (or presentation complex) associated to this presentation.

The proof of this statement extends without change to the following slightly more general setting: assume that K is a subgroup of G such that $S = KSK$, and consider the Cayley-Abels graph (G, K, S). Let v_0 the vertex corresponding to K in (G, K, S).

Consider the polygonal 2-complex $X = X(G, S, R)$, whose 1-skeleton is the Cayley-Abels graph (G, K, S), and where a k-gone is attached to every k-loop which is obtained as the projection in X of a k-loop labelled by some element of R in (G, S). Once again, one checks X is simply-connected if and only if R generates $\ker \phi$. In this case, $\langle S; R\rangle$ defines a presentation of G, and we call X the Cayley-Abels 2-complex (or presentation complex) associated to this presentation.

2.4. **Compact presentability and k-simple connectedness.** Recall that a locally compact group is compactly presentable if it admits a presentation $\langle S; R\rangle$, where S is a compact generating subset of G, and R is a set of words in S of length bounded by some constant k. Now let K be a compact open subgroup and let S be a such that $S = KSK$. We deduce from the previous paragraph that the morphism $\langle S; R\rangle \to G$ is an isomorphism if and only if the Cayley-Abels graph (G, K, S) is k-simply connected.

3. Large-scale simple connectedness and LG-rigidity

This section is dedicated to the proofs of the rather straightforward Propositions 1.3 and 1.5. It can be skipped by the reader only interested in our main results.

3.1. Proof of Proposition 1.3.

Lemma 3.1. *Let X and Z be two graphs, and let $R \geq 1$. Assume that X is vertex-transitive, that Z is R-locally X, and that $p : Z \to X$ is a covering map. Then p is an isometry in restriction to balls of radius R.*

Proof. Being a covering map, for all $z \in Z$, $p(B(z, R)) = B(p(z), R)$. Hence the fact that $B(z, R)$ and $B(p(z), R)$ have same cardinality implies that p must be injective in restriction to $B(z, R)$. Hence we are done. □

We obtain as an immediate corollary:

Corollary 3.2. *Let X be a vertex-transitive graph. Every self-covering map $p : X \to X$ is an automorphism.*

We shall use the following notion as well. If X is a graph and $k \in \mathbf{N}$, the *k-universal cover of X* is the 1-skeleton of the universal cover of $P_k(X)$. For example, if X is a Cayley graph (G, S), then the k-universal cover of X is the Cayley graph $(\widetilde{G}, S)$ where $\widetilde{G}$ is given by the presentation $\langle S|R\rangle$, with R the words of length at most k that are trivial in G.

Lemma 3.3. *Let X be a graph and $k \in \mathbf{N}$. The k-universal cover of X is k-simply connected.*

Proof. Let Q be the universal cover of $P_k(X)$ and Z its 1-skeleton, *i.e.* the k-universal cover of X. Observe that the 2-cells of Q consist of m-gons for some $m \leq k$, that are attached to simple loops of length m in Z. Hence $P_k(Z)$ is obtained from Q by possibly attaching more 2-cells. It follows that $P_k(Z)$ is simply connected. □

Let us turn to the proof of the proposition. Let Z by the $2R$-universal cover of X, and let $p : Z \to X$ be the covering map. Note that p has injectivity radius $\geq R$, from which it follows that Z is R-locally X. Hence we have a covering map $q : X \to Z$. By Corollary 3.2, $q \circ p$ is an automorphism, implying that p is injective and therefore is a graph isomorphism. Hence it follows that $X = Z$. By Lemma 3.3, Z (and therefore X) is $2R$-simply connected, so we are done.

3.2. **Proof of Proposition 1.5 and of Corollary 1.7 and 1.6.** Since X has a cocompact group of isometries, there are only finitely many orbits of vertices. Therefore, since $P_k(X)$ is simply connected, for all $R_1 \in \mathbf{N}$ there exists $R_2 \in \mathbf{N}$ such that every loop in X based at some vertex x and contained in $B(x, R_1)$ can be filled in inside $P_k(B(x, R_2)) \subset P_k(X)$. It turns out that Proposition 1.5 can be derived from a more general statement, which requires the following definition (which is a variant of Gromov's filling function [Gr93]).

Definition 3.4. We define the k-Filling function of a graph X as follows: for every $R_1 > 0$, $\mathrm{Fill}_X^k(R_1)$ is the infimum over all $R_2 \geq R_1$ such that every loop based at some vertex $x \in X$ and contained in $B(x, R_1)$ can be filled in inside $P_k(B(x, R_2))$.

Note that even if X is k-simply connected but does not have a cocompact isometry group, Fill_X^k can potentially take infinite values. Proposition 1.5 is now a corollary of

Proposition 3.5. *Let X be k-simply connected graph with finite k-Filling function, then X is LG-rigid if and only there exists R such that every k-simply connected graph which is R-locally X is isometric to it (a similar statement holds for SLG and USLG-rigidity).*

This proposition will from Lemma 3.3 and

Lemma 3.6. *Let X be k-simply connected graph with finite k-Filling function. For every $R_1 > 0$, there exists R_2 such that if a graph is R_2-locally X then its k-universal cover is R_1-locally X.*

Proof. Let $R_1 > 0$. Take $R_2 > \mathrm{Fill}_X^k(R_1)$, and assume that a graph Y is R_2-locally X. Let $p\colon Z \to Y$ be its k-universal cover. We claim that p is injective in restriction to balls of radius R_1: this implies that Z is R_1-locally Y, and hence R_1-locally X because $R_2 \geq R_1$, and we are done. Indeed, let $y \in Y$, and $z \in Z$ such that $p(z) = y$. Now let z_1 and z_2 two elements of $B(z, R_1)$ such that $p(z_1) = p(z_2) = y'$. We let γ_1 and γ_2 two geodesic paths joining z respectively to z_1 and z_2, and we let $\bar{\gamma}_1$, and $\bar{\gamma}_2$ be the corresponding paths in Y, both joining y to y'. The concatenation of $\bar{\gamma}_1$ with the inverse of $\bar{\gamma}_2$ defines a loop α based at y and contained in $B(y, R_1)$. But since Y is R_2-locally X, α can be filled in inside $P_k(B(y, R_2))$, and in particular inside $P_k(Y)$. From the assumption that $p\colon z \to Y$ is the k-universal cover, we deduce that $z_1 = z_2$. Hence the claim is proved. □

Lemma 3.7. *Let $\phi : X \to Y$ be a covering map from a graph X to a k-simply connected graph Y. If ϕ is injective in restriction to balls of radius $[k/2]+1$, then it is an isomorphism.*

Proof. The assumption on the injectivity radius implies that ϕ induces a covering map $\tilde{\phi} : P_k(X) \to P_k(Y)$. The conclusion follows from the fact that, $P_k(Y)$ being simply connected, $\tilde{\phi}$ must be a homeomorphism. □

Proof of Proposition 3.5. We shall only prove the first statement, the other two being very similar. Let us assume first that X is R-LC rigid for $R \geq [k/2]+1$, and let Y be k-simply connected and R-locally X. Then Y is covered by X, and it follows from Lemma 3.7 that this covering map is an isomorphism. This proves the first implication.

Let us turn to the (more subtle) converse implication. Assume that X is k-simply connected, and that there exists R such that the following holds: every k-simply connected graph which is R-locally X is isometric to it. Let $R_1 = R$, and let R_2 as in Lemma 3.6. If Y is R_2-locally X, then its k-universal cover is R-locally X, and hence is isometric to X. This gives a covering $X \to Y$ and concludes the proof. □

Let us prove Corollary 1.7. Let $X = (G, S)$ the Cayley graph of a finitely presented group. Let $k \in \mathbf{N}$ sucht that X is k-simply connected. Observe that the number of isometry classes of Cayley graphs $Z = (H, S')$ where H is given by a presentation $\langle S', R\rangle$ with $|S| = |S'|$ and with relations of length at most k is bounded by a function of $|S|$ and k. Hence, it follows from an easy compactness argument that for R_1 large enough, if such a Z is R_1-locally X, it is isometric to X.

Let $R_2 > 0$ given by Lemma 3.6 for X. Let $Y = (H_0, S')$ be a Cayley graph R_2-locally X. Then its k-universal cover is R_1-locally X, and is the Cayley graph (H, S') for the group H given by the presentation $\langle S'|R\rangle$ where R is the set of words of length less than k that are trivial in H_0. It is therefore isometric to X. This implies that X covers Y and proves Corollary 1.7.

Finally we prove Corollary 1.6. By Proposition 1.5 there exists $R \geq 2$ such that every k-simply connected graph which is R-locally X is isomorphic to X. We prove Corollary 1.6 for this R. Let $B_n \subset Y_n$ denote the set of (bad) vertices $y \in Y_n$ such that $B(y, R)$ is not isometric to a ball in X. If d is the maximum degree of a vertex in X, then the set of points at distance 1 from B_n is at most $d|B_n|$, because every such point has a neighbour in B_n, and (because $R \geq 2$) this neighbour has degree at most d. By the same argument, the cardinality of the set of points in Y_n at distance at most r from B_n is bounded above by $|B_n|(1+rd^r)$. In particular there exists a sequence r_n going to infinity such that, for a proportion $1-o(1)$ of the vertices in $y \in Y_n$, $B(y, r_n)$ does not intersect B_n. Denote by C_n the set of all such vertices. We claim that for every $R' > 0$, there exists $n(R')$ such that $B(y, R')$ is isometric to a ball in X for every $y \in C_n$ and every $n \geq n(R')$. If this was not true, we could find a sequence n_k going to infinity, a vertex $y_k \in C_{n_k}$ such that $B(y_k, R')$ is constant and different from a ball in X. By extracting a subsequence, we can even assume that $B(y_k, R'')$

converges for every $R'' > R'$, to the ball of radius R'' around y of some graph that we denote by Y. Then Y is k-simply connected as a limit of asymptotically k-simply connected graphs. Also, Y is R-locally X because every ball $B(y', R) \subset Y$ is isometric to a ball $B(y'_k, R)$ around a point y'_k at distance at most $d(y, y')$ from y_k for infinitely many k's; in particular, taking a k such that $r_{n_k} \geq d(y, y')$, this ball is isometric to a ball of radius R in X by the definition of C_n. Therefore Y is isometric to X, and in particular $B(y, R')$ (which coincides with $B(y_k, R')$ for every k) is isometric to a ball in X. This is a contradiction, and proves the corollary.

4. The case of quasi-trees: proof of Theorem A

We start with an elementary Lemma.

Lemma 4.1. *Let X be a graph with cocompact isometry group. Given some $r \geq 0$, there exists r_2 such that :*

- *for every $x \in X$, the restriction to $B_X(x, r)$ of an isometry $f\colon B_X(x, r_2) \to X$ coincides with the restriction of an element of $Isom(X)$.*
- *if $R > r_2$ and if Y is R-locally X and $x \in X$, then the restriction to $B_X(x, r)$ of an isometry $f\colon B_X(x, r_2) \to Y$ coincides with the restriction of an isometry $B_X(x, R) \to Y$.*

Proof. By the assumption that $\mathrm{Isom}(X)$ acts cocompactly there are finitely many orbits of vertices, and we can restrict ourselves to the case when x and $f(x)$ belong to some fixed finite subset of the vertices in X. Then the statement follows from a straightforward compactness argument: if this was not true, there would exist a sequence of isometries $f_n\colon B_X(x, n) \to X$ such that $(f_n(y))_n$ is a stationary sequence for all y, but f_n does never coincide on $B_X(x, r_1)$ with an element of G. Then $f = \lim_n f_n$ is a well-defined isometry of X, a contradition.

The second statement follows from the first. □

This lemma is the starting point of our approach for building a covering $X \to Y$ if Y is R-locally X in Theorem A and C. Indeed, we can start from an isometry $f_0\colon B_X(x_0, R) \to Y$. By the Lemma if $d(x, x') \leq R - r_2$, we can define another isometry $B_X(x', R) \to Y$ that coincides with f_0 on $B_X(x', r)$. If we have a sequence $x_0, \ldots, x_n$ in X with $d(x_i, x_{i-1}) \leq R - r_2$, we can therefore define $f_i\colon B_X(x_i, R) \to Y$ such that f_i and f_{i-1} coincide on $B(x_i, r)$. In this way, by choosing a path from x_0 to x we can define an isometry $f_x\colon B(x, R) \to Y$ for each $x \in X$, but such a construction depends on the choice of the path. We will be able to make this idea work in two cases. The first and easiest case is when X is a quasi-tree (Theorem A), in which case we can define a prefered path between any two points. The second harder case will be the situation in which f_x does not depend on the path; it is Theorem C.

Lemma 4.2. *Let X be a connected simplicial graph that is quasi-isometric to a tree. Then there exists $r_1 > 0$, a tree T, and a open covering $X = \cup_{u \in V(T)} O_u$ such that for each $u \neq v \in V(T)$,*

- *O_u has diameter less than r_1 (for the distance in X).*
- *$O_u \cap O_v \neq \emptyset$ if and only (u, v) is an edge in T.*

Proof. Consider $V(X)$ the 0-squeleton (the set of vertices) of X. There is a simplicial tree T and a *surjective* quasi-isometry $q \colon V(X) \to V(T)$ (see [KM08] for an explicit construction). Extend q to a continuous quasi-isometry $X \to T$, by sending an edge to the geodesic between the images by q of their endpoints. Define O_u as the preimage of $B_T(u, 2/3)$ by q. We leave it to the reader to check the required properties. □

Proof of Theorem A. Let X be a connected graph that is quasi-isometric to a tree and with cocompact isometry group. Let r_1, T and $(O_u)_{u \in T}$ given by Lemma 4.2.

Let $r \geq r_1$ and let r_2 given by Lemma 4.1 for this value of r.

We define $R = r + r_2$ and we will prove that X is SLG-rigid at scales (r, R).

Let Y be a space R-locally X. Let ϕ_0 be an isometry from $B(x_0, R)$ to Y. Let $O_u^{r_2} = \{x \in X, d(x, O_u) \leq r_2\}$ be the r_2-neighborhood of O_u. Our goal is to construct isometries $\phi_u \colon O_u^{r_2} \to Y$ such that for all $u, v \in V(T)$, (1) ϕ_u coincides with ϕ_0 on $O_u \cap B(x_0, r)$ (2) ϕ_u and ϕ_v coincide on $O_u \cap O_v$. This will prove the Theorem, since then the map ϕ defined by $\phi(x) = \phi_u(x)$ if $x \in O_u$ is a covering that is well defined by (2) and that coincides with ϕ_0 on $B(x_0, r)$ by (1).

Consider $S_0 = \{u \in V(T), B(x_0, r) \cap O_u \neq \emptyset\}$. Using that $B(x_0, r)$ is connected and that $O_u \cap O_v$ intersect only when u and v are adjacent in T, we see that S_0 is connected. We start from this subtree and take $(S_n)_{n \geq 0}$ an increasing sequence of connected subtrees of T that covers T, such that S_n is obtained from S_{n-1} by adding a vertex. We construct by induction maps ϕ_u for $u \in S_n$, that satisfy (1) and (2) for all $u, v \in S_n$.

We start with $n = 0$. For $u \in S_0$, we have $O_u^{r_2} \subset B(x_0, R)$ and we can define ϕ_u as the restriction of ϕ_0 to $O_u^{r_2}$. It is clear that (1) and (2) hold for all $u, v \in S_0$.

If $n \geq 1$ and $S_n = \{v\} \cup S_{n-1}$, take $u \in S_{n-1}$ the vertex adjacent to v. To ensure that (1) and (2) hold on S_n, we only have to construct $\phi_v \colon O_v^{r_2} \to Y$ that coincides with ϕ_u on $O_u \cap O_v$, because O_v intersects neither $B(x_0, r)$ nor $O_{u'}$ for the others $u' \in S_{n-1}$. Let $x \in O_u \cap O_v$. By Lemma 4.1, there is an isometry $\widetilde{\phi} \colon B(x, R) \to Y$ that coincides with ϕ_u on $B(x, r)$, and in particular on O_u because $r_1 \leq r$. We define ϕ_v as the restriction of $\widetilde{\phi}$ to $O_v^{r_2}$, which makes sense because $O_v^{r_2} \subset B(x, R)$. □

5. USLG-rigidity

The goal of this section is to study USLG-rigidity. If a graph X is USLG-rigid at scales (r, R), in particular two isometries of X that coincide on a ball of radius r must be equal. In other words the isometry group is discrete. Theorem C, that we prove later in this section, is a reciprocal of this. Before that we notice that covers in USLG-rigid graphs have a very special form.

Proposition 5.1. *If X is a USLG-rigid at scales (r, R), then for every graph Y that is R-locally X, there is a group H acting freely by isometries on X and an isomorphism $H\backslash X \to Y$ that is injective on balls of radius R.*

Proof. Let $p\colon X \to Y$ be a covering as given by LG-rigidity. Define the group $H = \{g \in \mathrm{Aut}(X), p(gx) = p(x) \forall x \in X\}$. Clearly p induces $H\backslash X \to Y$. Let us show that this map is injective. Let $x_1, x_2 \in X$. Assume that $p(x_1) = p(x_2) = y$. We want to find $g \in H$ such that $gx_1 = x_2$. Let $\psi\colon B_Y(y, R) \to X$ be an isometry. Using that X is R-locally X and that X is SLG-rigid, we see that there exist $g_1, g_2 \in \mathrm{Aut}(X)$ which coincide with $\psi \circ p$ on $B_X(x_i, r)$. In particular $g = g_2^{-1} g_1$ in an element of $\mathrm{Aut}(X)$ such that $gx_1 = x_2$. To see that g belongs to H and conclude the proof of the proposition, notice that p and $p \circ g$ are coverings of Y by X that coincide on $B_X(x_1, r)$. By the uniqueness of such a covering, $p = p \circ g$ as desired. ☐

We record here the following consequence of Proposition 5.1, that will be used in Corollary I.

Lemma 5.2. *Let (Γ, S) be a Cayley graph which is USLG-rigid. If there exists a sequence of finite graphs $(Y_n)_{n \in \mathbf{N}}$ that are n-locally (Γ, S), then Γ is residually finite.*

Proof. Let $0 < r \leq R$ such that $X = (\Gamma, S)$ is USLG-rigid at scales (r, R).

To prove that Γ is residually finite, for every finite set F in Γ we construct an action of Γ on a finite set such that the elements in $F \setminus \{1_\Gamma\}$ have no fixed point. To do so take a finite set F in Γ, and pick $n \geq R$ such that F is contained in the ball of radius $2n$ around the identity in (Γ, S). By the assumption there is a finite graph Y that is n-locally X. Since X is USLG-rigid at scales (r, R) and $R < n$, by Proposition 5.1 there is a subgroup $H \subset G$ that acts freely on X such that Y identifies with $H\backslash X$. In particular the action of Γ by right-multiplication on the vertex set of X passes to the quotient $H\backslash X$, and non-trivial elements of length less than $2n$ in Γ have no fixed point. In particular no element of $F \setminus \{1_\Gamma\}$ has a fixed point. This shows that Γ is residually finite. ☐

5.1. **Proof of Theorem C.** Let X be as in Theorem C. Let $k > 0$ such that X is k-simply connected. Denote G the isomorphism group of X. By the assumption that G is discrete and cocompact, there exists $r_c \geq 0$ such that if two isometries g and g' in G coincide on a ball $B_X(x, r_c)$ of radius r_c, then they are equal.

We shall prove the following precise form of Theorem C.

Proposition 5.3. *There exists $C > 0$ such that X is USLG-rigid at scales $(r, r + C)$ for every $r \geq r_c$.*

We shall need the following lemma.

Lemma 5.4. *Given $r_1 \geq r_c$, there exists $r_2 \geq r_1$ such that the following holds:*

- *for every $x \in X$, the restriction to $B_X(x, r_1)$ of an isometry $f\colon B_X(x, r_2) \to X$ coincides with the restriction of an element of G;*
- *the restriction to $B_X(x, r_1)$ of an isometry $f\colon B_X(x, r_2) \to X$ is uniquely determined by its restriction to $B_X(x, r_c)$.*

Proof. The first part is Lemma 4.1.

For the second part, let $f, g\colon B_X(x, r_2) \to X$ be two isometries which coincide on $B_X(x, r_c)$. By the first part there exists $f', g' \in G$ which coincide with f and g respectively on $B_X(x, r_1)$. Since $f' = g'$ on $B_X(x, r_c)$, we get $f' = g'$, and in particular $f = g$ on $B_X(x, r_1)$. ☐

Remark 5.5. This lemma applied to $r_c + 1$ provides us with $r_2^{(r_c+1)}$ such that if Y is $r_2^{(r_c+1)}$-locally X and $\phi_1, \phi_2\colon X \to Y$ are covering maps that coincide on $B(x, r_c)$, then they coincide on $B(x, r_c + 1)$ (and hence everywhere since X is connected). This implies the following : if we are able to prove that X is SLG-rigid at some scales (r, R) for $r \geq 1 + r_c$, then X is USLG-rigid at scales $(r_c + \delta, \max(R + \delta, r_2^{(r_c+1)}))$ for all $\delta \geq 0$. Indeed, if $\phi\colon B(x, \max(R+\delta, r_2^{(r_c+1)})) \to Y$ is an isometry, we can apply that X is SLG-rigid at scales (r, R) to the restriction of ϕ to $B(x', R)$ for every $x' \in B(x, \delta)$, and get a covering $\widetilde{\phi}_{x'}\colon X \to Y$ that coincides with ϕ on $B(x, r)$. If $x', x'' \in B(x, \delta)$ satisfy $d(x', x'') \leq 1$, the covering $\widetilde{\phi}_{x''}$ coincides with ϕ on $B(x'', r)$, and in particular on $B(x', r_c)$ because $r \geq r_c + 1$. By our property defining $r_2^{(r_c+1)}$, we have $\widetilde{\phi}_{x''} = \widetilde{\phi}_{x'}$. Since $B(x, \delta)$ is geodesic we get that $\widetilde{\phi}_x = \widetilde{\phi}_{x'}$ for all $x' \in B(x, \delta)$, and in particular $\widetilde{\phi}_x$ coincides with ϕ on $B(x, r + \delta)$. This proves that there exists a covering $\widetilde{\phi}\colon X \to Y$ which coincides with ϕ on $B(x, r_c + \delta)$. It is unique as the unique covering that coincides with ϕ on the smaller ball $B(x, r_c)$.

Take now $r_1 = r_c + t$ for some $t > 0$ to be determined later, and $r_2 \geq r_1$ the radius given by Lemma 5.4. Take Y a space R-locally X with $R \geq r_2 + t$. For every $x \in X$ denote by $\mathrm{germ}(x)$ the set of all isometries $\phi\colon B_X(x, r_1) \to Y$ that are restrictions of an isometry $B_X(x, r_2) \to Y$.

Lemma 5.6. *Let $x, x' \in X$ with $d(x, x') \leq t$ and $\phi \in germ(x)$. Then there is one and only one element of $germ(x')$ that coincides with ϕ on $B(x', r_c)$, and it coincides with ϕ on $B(x, r_1) \cap B(x', r_1)$.*

Proof. For the existence, by Lemma 5.4 and the fact that balls of radius R in Y are isometric to balls of radius R in X, $\phi \in \mathrm{germ}(x)$ is the restriction to $B(x, r_1)$ of (at least) one isometry $\widetilde{\phi}\colon B(x, R) \to Y$. Then the restriction of $\widetilde{\phi}$ to $B(x', r_2)$ is an isometry and hence defines an element of $\mathrm{germ}(x')$ that coincides with ϕ on $B(x, r_1) \cap B(x', r_1)$.

The uniqueness also follows from Lemma 5.4, which implies that every element of $\mathrm{germ}(y)$ is determined by its restriction to $B(y, r_c)$. ☐

Proposition 5.7. *Assume that $t \geq \frac{k}{2}$. There is a unique family $(F_{x,x'})_{x,x' \in V(X)}$ where*

(1) $F_{x,x'}$ is a bijection from germ$(x) \to$ germ(x').
(2) If $d(x,x') \leq t$ and $\phi \in$ germ(x), then $F_{x,x'}(\phi)$ is the unique element of germ(x') that coincides with ϕ on $B(x', r_c)$.
(3) $F_{x',x''} \circ F_{x,x'} = F_{x,x''}$ for all $x, x', x'' \in X$.

Proof. If $(x,x') \in X$ satisfy $d(x,x') \leq t$ Lemma 5.6 provides a bijection

$$F^{(0)}_{x,x'} \colon \mathrm{germ}(x) \to \mathrm{germ}(x')$$

satisfying (2).

For every sequence $(x_1, \dots, x_n)$ of vertices of X where $d(x_i, x_{i+1}) \leq t$ we define $F_{(x_1,\dots,x_n)} \colon \mathrm{germ}(x_1) \to \mathrm{germ}(x_n)$ by composing the bijections $F^{(0)}_{x_i,x_{i+1}}$ along the path. Moreover Lemma 5.6 implies that

$$F_{(x_1,\dots,x_n)} = F^{(0)}_{x_1,x_n} \text{ if } \mathrm{diam}(\{x_1,\dots,x_n\}) \leq t. \tag{5.1}$$

Let $\gamma \colon [a,b] \to X$ be an isometry. For every subdivision $a = a_1 \leq a_2 \leq \dots a_n = b$ with $a_{i+1} - a_i \leq t$, we can consider $F_{\gamma(a_1),\dots,\gamma(a_n)} \colon \mathrm{germ}(\gamma(a)) \to \mathrm{germ}(\gamma(b))$, and by (5.1) $F_{\gamma(a_1),\dots,\gamma(a_n)}$ is unchanged if one passes to a finer subdivision, and hence does not depend on the subdivision. Denote this map by $F_\gamma \colon \mathrm{germ}(\gamma(a)) \to \mathrm{germ}(\gamma(b))$.

But again by (5.1), F_γ is invariant under homotopy. Therefore the map $\gamma \mapsto F_\gamma$ induces a map on the fundamental groupoid $\Pi_1(X)$. By the definition of F, F_γ is the identity of germ$(\gamma(a))$ if $|b-a| \leq 2t$ and $\gamma(a) = \gamma(b)$. By the inequality $k \leq 2t$ and the fact that X is k-simply connected, we get that F_γ is the identity of germ$(\gamma(a))$ for all paths γ such that $\gamma(a) = \gamma(b)$. This implies that F_γ depends only on the endpoints $\gamma(a)$ and $\gamma(b)$. We can define $F_{x,x'}$ as the common value of F_γ for all such γ with $\gamma(a) = x$ and $\gamma(b) = x'$, and the existence of F satisfying 1,2,3 in Lemma is proved. The uniqueness is clear since X is connected. ☐

We are ready to prove that X is USLG-rigid. We now fix the value of t to $t = \frac{k}{2}$, so that $r_1 = r + \frac{k}{2}$. Let $f \colon B_X(x_0, R) \to Y$ an isometry. The restriction of f to $B_X(x_0, r_1)$ defines $\phi_0 \in \mathrm{germ}(x_0)$. For every $x \in X$ we define $\phi_x = F_{x_0,x}(\phi_0)$ and $\pi(x) = \phi_x(x)$ for the map given by Proposition 5.7. Then by (2) in the Proposition, π coincides with ϕ_x on $B_X(x, r_1)$ for every $x \in X$. In particular π is a covering map and coincides with f on $B(x_0, r)$. This proves that X is SLG-rigid at scales (r_1, R). This implies Proposition 5.3 by Remark 5.5.

6. Groups whose Cayley graphs all have discrete isometry group

We first observe that a necessary condition on a finitely generated group Γ to have all Cayley graphs with a discrete isometry group (or equivalenty for Γ to be USLG-rigid by Theorem C) is that this group is torsion-free.

Lemma 6.1. *An infinite finitely generated group with a non-trivial torsion element has a Cayley graph, the isometry group of which contains an infinite compact subgroup.*

Proof. Let Γ be infinite and finitely generated, with finite symmetric generating set S. If Γ is not torsion-free, it has a non-trivial finite subgroup F. Then $FSF = \{fsf' | f, f' \in F, s \in S\}$ is an F-biinvariant finite symmetric generating set and we claim that (Γ, FSF) does not have a discrete isometry group. Indeed, any permutation of Γ which preserves all left F-cosets is an isometry of (Γ, FSF). This shows that the isometry group of (Γ, FSF) contains the compact infinite group $\prod_{x \in \Gamma/F} \mathrm{Sym}(x)$. □

We will see in Corollary 6.5 that for a large class of groups (the groups appearing in Corollary D), being torsion-free is also a sufficient condition for all their Cayley graphs to have a discrete isometry group.

In a slightly different direction (Proposition G and Theorem H) we prove that many groups admit a Cayley graph with discrete isometry group.

Let us now turn our attention to the case of lattices in simple Lie groups and groups of polynomial growth. Our goal is to prove Corollary D. Let Γ be as in Corollary D. In order to apply Theorem C, one needs to show that the isometry group of any Cayley graph of Γ is discrete.

We shall use the following easy fact, showing a converse to Lemma 6.1.

Lemma 6.2. *Let Γ be an infinite, torsion-free finitely generated group, and let S be a finite symmetric generating subset of Γ. Then the isometry group of $X = (\Gamma, S)$ has no non-trivial compact normal subgroup.*

Proof. Let $G = \mathrm{Isom}(X)$, and assume by contradiction that G admits a non-trivial compact normal subgroup K. Then there exists a vertex x whose K-orbit Kx contains a vertex y distinct from x. Since Γ acts transitively, there exists $g \in \Gamma$ such that $gx = y$. Since K is normalized by g, we deduce that $gKx = Kgx = Ky = Kx$. Since Γ acts freely, this implies that g has finite order: contradiction. □

Let us denote by $\mathcal{C}$ the class of finitely generated groups satisfying the following property: $\Gamma \in \mathcal{C}$ if every locally compact totally disconnected group G containing Γ as a uniform lattice has an open compact normal subgroup. Observe that a group $\Gamma \in \mathcal{C}$ which is torsion free is USLG-rigid by Theorem C and Lemma 6.2.

Recall the following result of Furman.

Theorem 6.3. [F01] *Let Γ be an irreducible lattice in a connected semisimple real Lie group G with finite center (in case G is locally isomorphic to $PSL(2, \mathbf{R})$, we assume that Γ is uniform). Let H be a locally compact totally discontinuous group such that Γ embeds as a lattice in H. Then there exists a finite index subgroup H_0 of H containing Γ, and a compact normal subgroup K of H_0 such that $H_0/K \simeq \Gamma$. In particular, Γ belongs to $\mathcal{C}$.*

Regarding groups with polynomial growth, we have the following result of Trofimov.

Theorem 6.4. [T85] *Let X be a vertex-transitive graphs with polynomial growth. Then its isometry group has a compact open normal subgroup. In particular, finitely generated groups with polynomial groups belong to $\mathcal{C}$.*

Together with Lemma 6.2, we obtain

Corollary 6.5. *Let X be a Cayley graph of some finitely generated group torsion-free Γ which either has polynomial growth, or is as in Theorem 6.3. Then the isometry group of X is discrete.*

Remark 6.6. In [F01, Corollary 1.5] this Corollary for Γ as in Theorem 6.3 was stated without the hypothesis that is it torsion-free. This hypothesis is necessary as explained in Lemma 6.1.

Corollary D now follows from Theorem C.

7. Graphs that are not LG-rigid: Theorem E

The idea is that the assumption that $H^2(G, \mathbf{Z}/2\mathbf{Z})$ is infinite implies that there are many 2-coverings of Cayley graphs of G. Before we explain this in details, we briefly recall a definition of $H^2(G, \mathbf{Z}/2\mathbf{Z})$ and its connection with central extensions (see [B82]).

Let A be an abelian group (denoted additively). A central extension of a group G (denoted multiplicatively) by A is an extension

$$1 \to A \to E \to G \to 1$$

where the image of A lies in the center of E. Let us recall that two extensions

$$1 \to A \to E_1 \xrightarrow{\tau_1} G \to 1, \quad 1 \to A \to E_2 \xrightarrow{\tau_2} G \to 1$$

are called isomorphic if there is a group isomorphism $\varphi \colon G_1 \to G_2$ such that $\tau_2 \circ \varphi = \tau_1$.

Let us recall how the cohomology group $H^2(G, A)$ parametrizes the central extensions of G by A. The group $H^2(G, A)$ is defined as the quotient of $Z^2(G, A)$, the set of functions $\varphi \colon G^2 \to A$ such that $\varphi(g_1, g_2g_3) + \varphi(g_2, g_3) = \varphi(g_1g_2, g_3) + \varphi(g_1, g_2)$, viewed as an abelian group for pointwise operation, by its subgroup $B^2(G, A)$ of coboundaries, *i.e.* maps of the form $(g_1, g_2) \mapsto \psi(g_1)+\psi(g_2)-\psi(g_1g_2)$ for some $\psi \colon G \to A$. Every $\varphi \in Z^2(G, A)$ gives rise to a central extension

$$1 \to A \to E \to G \to 1$$

together with a (set-theoretical) section $s \colon G \to E$ by setting $E = A \times G$ for the group operation $(a, g_1)(b, g_2) = (a + b + \varphi(g_1, g_2), g_1g_2)$, and $s(h) = (0, h)$. Reciprocally every such central extension and section give rise to an element of $Z^2(G, A)$, by setting $\varphi(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$. Lastly two elements in

$Z^2(G,A)$ give isomorphic extensions if and only if they differ by an element in $B^2(G,A)$.

Lemma 7.1. *Let G be a group with a finite symmetric generating set S. If $H^2(G,\mathbf{Z}/2\mathbf{Z})$ is infinite, there is a sequence of 2-coverings $q_n\colon Y_n \to (G,S)$ such Y_n is connected but $q_n^{-1}(B_S(x,n))$ is disconnected for all $x \in G$.*

Remark 7.2. Actually the graphs Y_n in this Lemma are Cayley graphs of extensions of G by $\mathbf{Z}/2\mathbf{Z}$.

Proof. First we claim that for all $n \geq 1$ there exists $\varphi_n \in Z^2(G,\mathbf{Z}/2\mathbf{Z})$ which is not a coboundary and such that $\varphi_n(g_1,g_2) = 0$ if $|g_1|_S + |g_2|_S \leq n$. This follows from linear algebra considerations: $Z^2(G,\mathbf{Z}/2\mathbf{Z})$ can be viewed as vector space over the field with 2 elements, and our assumption that $H^2(G,\mathbf{Z}/2\mathbf{Z})$ is infinite means that $B^2(G,\mathbf{Z}/2\mathbf{Z})$ is an infinite codimensional subspace. It does therefore not contain the finite codimensional subspace made of the elements $\varphi \in Z^2(G,\mathbf{Z}/2\mathbf{Z})$ that vanish on $\{(g_1,g_2), |g_1|_S + |g_2|_S \leq n\}$.

If $n \geq 2$ and φ_n is as above, consider E_n the central extension of G by $\mathbf{Z}/2\mathbf{Z}$ constructed from φ_n and $S_n = \{(0,s), s \in S\}$. If $s \in S$, since $\varphi_n(1_H,s) = 0$, the unit of E_n is $(0,1_G)$ and since $\varphi(s,s^{-1}) = 0$, we have that $(0,s)^{-1} = (0,s^{-1})$. The set S_n is therefore a finite symmetric set in E_n, and the quotient map $q_n\colon E_n \to G$ induces a 2-covering $q_n\colon Y_n \to (H,S)$. The assumption on φ_n implies that $q_n^{-1}(B_S(1_H,n))$ is the disjoint union of $B_S(1_H,n) \times \{0\}$ and $B_S(1_H,n) \times \{1\}$; in particular it is disconnected. By transitivity $q_n^{-1}(B_S(x,n))$ is disconnected for all $x \in G$. To prove the lemma it remains to observe that Y_n is connected because φ_n is not a coboundary. $\square$

Theorem E now follows from the more general proposition

Proposition 7.3. *Let G be a group with a finite symmetric generating set, and assume that there is a sequence of 2-coverings $q_n\colon Y_n \to (G,S)$ satisfying the conclusion of Lemma 7.1. Then for every finitely generated group H containing G as a subgroup, there is a Cayley graph X_0 of $H \times \mathbf{Z}/2\mathbf{Z}$ that is not LG-rigid.*

To prove the Proposition, we complete S into a finite generating set T of H by adding elements of $H \setminus G$ in a way that will be made precise in Lemma 7.7. This allows to identify the Cayley graph (G,S) as a subgraph of the Cayley graph (H,T). We measure the distorsion of (G,S) in (H,T) by the function $\rho(R) = \sup\{|g|_S \,|g \in G, |g|_T \leq R\}$.

Consider X_0, the Cayley graph of $H \times \mathbf{Z}/2\mathbf{Z}$ for the finite generating set

$$T' = \{(e_H,1)\} \cup (S \times \{0\}) \cup (T \setminus S \times \{0,1\}).$$

Observe that the subgraph with vertex set $G \times \mathbf{Z}/2\mathbf{Z}$ of X_0 is the union of two copies of (G,S) where we added edges between pairs of same vertices.

Now if $q\colon Y \to (G,S)$ is another 2-covering, we can get a new graph denoted X_q, by replacing $G \times \mathbf{Z}/2\mathbf{Z}$ inside X_0 by Y. This means that the vertex set of

X_q is the disjoint union of $(H \setminus G) \times \mathbf{Z}/2\mathbf{Z}$ and Y, equipped with the natural 2-to-1 map $p\colon X_q \to H$. Two vertices in $(H \setminus G) \times \mathbf{Z}/2\mathbf{Z}$ (two vertices in Y) are connected by an edge if they were connected by an edge in X_0 (respectively if there were connected by an edge in Y or if they have the same image in G), and there is an edge between a vertex in $(H \setminus G) \times \mathbf{Z}/2\mathbf{Z}$ and a vertex in Y if there was an edge between their images in (H,T).

We denote by $\sim_q$ the equivalence relation on the vertex set of X_q where $x \sim_q y$ if $p(x) = p(y)$.

We start by a lemma showing that for each $R > 0$, X_{q_n} is R-locally X_0 for n large enough.

Lemma 7.4. *Let $R \in \mathbf{N}$. If the graph $q^{-1}(B_S(x, \rho(2R)))$ is disconnected, then X_q is R-locally X_0.*

Proof. Consider a ball of radius R in X_q. If it does not contain any vertex in Y, it is obvioulsy isometric to the corresponding ball in X_0. Otherwise it contains a point x in Y, and is therefore contained in the ball of radius $2R$ around x. By the definition of ρ its intersection with Y is contained in $q^{-1}(B_S(q(x), \rho(2R))$, which is two disjoint copies of $B_S(q(x), \rho(2R))$ by our assumption. This gives an isometry between the ball of radius $2R$ around x in X_q and a corresponding ball in X_0 and proves that X_q is R-locally X_0. $\square$

Remark 7.5. The proof shows that there is an isometry from every ball of radius R in X_q to X_0 which sends $\sim_0$ to $\sim_q$.

The next observation allows to distinguish in some weak sense the graphs X_{q_n} and X_0.

Lemma 7.6. *If Y is connected, there is no isomorphism between X_q and X_0 sending $\sim_q$ to $\sim_0$.*

Proof. Let us say that a subset E of the edge set of X_q is admissible if it has the property that for every vertex $x \in X_q$, every neighbor of $p(x)$ in (H,T) has a preimage y by p such that $\{x, y\} \in E$.

We claim that X_0 admits an admissible edge set which makes X_0 disconnected, but that X_q does not admit such an admissible edge set if Y is connected. This claim implies the Lemma because an isomorphism between X_q and X_0 sending $\sim_q$ to $\sim_0$ would send an admissible subset of edges to an admissible subset of edges.

The first claim is very easy, as we can just take for E the set

$$E = \{\{(x,i),(y,j)\} \text{ edge of } X_0 | i = j\}.$$

For the second claim, take an admissible edge subset E. Since (H,T) is connected, every vertex in X_q can be connected to an edge of Y by a sequence of edges in E. Also, observe that if $\{x, y\}$ is an edge in X_q that corresponds to an edge S in (H,T), *i.e.* if $p(x)^{-1}p(y) \in S$, then $\{x, y\}$ is the only edge between x and an element of $p^{-1}(p(y))$. This implies that $\{x,y\} \in E$ because E is admissible. In particular E contains all edges in Y. This shows that if Y is connected, X_q with edge set E remains connected, as announced. □

The last step is to observe that for a well-chosen T, an isomorphism between X_0 and X_q necessarily sends $\sim_0$ on $\sim_q$ (at least if q has a large injectivity radius). We start by

Lemma 7.7. *Let $G =< S >\subsetneq H$ as in Proposition 7.3. There is a symmetric $T \subset H$ such that $T \cap G = S$ and every isometry of X_0 preserves $\sim_0$, where X_0 is the Cayley graph of $H \times \mathbf{Z}/2\mathbf{Z}$ for the finite generating set*

$$T' = \{(e_H, 1)\} \cup (S \times \{0\}) \cup (T \setminus S \times \{0,1\}).$$

.

Proof. In the proof, for an arbitrary finite symmetric generating set $T \subset H$ and $t \in T \setminus \{e_H\}$, we will denote (as later in the proof of Theorem H) by $N_3(t,T)$ the number of triangles in the Cayley graph (H,T) containing the vertices e_H and t. To lighten the notation, let us denote $T^* = T \setminus \{e_H\}$.

First pick an arbitrary finite symmetric generating set $T_1 \subset H$ such that $T_1 \cap G = S$. Let $M = \max_{t\in T_1} N_3(t,T_1)$. Observe that replacing T_1 by $T_1 \cup \{h, h^{-1}\}$ for $h \in H \setminus G$ of word-length $|h|_{T_1} > 3$ does not change the function $N_3(\cdot, T_1)$ but increases the cardinality of $|T_1 \setminus G|$. Also, such an h exists because our assumption on G implies that G is infinite, and therefore $H \setminus G$ also. Therefore there exists a finite symmetric subset $T \subset H$ such that $T \cap G = S$ and such that $\max_{t\in T^*} N_3(t,T) + 1 < |T \setminus G|$.

On the other hand, one checks that $N_3((e_H,1),T') = 2|T\setminus S|$, whereas $N_3((t,\varepsilon),T') \leq 2+2N_3(t,T)$ for every $(t,\varepsilon) \in T^* \times \{0,1\}$. The previous formula therefore implies that $N_3((e_H,1),T') > N_3(t',T')$ for every $t' \in T' \setminus \{(e_H,1),(e_H,0)\}$. This means that the $\mathbf{Z}/2\mathbf{Z}$ cosets in $H \times \mathbf{Z}/2\mathbf{Z}$ are characterized in X_0 as the pairs of vertices that belong to exactly $2|T \setminus S|$ triangles in X_0. The conclusion follows. □

We deduce by a straightforward compactness argument from the previous lemma that given some $r > 0$, there exists $R > 0$ such that for every partial isometry between two balls $\phi : B(x,R) \to B(x',R)$, the restriction of ϕ to $B(x,r)$ preserves $\sim_0$. This implies the following

Corollary 7.8. *Let G, H, S, T as in Lemma 7.7. For all $r > 0$, there exists $R > 0$ such that for all Y which is R-locally X_0, there exists a unique equivalence relation $\sim$ on the vertex set of Y such that for all $x \in X$ and $y \in Y$, the restriction to $B(x,r)$ of some partial isometry $\phi : B(x,r) \to B(y,r)$ interwines $\sim_0$ and $\sim$.*

We can now complete the proof of Proposition 7.3. Consider X_{q_n}, the graph constructed from the 2-covering $q_n \colon Y_n \to (G,S)$ given by the assumption of Proposition 7.3, with T given by Lemma 7.7. Lemma 7.4 implies that $X_{q_{\rho(2n)}}$ is

n-locally X_0. Hence any covering map from $\phi_n : X_0 \to X_{q_n}$ must be injective in restriction to the ball of radius n.

Observe that the preimages of the surjective graph morphism $p_n : X_{q_n} \to (H,T)$ have diameter 1. It follows that p_n is a $(0,1)$-quasi-isometry, so that by Theorem 2.1, there exists $k \in \mathbf{N}$ such that X_{q_n} is k-simply connected for all n. Hence, by Lemma 3.7, for n large enough, ϕ_n is an isomorphism. By Remark 7.5 and Corollary 7.8, ϕ_n must send $\sim_0$ to $\sim_q$. This is a contradiction with Lemma 7.6. This implies that X_0 is not LG-rigid and concludes the proof.

8. Graphs that are not LG-rigid: Theorem F

We now move to Theorem F, which will follow from the results in §9 and from

Theorem 8.1. *Let $G \lhd H$ be finitely generated groups, and T a finite generating set of H such that $S := G \cap T$ generates G. Assume moreover that H splits as a semi-direct product $G \rtimes H/G$.*

There exists $C \in \mathbf{N}$ such that the following holds. For every extension

$$1 \to \mathbf{Z}/2\mathbf{Z} \to G_\tau \xrightarrow{\tau} G \to 1$$

and symmetric subset $S_\tau \subset G_\tau$ such that τ maps $\widetilde{S}_\tau$ bijectively onto S, we can associate a graph X_τ such that

1. *X_{τ_0} is a Cayley graph of $H \times \mathbf{Z}/2\mathbf{Z}$ if τ_0 is the trivial extension and $S_{\tau_0} = S \times \{0\}$ (note that it does not generate $G_{\tau_0} = G \times \mathbf{Z}/2\mathbf{Z}$).*
2. *For any two extensions τ and τ', X_τ and $X_{\tau'}$ are 4-Lipschitz equivalent.*
3. *For every $R \in \mathbf{R}_+$, there exists $R_1 \in \mathbf{R}_+$ such that for all extensions τ and τ', the graph $X_{\tau'}$ is R-locally X_τ whenever the covering $(G_{\tau'}, S_{\tau'}) \to (G,S)$ is R_1-locally[1] the covering $(G_\tau, S_\tau) \to (G,S)$.*
4. *If $\max_{t\in T} |tT \cap T| < |T| - |S| - 1$ and (G,S) has a discrete isometry group, then the number of isomorphism classes of extensions τ' such that $X_{\tau'}$ is isomorphic to some given X_τ is at most C.*
5. *If $\max_{t\in T} |tT \cap T| < |T| - |S| - 1$ and (H,T) has a discrete isometry group, then the isometry group of X_τ acts transitively, and has a finite index subgroup which is isomorphic as topological group to an extension of H by the compact group $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$.*

Remark 8.2. In (3), we exceptionally allow a less restrictive notion of graph than in the rest of the paper, as we do not request that S_τ generates G_τ. In that case (G_τ, S_τ) is the disconnected simplicial graph without multiple edges nor loops with vertex set G_τ and with a vertex between x, y if $x^{-1}y \in S_\tau$.

In (3) for a graph Y and two coverings $q_1 \colon \widetilde{Y}^{(1)} \to Y$ and $q_2 \colon \widetilde{Y}^{(2)} \to Y$ we say that q_1 is R_1-locally q_2 if for every ball B of radius R_1 in Y, there is an isometry ϕ between $q_1^{-1}(B)$ and $q_2^{(-1)}(B)$ such that $q_2 \circ \phi = q_1$.that commutes with the projections that two coverings $q_1, q_2 \colon \widetilde{Y}^{(1)} \to Y$.

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[1] See Remark 8.2.

In particular, it follows from (1) and (2), and Theorem 2.1 that there exists k such that all X_τ are k-simply connected.

The rest of this section is devoted to the proof of this theorem, which is very similar to the proof of Proposition 7.3, but involves significantly more work to ensure that items (4) and (5) hold. For the trivial extension, X_{τ_0} coincides with the graph X_0 from §7. For general τ, the graph X_τ is obtained by copying above *every* G-coset in H a copy of the Cayley graph $(\widetilde{G}_\tau, \widetilde{S}_\tau)$, and adding in a suitable way edges (that we call *outer* and *vertical* edges) between different copies. We first study this construction for general graphs, and then specialize to Cayley graphs.

8.1. **The construction in terms of graphs.** Let X, Y be connected graphs, and assume that the vertex set of X is partionned as $X = \sqcup_{i\in I} Y_i$ into subgraphs that are each isomorphic to Y, and fix an isomorphism $f_i \colon Y \to Y_i$ for each $i \in I$. Assume that we are given a 2-covering $q\colon \widetilde{Y} \to Y$. Note that $\widetilde{Y}$ does not need to be connected: in other words, the covering can be trivial. We define a graph $\widetilde{X}$ by putting over each Y_i a copy $\widetilde{Y}_i$ of $\widetilde{Y}$, and connecting two vertices in $\widetilde{Y}_i$ and $\widetilde{Y}_j$ for $i \neq j$ either if their images in X are equal, or if $k = \ell$ and their images in X are connected. Formally, the set of vertices of $\widetilde{X}$ is $\widetilde{Y} \times I$, and there are three types of edges:

1. *inner edges*: there is an edge between $(\widetilde{y}, i)$ and $(\widetilde{y}', i)$ if there is an edge between $\widetilde{y}$ and $\widetilde{y}'$ in $\widetilde{Y}$.
2. *outer edges*: if $i \neq j$, there is an edge between $(\widetilde{y}, i)$ and $(\widetilde{y}', j)$ if and only if there is an edge in X between $f_i(q(\widetilde{y}))$ and $f_j(q(\widetilde{y}'))$.
3. *vertical edges:* We put an edge betweeen $(\widetilde{y}, i)$ and $(\widetilde{y}', i)$ if $\widetilde{y} \neq \widetilde{y}'$ and $q(\widetilde{y}) = q(\widetilde{y}')$.

Then $\widetilde{Y}_i$ is $\widetilde{Y} \times \{i\}$, and there is a natural "projection" map $\widetilde{X} \to X$ sending $(\widetilde{y}, i)$ to $f_i(q(y))$.

We start by a lemma that will be used to show (2) in Theorem 8.1. The rest of this subsection will be a series of Lemma studying the isometries of $\widetilde{X}$.

Lemma 8.3. *If $\widetilde{Y}$, $\widetilde{Y}'$ are 2-coverings of Y and $\widetilde{X}, \widetilde{X}'$ are obtained by the above contruction, then any bijection $f\colon \widetilde{X} \to \widetilde{X}'$ which commutes with the projections $\widetilde{X} \to X$ and $\widetilde{X}' \to X$ is 2-Lipschitz.*

Proof. Let $\widetilde{x}_1$ and $\widetilde{x}_2$ be neighbors in $\widetilde{X}$. Let x_1, x_2 be their images in X, which by assumption are also the images of $f(\widetilde{x}_1), f(\widetilde{x}_2)$ by the projection $\widetilde{X}' \to X$. We have to show that $d(f(\widetilde{x}_1), f(\widetilde{x}_2)) \leq 2$.

If $x_1 = x_2$, then $f(\widetilde{x}_1)$ and $f(\widetilde{x}_2)$ are linked by a vertical edge: $d(f(\widetilde{x}_1), f(\widetilde{x}_2)) = 1$.

If $x_1 \neq x_2$, then the edge between $\widetilde{x}_1$ and $\widetilde{x}_2$ is an inner or an outer edge, and there is an edge between x_1 and x_2 in X. In particular there is at least one edge (and two if the edge is an outer edge) between $f(\widetilde{x}_1)$ and some point $\widetilde{x}' \in \widetilde{X}'$

that projects on x_2. If $\widetilde{x}' = f(\widetilde{x}_2)$ then $d(f(\widetilde{x}_1), f(\widetilde{x}_2)) = 1$. Otherwise there is a vertical edge between $\widetilde{x}'$ and $f(\widetilde{x}_2)$ and $d(f(\widetilde{x}_1), f(\widetilde{x}_2)) = 2$. $\square$

We will need a simple condition on Y, X ensuring that the isometries of $\widetilde{X}$ commute with the projection $\widetilde{X} \to X$. This condition is in terms of triangles, where a triangle in a graph is a set consisting of 3 vertices that are all connected by an edge. The condition is

(8.1) Every edge in X belongs to strictly less than $m_X - M_Y - 1$ triangles,

where m_X is the minimal degree of X and M_Y the maximal degree of Y.

Lemma 8.4. *Assume that* (8.1) *holds. Then for every* 2*-coverings* $q_1 \colon \widetilde{Y}^{(1)} \to Y$ *and* $q_2 \colon \widetilde{Y}^{(2)} \to Y$ *of* Y *and every isometry* $f \colon \widetilde{X}^{(1)} \to \widetilde{X}^{(2)}$*, there is an isometry* $g \colon X \to X$ *which permutes the* Y_i*'s, and such that the projections* $\widetilde{X}^{(1)} \to X$ *and* $\widetilde{X}^{(2)} \to X$ *intertwine* f *and* g*.*

In particular, if the graphs $\widetilde{X}^{(1)}$ *and* $\widetilde{X}^{(2)}$ *are isometric, then the* 2*-coverings are isomorphic: there are isomorphisms* $\phi \colon Y \to Y$ *and* $\widetilde{\phi} \colon \widetilde{Y}^{(1)} \to \widetilde{Y}^{(2)}$ *such that* $\phi \circ q_1 = q_2 \circ \widetilde{\phi}$*.*

Proof. Let $k = 1$ or 2. By construction, for every vertical edge between $(\widetilde{y}, i)$ and $(\widetilde{y}', i)$, in $\widetilde{X}^{(k)}$ there are as many triangles in $\widetilde{X}^{(k)}$ containing this edge as outer edges containing $(\widetilde{y}, i)$. This number is equal to twice the number of neighbors of $f_i(q_k(\widetilde{y}))$ in X which are not in Y_i; in particular this number is at least $2(m_X - M_Y)$. On the other hand, the number of triangles containing an outer or inner edge is at most 2 (a bound for the number triangles also containing a vertical edge) plus twice the number of triangles in X containing the image of this edge. Hence by our assumption the number of triangles containing an outer or inner edge is strictly less than $2(m_X - M_Y - 1)$.

If $f \colon \widetilde{X}^{(1)} \to \widetilde{X}^{(2)}$ is an isometry, it sends an edge to an edge belonging to the same number of triangles. By the preceding discussion it sends vertical edges to vertical edges. Therefore f induces an isomorphism g of X. It also sends bijectively outer edges to outer edges because the outer edges in $\widetilde{X}^{(k)}$ are the edges with the property that there are 3 other edges in $\widetilde{X}^{(k)}$ corresponding to the same edge in X. This implies that f preserves the partition of $X = \sqcup_{i \in I} Y_i$. Restricting f to the any $\widetilde{Y}_i$ gives the desired isomorphism. $\square$

The preceding lemma allows to describe the isometry group of $\widetilde{X}$ as an extension of a subgroup of the isometry group of X by a compact group defined in terms of the deck transformation group of $q \colon \widetilde{Y} \to Y$, ie the group of automorphisms φ of $\widetilde{Y}$ such that $q \circ \varphi = q$. Here $\widetilde{Y}$ is a 2-covering of a connected graph, hence the deck transformation group is either $\mathbf{Z}/2\mathbf{Z}$ or trivial.

Lemma 8.5. *Assume that* (8.1) *holds. Let* $\widetilde{Y}$ *be a* 2*-covering of* Y *and* $\widetilde{X}$ *obtained by the previous construction.*

If f is an isometry of $\widetilde{X}$, there is a unique isometry g of X such that the projection $\widetilde{X} \to X$ intertwines f and g. If we set $\pi(f) = g$, π is a morphism from the isometry group of $\widetilde{X}$ to the isometry group of X whose kernel is F^I, where F is the deck transformation group of $\widetilde{Y} \to Y$.

Proof. First, there is a subgroup of the isometry group of $\widetilde{X}$ isomorphic to F^I, where F^I acts by $(\varphi_i)_{i\in I} \cdot (\widetilde{y}, j) = (\varphi_j(\widetilde{y}), j)$.

The existence of g is Lemma 8.4, its uniqueness is clear, as is the fact that π is a group morphism. It remains to understand the kernel of π. If f belongs to the kernel of π, for every i the restriction of g to $\widetilde{Y}_i$ is a deck transformation of the cover $\widetilde{Y}_i \to Y_i$. This shows that the kernel of π_0 is contained in $(F^I)^N$. The reverse inclusion is obvious. This shows the lemma. $\square$

The last two lemmas isolate conditions on X or on the 2-covering $\widetilde{Y} \to Y$ that translate into transitivity properties of the graph $\widetilde{X}$.

Lemma 8.6. *Assume that* (8.1) *holds. If there is a group G acting transitively on I and acting by isometries on X such that $g \circ f_i = f_{gi}$ for all $g \in G, j \in I$, then there is a subgroup G' in the isometry group of $\widetilde{X}$ such that $\pi(G') = G$ and such that each orbit of $\widetilde{X}$ under G' meets each $\widetilde{Y}_i$.*

Proof. For $g \in G'$, the map $(\widetilde{y}, i) \mapsto (\widetilde{y}, gi)$ is an isomorphism of $\widetilde{X}$, sends $\widetilde{Y}_i$ to $\widetilde{Y}_{gi}$ and belongs to $\pi^{-1}(g)$. One concludes by the assumption that the action of G' on I is transitive. $\square$

Lemma 8.7. *Assume that* (8.1) *holds. Let G_1 be a group of isometries of Y and G_2 a group of isometries of X with the property that for all i and all $g \in G_1$, there is an isometry $g' \in G_2$ of X that preserves each Y_j, such that $f_j^{-1} \circ g' \circ f_j \in G_1$ for all j, and $f_i^{-1} \circ g' \circ f_i = g$.*

Assume also that there exists a transitive group $\widetilde{G}_1$ of isometries of $\widetilde{Y}$ and a surjective group homomorphism $\widetilde{G}_1 \to G_1$ such that the covering $\widetilde{Y} \to Y$ intertwines the actions.

Then there is a subgroup G_2' in the isometry group of $\widetilde{X}$ such that $\pi(G_2') = G_2$ and which acts transitively on $\widetilde{Y}_i$ for each i.

Proof. Fix $(\widetilde{y}, i)$ and $(\widetilde{y}', i) \in \widetilde{Y}_i$. We construct an element of $\pi^{-1}(G_2)$ which sends $(\widetilde{y}, i)$ to $(\widetilde{y}', i)$. Since $\widetilde{G}_1$ acts transitively on $\widetilde{Y}$, there is $\widetilde{g} \in \widetilde{G}_1$ such that $\widetilde{g}\widetilde{y} = \widetilde{y}'$. Let g be its image in G_1. By the first assumption there is an isometry $g' \in G_2$ that acts as an element g_j of G_1 on each Y_j, and as g on Y_i. Pick $\widetilde{g}_j \in \widetilde{H}$ in the preimage of the morphism $\widetilde{G}_1 \to G_1$, with $\widetilde{g}_i = \widetilde{g}$. Then the map $(\widetilde{y}, j) \mapsto (\widetilde{g}_j\widetilde{y}, j)$ is an isometry of $\widetilde{X}$ that preserves each $\widetilde{Y}_j$ and sends $(\widetilde{y}, i)$ to $(\widetilde{y}', i)$, as requested. By construction it belongs to $\pi^{-1}(g')$. $\square$

8.2. **The construction for Cayley graphs.** A particular case of this construction is the following situation. Let H be a finitely generated group, with finite symmetric generating set T not containing 1. Take $G < H$ be a subgroup such that $S = T \cap H$ is generating. Take X the Cayley graph (H, T) and Y the Cayley graph (G, S). The partition of H into left G-cosets gives a partition of X into graphs isomorphic to Y, and every (set-theoretical) section $\alpha\colon H/G \to H$ gives rise to a family of isomorphisms $(f_i\colon (G,S) \to (H,T))_{i\in H/G}$ given by $f_i(y) = \alpha(i)y$.

If $\{h, ht\}$ (for $h \in H$ and $t \in T$) is an arbitrary edge in X, the number of triangles in X containing this edge equal to the number of $h' \in H$ such that $h^{-1}h'$ and $t^{-1}h^{-1}h'$ belong to T, i.e. is equal to the cardinality of $tT \cap T$. Also, every edge in X (Y) has degree $|T|$ (respectively $|S|$). Therefore the condition (8.1) holds if and only if $\max_{t\in T} |tT \cap T| < |T| - |S| - 1$.

We get a 2-covering $q = q_\tau\colon \widetilde{Y}_\tau \to Y$ as above, for every extension

$$1 \to \mathbf{Z}/2\mathbf{Z} \to G_\tau \xrightarrow{\tau} G \to 1$$

together with a symmetric subset $S_\tau \subset G_\tau$ mapping bijectively to S, by taking $\widetilde{Y}_\tau$ to be the Cayley graph (G_τ, S_τ).

Remark 8.8. Once again, we remark that $S_{\tau_0} = S \times \{0\}$ *is not* a generating subset of G_{τ_0}, in which case (G_{τ_0}, S_{τ_0}) is disconnected.

Denote by X_τ the graph obtained from $q_\tau\colon \widetilde{Y}_\tau \to Y$ with the above construction.

Let us assume that $\max_{t\in T} |tT \cap T| < |T| - |S| - 1$. Then we can apply Lemma 8.5, 8.6 and 8.7. This is the content of the next lemmas.

Let π be the group morphism from the isometry group of X_τ to the isometry group of X given by Lemma 8.5. We regard H as a subgroup of the isometry group of X, acting by translation.

Lemma 8.9. *If G is a normal subgroup and H splits as a semi-direct product $G \rtimes H/G$, and if α is a group homomorphism, then X_τ is a transitive graph. More precisely, $\pi^{-1}(H)$ acts transitively on X_τ.*

Proof. We first observe that there is a group G' of isometries of X_τ acting transitively on each $\widetilde{Y}_{i,k}$ and such that $\pi(G') = G$. This follows from Lemma 8.7 and does not use that H splits as a semi-direct product.

Since α is a group homomorphism, we have that $\alpha(i)f_j(y) = f_{ij}(y)$ for all $y \in Y$ and $i, j \in Q$. By Lemma 8.6 there is a group G_2' of isometries of X_τ such that each G_2'-orbit meets each $\widetilde{Y}_{i,k}$, and such that $\pi(G_2') = \alpha(G/H)$.

The group generated by G' and G_2' therefore acts transitively on X_τ, and its image by π is the group generated by G and $\alpha(G/H)$, which is H. This concludes the proof of the lemma. □

Lemma 8.10. *Assume that the isometry group of (G, S) is discrete. Let G_τ, S_τ be as above.*

There are finitely many different isomorphism classes of extensions

$$1 \to \mathbf{Z}/2\mathbf{Z} \to G_{\tau'} \xrightarrow{\tau'} G \to 1$$

and symmetric preimages $S_{\tau'} \subset G_{\tau'}$ of S such that X_τ is isomorphic to $X_{\tau'}$.

Proof. By Lemma 8.4 we only have to prove that there are finitely many different isomorphism classes of extensions

$$1 \to \mathbf{Z}/2\mathbf{Z} \to G_{\tau'} \xrightarrow{\tau'} G \to 1$$

and symmetric preimages $S_{\tau'} \subset G_{\tau'}$ of S such that the resulting 2-covering $(G_{\tau'}, S_{\tau'}) \to (G, S)$ is isomorphic to $(G_\tau, S_\tau) \to (G, S)$.

By definition, $(G_{\tau'}, S_{\tau'}) \to (G, S)$ is isomorphic to $(G_\tau, S_\tau) \to (G, S)$ if and only if there are isomorphisms $\widetilde{\phi}\colon (G_{\tau'}, S_{\tau'}) \to (G_\tau, S_\tau)$ and $\phi\colon (G, S) \to (G, S)$ such that $\tau' \circ \widetilde{\phi} = \phi \circ \tau$. Moreover since $G_{\tau'}$ acts transitively on $(G_{\tau'}, S_{\tau'})$ we can always assume that $\widetilde{\phi}(1_{G_{\tau'}}) = 1_{G_\tau}$. In particular ϕ belongs to the stabilizer of the identity in the isometry group of (G, S), which by assumption is finite. The Lemma therefore reduces to the observation that if ϕ is the identity, then $\widetilde{\phi}$ is a group isomorphism. Actually, $\widetilde{\phi}$ is even an isomorphisms of rooted oriented marked Cayley graphs: since τ' and τ are bijections in restriction to $S_{\tau'}$ and S_τ, we can label the oriented edges in $G_{\tau'}$ and G_τ by S, and the map $\widetilde{\phi}$ respects this labelling because $\phi = \mathrm{id}$ does. □

8.3. **Proof of Theorem 8.1.** It remains to collect all the previous lemmas. Let G, H, T as in Theorem 8.1. As H splits as a semidirect product $G \rtimes H/G$, there is a section $\alpha\colon H/G \to H$ that is a group homomorphism.

For every extension

$$1 \to \mathbf{Z}/2\mathbf{Z} \to G_\tau \xrightarrow{\tau} G \to 1$$

and a symmetric set $S_\tau \subset G_\tau$ such that τ is a bijection $S_\tau \to S$, we define X_τ as the graph defined in § 8.2 for this α. If $\tau = \tau_0$ is the trivial extension ($G_\tau = G \times \mathbf{Z}/2\mathbf{Z}$), we take $\widetilde{S}_{\tau_0} = S \times \{0\}$.

It follows from its definition that X_{τ_0} coincides with X_0, the Cayley graph of $H \times \mathbf{Z}/2\mathbf{Z}$ for the symmetric generating set consisting of the union of $S \times \{0\}$, of $(T \setminus S) \times \{0, 1\}$ and of $\{e_H\} \times ((\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}) \setminus \{0\})$. This proves (1).

Then (2) is Lemma 8.3, and (4) is Lemma 8.10. We leave to the reader the easy task to check (3), where R_1 is the maximum of $|g|_S$ over all $g \in G$ with $|g|_T \leq R$.

Finally we prove (5). The fact that X_τ is a transitive graph follows from Lemma 8.9. Let π be the morphism from the isometry group of X_τ to the isometry group of (H, T), as given by Lemma 8.5. Since this latter group is discrete, the subgroup H acting by translation of the isometry group of (H, T) is a finite index subgroup, and $H' = \pi_0^{-1}(H)$ is a finite index subgroup of the isometry group of X_τ. By restriction we have a group morphism $\pi\colon H' \to H$, which is surjective by Lemma 8.9, and whose kernel is isomorphic to the compact group $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$ by Lemma 8.9.

8.4. **Concluding step in the proof of Theorem F.** We start by a proposition, the proof of which will be given in §9.4.

Proposition 8.11. *Let $G \subsetneq H$ be finitely generated groups, and assume that G contains an element of infinite order. Then there is a finite symmetric generating set T of $H \setminus \{e_H\}$ such that*

- *The Cayley graph (H, T) has a discrete isometry group.*
- *$S = G \cap T$ generates G and the Cayley graph (G, S) has a discrete isometry group.*
- $\max_{t\in T} |tT \cap T| < |T| - |S| - 1$.

Theorem F is a direct consequence of this proposition, of Theorem 8.1 and of the following Lemma.

Lemma 8.12. *Let G be a finitely generated group with finite generating set S. Let $R_1 > 0$. Assume that $H^2(G, \mathbf{Z}/2\mathbf{Z})$ is infinite. Then there is a continuum family $(\tau_i, S_i)_{i\in\mathbf{R}}$ where*

$$1 \to \mathbf{Z}/2\mathbf{Z} \to G_i \to G \xrightarrow{\tau_i} 1$$

are pairwise non isomorphic extensions, $S_i \subset G_i$ is a symmetric subset such that τ_i is a bijection $S_i \to S$, and where (G_i, S_i) if R_1-locally $(G \times \mathbf{Z}/2\mathbf{Z}, S \times \{0\})$ for all i.

Proof. It is easy to see that $H^2(G, \mathbf{Z}/2\mathbf{Z})$ has the cardinality of the continuum. One way to argue is by using that an infinite compact Hausdorff topological group has always at least continuum many elements. In particular $H^2(G, \mathbf{Z}/2\mathbf{Z})$, which is assumed to be infinite and which has a natural compact Hausdorff group topology as the quotient of the closed subgroup $Z^2(G, \mathbf{Z}/2\mathbf{Z})$ of the compact Hausdorff group $(\mathbf{Z}/2\mathbf{Z})^{G\times G}$ by its closed subgroup $B^2(G, \mathbf{Z}/2\mathbf{Z})$, has (at least, but also clearly at most) the cardinality of the continuum.

In particular by the same linear algebra consideration as in Lemma 7.1 we see that there are continuum many elements $\varphi_i \in Z^2(G, \mathbf{Z}/2\mathbf{Z})$ which are all distinct in $H^2(G, \mathbf{Z}/2\mathbf{Z})$ and which vanish on $\{(g_1, g_2), |g_1|_S + |g_2|_S \leq R_1\}$. We conclude as in Lemma 7.1. $\square$

9. On Cayley graphs with discrete isometry group

This section is dedicated to the proofs of Theorems G and H. We start with a preliminary result dealing with marked Cayley graphs.

9.1. **The case of marked Cayley graphs.** For the proof of Theorem H and Proposition G we introduce the notion of *marked Cayley graph.* If Γ is a group with finite symmetric generating set S, the marked Cayley graph (G, S) is the unoriented labelled graph in which each unoriented edge $\{\gamma, \gamma s\}$ is labelled by $\{s, s^{-1}\}$.

Lemma 9.1. *Let Γ be a finitely generated group. There is a finite symmetric generating set S such that the group of isometries of the marked Cayley graph (Γ, S) is discrete.*

Proof. With our notion of marked Cayley graph, by an isometry of the marked Cayley graph (Γ, S) we mean a bijection f of Γ such that $f(\gamma)^{-1}f(\gamma s) \in \{s, s^{-1}\}$ for all $s \in S$ and $\gamma \in \Gamma$.

Let S_1 be a symmetric finite generating set of Γ. Denote by $|\cdot|_1$ the word-length associated to S_1. Let N be an integer strictly larger than the cardinality of S_1. Denote $S_N = \{\gamma \in \Gamma, |\gamma|_1 \in \{1, 2, \ldots, N\}\}$. We claim that the isometry group of the marked Cayley graph (G, S_N) is discrete. For this we prove that an isometry f of the marked Cayley graph (G, S_N) that is the identity on the $|\cdot|_1$-ball of radius $N-1$ is the identity on (G, S_N). We prove by induction on $n \geq N-1$ that f is the identity on the $|\cdot|_1$-ball of radius n. Assume that the induction hypothesis holds for some $n \geq N-1$. Suppose for contradiction that there exists $|\gamma|_1 = n+1$ such that $f(\gamma) \neq \gamma$. Then for every decomposition $\gamma = \gamma' s$ with $s \in S_N$ and $|s|_1 + |\gamma'|_1 = n+1$, the fact that f is an isometry of marked Cayley graph (Γ, S_N) says that $f(\gamma) \in f(\gamma')\{s, s^{-1}\}$. By the induction hypothesis $f(\gamma') = \gamma'$, and $f(\gamma) = \gamma' s^{-1}$ because $f(\gamma) \neq \gamma$. Also $f(f(\gamma)) = \gamma$ because $f(f(\gamma)) \in \gamma'\{s, s^{-1}\}$ and $f(f(\gamma)) \neq f(\gamma)$.

Let us write $\gamma = \gamma_0 s_1 \ldots s_N$ for $s_1, \ldots, s_N \in S_1$ and $|\gamma_0|_1 = n+1-N$. Since $N > |S_1|$, there exists $k < l$ with $s_k = s_l$. By the preceding discussion for the decomposition $\gamma = (\gamma_0 s_1 \ldots s_{k-1})(s_k \ldots s_N)$, we obtain $f(\gamma) = \gamma_0 s_1 \ldots s_{k-1} s_N^{-1} \ldots s_k^{-1}$. By the same reasoning for the decomposition

$$f(\gamma) = (\gamma_0 s_1 \ldots s_{k-1} s_N^{-1} \ldots s_l^{-1})(s_{l-1}^{-1} \ldots s_k^{-1}),$$

and using that $f(f(\gamma)) = \gamma \neq f(\gamma)$, we have

$$\gamma = \gamma_0 s_1 \ldots s_{k-1} s_N^{-1} \ldots s_l^{-1} s_k \ldots s_{l-1}.$$

Since $s_k = s_l$, we obtain that $|\gamma|_1 \leq |\gamma_0|_1 + N - 2 = n - 1$, a contradiction. The map f is therefore the identity on the $|\cdot|_1$-ball of radius $n+1$. This concludes the proof of the induction, and of the Lemma. ☐

9.2. **Proof of Theorem G.** By Lemma 9.1 there is a finite symmetric generating set S_0 of Γ such that the marked Cayley graph (Γ, S_0) has a discrete isometry group. For redactional purposes we also make sure that $1_\Gamma \notin S_0$.

Take S a larger finite symmetric generating set containing S_0 but not 1_Γ, with the property that for all $s \in S_0$, there exists $s' \in S$ such that $ss' \in S$ and

$s \notin \{s', s'^{-1}, ss', (ss')^{-1}\}$. Such an S exists unless Γ is finite, in which case there is nothing to prove.

Since S contains S_0, the marked Cayley graph (Γ, S) *a fortiori* has a discrete isometry group.

We can assume that S has at least three elements. Let R be the size of the largest clique (=complete subgraph) in (Γ, S). Decompose S as a disjoint union $S = S_1 \cup S_2 \cup S_2^{-1}$, where S_1 is the elements of S of order 2. Enumerate $S_1 \cup S_2$ as $s_1 \ldots, s_n$, with $n \geq 2$. Let $p_1, \ldots, p_n$ be distinct integers, all strictly greater than R, and $F = \prod_{i=1}^n \mathbf{Z}/p_i\mathbf{Z}$, denoted additively. If the p_i are prime, F is a cyclic group. Consider the following symmetric generating set $\widetilde{S}$ of $\Gamma \times F$:

$$\widetilde{S} = \bigcup_{i=1}^n (\{s_i, s_i^{-1}\} \times \mathbf{Z}/p_i\mathbf{Z}) \cup (\{1_\Gamma\} \times (F \setminus \{0_F\})) .$$

Let $X = (\Gamma \times F, \widetilde{S})$, and $q \colon X \to (\Gamma, S)$ the projection. For each $\gamma \in \Gamma$, $\{\gamma\} \times F$ is a clique with $|F|$ vertices, and observe that these are the only cliques with $|F|$ vertices. Indeed, let K be a clique in X. Its image $q(K)$ is a clique in (Γ, S), and therefore has cardinality at most R. By the fact that the preimage by q of an edge in (Γ, S) has cardinality at most $\max_i p_i$, we see that if $q(K)$ contains at least two points, then K has cardinality less than $R \max_i p_i$, which is strictly less than $|F|$ because $n \geq 2$ and $R < \min_i p_i$.

Let f be an isometry of X. It sends cliques to cliques, and therefore there is an isometry f_0 of (Γ, S) such that $f_0 \circ q = q \circ f$. Since the number of edges between $\{\gamma\} \times F$ and $\{\gamma s\} \times F$ caracterizes $\{s, s^{-1}\}$, we see that f_0 is an isomorphism of marked Cayley graphs. This defines a group homomorphism from the isometry group of X to the isometry group the the marked Cayley graph (Γ, S), which is discrete. To prove that the isometry group of X is discrete we are left to prove that the kernel of this homomorphism is finite. Let f such that f_0 is the identity. This means that we can write $f(\gamma, x) = (\gamma, f_\gamma(x))$ for a family f_γ of bijections of F. If $s \in S$, there is a unique i such that $s \in \{s_i, s_i^{-1}\}$; denote by F_s the subgroup $\mathbf{Z}/p_i\mathbf{Z}$ of F, so that there is an edge between (γ, x) and $(\gamma s, x')$ if and only if $x - x' \in F_s$. In particular, there is an edge between (γ, x) and $(\gamma s, x)$, and therefore also between their images by f. This means that $f_{\gamma s}(x) - f_\gamma(x) \in F_s$. Now take $s \in S_0$, and $s' \in S$ such that $ss' \in S$ and $s \notin \{s', s'^{-1}, ss', (ss')^{-1}\}$, as made possible by our choice of S. Writing $f_{\gamma s}(x) - f_\gamma(x) = f_{\gamma s}(x) - f_{\gamma ss'}(x) + f_{\gamma ss'}(x) - f_\gamma(x)$, we see that $f_{\gamma s}(x) - f_\gamma(x) \in F_s \cap (F_{s'} + F_{ss'}) = \{0\}$. This proves that for all $s \in S_0$ and $\gamma \in \Gamma$, $f_\gamma = f_{\gamma s}$. Since S_0 generates Γ, we have that f_γ does not depend on Γ. This proves that the set of isometries f of X such that f_0 is trivial is finite. This implies that the isometry group of X is discrete, and proves Theorem G.

9.3. **Proof of Theorem H.** Let Γ be a finitely generated group with an element of infinite order. By Lemma 9.1 there is a finite symmetric generating set S_0 of Γ such that the marked Cayley graph (Γ, S_0) has a discrete isometry group. Our strategy is to find a larger generating set S such that we can recognize the marked Cayley graph (Γ, S_0) from the triangles in (Γ, S). For this, if S is a symmetric subset of $\Gamma \setminus \{e\}$ and $s \in S$, we denote by $N_3(s, S)$ the number of triangles in the Cayley graph (Γ, S) containing the two vertices e and s. In formulas,

$$N_3(s, S) = \left|\{t \in S, s^{-1}t \in S\}\right|.$$

We will also denote $N_3(s, S) = 0$ if $s \notin S$. By the invariance of (Γ, S) by translations, $N_3(s, S)$ is also equal, for every $\gamma \in \Gamma$, to the number of triangles in (Γ, S) containing the two vertices γ and γs. In particular, for $\gamma = s^{-1}$ we see that $N_3(s, S) = N_3(s^{-1}, S)$. The main technical result if the following.

Lemma 9.2. *Let $S \subset \Gamma \setminus \{e\}$ be a finite symmetric set and $s_0 \in S$. There exists a finite symmetric set $S' \subset \Gamma \setminus \{e\}$ containing S such that*

(1) $S' \setminus S$ does not intersect $\{s^2, s \in S\}$.
(2) $N_3(s, S') \leq 6$ for all $s \in S' \setminus S$.
(3) $N_3(s, S) = N_3(s, S')$ for all $s \in S \setminus \{s_0, s_0^{-1}, s_0^2, s_0^{-2}\}$.
(4) The couple $(N_3(s_0, S') - N_3(s_0, S), N_3(s_0^2, S') - N_3(s_0^2, S))$ belongs to

$$\begin{cases} \{(2,0),(4,0)\} & \text{if } s_0 \text{ has order } 2. \\ \{(1,1),(2,2),(3,3)\} & \text{if } s_0 \text{ has order } 3. \\ \{(1,0),(2,0),(2,2)\} & \text{if } s_0 \text{ has order } 4. \\ \{(1,0),(2,0),(2,1)\} & \text{if } s_0 \text{ has order} \geq 5. \end{cases}$$

Proof. Let $\gamma \in \Gamma$ be an element of infinite order. We define a finite symmetric set by $S' = S \cup \Delta$ where $\Delta = \{\gamma^n, \gamma^{-n}, s_0^{-1}\gamma^n, \gamma^{-n}s_0\}$ for an integer n that we will specify later. Since all the γ^n are distinct, for all n large enough (say $|n| \geq n_0$) all the elements in Δ have word-length with respect to S at least 3, and the three elements $\gamma^n, \gamma^{-n}, s_0^{-1}\gamma^n$ are distinct. This means that Δ has 4 elements unless $s_0^{-1}\gamma^n = \gamma^{-n}s_0$, in which case Δ has 3 elements.

Assume that $n \geq n_0$. Then the first condition clearly holds because an element of $\{s^2, s \in S\}$ has word length at most 2, which is strictly smaller than 3. Also, by the triangle inequality for the word-length with respect to S, a triangle in (G, S') either is a triangle in (G, S), or has at least two edges coming from $S' \setminus S = \Delta$. This shows the second item. Indeed, if $s \in S' \setminus S = \Delta$ and $t \in S'$ satisfies $s^{-1}t \in S'$, then either $t \in \Delta \setminus \{s\}$ or $s^{-1}t \in \Delta \setminus \{s^{-1}\}$, which leave at most $3 + 3 = 6$ possible triangles containing e and t. This also shows that for $s \in S$,

$$N_3(s, S') - N_3(s, S) = \left|\{t \in \Delta, s^{-1}t \in \Delta\}\right| = |\Delta \cap s\Delta|.$$

It remains to find $|n| \geq n_0$ such that (3) and (4) hold.

Let us first consider the simpler case when there exists infinitely many n's such that $s_0^{-1}\gamma^n = \gamma^{-n}s_0$. Then for such an n, $\Delta = \{\gamma^n, \gamma^{-n}, s_0^{-1}\gamma^n\}$ and if $|n| \geq n_0$ the previous formula means that for $s \in S$, $N_3(s, S') - N_3(s, S)$ is the number of elements equal to s in the list

$$s_0, s_0^{-1}, \gamma^{2n}, \gamma^{-2n}, s_0^{-1}\gamma^{2n}, \gamma^{-2n}s_0.$$

For $|n|$ large enough the terms $\gamma^{2n}, \gamma^{-2n}, s_0^{-1}\gamma^{2n}, \gamma^{-2n}s_0$ do not belong to S, which proves that $N_3(s, S') - N_3(s, S) = 0$ if $s \notin \{s_0, s_0^{-1}\}$, and that $N_3(s_0, S') - N_3(s_0, S') \in \{1, 2\}$ depending on whether s_0 has order 2 or not. This proves (3) and (4).

We now move to the case when $s_0^{-1}\gamma^n \neq \gamma^{-n}s_0$, *i.e.* Δ has 4 elements for all $|n|$ large enough. This means that $N_3(s, S') - N_3(s, S)$ is the number of elements equal to s in the list

$$s_0, s_0^{-1}, \gamma^n s_0 \gamma^{-n}, \gamma^n s_0^{-1}\gamma^{-n}, \gamma^{-n}s_0\gamma^{-n}, \gamma^n s_0^{-1}\gamma^n, \gamma^{-n}s_0\gamma^{-n}s_0, s_0^{-1}\gamma^n s_0^{-1}\gamma^n, \gamma^{2n}, \gamma^{-2n}.$$

If n is large enough we can forget the last two elements, which do not belong to S.

We have two actions of $\mathbf{Z}$ on G given by $\alpha_n g = \gamma^n g \gamma^{-n}$ and $\beta_n g = \gamma^n g \gamma^n$. With this notation, the previous list becomes

$$s_0, s_0^{-1}, \alpha_n s_0, (\alpha_n s_0)^{-1}, \beta_{-n}s_0, (\beta_{-n}s_0)^{-1}, (\beta_{-n}s_0)s_0, s_0^{-1}(\beta_{-n}s_0)^{-1}.$$

Denote by $T_1 \in \mathbf{N} \cup \{\infty\}$ and $T_2 \in \mathbf{N} \cup \{\infty\}$ the cardinality of the α-orbit and the β-orbit of s_0 respectively, so that $\alpha_n s_0 = s_0$ if and only if n is a multiple of T_1, and $\beta_n s_0 = s_0$ if and only if n is a multiple of T_2 (with the convention that the only multiple of ∞ is 0). If n is a multiple of T_1 and T_2, then $\alpha_n s_0 = \beta_n s_0$, and hence $\gamma^{2n} = 1$, which holds only if $n = 0$. This implies that T_1 and T_2 cannot both be finite. Also, note that $T_2 < \infty$ prevents s_0 from having order 2, because we assumed that $s_0^{-1}\gamma^n \neq \gamma^{-n}s_0$ for n large enough.

Case 1: $T_1 = T_2 = \infty$. Then all the terms in the previous list except s_0, s_0^{-1} escape from S as $n \to \infty$. This implies that for n large enough $N_3(s, S') - N_3(s, S) = 0$ if $s \notin \{s_0, s_0^{-1}\}$, and that $N_3(s_0, S') - N_3(s_0, S) \in \{1, 2\}$ depending on whether s_0 is of order 2. This proves (3), and that $(N_3(s_0, S') - N_3(s_0, S), N_3(s_0^2, S') - N_3(s_0^2, S))$ is equal to $(2, 0)$ if s_0 has order 2, $(1, 1)$ if s_0 has order 3, and $(1, 0)$ otherwise. This proves also (4).

Case 2: $T_1 < \infty$, $T_2 = \infty$. Take n a large multiple of T_1. Then the terms containing $\beta_{-n}s_0$ in the previous list are not in S, and the elements of the list that can belong to S are

$$s_0, s_0^{-1}, \alpha_n s_0 = s_0, (\alpha_n s_0)^{-1} = s_0^{-1}.$$

This implies that $N_3(s, S') - N_3(s, S) = 0$ if $s \notin \{s_0, s_0^{-1}\}$, and that $N_3(s_0, S') - N_3(s_0, S) \in \{2, 4\}$ depending on whether s_0 is of order 2. This proves (3) and (4) as in the first case.

Case 3: $T_1 = \infty$, $T_2 < \infty$. Take n a large multiple of T_2. Similarly the elements in the previous list that can belong to S are

$$s_0, s_0^{-1}, s_0, s_0^{-1}, s_0^2, s_0^{-2}.$$

This proves (3). If $s_0^2 \notin S$, by convention $N_3(s_0^2, S) = N_3(s_0^2 S') = 0$, and we get as above that $N_3(s_0, S') - N_3(s_0, S) = 2$ (recall that $s_0 \neq s_0^{-1}$ because $T_2 < \infty$), which proves (4). If $s_0^2 \in S$, we get that $(N_3(s_0, S') - N_3(s_0, S), N_3(s_0^2, S') -$

$N_3(s_0^2, S))$ is equal to $(3,3)$ if s_0 has order 3, and $(2,2)$ if s_0 has order 4 and $(2,1)$ otherwise. This proves also (4). $\square$

We now prove

Lemma 9.3. *There exists a finite symmetric generating set $S \subset \Gamma \setminus \{e\}$ containing S_0 such that for every $s \in S_0$ and $s' \in S$, $N_3(s, S) = N_3(s', S)$ if and only if $s' \in \{s, s^{-1}\}$.*

Since an isometry of (Γ, S) preserves the number of triangles adjacent to an edge, this proposition implies that the isometry group of (Γ, S) is a subgroup of the marked Cayley graph (Γ, S_0), which is discrete. This implies Theorem H.

Proof of Lemma 9.3. For a finite sequence $\underline{u} = u_1, \dots, u_N$ of elements in S_0, we define a finite symmetric generating sets $S(\underline{u}) \subset \Gamma \setminus \{e\}$ inductively as follows : if $N = 0$ (there is zero term in the sequence), $S(\underline{u}) = S_0$, and if $N > 0$ $S(\underline{u})$ is the set S' given by Lemma 9.2 for $S = S(u_1, \dots, u_{N-1})$ and $s_0 = u_N$.

By the first three items in Lemma 9.2, we have that $N_3(s, S(\underline{u})) \leq 6$ for all $s \in S(\underline{u}) \setminus S_0$.

We claim that the conclusion of the Lemma holds for a good choice of $\underline{u}$. For this we consider $T_0 = \emptyset \subset T_1 \subset \dots T_K = S_0$ a maximal strictly increasing sequence of symmetric subsets T_i of S_0 with the property that for all $s \in S_0$, $s^2 \in T_i \implies s \in T_i$. We prove by induction on i that there is a sequence $\underline{u}$ in T_i such that for all $s, s' \in T_i$, $N_3(s, S(\underline{u})) \geq 7$ and $N_3(s, S(\underline{u})) = N_3(s', S(\underline{u}))$ if and only if $s' \in \{s, s^{-1}\}$. For $i = 0$ there is nothing to prove. Assume that there exists $\underline{u}$ in T_i such that the conclusion holds for T_i. We will find a sequence $\underline{u'}$ in $T_{i+1} \setminus T_i$ such that for the sequence $\underline{u}, \underline{u'}$, the conclusion holds for T_{i+1}. Consider $t \in T_{i+1} \setminus T_i$. We consider two cases.

If $t^2 \notin T_{i+1}$ or $t^2 = t^{-1}$, then by maximality, $T_{i+1} = T_i \cup \{t, t^{-1}\}$ (otherwise $T_{i+1} \setminus \{t, t^{-1}\}$ could be added between T_i and T_{i+1}). We then define $\underline{u'} = t, \dots, t$ repeated $\max(n, 7)$ times for $n > \max_{s \in T_i} N_3(s, S(\underline{u}))$, and we see that $N_3(s, S(\underline{u}, \underline{u'})) = N_3(s, S(\underline{u}))$ if $s \in T_i$ because $s \notin \{t, t^2, t^{-1}, t^{-2}\}$, and $N_3(t, S(\underline{u}, \underline{u'})) \geq \max(n, 7) > \max_{s \in T_i} N_3(s, S(\underline{u}, \underline{u'}))$. This proves the assertion for T_{i+1}.

If $t^2 \in T_{i+1}$ and $t^2 \neq t^{-1}$, observe that for all j, $t^{2^j} \in T_{i+1} \setminus T_i$ (otherwise if $j \geq 2$ is the smallest integer such that $t^{2^j} \notin T_{i+1} \setminus T_i$, then $t^{2^j} \notin T_i$ because it is the square of $t^{2^{j-1}} \notin T_i$, and hence $T_{i+1} \setminus \{t^{2^{j-1}}, t^{-2^{j-1}}\}$ could be added between T_i and T_{i+1}, contradicting the maximality). Since T_{i+1} is finite, there is a smaller j such that $t^{2^j} \in \cup_{k=0}^{j-1} \{t^{2^k}, t^{-2^k}\}$, and by maximality necessarily $t^{2^j} \in \{t, t^{-1}\}$ and $T_{i+1} \setminus T_i = \{t^{2^k}, k = 0 \dots j-1\} \cup \{t^{-2^k}, k = 0 \dots j-1\}$. In particular, $2^{2j} - 1$ is a multiple of the order of t, which is therefore odd and hence at least 5 (we assumed that $t^3 \neq e$). Take a sequence $n_0 > n_1 > \cdots > n_j$, and take for $\underline{u'}$ the sequence containing n_k times t^{2^k} for all $k = 0, \dots, j$. Then by (3) $N_3(s, S(\underline{u}, \underline{u'})) = N_3(s, S(\underline{u}))$ if $s \in T_i$. Also, since by (4) for each occurence of t^{2^j}, $N_3(t^{2^j}, \cdot)$ is increased by at least 1 (and at most 2), we see that $N_3(t^{\pm 2^j}, S(\underline{u}, \underline{u}')) \geq n_j$, which can be made strictly larger than $\max_{s\in T_i} N_3(s, S(\underline{u}))$ and 7 if n_j is large enough. Finally, consider $k < j$. For each of the n_k occurences of t^{2^k} in $\underline{u}'$, only $N_3(t^{\pm 2^k}, \cdot)$ and $N_3(t^{\pm 2^{k+1}}, \cdot)$ can increase (by one or two), but necessarily $N_3(t^{\pm 2^k}, \cdot)$ increases by at least one unit more than $N_3(t^{\pm 2^{k+1}}, \cdot)$. This implies that

$$N_3(t^{\pm 2^k}, S(\underline{u}, \underline{u}')) - N_3(t^{\pm 2^k}, S(\underline{u})) \\ \geq n_k + N_3(t^{\pm 2^{k-1}}, S(\underline{u}, \underline{u}')) - N_3(t^{\pm 2^{k+1}}, S(\underline{u})) - 2n_{k+1}.$$

This implies that if n_k is large enough compared to n_{k+1}, we have

$$N_3(t^{\pm 2^k}, S(\underline{u}, \underline{u}')) > N_3(t^{\pm 2^{k-1}}, S(\underline{u}, \underline{u}')).$$

In particular there is a choice of $n_0, \ldots, n_j$ such that the induction hypothesis holds at step $i + 1$.

Finally the induction hypothesis holds for $T_K = S_0$, which concludes the proof of the Lemma. $\square$

Remark 9.4. An examination of the proof of Theorem H would give the following improvement : there is an explicit function $f \colon \mathbf{N} \to \mathbf{N}$ such that if Γ is a group with N generators and an element of order at least $f(N)$, then Γ has a Cayley graph with discrete isometry group.

9.4. **Proof of Proposition 8.11.** We can adapt the proof of Theorem H to prove a slightly stronger statement: Proposition 8.11 that was used in the proof of Theorem F.

Let $G \subsetneq H$ as in Proposition 8.11. It follows from Lemma 9.1 that H has a finite symmetric generating set T_0 such that $S_0 := G \cap T_0$ generates G, and such that the isometry groups of the marked Cayley graphs (G, S_0) and (H, T_0) are discrete (just take for T_0 the union of a finite generating set of G and of H given by Lemma 9.1). By applying the proof of Lemma 9.3 first in H, we see that there is a finite symmetric generating set $T \subset H$ such that (1) $N_3(t, T) \leq 6$ for $t \in T \setminus T_0$, (2) if $t, t' \in T_0$, $N_3(t, T) = N_3(t', T)$ if and only if $t' \in \{t, t^{-1}\}$ and (3) $N_3(t, T) > 6$ if $t \in T_0$. Now observe that adding to T elements of $G \setminus T^2$ does not change the function $N_3(\cdot, T)$ on $H \setminus G$, whereas on H it increases the functions $N_3(\cdot, T)$ and $N_3(\cdot, T \cap G)$ be the same amount. By applying the proof of Lemma 9.3 to G, we therefore see that we can enlarge T by adding elements of G such that (1) (2) (3) still hold but also (2') if $s, s' \in S_0$, $N_3(s, T \cap G) = N_3(s', T \cap G)$ if and only if $t' \in \{t, t^{-1}\}$. Finally, we observe that we can moreover assume that (4) $\max_{t\in T} |tT \cap T| < |T \setminus G| - 1$. This is because replacing T by $T \cup \{h, h^{-1}\}$ for $h \in H \setminus G$ of word-length $|h|_T > 3$ does not change the value of $\max_{t\in T} |tT \cap T|$ but increases the cardinality of $|T \setminus G|$; we can therefore repeat this as many times as necessary to ensure (4).

It follows from (1) (2) (3) (from (1) (2') (3)) that (H, T) (respectively $(G, T \cap G)$) has a discrete isometry group. (4) is exactly the last point to be proved. This concludes the proof of Proposition 8.11.

9.5. **Proof of Corollary I.** By Theorem H, Γ has a Cayley graph X with discrete isometry group. By Theorem C, X is USLG-rigid. We conclude by Lemma 5.2.

10. Proof of Theorem J

Lemma 10.1. *For each positive integer n, there exist geodesic contractible compact metric spaces C_n^0, C_n^1, C_n^2 with isometries i_n^k from $[0, 2^n]$ onto a segment $I_n^k \subset C_n^k$ such that*

- *The isometry group of C_n^k is trivial if $k = 0, 1$.*
- *The isometry group of C_n^2 is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and acts as the identity on I_n^2.*
- *For $k \neq l$, two connected components of $C_n^k \setminus I_n^k$ and $C_n^l \setminus I_n^l$ are not isometric.*
- *Every point in C_n^k is at distance less than 2^{-n} from I_n^k, and every connected component of $C_n^k \setminus I_n^k$ contains a point at distance 2^{-n} from I_n^k.*
- *For $k \neq l$ and every $x \in C_n^k$, there is an isometry from $B(x, 2^{n-2}) \cup I_n^k$ to C_n^l that maps $i_n^k(t)$ to $i_n^l(t)$ for all t.*

Proof. We start by constructing, for each integer $n \geq 1$, and each pair partition π of $\{1, 2, 3, 4, 5, 6\}$, a metric space C_n^π as follows. We start from 6 rectangles $[0, 2^n] \times [0, 2^{-n}]$, of length 2^n and height 2^{-n}. We remove from the first and the third rectangles a ball of radius 3^{-n} and 4^{-n} respectively around the point $(2^{-n}, 2^{-n})$. We glue all the rectangle along the long edge $[0, 2^n] \times \{0\}$. We also glue together the first and the second rectangle along the left segment $\{0\} \times [0, 2^{-n}]$. We do the same for the third and fourth rectangles, and for the fifth and sixth rectangle. Finally for each class $\{i, j\}$ in the partition π, we glue together to right segments $2^n \times [0, 2^{-n}]$ of the i-th and the j-th rectangle. The resulting space is C_n^π, that we equip with the unique geodesic metric that coincides with the euclidean metric on each (punctured) rectangle. See Figure 1.

Then one defines C_n^0 as C_n^π for $\pi = \{\{1, 6\}, \{2, 3\}, \{4, 5\}\}$, C_n^1 as C_n^π for $\pi = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\}$ and C_n^2 as C_n^π for $\pi = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$. By construction the exchange of the fifth and sixth rectangles gives an isometry of C_n^2. There is no difficulty to check that there are no other non-trivial isometries, and that C_n^0 and C_n^1 have trivial isometry groups. The reason is that such an isometry must preserve the common long side of all the rectangles, and also the two small balls that have been removed, and hence must be the identity on the first and third rectangles. The rest of the properties are easy to check; we only give a brief justification for the last one: a ball of radius $R < \frac{2^n - 2^{-n} - 3^{-n}}{2}$ around a point in C_n^π cannot simultaneously see one of the small balls that have been

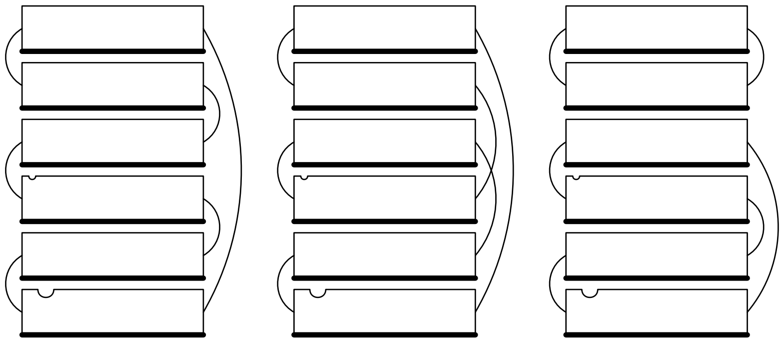

FIGURE 1. The spaces C_n^0 (left) and C_n^1 (middle) and C_n^2 (right), obtained by identifying the bottom side of all rectangles, and identifying each pair of vertical sides linked by an arc.

removed and a right side of a rectangle. The last point follows from the inequality $\frac{2^n-2^{-n}-3^{-n}}{2} > 2^{n-2}$. □

Given the Lemma, we construct the space X as follows. We start from a real line $\mathbf{R}$, and for each integer $n \geq 1$, $m \in \mathbf{Z}$ we glue a copy of C_n^0 to $\mathbf{R}$ by identifying the segment $[m - 2^{n-1}, m + 2^{n+1}]$ with $i_n^0([0, 2^n])$ (through $t \mapsto i_n^0(t - m + 2^{n-1})$. We equip X with the unique euclidean metric that coincide with the metric on each copy of C_n. The properties (i) (ii) and (iii) are easy to verify from Lemma 10.1, once we realize that we can recover $\mathbf{R}$ as the unique biinfinite geodesic in X.

Now for an arbitrary function $\sigma\colon \mathbf{N}\times\mathbf{Z} \to \{0, 1, 2\}$ we can modify the definition of X by gluing to $[m - 2^{n-1}, m + 2^{n+1}]$ a copy of $C_n^{\sigma(n,m)}$, to get a space Y_σ. Then the isometry group of Y_σ is the semidirect product of $\prod_{m\in\mathbf{Z}}(\prod_{n,\sigma(m,n)=2} \mathbf{Z}/2\mathbf{Z}))$ by the subgroup of $\mathbf{Z}$ consisting of the elements k satisfying $\sigma(m + k, n) = \sigma(m, n)$ for all m, n. Also Y_R is R-locally X if $\sigma(m, n) = 0$ for all (m, n) such that $2^{n-2} \leq R$. It is straightfoward that, taking appropriate choices for σ, we can find a continuum of non isometric metric spaces satisfying (iv) (respectively (v), respectively (vi))).

APPENDIX A. UNCOUNTABLE SECOND COHOMOLOGY GROUP $H^2(H, \mathbf{Z}/2\mathbf{Z})$, BY JEAN-CLAUDE SIKORAV

Proposition A.1. *Let $u : G \to \mathbf{Z}$ be a nonzero group homomorphism. We assume that G is the fundamental group of an acyclic CW-complex X with one 0-cell, p 1-cells, q 2-cells and r 3-cells, and that $q \geq p + r$. [Algebraically, G has a presentation with p generators, q relations and r "relations between relations"; we say that G is of type (p, q, r)].*

Then

- *(i) The second homology group* $H_2(\ker u; \mathbf{Z}/2\mathbf{Z})$ *is infinite dimensional as a vector space under* $\mathbf{Z}/2\mathbf{Z}$*, ie isomorphic to* $(\mathbf{Z}/2\mathbf{Z})^{(\mathbf{N})}$*.*
- *(ii) The second cohomology group* $H^2(\ker u; \mathbf{Z}/2\mathbf{Z})$ *is uncountable. Thus* $\ker u$ *is not finitely presented.*

Proof. For any group Γ, any field k and any $q \in \mathbf{N}$, $H^q(\Gamma; k)$ is naturally isomorphic as a k-vector space to the dual $(H_q(\Gamma; k))^*$. Thus if (i) holds, one has

$$H^2(\ker u; \mathbf{Z}/2\mathbf{Z}) \approx \left((\mathbf{Z}/2\mathbf{Z})^{(\mathbf{N})}\right)^* \approx (\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$$

and (ii) also holds.

It thus suffices to prove (i). By [B82], $H_2(\ker u; \mathbf{Z}/2\mathbf{Z})$ can be computed as follows.

1) Consider the chain complex $C_*(\widetilde{X})$ giving the homology $H_*(\widetilde{X}; \mathbf{Z}/2\mathbf{Z})$ as a left $(\mathbf{Z}/2\mathbf{Z})[G]$-module. Since $\widetilde{X}$ is contractible by the acyclicity of X, the lower part of this complex gives an exact sequence

$$(\mathbf{Z}/2\mathbf{Z})[G]^r \xrightarrow{\partial_3} (\mathbf{Z}/2\mathbf{Z})[G]^q \xrightarrow{\partial_2} (\mathbf{Z}/2\mathbf{Z})[G]^p \xrightarrow{\partial_1} (\mathbf{Z}/2\mathbf{Z})[G].$$

We shall use the fact that $\partial_1(\lambda_1, \cdots, \lambda_p) = \displaystyle\sum_{i=1}^{p} \lambda_i(x_i - 1)$ where $x_1, \cdots, x_p$ are the generators of G associated to the 1-cells.

2) Let $\widehat{X}$ be the covering of X such that $\pi_1(\widehat{X}) = \ker u$. Since its universal cover is still $\widetilde{X}$ thus contractible, we have $H_q(\widehat{X}; R) \approx H_q(\ker u; R)$ for any q and any coefficient ring R, in particular $H_2(\widehat{X}; \mathbf{Z}/2\mathbf{Z}) \approx H_2(\ker u; \mathbf{Z}/2\mathbf{Z})$. Moreover, since $\widehat{X}$ is a Galois covering of X of group $G/\ker u \approx \mathbf{Z}$, $H_*(\widehat{X}; \mathbf{Z}/2\mathbf{Z})$ is a module over

$$(\mathbf{Z}/2\mathbf{Z})[G/\ker u] \approx (\mathbf{Z}/2\mathbf{Z})[\mathbf{Z}] \approx (\mathbf{Z}/2\mathbf{Z})[t, t^{-1}]$$

which is given by the complex $(\mathbf{Z}/2\mathbf{Z})[G/\ker u] \otimes_{(\mathbf{Z}/2\mathbf{Z})[G]} C_*(\widetilde{X})$.
Thus $H_2(\ker u; \mathbf{Z}/2\mathbf{Z}) \approx \frac{\ker D_2}{\mathrm{im}\ D_3}$ where D_i is the image of ∂_i under the natural morphism $(\mathbf{Z}/2\mathbf{Z})[G] \to (\mathbf{Z}/2\mathbf{Z})[G/\ker u]$.

Denote $R = (\mathbf{Z}/2\mathbf{Z})[t, t^{-1}]$, which is an integral domain, and $F = (\mathbf{Z}/2\mathbf{Z})(t)$ its fraction field. We therefore have a sequence of R-linear maps

$$R^r \xrightarrow{D_3} R^q \xrightarrow{D_2} R^p \xrightarrow{D_1} R,$$

with $D_i D_{i+1} = 0$, which induces a sequence of F-linear maps

$$F^r \xrightarrow{D_3^F} F^q \xrightarrow{D_2^2F} F^p \xrightarrow{D_1^F} F.$$

Since R is a principal ideal domain, we have $\frac{\ker D_2}{\mathrm{im}\ D_3} \approx R^n \oplus T$ where T is a torsion R-module, and $\frac{\ker D_2^F}{\mathrm{im}\ D_3^F} \approx F^n$. Since $R \approx (\mathbf{Z}/2\mathbf{Z})^{(\mathbf{Z})}$ as a $(\mathbf{Z}/2\mathbf{Z})$-vector space, to finish the proof it suffices to show that $n > 0$.

The image of $g \in G \subset (\mathbf{Z}/2\mathbf{Z})[G]$ in R is $t^{u(g)}$, thus

$$D_1(\lambda_1, \cdots, \lambda_p) = \sum_{i=1}^{p} \lambda_i (t^{u(x_i)} - 1).$$

Since $u \neq 0$, $D_1 \neq 0$ thus $D_1^F \neq 0$. Since F is a field, this implies (by an easy exercize of linear algebra, used in the proof of the Morse inequalities)

$$n = \dim_F \big(\frac{\ker D_2^F}{\mathrm{im}\ D_3^F}\big) \geq -r + q - p + 1 = q + 1 - (r + p).$$

By the hypothesis, $n > 0$, qed. □

Example A.2. 1) Let $G = \mathbf{F}_{p_1} \times \mathbf{F}_{p_2}$, p_1 and $p_2 \geq 2$, where $\mathbf{F}_k$ is the free group on k generators. Here X is the product of two bouquets of circles [or roses], and $p = p_1 + p_2$, $q = p_1 p_2$, $r = 0$. Thus

$$q - (p + r) = (p_1 - 1)(p_2 - 1) - 1 \geq (2 - 1)(2 - 1) - 1 = 0.$$

Remark: here, $\ker u$ is finitely generated if u is nonzero on each factor. More generally, if $G = G_1 \times G_2$ with G_1, G_2 finitely generated and u nonzero on each factor, $\ker u$ is finitely generated.

2) Let $G = \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2})$, where Σ_g is a surface of genus g (closed, orientable) and $g_1, g_2 \geq 2$. Here $X = \Sigma_{g_1} \times \Sigma_{g_2}$, $p = r = 2g_1 + 2g_2$, $q = 4g_1g_2 + 2$, thus

$$q - (p + r) = 4g_1g_2 + 2 - (4g_1 + 4g_2) = 4(g_1 - 1)(g_2 - 1) - 2 > 0.$$

3) In general, if $G = G_1 \times G_2$ with G_i of type $(p_i, q_i, 0)$, then G is of type

$$(p, q, r) = (p_1 + p_2, p_1p_2 + q_1 + q_2, p_1q_2 + q_1p_2),$$

thus

$$q - (p + r) = (1 - p_1 + q_1)(1 - p_2 + q_2) - 1.$$

Thus the hypothesis is satisfied if ($q_1 < p_1$ and $q_2 < p_2$), or ($q_1 \geq p_1$ and $q_2 \geq p_2$). There are many examples for the first case (for instance all groups with at least two generators and a unique relator which is primitive), which also ensures $\mathrm{Hom}(G, \mathbf{Z}) \neq 0$. For the second case, I do not see any obvious example.

Remark A.3. The hypothesis on G can be weakened to: $\mathbf{Z}/2\mathbf{Z}$ has a free resolution (C_i) over $(\mathbf{Z}/2\mathbf{Z})[G]$ such that $\dim C_0 = 1$ and $\dim C_2 \geq \dim C_1 + \dim C_3$. Or more generally such that

$$\chi(C_{\leq 3}) := \sum_{i=0}^{3} (-1)^i \dim C_i > 0.$$

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