# EXACTNESS OF LINEAR GROUPS AND THE NOVIKOV CONJECTURE (AFTER AN ARTICLE OF GUENTNER, HIGSON AND WEINBERGER).

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## 1. Introduction

In a celebrated article, Guentner, Higson and Weinberger proved the Novikov conjecture for Linear groups [GHW]. More precisely, they showed that given a field $K$, any countable subgroup of $GL(d, K)$ uniformly embeds into a Hilbert space, and then deduced the topological implication from a deep result of Yu [Y]. When $d = 2$, they obtained an even stronger statement, namely that the group is a-T-menable and hence satisfies the Baum-Connes conjecture [HK]. Their approach relies on a classical construction inducing a left-invariant pseudo-metric on the group from a valuation on the field $K$. Their central discovery in this article is that any finitely generated field possesses, in a very strong sense, a lot of such metrics. They combine this observation with the fact that the group is exact with respect to each one of these metrics. Note that being exact is a sufficient condition to uniformly embed into a Hilbert space (see Section 3).

These notes correspond to a mini-course the author gave in Bloomington in August 2010 during a summer school on the Baum-Connes conjecture organized by David Fisher, Erik Guentner and Guoliang Yu. We will start by recalling a few definitions and basic facts about exactness, amenablity, uniform embeddings and a-T-menability. We will then state the main results about linear groups in Section 5. Finally we will give an entire proof in the special case[1] $G = GL(d, \mathbb{Z}[X])$. Our approach slightly differs from the original article. For instance, in Section 4, we introduce the notion, which is only implicit in the paper, of uniformly discrete sequence of pseudo-metrics on a metric space. We believe that it might have applications elsewhere.

## 2. preliminaries

Let us start with some basic definitions.

*Date*: September 1, 2010.

[1]We believe that working out this concrete example gives a fair idea of the general phenomenon.

2.1. **Hilbert kernel.** Let $X$ be a set. A *Hilbert kernel* on $X$ is a map $k : X \times X \to \mathbb{R}$ such that there exist a Hilbert space $\mathcal{H}$ and a map $\phi : X \to \mathcal{H}$ such that $k(x,y) = \|\phi(x) - \phi(y)\|^2$ for all $x, y \in X$. If $X = G$ is a group, and if $k$ is left-invariant, in the sense that $k(gx, gy) = k(x,y)$ for all $x, y, g \in G$, then the function $f(g) = k(1_G, g)$ will be called a *Hilbert function*.

2.2. **Positive-definite kernel.** Let $X$ be a set. A *positive-definite (PD) kernel* on $X$ is a map $k : X \times X \to \mathbb{R}$ such that there exist a Hilbert space $\mathcal{H}$ and a map $\phi : X \to \mathcal{H}$ such that $k(x,y) = \langle \phi(x), \phi(y) \rangle$ for all $x, y \in X$. Moreover if $X = G$ is a group, and if $k$ is left-invariant, then $f(g) = k(1_G, g)$ will be called a *PD function*. If in addition, $\|\phi(x)\| = 1$ for all $x \in X$, then we will say that $k$ (resp. $f$) is *normal*.

**Remark.** *Note that Hilbert kernels on a set $X$ (resp. Hilbert functions on a group $G$) form a convex subcone of the set of positive functions on $X \times X$ (resp. on $G$), and that PD kernels on $X$ (resp. PD functions on $G$) form a convex multiplicative subcone of the set of real functions on $X \times X$ (resp. on $G$). To see why, just remark that if $k_1(x,y) = \langle \phi_1(x), \phi_1(y) \rangle$, and $k_2(x,y) = \langle \phi_2(x), \phi_2(y) \rangle$, then one gets $k_1 + k_2$ by taking the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$ and of $\phi_1$ and $\phi_2$. The same argument applies to Hilbert kernels, and for multiplicativity, one just needs to replace the direct sum by a tensor product. Stability under positive scalar multiplication is trivial. These two convex cones are also closed for the topology of pointwise convergence. Indeed, if $k_n(x,y)$ is a sequence converging to some kernel $k_\infty$, then $k_\infty$ is associated to the ultralimit of $(\mathcal{H}_n, \phi_n)$ for some free ultrafilter on $\mathbb{N}$.*

2.3. **Example.** An important example of Hilbert kernel is the graph metric on the vertex set $\mathcal{V}$ of a simplicial tree $T$. Let us choose an orientation of $T$, and let us consider the Hilbert space $\ell^2(\mathcal{E})$, where $\mathcal{E}$ denote the set of oriented edges. Fix some vertex $o$, and for every vertex $x$, let $\Gamma_x$ be the set of oriented edges contained in the geodesic segment from $o$ to $x$. Define $\phi_x = 1_{\Gamma_x}$ if these edges are positively oriented, and $\phi_x = -1_{\Gamma_x}$ otherwise. An easy computation shows that $d(x,y) = \|\phi_x - \phi_y\|^2$.

We deduce from this that if a group $G$ acts on $T$, then the map $g \to d(o, g \cdot o)$ defines a Hilbert length[2] on $G$.

We will now define the properties of uniform embeddability into a Hilbert space and of exactness for a pseudo-metric space, and of a-T-menability and amenability for a group equipped with a pseudo-length function. In the sequel, $(X,d)$ shall denote a set equipped with a pseudo-distance $d$, i.e. a function $d : X \times X \to \mathbb{R}_+$ such that $d(x,x) = 0$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. If $X = G$ is a group, then we will suppose

[2]Note moreover that the map $g :\to \phi_{g \cdot o}$ defines a 1-cocycle with values in the unitary representation of $G$ on $\ell^2(\mathcal{E})$ (by left-translations).

in addition that $d$ is left-invariant. In this case, we will denote by $\ell(g) = d(1_G, g)$ the corresponding pseudo-length function on $G$.

2.4. **Uniform embeddability.** The (pseudo-)metric space $(X, d)$ is *uniformly embeddable* in a Hilbert space if there exists a Hilbert kernel $k$ on $X$, two unbounded non-decreasing functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x, y \in X$,

$$\rho_1(d(x, y)) \leq k(x, y) \leq \rho_2(d(x, y)).$$

If $G$ is a group, and if we add in the previous definition the requirement that the $k_n$'s are invariant, then we obtain the following property

2.5. **A-T-menability.** The metric group $(G, \ell)$ is *a-T-menable* if there exists a Hilbert function $f$ and $\rho_1, \rho_2$ as above such that for all $g \in G$,

$$\rho_1(\ell(g)) \leq k(g) \leq \rho_2(\ell(g)).$$

2.6. **Exactness.** The metric space $(X, d)$ is exact if there exists a sequence $(k_n)$ of normal PD kernels on $X$, together with a sequence $(R_n)$ such that

(i) $k_n(x, y) = 0$ if $d(x, y) \geq R_n$
(ii) $k_n$ converges to 1 uniformly on $\{(x, y), d(x, y) \leq C\}$ for every $C \geq 0$.

The "equivariant" version of exactness is amenability.

2.7. **Amenability.** The metric group $(G, \ell)$ is amenable if there exists a sequence of PD functions $(f_n)$ and a sequence $(R_n)$ such that

(i) $f_n(g) = 0$ if $\ell(g) \geq R_n$
(ii) $f_n$ converges to 1 uniformly on bounded sets.

## 3. Exactness implies uniform embeddability

In this section, we briefly recall why exactness (resp. amenability) implies uniform embeddability into a Hilbert space (resp. a-T-menability). Since the proofs are identical, we will only prove the non-equivariant statement.

Given a PD-kernel $k$, we can define a Hilbert kernel $k'$ (corresponding to the same function $\phi : X \to \mathcal{H}$) by

$$k'(x, y) = k(x, x) + k(y, y) - 2\Re(k(x, y)).$$

Suppose that $(k_n)$ verifies definition 2.6. Then $(k'_n)$ satisfies

(i) $k'_n(x, y) = 2$ if $d(x, y) \geq R_n$
(ii) $k'_n$ converges to 0 uniformly on $\{(x, y), d(x, y) \leq C\}$ for every $C \geq 0$.

Note that we can assume without loss of generality that the sequence $(R_n)$ is increasing and unbounded.

Let $(C_n)$ be an increasing sequence going to $\infty$ slowly enough so that $\sup_{d(x,y)\leq C_n} k'_n(x,y)$ goes to zero. Up to taking a subsequence, we can suppose that the sum $\sum_n \sup_{d(x,y)\leq C_n} k'_n(x,y)$ converges. In particular, $k'(x,y) := \sum_n k'_n(x,y)$ is a well-defined PD kernel on $X$, and the function $\rho_2(C) = \sum_n \sup_{d_n(x,y)\leq C} k'_n(x,y)$ is also well-defined for all $C \geq 0$. On the other hand, for all $j \in \mathbb{N}$, define $\rho_1(C) := j$ for all $R_j \leq C \leq R_{j+1}$. Then, by (i), and for all $x, y$ such that $d(x,y) \leq C$, we have

$$\rho_1(d(x,y)) \leq j \leq \sum_n k'_n(x,y).$$

So $k'$ verifies Definition 2.5.

## 4. Uniformly discrete sequences of pseudo-metrics and exacness

In this section, we define a very general notion of *uniform discreteness* for sequences of pseudo-metrics on a metric space. Although not explicitly defined there, this notion plays a crucial role in [GHW]. Let $(X,d)$ be a (pseudo-)metric space, and let $(d_1, d_2, \ldots)$ be a sequence of pseudo-metrics on $X$.

4.1. **Uniformly discrete sequence of metrics.** *We say that the sequence $(d_n)$ is uniformly discrete, if*

(i) $d_n \leq d$ *for all* $n \geq 1$,
(ii) *for all* $(R_n)$*, there exists* $R \geq 0$ *such that for all* $x \in X$*,*

$$\bigcap_{n\geq 1}\{(x,y) \in X^2, d_n(x,y) \leq R_n\} \subset \{(x,y) \in X^2, d(x,y) \leq R\}.$$

4.2. **Proposition.** *Suppose that $(d_n)$ is uniformly discrete, and that $(X, d_n)$ is exact (resp. is uniformly embeddable into a Hilbert space) for each $n \in \mathbb{N}$. Then so is $(X,d)$.*

*Proof.* Let us start with exactness. First we shall reformulate the definition of exactness as follows: for every $\varepsilon > 0$ and every $C \geq 0$, there exists $R \geq 0$ and a normal PD kernel $k$ on $X$ such that

(i) $k(x,y) = 0$ if $d(x,y) \geq R$
(ii) $|1-k| \leq \varepsilon$ on $\{(x,y), d(x,y) \leq C\}$.

Note that this is trivially equivalent to the original definition. Moreover, up to replacing $k$ by $k\overline{k} = |k|^2$, we can also suppose that $k$ is non-negative. Now, let $\varepsilon > 0$ and $C \geq 0$. Let $\varepsilon_n > 0$ be such that $\prod_n (1-\varepsilon_n) \geq 1-\varepsilon$. By assumption, for every $n$, there exists a non-negative normal PD kernel $k_n$ on $X$ and let $R_n \geq 0$ such that

(i) $k_n(x,y) = 0$ if $d_n(x,y) \geq R_n$

(ii) $1 - k_n \leq \varepsilon_n$ on $\{(x,y), d_n(x,y) \leq C\}$.

Now, define $k(x,y) = \prod_{n\geq 1} k_n(x,y)$. Note that since $0 \leq k(x,y) \leq 1$, the product converges, and therefore defines a normal PD kernel on $X$. The support of $k$ is contained in $\bigcap_{n\geq 1}\{(x,y) \in X^2, d_n(x,y) \leq R_n\}$ and therefore in $\{(x,y) \in X^2, d(x,y) \leq R\}$ for some $R \geq 0$. It remains to show that $1 - k(x,y) \leq \varepsilon$ for all $\{(x,y) \in X^2, d(x,y) \leq C\}$. On the other hand, if $d(x,y) \leq C$,

$$1 - k(x,y) \leq 1 - \prod_{n\geq 1}(1 - \varepsilon_n) \leq \varepsilon.$$

Now, let us consider the case of uniform embeddings. For every $n$, let $k_n$ be a Hilbert kernel satisfying Definition 2.4 with $\rho_{1,n}$ and $\rho_{2,n}$. Let $(\alpha_n)$ be a sequence of positive number such that $\sum_n \alpha_n \rho_{2,n}(n) < \infty$. Define $k = \sum_n \alpha_n k_n$, and $\rho_2 = \sum_n \alpha_n \rho_{2,n}$. We have, for all $x, y \in X$,

$$k(x,y) = \sum_n \alpha_n k_n(x,y) \leq \sum_n \alpha_n \rho_{2,n}(d_n(x,y)) \leq \sum_n \alpha_n \rho_{2,n}(d(x,y)) = \rho_2(d(x,y)) < \infty.$$

In particular, $k$ is a well-defined Hilbert kernel on $X$. Define for all $t > 0$, $\rho_1(t) = \inf_{d(x,y)\leq t} k(x,y)$. Clearly, $\rho_1$ is non-decreasing. Let us prove that it goes to infinity. In other words, we need to show that for every $C > 0$, there exists $R$ such that the set $A_C = \{(x,y), k(x,y) \leq C\}$ is contained in $\{(x,y), d(x,y) \leq R\}$. But observe that for every $n$, $A_C$ is contained in $\{(x,y), d_n(x,y) \leq R_n\}$, where $R_n$ satisfies $\alpha_n \rho_{1,n}(R_n) \geq C$ (such a number exists due to the properness of $\rho_{1,n}$). We therefore conclude thanks to the uniform discreteness of the sequence $(d_n)$. □

Note that if we replace the words “pseudo metrics on a metric space”, by “pseudo-length on a group equipped with length metric” in the previous demonstration, we obtain the following

4.3. **Proposition.** *Suppose that $(\ell_n)$ is a uniformly discrete sequence of pseudo-length on $(G,\ell)$, and that $(G,\ell_n)$ is amenable (resp. a-T-menable) for each $n \in \mathbb{N}$. Then so is $(G,\ell)$.* □

## 5. Linear groups are exact

Let us state the two main results of [GHW]. Recall that any countable group can be equipped with some proper left-invariant metric. Note that none of the properties that we have defined in the previous sections is sensitive to this choice of metric. In other words, in the context of countable groups, they are just properties of the group itself.

5.1. **Theorem.** *Let $K$ be a field. Any countable subgroup of $GL(d,K)$ is exact.*

5.2. **Theorem.** *Let $K$ be a field. Any countable subgroup of $GL(2,K)$ is a-T-menable.*

Since these two properties are stable under direct limit, we can restrict ourselves to finitely generated groups. On the other hand, we have the following short exact sequence

$$1 \to SL(2,K) \to GL(2,K) \to K^* \to 1$$

where the surjective map is the determinant. Since a-T-menability for countable groups is stable under extension by an amenable group, this reduces the problem to showing that finitely generated subgroups of $SL(2,K)$ are a-T-menable.

The above theorems will follow from Proposition 4.2 and the following

5.3. **Theorem.** *Let $K$ be a field. Let $G$ be a finitely generated subgroup of $GL(d,K)$ (resp. $SL(2,K)$). Equip $G$ with a word length $\ell_S$ associated to some finite generating set $S$. Then there exists a sequence $(\ell_n)$ of uniformly discrete pseudo-lengths on $G$, such that for every $n$, $(G,\ell_n)$ is exact (resp. a-T-menable).*

Let us reformulate the fact that $(\ell_n)$ is uniformly discrete: for any sequence $R_n > 0$, there exists $R > 0$ such that

$$\bigcap_n B_n(R_n) \subset B(R),$$

where $B_n(R_n) = \{g,\ \ell_n(1,g) \leq R_n\}$, and $B(R) = \{g,\ l_S(g) \leq R\}$. Note that we can rescale the lengths $\ell_n$ so that they also satisfy $\ell_n(s) \leq \ell_S(s)$ for all $s \in S$. It follows at once that this inequality holds for all elements in $G$ (in other words, up to a rescaling factor, the word length is always bigger than any other length function).

The next three sections are devoted to the proof of Theorem 5.3. In the next one, we introduce the notion of strongly embeddable fields. As already mentioned in the introduction, the central observation of [GHW] is that finitely generated fields are discretely embeddable, which roughly means that they admit a "uniformly discrete" sequence of valuations. Instead of proving this result in full generality, we will only consider a special case, keeping in mind that our main goal is to get a full proof for the interesting case $G = GL(m,\mathbb{Z}[X])$. In Section 7, we will explain how to associate a length on $GL(m,K)$, to a valuation on $K$, and we will prove that equipped with such a length, $GL(m,K)$ is exact and that $GL(2,K)$ is a-T-menable. Finally, in the last section, we will gather the conclusions of the previous sections and prove Theorem 5.3.

## 6. Discretely embeddable fields

A *valuation* on a field $K$ is a map $d : K \to [0,\infty)$ satisfying, for all $x$, $y \in K$

(1) $d(x) = 0 \Leftrightarrow x = 0$
(2) $d(xy) = d(x)d(y)$
(3) $d(x + y) \leq d(x) + d(y)$

A valuation obtained as the restriction of the usual absolute value on $\mathbb{C}$ via a field embedding $K \to \mathbb{C}$ is *archimedian*. A valuation satisfying the stronger *ultra-metric inequality*

(4) $d(x + y) \leq \max\{ d(x), d(y) \}$

in place of the triangle inequality (c) is *non-archimedian*. If in addition the range of $d$ on $K^\times$ is a discrete subgroup of the multiplicative group $(0, \infty)$ the valuation is *discrete*.

6.1. **Definition.** [GHW] A field $K$ is discretely embeddable if for every finitely generated subring $A$ of $K$ there exists a sequence $(d_n)$ of valuations on $K$ with the following property: For every sequence $R_n > 0$, the subset

$$\{a \in A,\ d_n(a) \leq R_n, \forall n \in \mathbb{N}\}$$

is finite.

The main observation in [GHW] is that a finitely generated field is discretely embeddable. Instead of proving this result in its full generality here, we will exhibit a sequence $d_n$ in the special case where $K = \mathbb{Q}(X)$ and $A = \mathbb{Z}[X]$. As a consequence, we will obtain a proof of Theorem 5.3 for $G = GL(m, \mathbb{Z}[X])$. Consider the following valuations on $K = \mathbb{Q}(X)$. Write every element $x \in K$ as an irreducible fraction $P/Q$, with $P, Q \in \mathbb{Q}[X]$, and denote

$$d_0(x) = 2^{deg(P)-deg(Q)}.$$

Choose a sequence $(a_1, a_2, \ldots)$ of pairwise distinct, transcendental complex numbers. For every $n \geq 1$, let $d_n$ be the archimedian valuation on $K$ corresponding to the embedding of $K$ into $\mathbb{C}$ sending $X$ to $a_n$.

6.2. **Proposition.** *For every sequence $R_n > 0$, the subset*

$$\{P \in \mathbb{Z}[X],\ d_n(P) \leq R_n, \forall n \in \mathbb{N}\}$$

*is finite.*

*Proof.* The inequality $d_0(P) \leq R_0$ implies that the degree of $P$ is at most $k \in \mathbb{N}$, for some integer $k$ satisfying $2^k \geq R_0$. Now consider the map $\varphi : \mathbb{Z}[X]_k \to \mathbb{C}^{k+1}$ defined by $\varphi(P) = (P(a_1), P(a_2), \ldots, P(a_{k+1})) = (d_1(P), \ldots, d_{k+1}(P))$. Note that $\varphi$ is the restriction to $\mathbb{Z}[X]_k$ (the set of polynomials of degree at most $k$) of a linear isomorphism between $\mathbb{C}[X]_k$ and $\mathbb{C}^{k+1}$. Since $\mathbb{Z}[X]_k$ is a discrete subset of $\mathbb{C}[X]_k$, we conclude that for every sequence $R_1, \ldots R_{k+1}$ of positive numbers, the subset

$$\{P \in \mathbb{Z}[X]_k,\ d_n(P) \leq R_n, \forall 1 \leq n \leq k + 1\}$$

is finite, which finishes the proof of the proposition. □

## 7. Metrics on linear groups associated with valuations

Let $d$ be a valuation on a field $K$. Guentner-Higson-Weinberger define a pseudo-length function $\ell_d$ on $GL(m,K)$ as follows: if $d$ is discrete

$$\ell_d(g) = \log \max_{ij}\{\, d(g_{ij}), d(g^{ij})\,\}, \tag{7.1}$$

where $g_{ij}$ and $g^{ij}$ are the matrix coefficients of $g$ and $g^{-1}$, respectively; if $d$ is archimedean, arising from an embedding $K \hookrightarrow \mathbb{C}$ then

$$\ell_d(g) = \log \max\{\, \|g\|, \|g^{-1}\|\,\}, \tag{7.2}$$

where $\|g\|$ is the norm of $g$ viewed as an element of $GL(m,\mathbb{C})$, and similarly for $g^{-1}$.

7.1. **Proposition.** *Let $d$ be an archimedean or a discrete valuation on a field $K$. The group $GL(m,K)$, equipped with the left-invariant pseudo-metric induced by $\ell_d$, is exact.*

*Proof.* The result in the archimedean case follows immediately from the corresponding result for $GL(m,\mathbb{C})$; indeed, the metric on $GL(m,K)$ is the subspace metric inherited from an embedding into $GL(m,\mathbb{C})$. Observe that $GL(m,\mathbb{C})$ has a cocompact, solvable subgroup, namely the subgroup of triangular matrices. Since this group is amenable, it is exact, and so is $GL(m,\mathbb{C})$. Indeed, one has a decomposition $GL(m,\mathbb{C}) = TK$, where $T$ is the group of triangular matrices, and $K = U(m)$ is compact (here we do not really use the fact that $U(m)$ is a group). Consider the projection map $TK \to T$. Now, to prove exactness for $GL(m,\mathbb{C})$, one can simply define a sequence of PD kernel on $T$, and pull it back on $GL(m,\mathbb{C})$. It is easily checked to have the required properties.

The discrete case is more subtle than the archimedean case, primarily because we do not assume that $K$ is locally compact. If $d$ is a discrete norm on a field $K$ the subset

$$\mathcal{O} = \{\, x \in K : d(x) \le 1\,\}$$

is a subring of $K$, the *ring of integers of* $d$; the subset

$$\mathfrak{m} = \{\, x \in K : d(x) < 1\,\}$$

is a principal ideal in $\mathcal{O}$; a generator for $\mathfrak{m}$ is a *uniformizer*. In our special case $K = \mathbb{Q}(X)$, $d = d_0$, a uniformizer is $X$.

For the proof of the proposition, let $d$ be a discrete norm on a field $K$ and fix a uniformizer $\pi$. For the proof we shall introduce some subgroups of $\mathrm{GL}(m,K)$. Let $D$ denote the subgroup of diagonal matrices with powers of the uniformizer on the diagonal and let $U$ denote the unipotent upper triangular matrices. Observe that $D$ normalizes $U$ so that $T = DU$ is also a subgroup (namely the group upper triangular matrices). Restrict the length function $\ell_\gamma$ to each subgroup and equip each with the associated (left-invariant pseudo-)metric (which is in fact the subspace pseudo-metric from $G$). The inclusion of $T$ in $G$ is isometric. Further, it is metrically onto in the sense that every element of $G$ is at distance zero from an element of $T$. One easily checks[3] that $G = TGL(m, \mathcal{O})$ and elementary calculations show that every $h \in GL(m, \mathcal{O})$ has length zero. Hence, if $g = th$ then $d(t, g) = \ell(h) = 0$. It is therefore enough to show that $T$ is exact.

The *dilation by* $\theta \in K$ is the function $\Theta : U \to U$ defined by

$$\Theta(u)_{ij} = \theta^{j-i} u_{ij};$$

the entries on the $k^{th}$-superdiagonal of $n$ are multiplied by $\theta^k$. (For $k = 0, \ldots, n-1$ the $k^{th}$*-superdiagonal* of an $n \times n$ matrix consists of the positions $(i, j)$ for which $j - i = k$.) The formula for matrix multiplication shows that $\Theta$ is an endomorphism of $U$. Further, it is an automorphism with inverse the dilation by $\theta^{-1}$.

Fix $\theta = \pi^{-1}$, so that $d(\theta) > 1$. Let $U_0$ be the subgroup of $U$ comprised of elements of length zero, and define a sequence of subgroups of $U$ by $U_k = \Theta(U_{k-1})$. Clearly, $U_k$ is bounded and contains the ball of radius $k$.

Let $T$ act on $\ell^2(T/U_k)$ by the quasi-regular representation. Denote by $D_k$ the finite subset $D$ consisting of diagonal matrices $a$ for which $\ell_d(a) \leq k/4$. Let $\nu_k \in \ell^2(T/U_k)$ be the normalized characteristic function of $D_k$. Finally, define

$$\phi_k(g) = \langle \nu_k, g \cdot \nu_k \rangle_{\ell^2(T/U_k)}.$$

Let us prove that

(i) $\phi_k(g) = 0$ for $\ell_d(g)$ large enough.
(ii) $1 - \phi_k(g) \to 0$, uniformly on $\{g, \ell_d(g) \leq C\}$

which will imply the proposition.

The first statement follows easily from the fact that $U_k$ is bounded. Indeed, a large enough element $g$ translates $D_k U_k$ into a disjoint subset in $T$.

To prove the second statement, let $g = au$, with $a \in D$, $u \in U$. Note that $\ell_d(a) \leq \ell_d(g)$, so that by triangular inequality, $\ell_d(u) \leq 2\ell_d(g)$. Now, let $g = au \in T$ such that $\ell_d(g) \leq k/8$. In particular, $\ell_d(u) \leq k/4$. Let $b \in D_k$, we have $ubU_k = b(b^{-1}ub)U_k$. So $\ell_d(b(b^{-1}ub)) \leq k$, and since $U_k$ contains all elements of $U$ of length at most $k$, $ubU_k \subset D_k U_k$. Hence $uT_k = T_k$ modulo $U_k$, so that we can assume that $g = a$. On the other hand, we have

$$1 - \phi_k(a) = \langle \nu_k, \nu_k - a \cdot \nu_k \rangle = |D_k \smallsetminus aD_k|.$$

[3] This is a nice undergraduate exercise on elementary operations on matrices.

Note that $D = \mathbb{Z}^m$, and $D_k$ is a ball of radius $k$ in $\mathbb{Z}^m$. Hence (ii) amounts to the fact that an increasing sequence of balls gives a Følner sequence for the amenable group $\mathbb{Z}^m$. □

7.2. **Proposition.** *Let $d$ be an archimedean or a discrete valuation on a field $K$. The group $SL(2, K)$, equipped with the left-invariant pseudo-metric induced by $\ell_d$, is a-T-menable.*

*Proof.* The fact that $SL(2, \mathbb{C})$ is a-T-menable is due to Farraut ad Harzallah [FH]. The proof roughly goes like this: $SL(2, \mathbb{C})$ acts properly transitively by isometries on the real 3-dimensional Hyperbolic space, whose distance is a Hilbert kernel. Pulling this distance back on $SL(2, \mathbb{C})$ gives the desired left-invariant proper Hilbert kernel.

Let us consider a discrete valuation $d$. We will prove that the length $\ell_d$ on $SL(2, K)$ is a Hilbert length. For this, it is enough to prove that $\ell_d$ is a tree length (see example 2.3). As in the proof of the previous proposition, we have $SL(2, K) = T^1 SL(2, \mathcal{O})$ where here, $T^1$ is the subgroup of $T$ of matrices with determinant 1. The inclusion $T^1 \to SL(2, K)$ is again a metrically onto isometry. So that we just need to prove that $\ell_d$ is a tree length on $T^1$. Observe that $T^1 = D^1 \ltimes U$, where $D^1 \simeq \mathbb{Z}$ is the subgroup generated by the diagonal matrix $a$ with coefficients $(\pi, \pi^{-1})$. Let $U_0$ be the subgroup of unipotent elements with coefficients in $\mathcal{O}$, and let $p$ be the projection onto $D^1 \simeq \mathbb{Z}$.

Let $\ell_M$ be the word length associated to the subset $M = U_0\{a, a^{-1}\}U_0$. This defines a Cayley graph structure on $T^1$, which is invariant under right translations by $U_0$. Hence it induces a graph structure on the quotient $T^1/U_0$ on which $T^1$ acts transitively. The length induced by this action actually coincides with $\ell_d$. To see this, first observe that the two lengths are invariant under both right and left translations by $U_0$. In virtue of the decomposition $T^1 = U_0 D^1 U_0$, it is therefore enough to check that the two lengths coincide in restriction to $D^1$, in which case a simple calculation shows that $\ell_d(a^n) = \ell_M(a^n) = |n|$. Now we need to prove that this graph is actually a tree. Since $M$ generates $T^1$, it is connected. Note that if there is an injective loop, then this loop has a vertex $v$ for which $p(v) = k$ is minimum, and two vertices adjacent to $v$ whose images by $p$ equal $k+1$. We therefore need to prove that every vertex $v$ in our graph has only one adjacent vertex $v'$ such that $p(v') = p(v) + 1$. Let us see vertices as elements of $T^1/U_0$. By homogeneity, we can suppose that $v = U_0$. We first make the crucial observation that $a^{-1}U_0 a \subset U_0$. Suppose indeed that two vertices are adjacent to $U_0$ and both project to 1. This means that they are of the form $u_1 a U_0$ and $u_2 a U_0$, with $u_1, u_2 \in U_0$. We have

$$(u_1 a U_0)^{-1}(u_2 a U_0) \subset U_0 a^{-1} U_0 a U_0 = U_0.$$

Hence these two vertices are at distance 0 in the graph, so they coincide. □

## 8. End of the proof of Theorem 5.3 (for $G = GL(m, \mathbb{Z}[X])$)

Let us prove that any finitely generated subgroup $G$ of $GL(m, \mathbb{Z}[X])$ is exact and that it is a-T-menable if $m = 2$. Let $(d_n)$ be the sequence of valuation on $\mathbb{Q}(X)$ introduced in Proposition 6.2. It follows from the formulas (7.1) and (7.2) that the corresponding sequence of lengths $(\ell_{d_n})$ is uniformly discrete on $G$, equipped with a word metric. In the last section, we proved that for every $n$, $(GL(m, \mathbb{Z}[X]), \ell_n)$ is exact (and a-T-menable if $m = 2$). The conclusion for the exactness (resp. a-T-menability) now follows from Proposition 4.2 (resp. Proposition 4.3).

## References


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