MAS275 Probability Modelling

Solutions to Exercises

1. (a) The transition matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where the rows and columns should be labelled $0, 1, 2, 3$.

(b) The question implies that the distribution at time n, $\boldsymbol{\pi}^{(n)}$, is $(\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4})$. So we calculate

$$\boldsymbol{\pi}^{(n+1)} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{pmatrix}.$$

So the probability that there are no ducks in pond B, i.e. that $X_{n+1} = 3$, is $1/12$.

2. (a)

$$P = \begin{pmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{pmatrix},$$

where the rows and columns are labelled $1, 2, 3, 4, 5$.

(b) The required probability is

$$\pi_1^{(0)} P_{12} P_{23} P_{34} P_{45} = \frac{1}{5}(0.3)^4 = 0.00162,$$

so there is a non-zero probability of observing the appearance of this "deterministic motion" after 4 steps. With each successive step, this probability becomes smaller by a factor of 0.3.

3. (a)

$$\begin{aligned} P(X_0 = 3, X_1 = 2, X_2 = 1) &= \pi_3^{(0)} P_{32} P_{21} \\ &= \frac{1}{3} \times \frac{1}{4} \times 1 = \frac{1}{12}. \end{aligned}$$

(b)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

so the distribution of X_1 is the same as that of X_0.

4.

$$P^2 = \begin{pmatrix} \frac{7}{12} & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix},$$

so (a) is $(P^2)_{22} = \frac{2}{3}$ and (b) is $(P^2)_{13} = \frac{1}{6}$.

To get (c), we need to calculate $\boldsymbol{\pi}^{(0)} P^2 = \begin{pmatrix} \frac{19}{36} & \frac{11}{36} & \frac{1}{6} \end{pmatrix}$. So $P(X(2) = 1) = \frac{19}{36}$, $P(X(2) = 2) = \frac{11}{36}$, $P(X(2) = 3) = \frac{1}{6}$.

5. With the states in the order

$$000, 001, 010, 011, 100, 101, 110, 111,$$

the transition matrix is

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

6. (a) The state space is $\{0, 1, 2\}$. This is a special case of (b), and the transition matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) If there are i black particles in container A, then (by the description of the process) there are also i white particles in container B. So the probability that a black particle in container A swaps with a white particle in container B is $\frac{i^2}{N^2}$, the probability that a white particle in container A swaps with a black particle in container B is $\frac{(N-i)^2}{N^2}$, and the probability that the two particles swapped are the same colour (in which case there is no change) is $\frac{2i(N-i)}{N^2}$.

So, for $0 \leq i \leq N$,

$$\begin{aligned} p_{i,i+1} &= \frac{(N-i)^2}{N^2} \\ p_{i,i-1} &= \frac{i^2}{N^2} \\ p_{i,i} &= \frac{2i(N-i)}{N^2}, \end{aligned}$$

and $p_{i,j} = 0$ if $j \notin \{i-1, i, i+1\}$.

With the states $\{0, 1, 2, \ldots, N-1, N\}$, the transition matrix looks like

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \left(\frac{1}{N}\right)^2 & \frac{2(N-1)}{N^2} & \left(\frac{N-1}{N}\right)^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{2}{N}\right)^2 & \frac{4(N-2)}{N^2} & \left(\frac{N-2}{N}\right)^2 & \cdots & & & \\ 0 & 0 & \left(\frac{3}{N}\right)^2 & \frac{6(N-3)}{N^2} & \cdots & & & \\ \vdots & \vdots & & & & & & \vdots \\ 0 & 0 & & & & \frac{4(N-2)}{N^2} & \left(\frac{2}{N}\right)^2 & 0 \\ 0 & 0 & & & & \left(\frac{N-1}{N}\right)^2 & \frac{2(N-1)}{N^2} & \left(\frac{1}{N}\right)^2 \\ 0 & 0 & \cdots & & & 0 & 1 & 0 \end{pmatrix}$$

7. For gambler A, with states $\{0, 1, 2, 3, \ldots\}$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0.4 & 0 & 0.6 & 0 & 0 & \cdots \\ 0 & 0.4 & 0 & 0.6 & 0 & \cdots \\ 0 & 0 & 0.4 & 0 & 0.6 & \cdots \\ \vdots & \vdots & & & & \ddots \end{pmatrix}.$$

For gambler B, again with states $\{0, 1, 2, 3, \ldots\}$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0.55 & 0 & 0 & 0.45 & 0 & \ldots \\ 0 & 0.55 & 0 & 0 & 0.45 & \ldots \\ 0 & 0 & 0.55 & 0 & 0 & \ldots \\ \vdots & \vdots & & & & \ddots \end{pmatrix}.$$

8. Labelling the states as

1	player 2 has won
2	player 1 behind
3	scores level
4	player 1 ahead
5	player 1 has won

The transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is a Gambler's Ruin Markov chain with $N = 4$.

9. The states are:

1	scores level, player 1 serving
2	scores level, player 2 serving
3	player 1 ahead and serving
4	player 2 ahead and serving
5	player 1 behind and serving
6	player 2 behind and serving
7	player 1 has won
8	player 2 has won

With this labelling the transition matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1-p & 0 & p & 0 & 0 \\ 0 & 0 & 1-r & 0 & r & 0 & 0 & 0 \\ 1-p & 0 & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 1-r & 0 & 0 & 0 & 0 & 0 & r \\ p & 0 & 0 & 0 & 0 & 0 & 0 & 1-p \\ 0 & r & 0 & 0 & 0 & 0 & 1-r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

10. We have

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \neq P,$$

and

$$P^3 = P^2 P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = P.$$

So $P^4 = P^3 P = PP = P^2, P^5 = P^4 P = P^2 P = P^3 = P$, etc., and P^n does not converge, but alternates indefinitely between two different matrices. (Formally, the "etc." here represents a proof by induction: for the odd powers, the base case is that $P = P$, and the induction step is that if $P^{2k+1} = P$, then $P^{2k+3} = P^2 P^{2k+1} = P^2 P = P$. You can then deduce the even case from this, or do an induction for that too.)

11. The eigenvalues λ satisfy

$$\begin{aligned} \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} &= 0. \\ \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} &= 0. \\ (\lambda - 1)\left(\lambda^2 + \lambda + \frac{1}{4}\right) &= 0. \\ (\lambda - 1)\left(\lambda + \frac{1}{2}\right)^2 &= 0. \end{aligned}$$

So the roots are $\lambda = 1$ (as always for a stochastic matrix) and $\lambda = -\frac{1}{2}$ (twice). To find right eigenvectors corresponding to $\lambda = -\frac{1}{2}$ solve

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which gives three equations all amounting to $x + y + z = 0$. So any two linearly independent vectors satisfying this equation will do; for example take

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

and then

$$P^n = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-\frac{1}{2})^n & 0 \\ 0 & 0 & (-\frac{1}{2})^n \end{pmatrix} C^{-1} \to C \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} C^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

as $n \to \infty$.

12. The equations for the stationary distribution $\boldsymbol{\pi}$ are

$$\pi_1 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 \qquad (1)$$

$$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \qquad (2)$$

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2. \qquad (3)$$

Substituting (2) into (1) gives $\pi_1 = \pi_3$, and (2) then becomes $\pi_2 = \pi_1$. We must have $\pi_1 + \pi_2 + \pi_3 = 1$, so $\boldsymbol{\pi} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. It is easy to check that this does indeed satisfy the equations.

13. The equations for the stationary distribution are

$$\pi_1 = \frac{1}{3}\pi_1 + \pi_2 + \frac{1}{2}\pi_3 \qquad (4)$$

$$\pi_2 = \frac{1}{6}\pi_1 + \frac{1}{4}\pi_3 \qquad (5)$$

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_3. \qquad (6)$$

By (6), $\pi_3 = \frac{2}{3}\pi_1$, and substituting this into (5) we get $\pi_2 = \frac{1}{6}\pi_1 + \frac{1}{6}\pi_1 = \frac{1}{3}\pi_1$. We must have $\pi_1 + \pi_2 + \pi_3 = 1$, so $\pi_1 + \frac{1}{3}\pi_1 + \frac{2}{3}\pi_1 = 1$, giving $\pi_1 = \frac{1}{2}$ and hence $\pi_2 = \frac{1}{6}$, $\pi_3 = \frac{1}{3}$. Hence $\begin{pmatrix}\frac{1}{2} & \frac{1}{6} & \frac{1}{3}\end{pmatrix}$ is the unique stationary distribution. (That it is a stationary distribution was already shown in question 3(b).)

14. Looking down the column corresponding to state $j(\geq 1)$, there is $1/j$ in row $j-1$ and $(j+1)/(j+2)$ in row $j+1$, and zero everywhere else. Hence the general equilibrium equation is

$$\pi_j = \pi_{j-1}\frac{1}{j} + \pi_{j+1}\frac{(j+1)}{(j+2)}.$$

Substituting the given form for π_j on the RHS,

$$k\frac{j}{(j-1)!}\frac{1}{j} + k\frac{j+2}{(j+1)!}\frac{j+1}{j+2} = k\frac{1}{(j-1)!} + k\frac{1}{j!} = \frac{k}{j!}\{j+1\} = k\frac{j+1}{j!}$$

which is the LHS as required. (In fact $k = (2e)^{-1}$, since

$$\sum_{j=0}^{\infty}\frac{(j+1)}{j!} = \sum_{j=0}^{\infty}\frac{j}{j!} + \sum_{j=0}^{\infty}\frac{1}{j!} = \sum_{j=1}^{\infty}\frac{1}{(j-1)!} + e = e + e = 2e.)$$

15. There are eight possible ways in which a renewal can occur at time 4, listed below with their corresponding probabilities. (Renewals are marked by **x**.)

0	1	2	3	4	probability
x				x	f_4
x	x			x	$f_1 f_3$
x		x		x	f_2^2
x			x	x	$f_3 f_1$
x	x	x		x	$f_1^2 f_2$
x	x		x	x	$f_1 f_2 f_1$
x		x	x	x	$f_2 f_1^2$
x	x	x	x	x	f_1^4

So

$$u_4 = f_4 + 2f_1 f_3 + f_2^2 + 3f_1^2 f_2 + f_1^4.$$

Similarly

$$u_5 = f_5 + 2f_1 f_4 + 3f_1^2 f_3 + 4f_1^3 f_2 + 3f_1 f_2^2 + 2f_2 f_3 + f_1^5.$$

16. (a) Here $\sum_{n=1}^{\infty} f_n = 1/2$ (geometric series) so the renewal process is transient.

(b) Here $\sum_{n=1}^{\infty} f_n = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1$, by the hint, so the renewal process is recurrent. The expected time until the first renewal will be $\frac{6}{\pi^2}\sum_{n=1}^{\infty} n\frac{1}{n^2}$, but the sum here is infinite, so the renewal process is null recurrent.

Note: if we try to work out the generating function, we get

$$F(s) = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{s^n}{n^2}.$$

This probably isn't a series you've seen before, and its sum can't be expressed in terms of standard functions, though it can be written in terms of the *polylogarithm* function. We can, however, find $F'(s)$ by differentiating term by term:

$$F'(s) = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{s^{n-1}}{n}.$$

This *can* be related to a standard series, the logarithmic one, and if we multiply through by s we can see that

$$F'(s) = -\frac{6}{\pi^2 s}\log(1-s).$$

We can now see that $F'(s) \to \infty$ as $s \uparrow 1$, by considering the behaviour of $\log x$ as $x \downarrow 0$. This confirms null recurrence, though in this case the method above, without the generating functions, is easier.

(c) Here we work out the generating function $F(s)$:

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty}\frac{4}{5}\left(\left(\frac{1}{2}\right)^n - \left(-\frac{1}{3}\right)^n\right)s^n \\ &= \frac{4}{5}\left(\sum_{n=1}^{\infty}\left(\frac{1}{2}s\right)^n - \sum_{n=1}^{\infty}\left(-\frac{1}{3}s\right)^n\right) \\ &= \frac{4}{5}\left(\frac{\frac{1}{2}s}{1-\frac{1}{2}s} + \frac{\frac{1}{3}s}{1+\frac{1}{3}s}\right). \end{aligned}$$

Evaluating $F(s)|_{s=1}$ gives $F(1) = \frac{4}{5}(1+\frac{1}{4}) = 1$, so the process is recurrent. Differentiating gives

$$F'(s) = \frac{4}{5}\left(\frac{\frac{1}{2}}{(1-s/2)^2} + \frac{\frac{1}{3}}{(1+s/3)^2}\right),$$

and evaluating at $s = 1$ gives $F'(s)|_{s=1} = \frac{4}{5}\left(2 + \frac{3}{16}\right) = \frac{7}{4}$, so the process is positive recurrent.

17. (a) The generating function is

$$F(s) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} s^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}.$$

(b) i. The distribution of X has generating function $F_X(s) = e^{\lambda(s-1)}$, and that of Y has generating function $F_Y(s) = e^{\mu(s-1)}$. By independence, the distribution of $X+Y$ has generating function $F_{X+Y}(s) = e^{(\lambda+\mu)(s-1)}$, which is the generating function of a $Po(\lambda + \mu)$ distribution, and hence $X + Y$ has this distribution.

ii. If $F(s) = e^{\lambda(s-1)}$, then

$$\frac{d}{ds} F(s) = \lambda e^{\lambda(s-1)},$$

and we find the mean by evaluating this at $s = 1$, which gives λ.

18. Firstly, $F(s) = as + bs^2$. So

$$\begin{aligned} U(s) &= \frac{1}{1 - F(s)} \\ &= \frac{1}{1 - as - bs^2} \\ &= \frac{1}{(1-s)(1+bs)} \\ &\quad \text{(using the fact that } a + b = 1 \text{ to factorise the quadratic)} \\ &= \frac{1}{1+b}\left\{\frac{1}{1-s} + \frac{b}{1+bs}\right\} \quad \text{in partial fractions} \\ &= \frac{1}{1+b}\left\{\left(1 + s + s^2 + \ldots\right) + b\left(1 - bs + b^2 s^2 - \ldots\right)\right\}. \end{aligned}$$

Then u_n is the coefficient of s^n in $U(s)$, so

$$u_n = \frac{1 + b(-b)^n}{1+b} = \frac{1 - (-b)^{n+1}}{1+b}.$$

As $n \to \infty$,

$$u_n \to \frac{1}{1+b},$$

since $|-b| < 1$.

The mean inter-renewal time is $\mu = a + 2b = 1 + b$. So

$$u_n \to \frac{1}{\mu} \text{ as } n \to \infty.$$

19. (a) The event $\{R = r\}$ is the same as the event that the first r inter-renewal times are finite and the $r+1$th is infinite. The probability that an individual inter-renewal time is finite is f, so independence immediately gives us $P(R = r) = f^r(1-f)$.

(b) We require $P(R = 0)$ and $P(R = 2)$ where R is the total number of visits. By (a), $P(R = r) = f^r(1-f)$ for $r = 0, 1, 2, \ldots$ From the proof of Theorem 5,

$$U(1) = \frac{1}{|1-2p|} = \frac{1}{1-2p}$$

since $p < \frac{1}{2}$. Hence

$$f = F(1) = 1 - \frac{1}{U(1)} = 1 - (1-2p) = 2p.$$

So the answers are

i. $f^0(1-f) = 1 - 2p$.

ii. $f^2(1-f) = (2p)^2(1-2p)$.

20. If the chain is in state 0 at time n and has taken m upward steps, then it must have taken $2m$ downward steps. Hence $n = 3m$, so n must be a multiple of 3 if u_n is non-zero. We can check that $u_3 = 3p(1-p)^2 > 0$, so $d = 3$.

If the chain is at zero at time $3m$, then of the first $3m$ steps exactly m have been upward. The number of upward steps has a Binomial distribution with parameters $3m$ and p, so

$$u_{3m} = \binom{3m}{m} p^m (1-p)^{2m}.$$

It is possible to show that this renewal process is recurrent if and only if $p = 1/3$. The binomial expansion method used in the proof of Theorem 5 doesn't work with this example; however, we can use *Stirling's approximation* to deal with the binomial co-efficient. This says that $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$ (in the sense that the ratio between the two sides converges to 1 as $n \to \infty$), so applying it to $\binom{3m}{m} = \frac{(3m)!}{m!(2m)!}$ gives an asymptotic form of

$$\frac{\sqrt{2\pi}3^{3m+1/2}m^{3m+1/2}e^{-3m}}{\sqrt{2\pi}2^{2m+1/2}m^{2m+1/2}e^{-2m}\sqrt{2\pi}m^{m+1/2}e^{-m}},$$

which, simplifying, tells us

$$\binom{3m}{m} \sim \frac{1}{\sqrt{2\pi}}\frac{3^{3m+1/2}}{2^{2m+1/2}}m^{-1/2} = \frac{\sqrt{3}}{\sqrt{4\pi}}\left(\frac{27}{4}\right)^m m^{-1/2}.$$

Hence

$$u_{3m} \sim \frac{\sqrt{3}}{\sqrt{4\pi}}\left(\frac{27}{4}p(1-p)^2\right)^m m^{-1/2}.$$

If $p = 1/3$ then $\frac{27}{4}p(1-p)^2 = 1$, so in this case $u_{3m} \sim \frac{\sqrt{3}}{\sqrt{4\pi}}m^{-1/2}$, which gives a series which sums to infinity, so the process is recurrent. If p takes any other value then $\frac{27}{4}p(1-p)^2 < 1$, which means we have a series with a finite sum, so the process is transient.

21. (a) i. A renewal at time 1 is impossible as it requires at least two heads. For $n \geq 2$, for the first renewal to occur at time n means the second head occurs on the nth toss. There are then $n-1$ possibilities for the time of the first head, and each possible sequence for the first n tosses has probability $\left(\frac{1}{2}\right)^n$. Hence $f_n = (n-1)\left(\frac{1}{2}\right)^n$ as required.

ii. The mean time to the first renewal is

$$\sum_{n=1}^{\infty} nf_n = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \frac{2\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = 4,$$

using question **??**(c) from the series handout.

(b) Let A_n be the event that there are an odd number of heads in the first n tosses, and let H_n be the event that the nth toss is a head. We aim to prove $P(A_n) = \frac{1}{2}$ for all $n \geq 1$. For $n = 1$, there is one head with probability $1/2$ and no heads with probability $1/2$, so the statement is true for $n = 1$. Assume true for $n = k$; then by independence

$$\begin{aligned} P(A_{k+1}) &= P(H_{k+1})P(A_k^c) + P(H_{k+1}^c)P(A_k) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \end{aligned}$$

by the induction hypothesis and the assumption the coin is fair, so the statement is true for $n = k+1$ and hence for all n.

In the renewal process, again by independence, for $n \geq 2$

$$u_n = P(A_{n-1} \cap H_n) = P(A_{n-1})P(H_n) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

22. (a) Once there has been a renewal, the process behaves in exactly the same way as that in question 21. So the values of f_n and u_n are the same.

(b) Here, b_n is the probability that the first renewal occurs at time n, which is the same as the probability that the first head occurs at time n, which is the probability that the first $n-1$ tosses are tails and the nth is a head, which is $(1/2)^n$. (The delay has a geometric distribution.) Hence $B(s) = \sum_{n=1}^{\infty} b_n s^n = \frac{s}{2(1-s/2)} = \frac{s}{2-s}$ for $|s| < 2$.

(c) Theorem 7 tells us that $V(s) = B(s)U(s)$. From question 21, $u_n = 1/4$ for $s \geq 2$, and also $u_1 = 0$ and $u_0 = 1$ (as always). So

$$U(s) = 1 + \sum_{n=2}^{\infty} \frac{1}{4} s^n = 1 + \frac{s^2}{4(1-s)}$$

for $|s| < 1$. Hence

$$\begin{aligned}
V(s) &= \frac{s}{2-s}\left(1 + \frac{s^2}{4(1-s)}\right) \\
&= \frac{s}{2-s} + \frac{s^3}{4(1-s)(2-s)} \\
&= \frac{s}{2-s} + \frac{s^3}{4(1-s)} - \frac{s^3}{4(2-s)} \\
&\quad \text{(partial fractions)} \\
&= \frac{s^3}{4(1-s)} + \frac{1}{2-s}\left(s - \frac{s^3}{4}\right) \\
&= \frac{s^3}{4(1-s)} + \frac{s}{4(2-s)}(2-s)(2+s) \\
&= \frac{s^3}{4(1-s)} + \frac{1}{2}s + \frac{1}{4}s^2 \\
&= \frac{1}{2}s + \frac{1}{4}\sum_{n=2}^{\infty} s^n.
\end{aligned}$$

Hence $v_0 = 0$, $v_1 = 1/2$ and $v_2 = 1/4$ for $n \geq 2$. This makes sense: a renewal cannot happen before the coin has been tossed, a renewal happens on the first toss if it is a head (so $v_1 = 1/2$), a renewal happens on the second toss if and only if the first two are TH (so $v_2 = 1/4$), and for larger n $v_n = 1/4$ by a similar argument to that in question 21.

23. (a) i. We treat occurrences of HT as a non-delayed renewal process, as two occurrences cannot overlap. Then $u_0 = 1$ (usual convention), $u_1 = 0$ (renewal after one toss impossible), and $u_n = \frac{1}{4}$ for $n \geq 2$. So

$$U(s) = 1 + \frac{1}{4}\left\{s^2 + s^3 + s^4 + \dots\right\} = 1 + \frac{s^2}{4(1-s)} = \frac{4 - 4s + s^2}{4 - 4s}.$$

By the basic relation,

$$F(s) = 1 - \frac{1}{U(s)} = 1 - \frac{4 - 4s}{4 - 4s + s^2} = \frac{s^2}{4 - 4s + s^2}.$$

Differentiating,

$$F'(s) = \frac{(4 - 4s + s^2)(2s) - s^2(-4 + 2s)}{(4 - 4s + s^2)^2},$$

and so

$$E(T_1) = F'(1) = \frac{1 \times 2 - 1(-2)}{1^2} = 4.$$

ii. Occurrences of HH must be treated as a delayed renewal process, since two occurrences can overlap. Then $v_0 = v_1 = 0$ (renewals impossible after no steps or after 1 step), and $v_n = \frac{1}{4}$ for $n \geq 2$. So

$$V(s) = \frac{1}{4}\left\{s^2 + s^3 + s^4 + \dots\right\} = \frac{s^2}{4 - 4s}.$$

Next $u_0 = 1$ and $u_1 = \frac{1}{2}$ (if a renewal has just occurred at time k then another renewal occurs at $k + 1$ if the next toss is H), and $u_n = \frac{1}{4}$ if $n \geq 2$. So

$$U(s) = 1 + \frac{1}{2}s + \frac{s^2}{4 - 4s} = \frac{4 - 2s - s^2}{4 - 4s}.$$

Because $V(s) = B(s)U(s)$ (in the notes)

$$B(s) = \frac{V(s)}{U(s)} = \frac{s^2}{4 - 2s - s^2}.$$

Differentiating,

$$B'(s) = \frac{(4-2s-s^2)(2s) - s^2(-2-2s)}{(4-2s-s^2)^2},$$

so

$$E(\text{first } HH) = E(\text{delay}) = B'(1) = \frac{1 \times 2 - 1(-4)}{1^2} = 6.$$

(b) We have $v_n = 1/8$ for $n \geq 3$. Here $v_1 = 1/2$ (an H will give us a renewal) and similarly $v_2 = 1/4$, while $v_0 = 0$. So $V(s) = \frac{s}{2} + \frac{s^2}{4} + \frac{s^3}{8-8s}$.
Now $u_0 = 1$ (as always), $u_1 = 1/2$, $u_2 = 1/4$, and $u_n = 1/8$ for $n \geq 3$. So $U(s) = 1 + \frac{s}{2} + \frac{s^2}{4} + \frac{s^3}{8-8s}$. Hence

$$B(s) = \frac{V(s)}{U(s)} = \frac{8(1-s)(\frac{s}{2} + \frac{s^2}{4}) + s^3}{8(1-s)(1 + \frac{s}{2} + \frac{s^2}{4}) + s^3},$$

which has a derivative of 8 at $s = 1$. So the expected time until the next HHH is 8.

24. (a) We treat occurrences of $ABCDEFGHIJKLM$ as a non-delayed renewal process, as two occurrences cannot overlap. Then $u_0 = 1$ (usual convention), $u_n = 0$ for $1 \leq n \leq 12$, and $u_n = \frac{1}{26^{13}}$ for $n \geq 13$. So

$$U(s) = 1 + \frac{s^{13}}{26^{13}(1-s)} = \frac{26^{13} - 26^{13}s + s^{13}}{26^{13} - 26^{13}s}.$$

By the basic relation,

$$F(s) = 1 - \frac{1}{U(s)} = 1 - \frac{26^{13} - 26^{13}s}{26^{13} - 26^{13}s + s^{13}} = \frac{s^{13}}{26^{13} - 26^{13}s + s^{13}}.$$

Differentiating,

$$F'(s) = \frac{(26^{13} - 26^{13}s + s^{13})(13s^{12}) - s^{13}(-26^{13} + 13s^{12})}{(26^{13} - 26^{13}s + s^{13})^2},$$

and so

$$E(T_1) = F'(1) = 26^{13}.$$

(b) Occurrences of $TOBEORNOTTOBE$ must be treated as a delayed renewal process, since two occurrences can overlap. Then $v_n = 0$ for $n \leq 12$, and $v_n = \frac{1}{26^{13}}$ for $n \geq 13$. So

$$V(s) = \frac{s^{13}}{26^{13} - 26^{13}s}.$$

Next $u_0 = 1$ and $u_9 = \frac{1}{26^9}$ (the probability that the next nine letters after a renewal are $ORNOTTOBE$), $u_n = 0$ for $1 \leq n \leq 8$ and $10 \leq n \leq 12$, and $u_n = \frac{1}{26^{13}}$ if $n \geq 13$. So

$$U(s) = 1 + \frac{1}{26^9}s^9 + \frac{s^{13}}{26^{13} - 26^{13}s} = \frac{26^{13} - 26^{13}s + 26^4 s^9 - 26^4 s^10 + s^13}{26^{13} - 26^{13}s}.$$

As in the previous question

$$B(s) = \frac{V(s)}{U(s)} = \frac{s^{13}}{26^{13} - 26^{13}s + 26^4 s^9 - 26^4 s^{10} + s^{13}}.$$

Differentiating and setting $s = 1$,

$$E(\text{first } TOBEORNOTTOBE) = E(\text{delay}) = B'(1) = 26^{13} + 26^4.$$

(Via Maple, the answer to (a) is 2,481,152,873,203,736,576 and the answer to (b) is 2,481,152,873,204,193,552; both are 19-digit numbers. If the monkey types one letter every second the expected time until $TOBEORNOTTOBE$ appears is over 10^{10} years, greater than the age of the Earth.)

25. (Open ended question)

26. A diagram could be drawn of the possible one-step transitions, but, without doing this, we can argue as follows. As long as the set is still being played, it is possible to visit any other state. However, once the set has been completed by either player winning it, no other state is visited. So states $\{1, 2, 3, 4, 5, 6\}$ form a class, which is not closed, and $\{7\}$ and $\{8\}$ are closed (absorbing) classes, which are trivially aperiodic.

    It remains to note that to return to a state in $\{1, 2, 3, 4, 5, 6\}$ after a sequence of n points played, there must be equal numbers of points won by each player, so that n is even, and the server at the end must be the same as at the beginning, which means that n must be a multiple of 4. Since for example $1 \to 4 \to 2 \to 3 \to 1$ is possible, this confirms that this class is periodic with period 4.

27. (a) When a transition takes place, one of the co-ordinates of the point changes and the others stay the same. Hence the number of (say) "ones" in the co-ordinates always alternates between an odd number and an even number. So a return to a state can only happen after an even number of steps, and it can happen after two steps. Hence the period of any state is 2. (Another way of looking at this is that if the states $\{000, 011, 101, 110\}$ are painted one colour and the states $\{001, 010, 100, 111\}$ are painted another colour, then the random walk always alternates between one colour and the other.)

    (b) The simplest example is the random walk on a triangle, where it is possible to return after two steps, and also after three steps (by going right round the triangle). Since h.c.f.$\{2, 3, \ldots\} = 1$, the states are aperiodic. (In fact the transition matrix of question 11 is of this random walk.)

28. (a) If the chain starts in A, then after an odd number of steps it must always be in B, and it will then go to A or C at the next (even) step each with probability 1/2. (This can also be seen by considering the matrices in question 10.) So if n is odd $u_n = p_{AA}^{(n)} = 0$, and if n is even (and $n \geq 2$) $u_n = p_{AA}^{(n)} = 1/2$.

    (b) We have

$$U(s) = 1 + \sum_{m=1}^{\infty} u_{2m} s^{2m} = 1 + \frac{\frac{1}{2}s^2}{1 - s^2} = \frac{1 - \frac{1}{2}s^2}{1 - s^2}.$$

    By Theorem 4,

$$F(s) = 1 - \frac{1}{U(s)} = \frac{\frac{1}{2}s^2}{1 - \frac{1}{2}s^2}.$$

    (c) Differentiating gives

$$F'(s) = \frac{s}{(1 - \frac{1}{2}s^2)^2},$$

    and evaluating at $s = 1$ gives $F'(1) = 4$ so the expected time until the first return to A is 4.

29. The sequence $1 \to 2 \to 5 \to 1 \to \ldots$ is possible, showing that $1, 2$ and 5 communicate, but this set can be left, via $2 \to 3$, and it cannot be returned to, once left. Also, the only way of staying is to cycle round as indicated above. Hence $\{1, 2, 5\}$ is a class which is not closed and has period 3.

    The sequence $3 \to 4 \to 6 \to 3 \to \ldots$ is possible, and so these three states communicate, but it is impossible to leave this set. As $p_{44} > 0$, state 4 is aperiodic and so the class is aperiodic. So $\{3, 4, 6\}$ is a class which is closed and aperiodic.

Finally (similarly), $\{7, 8\}$ is a class which not closed and is periodic with period 2.

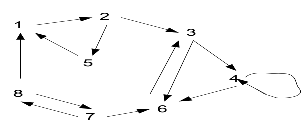
To summarise:

(a) The classes are $\{1, 2, 5\}, \{3, 4, 6\}$ and $\{7, 8\}$.

(b) The only closed class is $\{3, 4, 6\}$.

(c) $\{1, 2, 5\}$ has period 3, $\{3, 4, 6\}$ is aperiodic and $\{7, 8\}$ has period 2.

A diagram as shown can help with this:



30. (a) The chain is irreducible as it is possible to get from state 1 to any other state, and it is also possible to get from any other state back to state 1, which is enough to show that all states communicate with state 1 and hence with each other. It is aperiodic as it is possible to return to state 1 in one step, and so state 1 is aperiodic and by the solidarity theorem the chain is aperiodic.

(b) The equations are

$$\pi_1 = \frac{1}{3}\pi_1 + \pi_2 + \frac{1}{2}\pi_3 \tag{7}$$

$$\pi_2 = \frac{1}{6}\pi_1 + \frac{1}{4}\pi_3 \tag{8}$$

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_3. \tag{9}$$

By (9), $\pi_3 = \frac{2}{3}\pi_1$ and substituting this into (8) gives $\pi_2 = \frac{1}{3}\pi_1$. For a stationary distribution $\pi_1 + \pi_2 + \pi_3 = 1$, so we get $\boldsymbol{\pi} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$.

(c) The chain is irreducible and aperiodic and as it has finite state space is positive recurrent. Hence the convergence results apply, and so as $n \to \infty$, $\boldsymbol{\pi}^{(n)} \to \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$, regardless of $\boldsymbol{\pi}^{(0)}$, and $p_{ij}^{(}n) \to \pi_j$. Assuming $n = 20$ is large enough for $p_{13}^{(n)}$ to be close to its limit we have $P(X_{20} = 3|X_0 = 1) = p_{13}^{(20)} \approx \frac{1}{3}$. [In fact it is indistinguishable from 1/3 to 10dp.]

31. (a) The equations for the stationary distribution are

$$\begin{aligned}
\pi_A &= \pi_B\frac{1}{3} + \pi_C\frac{1}{3} && (10)\\
\pi_B &= \pi_A\frac{1}{2} + \pi_C\frac{1}{3} + \pi_D\frac{1}{4} && (11)\\
\pi_C &= \pi_A\frac{1}{2} + \pi_B\frac{1}{3} + \pi_D\frac{1}{4} && (12)\\
\pi_D &= \pi_B\frac{1}{3} + \pi_C\frac{1}{3} + \pi_E\frac{1}{3} + \pi_F\frac{1}{3} && (13)\\
\pi_E &= \pi_D\frac{1}{4} + \pi_F\frac{1}{3} + \pi_G\frac{1}{2} && (14)\\
\pi_F &= \pi_D\frac{1}{4} + \pi_E\frac{1}{3} + \pi_G\frac{1}{2} && (15)\\
\pi_G &= \pi_E\frac{1}{3} + \pi_F\frac{1}{3} && (16)
\end{aligned}$$

By symmetry and the uniqueness of the stationary distribution, $\pi_A = \pi_G$ and $\pi_B = \pi_C = \pi_E = \pi_F$. By (10), $\pi_B = \frac{3}{2}\pi_A$. By (10) and (13), $\pi_D = \frac{4}{3}\pi_B = 2\pi_A$. Everything is now expressed in terms of π_A. Because we are looking for a probability distribution,

$$\pi_A + \pi_B + \pi_C + \pi_D + \pi_E + \pi_F + \pi_G = 1,$$

which becomes

$$\pi_A + \frac{3}{2}\pi_A + \frac{3}{2}\pi_A + 2\pi_A + \frac{3}{2}\pi_A + \frac{3}{2}\pi_A + \pi_A = 1.$$

Hence $10\pi_A = 1$, so $\pi_A = \frac{1}{10}$, and so

$$\pi_A = \pi_G = \frac{1}{10}, \pi_B = \pi_C = \pi_E = \pi_F = \frac{3}{20}, \pi_D = \frac{1}{5},$$

or

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{1}{10} & \frac{3}{20} & \frac{3}{20} & \frac{1}{5} & \frac{3}{20} & \frac{3}{20} & \frac{1}{10} \end{pmatrix}.$$

Note: A bit more on why the symmetry assumptions can be made: it is possible to "reflect" the graph so that vertices A, B, C are mapped to G, E, F and vice versa, with the structure of the graph being otherwise unchanged. This means that if we can find a stationary distribution $\begin{pmatrix}\pi_A & \pi_B & \pi_C & \pi_D & \pi_E & \pi_F & \pi_G\end{pmatrix}$ then $\begin{pmatrix}\pi_G & \pi_E & \pi_F & \pi_D & \pi_B & \pi_C & \pi_A\end{pmatrix}$ would also be a stationary distribution. As the stationary distribution is unique, these must in fact be the same, so $\pi_A = \pi_G$, $\pi_B = \pi_E$, $\pi_C = \pi_F$. A similar argument with a different transformation gives $\pi_B = \pi_C$.)

(b) The chain is irreducible, because the graph is connected, so there is a possible route from each vertex to every other vertex. To see that it is aperiodic, consider state A. It is possible to return to A after two steps (go to B, then back to A) or after three (go to B, then to C, then back to A) and the hcf of 2 and 3 is 1, so A is aperiodic; by solidarity so are all other states. An irreducible chain with a finite state space is positive recurrent, so the results apply. Hence, as $n \to \infty$, $P(X_n = A) \to 1/10$, $P(X_n = B) \to 3/20$, $P(X_n = C) \to 3/20$, $P(X_n = D) \to 1/5$, $P(X_n = E) \to 3/20$, $P(X_n = F) \to 3/20$ and $P(X_n = G) \to 1/10$.

(c) Let d_i be the degree of vertex i. Then the transition probabilities of the symmetric random walk are given by $p_{ij} = 1/d_i$ if there is an edge between i and j, and zero otherwise.

Label the vertices 1 to N, and let $\boldsymbol{\pi}$ be defined by $\pi_i = d_i / \sum_{k=1}^{N} d_k$. Then

$$(\boldsymbol{\pi} P)_j = \sum_{i=1}^{N} (d_i / \sum_{k=1}^{N} d_k) p_{ij} = \sum_{i \leftrightarrow j} (d_i / \sum_{k=1}^{N} d_k)/d_i.$$

(Here $i \leftrightarrow j$ means that there is an edge between i and j in the graph.) Simplifying,

$$(\boldsymbol{\pi} P)_j = \sum_{i \leftrightarrow j} \frac{1}{\sum_{k=1}^{N} d_k}.$$

Because there are d_j vertices i which are connected to j, there are d_j terms in this sum, so

$$(\boldsymbol{\pi} P)_j = \frac{d_j}{\sum_{k=1}^{N} d_k} = \pi_j.$$

Hence $\boldsymbol{\pi} P = \boldsymbol{\pi}$, so $\boldsymbol{\pi}$ is a stationary distribution.

If the graph is connected, the chain is irreducible and the stationary distribution is unique. If the graph is not connected, the chain will not be irreducible and there will be other stationary distributions.

32. Checking the stationary distribution equations, for $i \in \mathbb{N}_0$

$$\begin{aligned}(\boldsymbol{\pi}P)_i &= \pi_0 f_{i+1} + \pi_{i+1} \\ &= \frac{1}{\mu} f_{i+1} + \frac{1}{\mu}(1 - f_1 - f_2 - \ldots - f_i - f_{i+1}) \\ &= \frac{1}{\mu}(1 - f_1 - f_2 - \ldots - f_i) \\ &= \pi_i.\end{aligned}$$

Hence $\boldsymbol{\pi}P = \boldsymbol{\pi}$.

33. By the definition of conditional probability

$$\begin{aligned}P(X_n = j | X_{n+1} = i) &= \frac{P(X_n = j, X_{n+1} = i)}{P(X_{n+1} = i)} \\ &= \frac{P(X_n = j)P(X_{n+1} = i | X_n = j)}{P(X_{n+1} = i)} \\ &= \frac{\pi_j p_{ji}}{\pi_i}\end{aligned}$$

since the Markov chain is in equilibrium. This is the typical "backwards" transition probability of the Markov chain. It will be the same as the corresponding "forwards" transition probability if and only if

$$\frac{\pi_j p_{ji}}{\pi_i} = p_{ij},$$

that is

$$\pi_i p_{ij} = \pi_j p_{ji}$$

for all i and j.

34. (a) The possible one-step transitions are $1 \to 1$, $1 \to 2$, $2 \to 3$, $3 \to 2$ and $3 \to 3$. So states 2 and 3 communicate with each other but state 1 does not communicate with any other state. Hence the classes are $\{1\}$ and $\{2, 3\}$; the latter is closed as it is not possible to leave, but the former is not closed as it can be left.

(b) The equations are

$$\pi_1 = \frac{1}{3}\pi_1 \tag{17}$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_3 \tag{18}$$

$$\pi_3 = \pi_2 + \frac{1}{2}\pi_3. \tag{19}$$

Then (17) immediately gives $\pi_1 = 0$, and (18) then gives $\pi_2 = \frac{1}{2}\pi_3$. For a stationary distribution $\pi_1 + \pi_2 + \pi_3 = 1$, so $\boldsymbol{\pi} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ is a unique stationary distribution.

(c) If we start in state 2, then the chain remains in $\{2, 3\}$ for ever, so we can consider it as a Markov chain on $\{2, 3\}$ with transition matrix

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The stationary distribution is $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, and this reduced chain is irreducible and aperiodic. So as $n \to \infty$ $p_{22}^{(n)} \to \frac{1}{3}$ and $p_{23}^{(n)} \to \frac{2}{3}$, and the approximate probabilities asked for in the question are 1/3 for state 2 and 2/3 for state 3.

35. (a) Obviously states 1 and 2 communicate and states 3 and 4 communicate, but no other pairs of different states do. So the classes are $\{1, 2\}$ and $\{3, 4\}$; neither class can be left so both are closed. It is possible to return to either 1 or 2 in one step so this class is aperiodic, while starting from state 3 the chain will return there only after 2 steps, so by solidarity this class has period 2.

(b) The equations for a stationary distribution are

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{2}{3}\pi_2 \tag{20}$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{3}\pi_2 \tag{21}$$

$$\pi_3 = \pi_4 \tag{22}$$

$$\pi_4 = \pi_3. \tag{23}$$

Clearly (22) and (23) are equivalent, and then (20) and (21) both imply $\pi_1 = \pi_2$. These equations give no more information, so the only remaining information we have is that $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. Hence we can get a stationary distribution $\boldsymbol{\pi}_\alpha = \begin{pmatrix} \alpha & \alpha & \frac{1}{2} - \alpha & \frac{1}{2} - \alpha \end{pmatrix}$ for any $\alpha \in [0, \frac{1}{2}]$.

36. (a) All states can be reached from state 1 (states 4 and 5 via state 2) and it is possible to reach state 1 from every other state (from states 2 and 3 via state 4). So, by transitivity, the chain is irreducible. Starting in state 1 the chain will always return to state 1 after exactly 3 steps (one of 1,2,4,1; 1,2,5,1; 1,3,4,1) so $p_{11}^{(n)}$ will be 1 if n is a multiple of 3 and zero otherwise. Hence state 1 has period 3, and hence, by the solidarity theorem, so does the chain.

(b) The initial distribution $\boldsymbol{\pi}^{(0)} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix}$. By matrix multiplication,

$$\begin{aligned}\boldsymbol{\pi}^{(1)} &= \begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\end{pmatrix}\\ \boldsymbol{\pi}^{(2)} &= \begin{pmatrix}0 & 0 & 0 & \frac{5}{8} & \frac{3}{8}\end{pmatrix}\\ \boldsymbol{\pi}^{(3)} &= \begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix}.\end{aligned}$$

Hence (formally by induction) for $n \geq 0$,

$$\boldsymbol{\pi}^{(n)} = \begin{cases} \begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix} & n \equiv 0 \pmod 3\\ \begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\end{pmatrix} & n \equiv 1 \pmod 3\\ \begin{pmatrix}0 & 0 & 0 & \frac{5}{8} & \frac{3}{8}\end{pmatrix} & n \equiv 2 \pmod 3\end{cases}$$

so there cannot be convergence to a stationary distribution.

37. (a) This renewal process is aperiodic and has mean inter-renewal time 7/4. So, by the renewal theorem, $u_n \to 4/7$ as $n \to \infty$.

(b) This renewal process has period 2 and mean inter-renewal time 14/3. So, by the renewal theorem, $u_{2n} \to 3/7$ as $n \to \infty$.

38. (Open ended question)

39. (a) The transition matrix is

$$\begin{pmatrix}0 & 1/2 & 1/2 & 0 & 0\\ 0 & 0 & 0 & 1/2 & 1/2\\ 0 & 0 & 0 & 1/2 & 1/2\\ 0 & 1/2 & 1/2 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}.$$

This chain is not irreducible, because state 1 cannot be reached from any other state.

(b) The equations for a stationary distribution are

$$\pi_1 = 0 \quad (24)$$

$$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_4 \quad (25)$$

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_4 \quad (26)$$

$$\pi_4 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 + \pi_5 \quad (27)$$

$$\pi_5 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 \quad (28)$$

Obviously $\pi_2 = \pi_3 = \frac{1}{2}\pi_4$, and from (27) we get $\pi_5 = \frac{1}{2}\pi_4$ too. Because a stationary distribution must sum to 1 we get that $\begin{pmatrix} 0 & 1/5 & 1/5 & 2/5 & 1/5 \end{pmatrix}$ is the unique stationary distribution.

(c) The transition matrix is

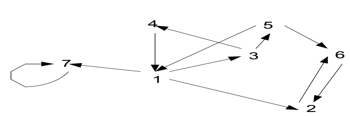
$$\frac{5}{11}\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} + \frac{6}{11}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & 4 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 4 & 4 & 1 & 1 \\ 1 & 1 & 1 & 7 & 1 \end{pmatrix}.$$

We can check that if this is P and $\boldsymbol{\pi} = \frac{1}{55}\begin{pmatrix} 5 & 11 & 11 & 17 & 11 \end{pmatrix}$ as given in the question, then

$$\boldsymbol{\pi}P = \frac{1}{55}\frac{1}{11}\begin{pmatrix} 55 & 121 & 121 & 187 & 121 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 5 & 11 & 11 & 17 & 11 \end{pmatrix} = \boldsymbol{\pi},$$

so $\boldsymbol{\pi}$ is a stationary distribution. By irreducibility, aperiodicity and positive recurrence, it is unique, so gives the PageRanks.

40. The one step transitions are shown in the diagram:



The closed classes are $\{2, 6\}$ and $\{7\}$.

Let q_i be the probability of absorption in $\{2, 6\}$ starting in i, with boundary conditions

$q_2 = q_6 = 1, q_7 = 0$:

$$\begin{aligned} q_1 &= \frac{1}{2}q_2 + \frac{1}{4}q_3 + \frac{1}{4}q_7 = \frac{1}{2} + \frac{1}{4}q_3 && (29)\\ q_3 &= \frac{1}{2}q_4 + \frac{1}{2}q_5 && (30)\\ q_4 &= q_1 && (31)\\ q_5 &= \frac{1}{3}q_1 + \frac{2}{3}q_6 = \frac{1}{3}q_1 + \frac{2}{3} && (32) \end{aligned}$$

Substitute (31) and (32) into (30):

$$q_3 = \frac{1}{2}q_1 + \frac{1}{6}q_1 + \frac{1}{3} = \frac{2}{3}q_1 + \frac{1}{3}.$$

Substitute this into (29):

$$q_1 = \frac{1}{2} + \frac{1}{6}q_1 + \frac{1}{12} = \frac{7}{12} + \frac{1}{6}q_1.$$

Rearranging this, we get $q_1 = \frac{7}{10}$ and hence $q_3 = \frac{4}{5}, q_4 = \frac{7}{10}, q_5 = \frac{9}{10}$

The probability of absorption in $\{7\}$ starting in i is then $1 - q_i$.

41. The equations are

$$\begin{aligned} q_1 &= (1-p)q_4 + pq_6 && (33)\\ q_2 &= (1-r)q_3 + rq_5 && (34)\\ q_3 &= (1-p)q_1 + p && (35)\\ q_4 &= (1-r)q_2 && (36)\\ q_5 &= pq_1 && (37)\\ q_6 &= rq_2 + (1-r) && (38) \end{aligned}$$

Substituting (35) to (38) into (33) and (34)

$$\begin{aligned} q_1 &= (1-p)(1-r)q_2 + prq_2 + p(1-r) = (1-p-r+2pr)q_2 + p(1-r)\\ q_2 &= (1-r)(1-p)q_1 + (1-r)p + rpq_1 = (1-p-r+2pr)q_1 + p(1-r) \end{aligned}$$

By symmetry or by subtracting one equation from the other, we see that $q_1 = q_2$ and then

$$(p + r - 2pr)q_1 = p(1-r)$$

giving the stated result.

42. Let q_i be the probability of visiting G first, starting at i. Then $q_A = 0, q_G = 1$ and, by symmetry, $q_B = q_C, q_E = q_F$.

$$q_B = \frac{1}{3}q_A + \frac{1}{3}q_C + \frac{1}{3}q_D = \frac{1}{3}q_B + \frac{1}{3}q_D \tag{39}$$

$$q_D = \frac{1}{4}q_B + \frac{1}{4}q_C + \frac{1}{4}q_E + \frac{1}{4}q_F = \frac{1}{2}q_B + \frac{1}{2}q_E \tag{40}$$

$$q_E = \frac{1}{3}q_D + \frac{1}{3}q_F + \frac{1}{3}q_G = \frac{1}{3}q_D + \frac{1}{3}q_E + \frac{1}{3} \tag{41}$$

From (41),

$$q_E = \frac{1}{2}q_D + \frac{1}{2}.$$

Substituting this into (40),

$$\begin{aligned} q_D &= \frac{1}{2}q_B + \frac{1}{4}q_D + \frac{1}{4} \\ q_D &= \frac{2}{3}q_B + \frac{1}{3} \end{aligned}$$

Substituting this into (39),

$$\begin{aligned} \frac{2}{3}q_B &= \frac{2}{9}q_B + \frac{1}{9} \\ q_B &= \frac{1}{4}, \end{aligned}$$

so the probability of reaching G before A starting at B is $\frac{1}{4}$.

43. Let e_i be the expected time to absorption in either $\{2, 6\}$ or $\{7\}$, starting from i. Then the boundary conditions are $e_2 = e_6 = e_7 = 0$.

$$e_1 = 1 + \frac{1}{4}e_3 \tag{42}$$

$$e_3 = 1 + \frac{1}{2}e_4 + \frac{1}{2}e_5 \tag{43}$$

$$e_4 = 1 + e_1 \tag{44}$$

$$e_5 = 1 + \frac{1}{3}e_1 \tag{45}$$

Substitute (44) and (45) into (43):

$$e_3 = 1 + \frac{1}{2} + \frac{1}{2}e_1 + \frac{1}{2} + \frac{1}{6}e_1 = 2 + \frac{2}{3}e_1.$$

Substitute this into (42):

$$e_1 = 1 + \frac{1}{2} + \frac{1}{6}e_1.$$

Rearranging gives

$$e_1 = \frac{9}{5}$$

and hence

$$e_1 = \frac{9}{5}, e_3 = \frac{16}{5}, e_4 = \frac{14}{5}, e_5 = \frac{8}{5}.$$

44. Let e_i be the expected time to absorption in either $\{7\}$ or $\{8\}$, starting from i. Then the boundary conditions are $e_7 = e_8 = 0$.

$$\begin{aligned}
e_1 &= 1 + (1-p)e_4 + pe_6 && (46)\\
e_2 &= 1 + (1-r)e_3 + re_5 && (47)\\
e_3 &= 1 + (1-p)e_1 && (48)\\
e_4 &= 1 + (1-r)e_2 && (49)\\
e_5 &= 1 + pe_1 && (50)\\
e_6 &= 1 + re_2 && (51)
\end{aligned}$$

Substituting (48) to (51) into (46) and (47):

$$\begin{aligned}
e_1 &= 1 + (1-p) + (1-p)(1-r)e_2 + p + pre_2\\
e_2 &= 1 + (1-r) + (1-r)(1-p)e_1 + r + rpe_1.
\end{aligned}$$

These simplify to

$$\begin{aligned}
e_1 &= 2 + (1-p-r+2pr)e_2\\
e_2 &= 2 + (1-p-r+2pr)e_1.
\end{aligned}$$

Again by symmetry or by subtracting one equation from the other, we see that $e_1 = e_2$ and then

$$(p + r - 2pr)e_1 = 2$$

giving the stated result.

45. This is a symmetric random walk on the described graph, so can be modelled as a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Let e_i be the expected time to reach D, starting in i. Then $e_D = 0$, and

$$e_A = 1 + e_B \quad (52)$$

$$e_B = 1 + \frac{1}{3}(e_A + e_C) \quad (53)$$

$$e_C = 1 + \frac{1}{4}(e_B + e_E + e_F) \quad (54)$$

$$e_E = 1 + e_C \quad (55)$$

$$e_F = 1 + \frac{1}{2}e_C \quad (56)$$

Substituting (55) and (56) into (54) gives $e_C = 1 + \frac{1}{4}e_B + \frac{1}{4} + \frac{1}{4}e_C + \frac{1}{4} + \frac{1}{8}e_C = \frac{3}{2} + \frac{1}{4}e_B + \frac{3}{8}e_C$, so

$$\frac{5}{8}e_C = \frac{3}{2} + \frac{1}{4}e_B.$$

Substituting (52) into (53) gives

$$\frac{2}{3}e_B = \frac{4}{3} + \frac{1}{3}e_C,$$

and solving for e_B gives $e_B = 4$. Hence $e_A = 5$.

46. (a) The state space $S = \{0, H, HT\}$ (0 representing no progress towards HT, H that the last toss was a head, and HT that HT has been completed) and the transition matrix is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Letting e_i be the expected time until reaching HT starting from i,

$$e_0 = 1 + \frac{1}{2}e_0 + \frac{1}{2}e_H,$$

which can be rearranged as

$$e_0 = 2 + e_H,$$

and

$$e_H = 1 + \frac{1}{2}e_H$$

giving $e_H = 2$ and hence $e_0 = 4$.

(b) The state space $S = \{0, H, HH\}$ (0 representing no progress towards HH, H that the last toss was a head not preceded by a head, and HH that HH has been completed) and the transition matrix is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Letting e_i be the expected time until reaching HH starting from i,

$$e_0 = 1 + \frac{1}{2}e_0 + \frac{1}{2}e_H,$$

giving $e_0 = 2 + e_H$, and

$$e_H = 1 + \frac{1}{2}e_0 = 1 + 1 + \frac{1}{2}e_H$$

giving $e_H = 4$ and hence $e_0 = 6$.

47. (a) After the nth toss, let X_n be the length of the current run of heads (setting $X_n = 0$ if the nth toss was a tail) and let Y_n be the length of the run of tails which preceded it (setting $Y_n = 0$ if there were no tails, i.e. the current run of heads started from the first toss). We then have a Markov chain on the following 12 states:

Label	Description	Notes
1	$X_n = 0, Y_n = 0$	(i.e. before the first toss)
2	$X_n = 1, Y_n = 0$	(i.e. after 1 toss, and it was a head)
3	$X_n = 2, Y_n = 0$	
4	$X_n \geq 3, Y_n = 0$	**HHH completed**
5	$X_n = 0, Y_n = 1$	(i.e. last toss was a tail not preceded by a tail)
6	$X_n = 1, Y_n = 1$	
7	$X_n = 2, Y_n = 1$	
8	$X_n \geq 3, Y_n = 1$	**HHH completed**
9	$X_n = 0, Y_n \geq 2$	(i.e. last toss was a tail preceded by a tail)
10	$X_n = 1, Y_n \geq 2$	
11	$X_n = 2, Y_n \geq 2$	**TTHH completed**
12	$X_n \geq 3, Y_n \geq 2$	**HHH completed, but TTHH completed previously**

The transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Let q_n be the probability that TTHH is completed before HHH, starting in state

n. Then $q_4 = q_8 = q_{12} = 0$, and $q_{11} = 1$, and we need q_1. We have

$$\begin{aligned}
q_{10} &= \frac{1}{2}(q_5 + 1) && (57)\\
q_9 &= \frac{1}{2}(q_9 + q_{10}), \text{ so } q_9 = q_{10} && (58)\\
q_7 &= \frac{1}{2}q_5 && (59)\\
q_6 &= \frac{1}{2}(q_5 + q_7) = \frac{3}{4}q_5 && (60)\\
q_5 &= \frac{1}{2}(q_6 + q_9) && (61)\\
q_3 &= \frac{1}{2}q_5 && (62)\\
q_2 &= \frac{1}{2}(q_3 + q_5) && (63)\\
q_1 &= \frac{1}{2}(q_2 + q_5) && (64)
\end{aligned}$$

Substituting (57) and (60) into (61), $q_5 = \frac{3}{8}q_5 + \frac{1}{4}(q_5 + 1)$, giving $q_5 = \frac{2}{3}$. Then the last three equations give $q_3 = \frac{1}{3}$, $q_2 = \frac{1}{2}$ and finally $q_1 = \frac{7}{12}$.

(b) We showed in question 23b that the expected time until a HHH occurs, given that a $TTHH$ has just done so, is 8. Similarly, it is possible to show that the expected time until a $TTHH$ occurs, given that a HHH has just done so, is 16, and it is also possible to show (by the same method as for question 23a) that the expected times from the start of a sequence are 16 for $TTHH$ and 14 for HHH. Let the times of the first occurrence of $TTHH$ and HHH be T_1 and T_2 respectively. Then we have $E(T_1) = 16$ and $E(T_2) = 14$. Now, consider the difference between T_1 and T_2. If $TTHH$ occurs first, then $T_2 - T_1$ is the time until an HHH occurs, given than $TTHH$ has just done so, which we saw above has expectation 8. So we can write

$$E(T_2 - T_1 | T_2 > T_1) = 8.$$

Similarly we can write

$$E(T_1 - T_2 | T_1 > T_2) = 16.$$

Let $p = P(T_2 > T_1)$ (so $P(T_1 > T_2) = 1 - p$, as $T_1 = T_2$ is impossible). Then

$$\begin{aligned}
E(T_2 - T_1) &= pE(T_2 - T_1 | T_2 > T_1) + (1 - p)E(T_2 - T_1 | T_2 < T_1)\\
&= 8p - 16(1 - p) = 24p - 16.
\end{aligned}$$

But $E(T_2 - T_1) = E(T_2) - E(T_1) = 14 - 16 = -2$, hence $-2 = 24p - 16$, giving $24p = 14$, giving $p = 7/12$.

(c) (Open ended question)

48. (a) This is

$$P(N_{0,1} = 3) = e^{-4}\frac{4^3}{6} = \frac{32}{3}e^{-4} = 0.195.$$

(b) By the Poisson process assumptions, $N_{0,3} - N_{0,1} = N_{1,3}$ has the $Po(8)$ distribution and is independent of $N_{0,1}$. So

$$\begin{aligned} P(N_{0,1} = 3, N_{0,3} = 12) &= P(N_{0,1} = 3, N_{1,3} = 9) \\ &= P(N_{0,1} = 3).P(N_{1,3} = 9) \\ &= \frac{32}{3}e^{-4}.e^{-8}\frac{8^9}{9!} = 0.024. \end{aligned}$$

(c) Using the definition of conditional probability,

$$\begin{aligned} P(N_{0,1} = 3 | N_{0,3} = 12) &= \frac{P(N_{0,1} = 3, N_{0,3} = 12)}{P(N_{0,3} = 12)} \\ &= \frac{\frac{4^3}{3!}e^{-4}.e^{-8}\frac{8^9}{9!}}{e^{-12}\frac{12^{12}}{12!}} \\ &= \frac{12!}{3!9!}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^9 = 0.212. \end{aligned}$$

[This is the probability that a $Bin(12, 1/3)$ random variable takes the value 3; this is a special case of the result we will prove in Theorem 23.]

(d) You can get the solution quickly by dividing the answer to (b) by that for (a). However, if you get something like this in the future, note that since the time at which we condition is prior to that of the main event, we have a nice cancellation which simplifies the calculation:

$$\begin{aligned} P(N_{0,3} = 12 | N_{0,1} = 3) &= \frac{P(N_{0,1} = 3, N_{0,3} = 12)}{P(N_{0,1} = 3)} \\ &= \frac{P(N_{0,1} = 3).P(N_{1,3} = 9)}{P(X(1) = 3)} \\ &\quad \text{(using independence assumption)} \\ &= P(N_{1,3} = 9) = 0.124 \end{aligned}$$

49. The rate is 0.7 per week, so the mean for 2 weeks is 1.4.

(a)

$$\begin{aligned} P(N_{0,2} > 2) &= 1 - P(N_{0,2} \leq 2) \\ &= 1 - 0.8335 \\ &= 0.1665. \end{aligned}$$

by `ppois(2,1.4)` in R.

(b) The mean for the first week is 0.7; mean for next 12 is 8.4.

$$\begin{aligned} P(N_{0,1} \geq 2, N_{1,13} \leq 8) &= (1 - P(N_{0,1} \leq 1))P(N_{1,13} \leq 8) \\ &= (1 - 0.8442)(0.5369) \\ &\quad \text{(using R)} \\ &= 0.0837. \end{aligned}$$

50. The number of events in the first hour is $N_{0,60} \sim Po(60) \sim N(60, 60)$ approx.

$$\begin{aligned} P(N_{0,60} < 50) &= P(N_{0,60} < 49.5) \\ &\approx \Phi\left(\frac{49.5 - 60}{\sqrt{60}}\right) \simeq \Phi(-1.356) \\ &= 0.0875. \end{aligned}$$

The time of the fiftieth occurrence is $U_{50} = T_1 + T_2 + \ldots + T_{50}$. Each T_i has the exponential distribution with mean 1 and variance 1, and they are independent, so U_{50} has mean 50 and variance 50. It is approximately normal by the Central Limit Theorem.

$$\begin{aligned} P(U_{50} > 60) &\simeq 1 - \Phi\left(\frac{60 - 50}{\sqrt{50}}\right) \simeq 1 - \Phi(1.414) \\ &= 0.0787. \end{aligned}$$

(Via R, the exact answer is 0.0844 (to 4dp).)

51. Let $F_{S_2}(s) = P(S_2 \leq s)$ be the distribution function of S_2. Then

$$\begin{aligned} F_{S_2}(s) &= P(N_{0,s} \geq 2) \\ &= 1 - P(N_{0,s} \leq 1) \\ &= 1 - (P(N_{0,s} = 0) + P(N_{0,s} = 1)) \\ &= 1 - (e^{-\lambda s} + \lambda s e^{-\lambda s}) \\ &= 1 - (1 + \lambda s)e^{-\lambda s}. \end{aligned}$$

Hence the probability density function of S_2 is

$$f_{S_2}(s) = F'_{S_2}(s) = \lambda(1+\lambda s)e^{-\lambda s} - \lambda e^{-\lambda s} = \lambda s e^{-\lambda s}.$$

52. (a) The observable supernovas, by the thinning property, form a Poisson process with rate 0.004, so the number in a period of length t will have a Poisson distribution with parameter $0.004t$.

(b) The number in a 100 year period will be $Po(0.4)$, so this is $1 - e^{-0.4} = 0.3297$ (4dp).

(c) Conditional on having 2 in a 600 year period, the times will look like a random sample of size 2 from a uniform distribution on the 600 year period. So this is $(1/3)^2 = 1/9$.

53. (a) The number who arrive in $(0, 1]$ is Poisson with parameter

$$\int_0^1 (1+2t)\ dt = \left[t + t^2\right]_0^1 = 2$$

and the number who arrive in $(7, 8]$ is Poisson with parameter

$$\int_7^8 (17-2t)\ dt = \left[17t - t^2\right]_7^8 = 136 - 64 - 119 + 49 = 2.$$

So

$$P(N_{0,1} = 0)P(N_{7,8} = 0) = e^{-2}e^{-2} = (0.1353)^2 = 0.0183.$$

(b) Similar integration gives parameters 2, 4, 6, 8, 8, 6, 4 and 2 for the eight periods. So the required probability is

$$\begin{aligned} &= (1-0.4060)^2(1-0.0916)^2(1-0.0174)^2(1-0.0030)^2 \\ &= \{0.5940 \times 0.9084 \times 0.9826 \times 0.9970\}^2 = 0.2794. \end{aligned}$$

(c) Each independently has probability

$$\frac{\int_0^1 (1+2t)\ dt}{\int_0^2 (1+2t)\ dt} = \frac{2}{6} = \frac{1}{3}$$

of being in the first hour. So the required probability is

$$1 - \left(\frac{2}{3}\right)^5 - 5\left(\frac{2}{3}\right)^4 \frac{1}{3} = \frac{131}{243} (\approx 0.5391).$$

54. (a) Let $N_{u,v}$ be the number of records broken in $(u, v]$.

$$\begin{aligned} P(N_{u,v} = 0) &= \exp\left\{-\int_u^v t^{-1}\,dt\right\} \\ &= \exp\{-\log v + \log u\} \\ &= \frac{u}{v}. \end{aligned}$$

The best performance in $(0, v]$ lies in $(0, u]$ if and only if there are no new records in $(u, v]$. If the time of the best performance is uniformly distributed, then this event has probability u/v as required.

(b) For $t \geq 0$,

$$\begin{aligned} P(T \leq t) &= P(N_{u,u+t} \geq 1) \\ &= 1 - \frac{u}{u+t} \\ &= \frac{t}{u+t}. \end{aligned}$$

Differentiating,

$$f_T(t) = \frac{u}{(u+t)^2},$$

The mean is

$$E(T) = \int_0^\infty \frac{tu}{(u+t)^2}dt = \int_u^\infty \frac{u}{x}dx - \int_u^\infty \frac{u^2}{x^2}$$

(substituting $x = u+t$), of which the second integral is finite but the first diverges at infinity, since it involves the logarithm of x. Hence the mean is not finite.

55. (a) Let N_A be the number of oaks in A and M_A be the number of ashes in A. Then $N_A \sim Po(a/50)$ and $M_A \sim Po(a/40)$, and they are independent, so

$$P(N_A \geq 1, M_A \geq 1) = (1 - e^{-a/40})(1 - e^{-a/50}) = 1 + e^{-9a/200} - e^{a/40} - e^{-a/50}.$$

(b) Using the same notation, $N_B \sim Po(80) \approx N(80, 80)$ and $M_B \sim Po(100) \approx N(100, 100)$. By independence, $N_B - M_B$ has an approximately $N(-20, 180)$ distribution, so

$$\begin{aligned} P(N_B > M_B) &\approx 1 - \Phi\left(\frac{0.5 + 20}{\sqrt{180}}\right) \\ &= 0.063\ldots. \end{aligned}$$

56. (a) There are n^2 small squares, and each has a "success" with probability θ independently of the others, so this distribution will be $Bin(n^2, \theta)$. This gives the expected number of trees (which should be λ) as $n^2\theta$, so we want $n^2\theta = \lambda$ and so $\theta = \lambda/n^2$.

(b) For large n, $Bin(n^2, \lambda/n^2)$ is approximately Poisson with parameter λ, so using the same idea as in the discrete-time approximation to the basic Poisson process we can use a Poisson(λ) random variable as a model for the total number of trees in the large square.

57. (a) It is impossible to complete the pattern $1, 2, 3, 4, 5, 6$ before the 6th roll, so $u_n = 0$ for $n \leq 5$. For $n \geq 6$ the probability of completing the pattern on the nth toss is $1/6^6$, so $u_n = 1/6^6$. Also $u_0 = 1$.

(b) From above

$$U(s) = \sum_{n=0}^{\infty} u_n s^n = 1 + \sum_{n=6}^{\infty} \frac{s^n}{6^6} = \frac{6^6(1-s) + s^6}{6^6(1-s)}.$$

By the basic relation,

$$F(s) = 1 - \frac{1}{U(s)} = 1 - \frac{6^6(1-s)}{6^6(1-s) + s^6} = \frac{s^6}{6^6(1-s) + s^6}.$$

Differentiating,

$$F'(s) = \frac{(6^6(1-s) + s^6)(6s^5) - s^6(-6^6 + 6s^5)}{(6^6(1-s) + s^6)^2},$$

and so the expected time of the first renewal is $F'(1) = 6^6$.

58. (a) The equations are

$$\pi_1 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_4 \qquad (65)$$

$$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2} + \pi_3 + \frac{1}{3}\pi_4 \qquad (66)$$

$$\pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_4 \qquad (67)$$

$$\pi_4 = \frac{1}{2}\pi_1 + \frac{1}{2} + \pi_3 + \frac{1}{3}\pi_2 \qquad (68)$$

Either by symmetry or by rearranging the equations, $\pi_1 = \pi_3$ and $\pi_2 = \pi_4$. Hence $\pi_1 = \frac{2}{3}\pi_2$, and we must have $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$, giving $\boldsymbol{\pi} = \left(\frac{1}{5}\,\frac{3}{10}\,\frac{1}{5}\,\frac{3}{10}\right)$. (This can also be obtained using the result of question 31(c).)

(b) The chain is irreducible, because the graph is connected, so there is a possible route from each vertex to every other vertex. To see that it is aperiodic, consider state A. It is possible to return to A after two steps (go to B, then back to A) or after three (go to B, then to D, then back to A) and the hcf of 2 and 3 is 1, so A is aperiodic; by solidarity so are all other states. An irreducible chain with a finite state space is positive recurrent, so the results apply. Hence, as $n \to \infty$, $P(X_n = A) \to 1/5$, $P(X_n = B) \to 3/10$, $P(X_n = C) \to 1/5$, and $P(X_n = D) \to 3/10$.

59. (a) Let q_i be the probability of reaching C before D, starting in i. Then $q_C = 1$ and $q_D = 0$, and

$$q_A = \frac{1}{2}q_B \qquad (69)$$
$$q_B = \frac{1}{3}q_A + \frac{1}{3}. \qquad (70)$$

Substituting (69) into (70) gives $q_B = \frac{1}{6}q_B + \frac{1}{3}$, so $q_B = \frac{2}{5}$ and $q_A = \frac{1}{5}$.

(b) Let e_i be the expected time to reach C, starting in i. Then $e_C = 0$, and

$$e_A = \frac{1}{2}(e_B + e_D) + 1 \qquad (71)$$
$$e_B = \frac{1}{3}(e_A + e_D) + 1 \qquad (72)$$
$$e_D = \frac{1}{3}(e_A + e_B) + 1 \qquad (73)$$

Substituting (73) into (72) gives $e_B = \frac{4}{9}e_A + \frac{1}{9}e_B + 2$ and hence $e_B = \frac{1}{2}e_A + \frac{3}{2}$. Similarly substituting (72) into (73) gives $e_D = \frac{1}{2}e_A + \frac{3}{2}$ (or alternatively symmetry implies $e_B = e_D$). Substituting these into (71) gives $e_A = \frac{1}{2}e_A + \frac{5}{2}$, and hence $e_A = 5$. (Also $e_B = e_D = 4$.)

60. (a) The number of occurrences in $(0, 1]$ is Poisson with parameter $\int_0^1 2t\ dt = 1$, so this is $e^{-1}/2 = 0.184$.

(b) Each of the two can be thought of as occurring in $(0, 3/4]$ independently of the others with probability $\frac{\int_0^3 /42t\ dt}{\int_0^1 2t\ dt} = 9/16$, so the number which do has a $Bi(2, 9/16)$ distribution, and the probability that it is 1 is thus $2(9/16)(7/16) = 126/256 = 0.492$.

(c) This will now be a Poisson process with variable rate $2t/3$, by the thinning property. So the number of retained occurrences in $(0, 1]$ is Poisson with parameter $\int_0^1 2t/3 \, dt = 1/3$, so the probability there are none is $e^{-1/3} = 0.717$.