The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Probability Modelling **2 hours**

Candidates should attempt ***ALL*** *four questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–30; Q2–21; Q3–25; Q4–24)

1 A group of four badgers share two setts, A and B. Each day, exactly one badger moves from the sett it is currently in to the other one, each badger being equally likely to move. Let X_n be the number of badgers in sett A on day n, and model this as a Markov chain with state space $\{0,1,2,3,4\}$. Assume that, on day 0, two of the badgers are in sett A.

(a)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b) Irreducible: all transitions from i to $i-1$ or $i+1$ within the state space have positive probability, so it is always possible to get from state i to state j in $|i-j|$ steps, so all pairs of states communicate, so the chain is irreducible. Also possible to single out a particular state and show that all states communicate with it. Periodicity: consider state 0 (or any other state). It is only possible to return to state 0 at time n by an equal number of "up" moves and "down" moves, and so $n = 2m$ for some m and must be even, so the period is a multiple of 2. It is possible to return in 2 steps (going to state 1 and then back again) so the period actually is 2. The solidarity theorem and irreducibility now tell us that all the other states also have period 2.

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1 (continued)

(c)

$$\begin{aligned}
\pi_0 &= \frac{1}{4}\pi_1 \\
\pi_1 &= \pi_0 + \frac{1}{2}\pi_2 \\
\pi_2 &= \frac{3}{4}\pi_1 + \frac{3}{4}\pi_3 \\
\pi_3 &= \frac{1}{2}\pi_2 + \pi_4 \\
\pi_4 &= \frac{1}{4}\pi_3
\end{aligned}$$

Using the given distribution, the right hand sides become

$$\begin{aligned}
\frac{1}{4}\frac{4}{16} &= \frac{1}{16} \\
\frac{1}{16} + \frac{1}{2}\frac{6}{16} &= \frac{4}{16} \\
\frac{3}{4}\frac{4}{16} + \frac{3}{4}\frac{4}{16} &= \frac{6}{16} \\
\frac{1}{2}\frac{6}{16} + \frac{1}{16} &= \frac{4}{16} \\
\frac{1}{4}\frac{4}{16} &= \frac{1}{16}
\end{aligned}$$

confirming that the equations are satisfied.

(d) (i) We have $\boldsymbol{\pi}^{(0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix}$. Hence $\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)}P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$, confirming the $n = 1$ case. Assume true for $n = k$. Then $\boldsymbol{\pi}^{(2k-1)} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$, from which

$$\boldsymbol{\pi}^{(2k)} = \boldsymbol{\pi}^{(2k-1)}P = \begin{pmatrix} \frac{1}{8} & 0 & \frac{3}{4} & 0 & \frac{1}{8} \end{pmatrix}$$

and thus

$$\boldsymbol{\pi}^{(2k+1)} = \boldsymbol{\pi}^{(2k)}P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

giving the result by induction.

(ii) If we were to have convergence, then $P(X_n = 1)$ would converge to $\frac{4}{16} = \frac{1}{4}$ as $n \to \infty$, but we have just shown it is $\frac{1}{2}$ for all odd n, so this is impossible.

 Question 1 continued on next page

1 (continued)

(e) Let p_i be the probability of reaching state 4 before state 0. Then $p_4 = 1$, $p_0 = 0$, and

$$\begin{aligned} p_1 &= \frac{1}{4}p_0 + \frac{3}{4}p_2 = \frac{3}{4}p_2 \\ p_2 &= \frac{1}{2}p_1 + \frac{1}{2}p_3 \\ p_3 &= \frac{3}{4}p_2 + \frac{1}{4}p_4 = \frac{3}{4}p_2 + \frac{1}{4} \end{aligned}$$

Thus $p_2 = \frac{3}{8}p_2 + \frac{1}{2}p_3$, so $p_2 = \frac{4}{5}p_3$, and now rearranging the last equation shows that $p_3 = 5/8$ and thus $p_2 = 1/2$ and $p_1 = 3/8$.

2 (a) A renewal at time 1 must be the first renewal, so $u_1 = f_1 = 1/12$. For a renewal at time 2, either there are renewals at both times 1 and 2, which happens with probability $f_1^2 = 1/144$, or the first renewal happens at time 2, which happens with probability $f_2 = 7/144$, so $u_2 = 1/144 + 7/144 = 8/144 = 1/18$. (Can also use the formulae $u_1 = f_1$ and $u_2 = f_2 + f_1^2$, which are in the notes.)

(b) We have

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \left(\frac{1}{3^n} - \frac{1}{4^n} \right) s^n \\ &= \frac{s/3}{1 - s/3} - \frac{s/4}{1 - s/4} \\ &= \frac{s}{3 - s} - \frac{s}{4 - s}. \end{aligned}$$

(c) A renewal process is recurrent if $\sum_{n=1}^{\infty} f_n = 1$ and transient otherwise. We have $\sum_{n=1}^{\infty} f_n = F(1) = 1/6 < 1$, so this renewal process is transient.

(d) By a result in the course,

$$U(s) = \frac{1}{1 - F(s)}.$$

We have

$$1 - F(s) = 1 - \frac{s}{3 - s} + \frac{s}{4 - s} = \frac{(3 - s)(4 - s) - s}{(3 - s)(4 - s)},$$

so

$$U(s) = \frac{(3 - s)(4 - s)}{(3 - s)(4 - s) - s} = 1 + \frac{s}{12 - 8s + s^2}.$$

Either using partial fractions or comparison with the given answer shows that this is equal to the given answer.

(e) Expanding the form for $U(s)$ from (c),

$$\begin{aligned} U(s) &= 1 + \frac{1}{2} \left(\frac{1}{2 - s} - \frac{3}{6 - s} \right) \\ &= 1 + \frac{1}{4} \sum_{n=0}^{\infty} (s/2)^n - \frac{1}{4} \sum_{n=0}^{\infty} (s/6)^n \\ &= 1 + \frac{1}{4} \sum_{n=0}^{\infty} s^n \left(\frac{1}{2^n} - \frac{1}{6^n} \right), \end{aligned}$$

and therefore $u_n = 1$ if $n = 0$ and $\frac{1}{4}\left(\frac{1}{2^n} - \frac{1}{6^n}\right)$ otherwise. As $\frac{1}{2^n}$ and $\frac{1}{6^n}$ both tend to zero as $n \to \infty$ so does u_n.

3 (a) Irreducible: from state 1, states 2 and 3 can be reached in one step, and state 4 can be reached in two steps (either via 2 or 3). State 1 can be reached from state 4 in one step, and from states 2 and 3 in two steps (via 4). Hence all states communicate with state 1, and hence all with each other by transitivity, so the chain is irreducible.

Aperiodic: consider state 1. It is possible to return to state 1 in 3 steps (via 3 and 4) or in 4 steps (via 2,3,4). The hcf of 3 and 4 is 1 so state 1 is aperiodic; hence all states are by the solidarity theorem.

(b) The chain is irreducible and aperiodic, and as it has a finite state space is thus positive recurrent. Hence, by results in the course, it has a unique stationary distribution and the distribution of the chain at time n converges to that distribution as $n \to \infty$.

The equations for the stationary distribution are

$$\begin{aligned} \pi_1 &= \pi_4 \\ \pi_2 &= \frac{1}{2}\pi_1 \\ \pi_3 &= \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 \\ \pi_4 &= \frac{1}{2}\pi_2 + \pi_3 \end{aligned}$$

which are satisfied by $\begin{pmatrix} \frac{4}{13} & \frac{2}{13} & \frac{3}{13} & \frac{4}{13} \end{pmatrix}$. This is unique, so we know from the results that the distribution of the state of the chain must converge to it. Hence, as $n \to \infty$,

$$\begin{aligned} P(X_n = 1) &\to \frac{4}{13} \\ P(X_n = 2) &\to \frac{2}{13} \\ P(X_n = 3) &\to \frac{3}{13} \\ P(X_n = 4) &\to \frac{4}{13} \end{aligned}$$

(c) (i) Let e_i be the length of time to reach state 2 given that we start at state i. Then $e_2 = 0$ and

$$\begin{aligned} e_1 &= \frac{1}{2}e_2 + \frac{1}{2}e_3 + 1 = \frac{1}{2}e_3 + 1 \\ e_3 &= e_4 + 1 \\ e_4 &= e1 + 1 \end{aligned}$$

So $e_1 = \frac{1}{2}(e_1 + 2) + 1$ giving $e_1 = 4$.

(ii) By a result in the course the expected time μ_i to return to state i given that we start there is $1/\pi_i$, where $\boldsymbol{\pi}$ is the unique stationary distribution. So here $\mu_2 = \frac{13}{2}$.

4 (a) The number within three light years has a Poisson distribution with parameter $\mu_1 = \frac{1}{100}\frac{4}{3}\pi 3^3 = 1.131$. The probability it is 1 is thus $\mu_1 e^{-\mu_1} = 0.365$.

(b) Given that there are exactly five star systems within five light years of a specific point, find

(i) By independence of numbers in disjoint regions, this is the probability that there are exactly three within 6ly but not within 5ly. The number within 6ly but not 5ly is Poisson with parameter $\mu_2 = \frac{1}{100}\frac{4}{3}\pi(6^3 - 5^3) = 3.81$, and the probability there are exactly 3 is $\mu_2^3 e^{-\mu_2}/6 = 0.204$.

(ii) Each of the five can be considered to be within 3ly independently of the others, with probability $3^3/5^3 = 27/125$. So the number which are has a Binomial distribution with parameters 5 and 27/125, and so the probability there are none is $(98/125)^5 = 0.296$

(c) Let R be the distance to the nearest system in ly. Then $F_R(r) = P(R \leq r) = P(N_r \geq 1)$ where N_r is the number of star systems within $r$ly. $N_t \sim Poisson\left(\frac{4}{300}\pi r^3\right)$ so

$$P(N_r \geq 1) = 1 - P(N_r = 0) = 1 - \exp\left(-\frac{4}{300}\pi r^3\right).$$

Differentiating, the probability density function

$$f_R(r) = \frac{1}{25}\pi r^2 \exp\left(-\frac{1}{75}\pi r^3\right).$$

(d) Assuming the presence of a habitable planet in a star system is independent of other systems and has probability 0.1 for all systems, then the thinning result for Poisson processes tells us that systems containing a habitable planet form a spatial Poisson process with rate $\frac{1}{1000}$. Hence, the number within 5ly has a Poisson distribution with parameter $\mu_3 = \frac{1}{1000}\frac{4}{3}\pi 5^3 = 0.524$. So the probability there is at least one is $1 - e^{-\mu_3} = 0.408$.

End of Question Paper