The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Probability Modelling **2 hours**

SOLUTIONS

1 (a) We have $u_1 = p$ as if a renewal has just occurred, another renewal occurs at the next step if and only if the next toss is a head, which has probability p. For $n \geq 2$ $u_n = p^2$ as a renewal occurs at time $t+n$ if and only if tosses $t+n-1$ and $t+n$ are both heads. We always have $u_0 = 1$ so

$$\sum_{n=0}^{\infty} u_n s^n = 1 + ps + \sum_{n=2}^{\infty} p^2 s^n = 1 + ps + \frac{p^2 s^2}{1-s},$$

as required.

(b) We have $v_0 = v_1 = 0$ as a renewal requires two tosses, so renewals at times 0 and 1 are impossible. For $n \geq 2$ $u_n = p^2$ as a renewal occurs at time n if and only if tosses $n-1$ and n, which are independent, are both heads. So

$$\sum_{n=0}^{\infty} v_n s^n = \sum_{n=2}^{\infty} p^2 s^n = \frac{p^2 s^2}{1-s},$$

as required.

(c) $f_2 = 0$, because a renewal can only take place at time $t+2$ if tosses $t+1$ and $t+2$ are both heads, but given that a renewal happened at t toss t was also a head, so a renewal must have occurred at time $t+1$, so the one at $t+2$ cannot be the next renewal.

(d) By the result given, the generating function

$$B(s) = \frac{V(s)}{U(s)} = \frac{p^2 s^2}{(1+ps)(1-s) + p^2 s^2}.$$

The mean time until the first renewal is $B'(1)$, so differentiate:

$$B'(s) = \frac{2p^2 s((1+ps)(1-s) + p^2 s^2) - p^2 s^2 (2p^2 s + p - 1 - 2ps)}{((1+ps)(1-s) + p^2 s^2)^2}.$$

Setting $s = 1$ gives the answer as $\dfrac{p^2 + p^3}{p^4} = \dfrac{1}{p^2} + \dfrac{1}{p}$.

2 (a) State 1 cannot be reached from any other state, so must form a class on its own; similarly state 2 must form a class on its own. States 3 and 4 communicate as each can be reached from the other, so the classes are $\{1\}; \{2\}; \{3,4\}$. Classes $\{2\}$ and $\{3,4\}$ cannot be left so these states are recurrent, but the class $\{1\}$ can be left so state 1 is transient.

(b) The equations are

$$\begin{aligned}
\pi_1 &= \frac{1}{3}\pi_1 \\
\pi_2 &= \frac{1}{3}\pi_1 + \pi_2 \\
\pi_3 &= \frac{1}{4}\pi_3 + \frac{1}{2}\pi_4 \\
\pi_4 &= \frac{1}{3}\pi_1 + \frac{3}{4}\pi_3 + \frac{1}{2}\pi_4
\end{aligned}$$

The first two equations give $\pi_1 = 0$ and no further information. The last two are both equivalent to $\frac{3}{4}\pi_3 = \frac{1}{2}\pi_4$ or $\pi_4 = \frac{3}{2}\pi_3$.

If $\pi_2 = \alpha$, then $\pi_3 + \pi_4$ must be $1 - \alpha$, and that the entries must be probabilities requires $\alpha \in [0,1]$, giving for any $\alpha \in [0,1]$,

$$\begin{pmatrix} 0 & \alpha & \frac{2}{5}(1-\alpha) & \frac{3}{5}(1-\alpha) \end{pmatrix}.$$

(c) (i) When $n = 0$, $P(Y_n = 1) = 1$ and $P(Y_n = 2) = 0$, which fits the claim. Assume true for $n = k$ and using the Markov property calculate

$$P(Y_{k+1} = 1) = \frac{1}{3}P(Y_k = 1) = \frac{1}{3}\left(\frac{1}{3}\right)^k = \left(\frac{1}{3}\right)^{k+1}$$

and

$$\begin{aligned}
P(Y_{k+1} = 2) &= \frac{1}{3}P(Y_k = 1) + P(Y_k = 2) \\
&= \left(\frac{1}{3}\right)^{k+1} + \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^k\right) \\
&= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^k + 2\left(\frac{1}{3}\right)^{k+1}\right) \\
&= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^{k+1}\right)
\end{aligned}$$

giving the claim for $n = k+1$, hence true for all n by induction.

(ii) By (i), $P(Y_n = 2) \to \frac{1}{2}$. The only stationary distribution from (b) which fits this is $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{5} & \frac{3}{10} \end{pmatrix}$, so, assuming the chain converges to a stationary distribution, this must be the limit.

3 (a) (i) The chain is irreducible: it is possible to reach state i from state 1 in $i-1$ steps, and also possible to reach state 1 from state i in $i-1$ steps, so all states communicate with state 1 and hence all with each other.

Considering state 1, it is possible to return in 2 steps (go to 2, then back to 1) and in 5 steps (2,3,4,5,1) and the highest common factor of 2 and 5 is 1, so state 1 has period 1. Because the chain is irreducible, all states have the same period by solidarity, so the chain is aperiodic.

(ii)

$$\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix},$$

and all entries are non-negative and sum to 1.

(iii) The chain is irreducible and has finite state space so is positive recurrent.

A chain which is irreducible, aperiodic and positive recurrent has a unique stationary distribution which the chain converges to. So the stationary distribution from (ii) must be the unique stationary distribution and so the chain converges to it, from which the conclusion follows.

(b) (i) Yes: it is still possible (in fact certain) to reach state i from state 1 in $i-1$ steps and also to reach 1 from i in $6-i$ steps, so all states communicate with 1 and hence all with each other.

(ii) No. Consider state 1. The only way the chain returns to state 1 is after 5 steps, via 2,3,4,5. The highest common factor of $\{5\}$ is 5, so this is the period of state 1.

4 (a) With the rows in the order 0 to 5,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Let p_i be the probability of running out of money, starting from state i; then we are looking for p_1. Then $p_0 = 1$ and $p_4 = p_5 = 0$, and

$$\begin{aligned} p_1 &= \frac{2}{3}p_0 + \frac{1}{3}p_3 = \frac{1}{3}(2 + p_3) \\ p_2 &= \frac{2}{3}p_1 + \frac{1}{3}p_4 = \frac{2}{3}p_1 \\ p_3 &= \frac{2}{3}p_2 + \frac{1}{3}p_5 = \frac{2}{3}p_2 \end{aligned}$$

So $p_3 = \frac{4}{9}p_1$, and hence $p_1 = \frac{2}{3} + \frac{4}{27}p_1$, giving the answer as $p_1 = \frac{18}{23}$.

(c) Let e_i be the expected number of games played, starting from state i; then we are looking for e_1. Then $e_0 = e_4 = e_5 = 0$, and

$$\begin{aligned} e_1 &= 1 + \frac{2}{3}e_0 + \frac{1}{3}e_3 = 1 + \frac{1}{3}e_3 \\ e_2 &= 1 + \frac{2}{3}e_1 + \frac{1}{3}e_4 = 1 + \frac{2}{3}e_1 \\ e_3 &= 1 + \frac{2}{3}e_2 + \frac{1}{3}e_5 = 1 + \frac{2}{3}e_2 \end{aligned}$$

So $e_3 = \frac{5}{3} + \frac{4}{9}e_1$, and hence $e_1 = \frac{14}{9} + \frac{4}{27}e_1$, giving the answer as $e_1 = \frac{42}{23}$.

5 (a) Poisson, with parameter $2 \cdot 4 = 8$.

(b) The number in the hour has a Poisson distribution with rate 4, so this is $e^{-4} = 0.0183$.

(c) Given that there are six in two hours, the number in the first hour has a $Bin(6, 1/2)$ distribution, so the probability there is exactly one is $6/64 = 0.0938$.

(d) We have, for $t > 0$, $F_{S_2}(t) = P(S_2 \leq t)$. The event $\{S_2 \leq t\}$ is the same as $\{N_t > 1\}$, where N_t is the number of emails arriving up to time t, which has a Poisson distribution with parameter $4t$. So $F_{S_2}(t) = P(N_t > 1) = 1 - e^{-4t}(1 + 4t)$.

(e) From a result in the course, if each occurrence in a Poisson process with rate λ is retained with probability p independently of the time and the retention of other points, the retained occurrences form a Poisson process with rate $p\lambda$. So we have a Poisson process with rate 1.

End of Question Paper