Solutions to series questions

1. This is a geometric series, with first term 1 and common ratio $\frac{2}{3}$. As $|\frac{2}{3}| < 1$, it is convergent, and the sum is $\frac{1}{1-\frac{2}{3}} = 3$.

2. This is similar to the previous question, except that the common ratio is now $\frac{2}{3}s^2$. Given that $|s| < 1$, $|\frac{2}{3}s^2| < 1$, so again it is convergent, and the sum is now
$$\frac{1}{1 - \frac{2}{3}s^2} = \frac{3}{3 - 2s^2}.$$

3. We have, by factorising the quadratic and using partial fractions
$$\begin{aligned} \frac{2}{2 - s - s^2} &= \frac{2}{(2+s)(1-s)} \\ &= \frac{2/3}{2+s} + \frac{2/3}{1-s} \\ &= \frac{1/3}{1+s/2} + \frac{2/3}{1-s}. \end{aligned}$$
Expanding both terms as geometric series, we get
$$\frac{1}{3}\sum_{k=0}^{\infty}(-s/2)^k + \frac{2}{3}\sum_{k=0}^{\infty} s^k,$$
which we can rearrange as a single sum
$$\sum_{k=0}^{\infty} \frac{1}{3}\left(2 + \left(-\frac{1}{2}\right)^k\right) s^k.$$
Both geometric series expansions will be valid as long as $|s| < 1$.

4. We know that $\sum_{n=1}^{\infty} 1/n^2$ converges. We also know that $\log(n) > 1$ for $n \geq 2$ so
$$\frac{1}{\log(n)} < 1$$
for $n \geq 2$. Hence
$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log(n)} < \sum_{n=2}^{\infty} \frac{1}{n^2}$$
which converges. Hence
$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log(n)}$$
converges by the comparison test.

5. This is quite similar to (11) in the handout. We can write the series as

$$x(4 + 7x^2 + 10x^4 + 13x^6 + \dots),$$

where the term in the brackets is $A_2(x^2)$, where A_2 is defined as in (11). So it becomes

$$x((4+4x^2+4x^4+4x^6+\dots)+(3x^2+6x^4+9x^6+\dots)) = x\left(\frac{4}{1-x^2} + \frac{3x^2}{(1-x^2)^2}\right),$$

which we can simplify to give

$$\frac{x(4(1-x^2)+3x^2)}{(1-x^2)^2} = \frac{x(4-x^2)}{(1-x^2)^2}.$$

6. By the Binomial Theorem, if $|-x| = |x| < 1$ then

$$(1-x)(1+x)^{-3} = (1-x)\sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(-3-i)}{k!} x^k$$

For the first four terms, we can write this as $(1-x)(1-3x+6x^2-10x^3+\dots)$, which gives $1 - 4x + 9x^2 - 16x^3 + \dots$.

For a general expression for the co-efficient of x^k, note that (from (9) in the handout) $\binom{-3}{k} = (-1)^k \frac{(k+2)(k+1)}{2}$, so we can write (again for $|-x| = |x| < 1$),

$$\begin{aligned}
(1-x)(1+x)^{-3} &= (1-x)\sum_{k=0}^{\infty}(-1)^k \frac{(k+2)(k+1)}{2} x^k \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{(k+2)(k+1)}{2} x^k - \sum_{k=0}^{\infty}(-1)^k \frac{(k+2)(k+1)}{2} x^{k+1} \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{(k+2)(k+1)}{2} x^k - \sum_{k=1}^{\infty}(-1)^{k-1} \frac{(k+1)k}{2} x^k
\end{aligned}$$

(relabelling the second sum by replacing k by $k-1$) which gives the co-efficient of x^k as

$$\begin{aligned}
(-1)^k \frac{(k+2)(k+1)}{2} - (-1)^{k-1}\frac{(k+1)k}{2} &= \frac{(-1)^k}{2}((k+2)(k+1) + (k+1)k) \\
&= (-1)^k (k+1)^2,
\end{aligned}$$

as you might have guessed from the first four terms.

The series converges for $|-x| = |x| < 1$, i.e. $-1 < x < 1$. The approximation from the first four terms when $x = 0.1$ is

$$1 - 0.4 + 0.09 - 0.016 = 0.674$$

compared with an actual value of $0.9/(1.1)^3 = 0.676\dots$.

7. (a)

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

(the exponential series).

(b) The series is

$$1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1 - x^2}$$

(geometric series with common ratio x^2.

(c)

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)x^n &= x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} \\ &= x^2 \sum_{n=0}^{\infty} n(n+1)x^n \\ &= \frac{2x^2}{(1-x)^3}. \end{aligned}$$

by the Binomial Theorem (see (9) in the handout) for $|x| < 1$.
[You can also do this by differentiating the geometric series $\sum_{n=0}^{\infty} x^n$ twice with respect to x.]