Series: revision

This document is intended to remind you about some important facts on series which are important in MAS275, particularly in chapter 2. The material should be familiar from MAS111.

Convergence and divergence

A series

$$\sum_{k=0}^{\infty} a_k$$

is **convergent** if its partial sums

$$\sum_{k=0}^{n} a_k$$

converge to a limit L as $n \to \infty$, in which case we write

$$\sum_{k=0}^{\infty} a_k = L.$$

If a series is not convergent then it is said to be **divergent**. (NB sometimes the labelling will start with a_1, or indeed perhaps a_2 or a_3 etc., rather than a_0.)

If the terms a_k do not tend to zero as $k \to \infty$ then the series cannot be convergent, but the converse is not true: you cannot conclude that $\sum_{k=0}^{\infty} a_k$ is convergent because $\lim_{k\to\infty} a_k = 0$. The most important counterexample is the **harmonic series**

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

which is divergent; more generally

$$\sum_{k=1}^{\infty} \frac{1}{k^s}$$

is convergent if $s > 1$ and divergent otherwise.

Geometric series

In a geometric series, the terms have a **common ratio**, in the sense that $a_{k+1}/a_k = r$ for all k. We can thus write a_k as ar^k, where $a = a_0$ is the **first term**.

For all r,

$$\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}. \tag{1}$$

If $|r| < 1$ then letting $n \to \infty$ in (1) gives

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}. \tag{2}$$

Tests for convergence

We start with the **comparison test**, which states that we can deduce information about convergence by comparing with a series whose behaviour is known.

- If $\sum_{k=0}^{\infty} c_k$ is known to be convergent, and for all k we have $|a_k| \leq c_k$, then $\sum_{k=0}^{\infty} a_k$ is also convergent.
- If $\sum_{k=0}^{\infty} d_k$ is known to be divergent, and for all k we have $a_k \geq d_k \geq 0$, then $\sum_{k=0}^{\infty} a_k$ is also divergent.

(In fact we don't need "for all k" here: "for $k \geq N$" for some fixed N is enough, as the initial terms don't affect convergence.)

The **ratio test** says that if we have a series $\sum_{k=0}^{n} a_k$ with non-negative terms and the ratios of consecutive terms tend to a limit, i.e. that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} \to \rho,$$

for some limit $\rho \geq 0$, then the series is convergent if $\rho < 1$ and divergent if $\rho > 1$. (But it doesn't say what happens if $\rho = 1$.)

Power series and differentiation

Frequently we will be working with **power series** of the form

$$A(s) = \sum_{k=0}^{\infty} a_k s^k,$$

where s is thought of as a variable.

A power series has a radius of convergence R (which may be zero or infinity) such that for $|s| < R$ it converges, and for $|s| > R$ it does not. (If $|s| = R$ it may converge, but it may not.)

Within the radius of convergence it is possible to differentiate the power series term by term:

$$\frac{d}{ds} A(s) = \sum_{k=1}^{\infty} k a_k s^{k-1}.$$

Important examples of power series

Most of these are obtained as Maclaurin series of a function. All are valid if $|s| < 1$; some are valid outside this range as well.

$$e^s = \sum_{k=0}^{\infty} \frac{s^n}{n!} = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \dots \quad (3)$$

$$\log(1+s) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{s^k}{k} = s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + \dots \quad (4)$$

$$\frac{1}{1-s} = \sum_{k=0}^{\infty} s^k \quad (5)$$

$$(6)$$

NB in (4) we mean natural log (also known as ln), not log to base 10, and (5) is just the infinite geometric series again.

Note that rational functions can often be written in terms of sums of geometric series, for example

$$\frac{3s-2}{2-3s+s^2}$$

can be expanded using partial fractions as

$$\frac{1}{1-s} - \frac{4}{2-s} = \frac{1}{1-s} - \frac{2}{1-\frac{s}{2}}.$$

Both terms can now be expanded as geometric series, giving

$$\sum_{k=0}^{\infty} s^k - 2\sum_{k=1}^{\infty} \left(\frac{s}{2}\right)^k$$

giving

$$\sum_{k=0}^{\infty} \left(1 - \frac{1}{2^{k-1}}\right) s^k = -1 + \frac{1}{2}s^2 + \frac{3}{4}s^3 + \frac{7}{8}s^4 + \dots,$$

valid as long as $|s| < 1$.

The Binomial Theorem

Recall that

$$(1+s)^n = \sum_{k=0}^{n} \binom{n}{k} s^k$$

when $n \in \mathbb{N}$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}.$$

If $|s| < 1$, then this can be extended to n not a positive integer. The sum is now an infinite series:

$$(1+s)^n = \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(n-i)}{k!} s^k. \tag{7}$$

(Note that the case when $n \in \mathbb{N}$ can also be written as an infinite sum in this way: the terms with $k > n$ are simply zero.)

So if we define

$$\binom{n}{k} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}$$

for **any** $n \in \mathbb{R}$, and $k \in \mathbb{N}_0$, we can write

$$(1+s)^n = \sum_{k=0}^{\infty} \binom{n}{k} s^k,$$

for $|s| < 1$.

For example (relevant in the proof of Theorem 5), if $0 \le s < 1$, $0 < p < 1$ and $q = 1 - p$ it is guaranteed that $0 \le 4pqs^2 < 1$, so we can write

$$(1 - 4pqs^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-4pqs^2)^k = \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1} \left(\frac{1-2i}{2}\right)}{k!} (-4pqs^2)^k.$$

Where n is a negative integer, $-a$ say, it is possible to write $\binom{n}{r} = \binom{-a}{r}$ in terms of a familiar binomial coefficient. We have

$$\binom{-a}{r} = \frac{\prod_{i=0}^{r-1}(-a-i)}{r!},$$

and taking out a factor of -1 in each term in the product gives

$$\binom{-a}{r} = (-1)^r \frac{\prod_{i=0}^{r-1}(a+i)}{r!} = (-1)^k \frac{(a+r-1)!}{(a-1)!r!} = (-1)^r \binom{a+r-1}{a-1}.$$

A useful special case (see the next section) is when $a = 2$:

$$\binom{-2}{r} = (-1)^r (r+1). \tag{8}$$

Similarly,

$$\binom{-3}{r} = (-1)^r \binom{r+2}{2} = (-1)^r (r+2)(r+1)/2. \tag{9}$$

Recognising binomial and related series

Frequently (for example when calculating the mean of a geometric random variable) we encounter power series where the co-efficients of s^k increase linearly with k, for example

$$A_1(s) = s + 2s^2 + 3s^3 + 4s^4 + \dots \tag{10}$$

or

$$A_2(s) = 4 + 7s + 10s^2 + 13s^3 + 16s^4 + \dots \tag{11}$$

Considering (7) with $n = -2$ gives, for $|s| < 1$,

$$(1 - s)^{-2} = 1 + 2s + 3s^2 + 4s^3 + \dots$$

(which can also be obtained by differentiating a geometric series term by term). Comparing with (10), we can now see that

$$A_1(s) = s(1-s)^{-2}.$$

Series like (11) can be summed by treating them as a sum of a geometric series and a series like (10):

$$A_2(s) = 4(1 + s + s^2 + s^3 + \ldots) + 3A_1(s) = \frac{4}{1-s} + \frac{3s}{(1-s)^2},$$

again for $|s| < 1$.

Other series where the co-efficients of s_k are polynomials in k can be summed in a similar way. For example, consider

$$A_3(s) = \sum_{k=0}^{\infty} (k+1)(k+2)s^k = 2 + 6s + 12s^2 + 20s^3 + 30s^4 + \ldots. \qquad (12)$$

By (9) and the Binomial Theorem, this is

$$\sum_{k=0}^{\infty} 2(-1)^k \binom{-3}{k} s^k = 2(1-s)^{-3},$$

again for $|s| < 1$.

Exercises

1. (From MAS111) Find the sum of the series

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots$$

2. Find the sum of the power series

$$1 + \frac{2}{3}s^2 + \frac{4}{9}s^4 + \frac{8}{27}s^6 + \frac{16}{81}s^8 + \ldots$$

   if $|s| < 1$.

3. Use partial fractions and expansion as geometric series to expand the function

$$A(s) = \frac{2}{2 - s - s^2}$$

   as a series, giving the range of s for which the expansion is valid.

4. (From MAS111) Use the fact that $\log(n) > 1$ for $n > 1$ and the comparison test to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log(n)}$$

converges.

5. Find a closed form expression for the sum of the infinite series

$$4x + 7x^3 + 10x^5 + 13x^7 + \dots$$

and state for what values of x the series converges.

6. Find the first four terms in the power series expansion of

$$\frac{(1-x)}{(1+x)^3}$$

and state for what values of x the series converges. Can you find a general formula for the co-efficient of x^k in this expansion?

Compare the sum of these first four terms with the actual value (to three decimal places) of the function when $x = 0.1$.

7. Find a closed form expression for the sum of the power series $\sum_{n=0}^{\infty} a_n x^n$ in each of the following cases. You may assume $|x| < 1$ if you need to.

(a)

$$a_n = \frac{1}{n!} \quad \text{for } n = 0, 1, 2, \dots$$

(b)

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

(c)

$$a_n = n(n-1) \quad \text{for } n = 0, 1, 2, \dots$$

(HINT: note the similarity to (12).)