Null recurrent Markov chains

In the main notes, we omitted the proofs of some results concerning null recurrent Markov chains, in particular the solidarity theorem (Theorem 9) and the lack of a stationary distribution. This document gives proofs of those results.

We first obtain a characterisation of null recurrence of a renewal process in terms of the u_n (or rather the generating function $U(s)$), which will be useful in the proofs.

Lemma A. *A recurrent renewal process is null recurrent if and only if* $\lim_{s\uparrow 1}[U(s)(1-s)] = 0$.

Proof. If the mean renewal time is finite, then by the definition of the derivative and the relationship between the mean and generating functions

$$F'(1) = \lim_{s\uparrow 1} \frac{F(s)-1}{s-1}.$$

In the null recurrent case, we can similarly conclude

$$\lim_{s\uparrow 1} \frac{F(s)-1}{s-1} = \lim_{s\uparrow 1} F'(s) = \infty$$

by l'Hôpital's rule. By Theorem 4, $U(s) = \frac{1}{1-F(s)}$, so

$$\lim_{s\uparrow 1} F'(s) = \lim_{s\uparrow 1} \frac{1}{U(s)(1-s)}.$$

We know that $F'(s) \to \infty$ as $s \uparrow 1$ if and only if the renewal process is null recurrent, giving the result. $\square$

Next, we show that Theorem 9 applies to null recurrence.

Theorem B. *Solidarity applies to null recurrence, so that if states i and j communicate and state i is null recurrent, then state j is null recurrent as well.*

Proof. Assume that i and j communicate, and that state i is null recurrent, and choose m and n as in the proof of Theorem 9 so that $p_{ij}^{(m)}$ and $p_{ji}^{(n)}$ are positive. We can then conclude that

$$\lim_{s\uparrow 1} \left[(1-s)\sum_{r=0}^{\infty} p_{ii}^{(r)} s^r\right] = 0$$

by Lemma A. Omitting the first $m+n$ terms, we then get

$$\lim_{s\uparrow 1}\left[(1-s)\sum_{r=0}^{\infty}p_{ii}^{(m+r+n)}s^{m+r+n}\right]=0.$$

By (1) in the proof of Theorem 9,

$$p_{ii}^{(m+r+n)}\geq p_{ij}^{(m)}p_{ji}^{(n)}p_{jj}^{(r)}$$

and thus

$$p_{jj}^{(r)}\leq\frac{1}{p_{ij}^{(m)}p_{ji}^{(n)}}p_{ii}^{(m+r+n)},$$

so

$$\begin{aligned}
\lim_{s\uparrow 1}\left[(1-s)\sum_{r=0}^{\infty}p_{jj}^{(r)}s^r\right] &\leq \lim_{s\uparrow 1}\left[(1-s)\frac{1}{p_{ij}^{(m)}p_{ji}^{(n)}}\sum_{r=0}^{\infty}p_{ii}^{(m+r+n)}s^r\right]\\
&= \frac{1}{p_{ij}^{(m)}p_{ji}^{(n)}}\lim_{s\uparrow 1}\left[(1-s)\sum_{r=0}^{\infty}p_{ii}^{(m+r+n)}\frac{s^{m+r+n}}{s^{m+n}}\right]\\
&= \frac{1}{p_{ij}^{(m)}p_{ji}^{(n)}}\lim_{s\uparrow 1}\left[\frac{1}{s^{m+n}}(1-s)\sum_{r=0}^{\infty}p_{ii}^{(m+r+n)}s^{m+r+n}\right]\\
&= \frac{1}{p_{ij}^{(m)}p_{ji}^{(n)}}\lim_{s\uparrow 1}\left[(1-s)\sum_{r=0}^{\infty}p_{ii}^{(m+r+n)}s^{m+r+n}\right]\\
&= 0.
\end{aligned}$$

Hence state j is also null recurrent. □

We now show that null recurrent chains, like transient ones, do not have stationary distributions.

Theorem C. *An irreducible null recurrent Markov chain has no stationary distribution.*

Proof. If we have an irreducible null recurrent Markov chain with a stationary distribution $\boldsymbol{\pi}$, then consider the chain (X_n) started in its stationary distribution, pick a state i, and consider the delayed renewal process where renewals are visits to state i. This renewal process will be null recurrent, so by Lemma A $\lim_{s\uparrow 1}U(s)(1-s)=0$.

However, because we started in a stationary distribution we can immediately see that $v_n = P(X_n=i)=\pi_i$. Hence the generating function $V(s)=\frac{\pi_i}{1-s}$, and so

$$\lim_{s\uparrow 1}V(s)(1-s)=\pi_i.$$

By Theorem 7 in the renewal processes chapter, $V(s) = B(s)U(s)$, so

$$\lim_{s\uparrow 1} V(s)(1-s) = \lim_{s\uparrow 1} B(s) \lim_{s\uparrow 1} U(s)(1-s) = 0,$$

by the above and by $\lim_{s\uparrow 1} B(s) = \sum_{k=0}^{\infty} b_k = 1$. Hence $\pi_i = 0$; there was nothing special about our choice of i, so $\pi_i = 0$ for all i, and as in the transient case this does not work as stationary distributions must sum to 1. Hence we have a contradiction. $\square$

This argument can be adapted to show that, for a null recurrent Markov chain (X_n), $P(X_n = i)$ cannot tend to a limit other than zero as $n \to \infty$: if this happens then for the (delayed) renewal process consisting of visits to i, $\lim_{s\uparrow 1} V(s)(1-s)$ would not be zero. Showing that $P(X_n = i)$ actually tends to zero requires a bit more analysis.