MAS275 Probability Modelling

Examples

1 Chapter 1

Example 1. *Wet and dry days*

Imagine that a wet day is followed by a dry day with probability α and otherwise by another wet day, and that a dry day is followed by a wet day with probability β and otherwise by another dry day.

This can be modelled (rather crudely) by a very simple Markov chain, with two states W and D (or 1 and 2). The transition matrix is

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Example 2. *Gambler's ruin*

A gambler is playing a game in which, on each turn, the player wins 1 unit with probability p and loses 1 unit with probability $q = 1 - p$. The gambler has a target N and will stop playing either when the money runs out or the target is reached. Here the state space $S = \{0, 1, 2, \ldots, N\}$ and the transition matrix looks like

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ q & 0 & p & 0 & \ldots & 0 & 0 & 0 \\ 0 & q & 0 & p & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & q & 0 & p \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

Example 3. *Gambler's ruin with no target*

This is the same as the previous example but now there is no target N. So the state space is now infinite, $S = \mathbb{N}_0$, and the transition probabilities are

$$p_{ij} = \begin{cases} p & j = i+1 \\ q & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

if $i > 0$, and $p_{00} = 1$.

Example 4. *Ehrenfest model for diffusion*

Two containers contain between them N particles. At each time point a particle is chosen at random and transferred to the other container. Let the state of the system be the number of particles in container A, say. Then we have a Markov chain with state space $\{0, 1, 2, \ldots, N\}$ and transition probabilities

$$\begin{aligned} p_{i,i+1} &= \frac{N-i}{n} \\ p_{i,i-1} &= \frac{i}{N} \\ p_{i,j} &= 0 \text{ if } j \notin \{i-1, i+1\}, \end{aligned}$$

Example 5. *Symmetric random walk on a graph*

Six vertices, A, B, C, D, E, F, with edges AB, BC, BD, CD, CE, DE, EF. The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let the walk start either at A or F, each with probability 1/2. This is represented by $\pi^{(0)} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$. Then $\pi^{(1)} = \pi^{(0)} P = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix}$ and $\pi^{(2)} = \pi^{(1)} P = \pi^{(0)} P^2 = \begin{pmatrix} 1/6 & 0 & 1/3 & 1/3 & 0 & 1/6 \end{pmatrix}$.

Example 6. *Diagonalisation*

Let

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues satisfy

$$\begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \lambda - \frac{1}{2} & 0 \\ 1 & 0 & \lambda - \frac{1}{2} \end{vmatrix} = 0,$$

so

$$\begin{aligned} \lambda((\lambda - \frac{1}{2})^2 - \frac{1}{16}) + \frac{1}{4}(\frac{1}{2} - \lambda) - \frac{1}{16} &= 0 \\ \lambda^3 - \lambda^2 - \frac{1}{16}\lambda + \frac{1}{16} &= 0 \\ (\lambda - 1)(\lambda^2 - \frac{1}{16}) &= 0 \\ (\lambda - 1)\left(\lambda + \frac{1}{4}\right)\left(\lambda - \frac{1}{4}\right) &= 0, \end{aligned}$$

so the eigenvalues are 1 (as always), 1/4 and −1/4. We know $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1; to find eigenvectors with eigenvalue 1/4 solve

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which implies $y = \frac{1}{4}z$ and $\frac{1}{2}x + \frac{1}{4}y + \frac{1}{4}z = \frac{1}{4}x$, from which $x = -\frac{5}{4}z$. Hence $\begin{pmatrix} -\frac{5}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1/4; to find eigenvectors with eigenvalue −1/4 solve

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which implies $y = -\frac{1}{4}z$ and $\frac{1}{2}x + \frac{1}{4}y + \frac{1}{4}z = -\frac{1}{4}x$, from which $x = -\frac{1}{4}z$. Hence $\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $-1/4$.

So if

$$C = \begin{pmatrix} 1 & -\frac{5}{4} & -\frac{1}{4} \\ 1 & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}$$

then $P = CDC^{-1}$ and so $P^n = CD^nC^{-1}$. We can calculate

$$C^{-1} = \begin{pmatrix} \frac{4}{15} & \frac{8}{15} & \frac{3}{15} \\ -\frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{5} & -\frac{6}{5} & \frac{4}{5} \end{pmatrix}.$$

As

$$D^n \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as $n \to \infty$, we also have

$$P^n \to C \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} C^{-1} = \begin{pmatrix} \frac{4}{15} & \frac{8}{15} & \frac{1}{5} \\ \frac{4}{15} & \frac{8}{15} & \frac{1}{5} \\ \frac{4}{15} & \frac{8}{15} & \frac{1}{5} \end{pmatrix}.$$

(Do not include full details of linear algebra in lecture.)

Example 7. *Stationary distribution for random walk on a graph*

We find the stationary distribution for the symmetric random walk on a graph with four vertices A, B, C, D and edges AB, AC, BC, AD. The states are $\{A, B, C, D\}$ and the transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The equations are

$$\begin{aligned}
\pi_A &= \frac{1}{2}\pi_B + \frac{1}{2}\pi_C + \pi_D &(1)\\
\pi_B &= \frac{1}{3}\pi_A + \frac{1}{2}\pi_C &(2)\\
\pi_C &= \frac{1}{3}\pi_A + \frac{1}{2}\pi_B &(3)\\
\pi_D &= \frac{1}{3}\pi_A &(4)
\end{aligned}$$

Obviously $\pi_A = 3\pi_D$, and then by (1) $\frac{1}{2}(\pi_B + \pi_C) = \frac{2}{3}\pi_A$. From (2), $\pi_B - \frac{1}{2}\pi_C = \frac{1}{3}\pi_A = \frac{1}{4}(\pi_B + \pi_C)$ giving $\pi_B = \pi_C$. Hence $\pi_B = \pi_C = \frac{2}{3}\pi_A = 2\pi_D$. Then $\sum_i \pi_i = 1$ becomes $8\pi_D = 1$, so $\pi_A = \frac{3}{8}$, $\pi_B = \pi_C = \frac{1}{4}$ and $\pi_D = \frac{1}{8}$. So $\boldsymbol{\pi} = \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$.

(Make symmetry comment after obtaining the answer.)

2 Chapter 2

Example 8. *Bernoulli trials*

We have a sequence of Bernoulli trials with "success" probability p, and we have a renewal every time there is a success.

Then $f_n = q^{n-1}p$ (where $q = 1 - p$) because once a success has occurred, to get the next success after a further n trials requires $n - 1$ failures followed by a success. So T_i has a geometric distribution and the renewal process is recurrent.

Example 9. *Examples of generating functions*

(a). Let $a_i = 1$ for $i = 0, 1, 2, \ldots$. Then the generating function $A(s)$ of this sequence is given by

$$A(s) = \sum_{k=0}^{\infty} 1 \cdot s^k = \frac{1}{1-s}$$

for $-1 < s < 1$ (geometric series).

(b). (Bernoulli distribution) Let X be a random variable with $P(X = 0) = 1-p$ and $P(X = 1) = p$. Then we can write $f_0 = 1-p$, $f_1 = p$, and the probability generating function $F(s) = (1-p)s^0 + ps^1 = ps + (1-p)$.

(c). (Binomial distribution) Let $X \sim Bin(n, p)$. Then $X = Z_1 + Z_2 + \ldots + Z_n$, where the Z_i have the Bernoulli distribution and are independent. So by Lemma 4 the probability generating function of X is found by multiplying together the p.g.f.s of the Z_i, so is $F_X(s) = (ps + (1-p))^n$. (Note: we can expand this using the Binomial Theorem as

$$F_X(s) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k,$$

from which we can recover $f_k = \binom{n}{k} p^k (1-p)^{n-k}$, as expected.)

(d). (Geometric distribution) Let X be a random variable with $P(X = n) = (1-p)^{n-1}p$ for $n \geq 1$ (NB this is the variant of the geometric distribution which counts the total number of trials needed for the first success, so zero is not a possible value). Then $f_n = q^{n-1}p$ (writing $q = 1-p$) and so $F_X(s) = \sum_{n=1}^{\infty} q^{n-1} p s^n = \frac{ps}{1-qs}$ for $|qs| < 1$ (geometric series).

Example 10. *Calculation of mean using generating function*

Let X be a random variable with $P(X = n) = (1-p)^{n-1}p$. Then (again writing q for $1-p$) for $|s| < 1/q$ $F_X(s) = \sum_{n=1}^{\infty} q^{n-1} p s^n = \frac{ps}{1-qs}$ by part (d) of Example 9. Differentiating,

$$F_X'(s) = \frac{p(1-qs) + qps}{(1-qs)^2}$$

and evaluating this at $s = 1$ (which is within the range of validity) gives $E(X) = p/(1-q)^2 = p/p^2 = 1/p$.

Example 11. *Theorem 4 for Bernoulli trials*

Here $F(s) = \frac{ps}{1-qs}$ (part (d) of Example 9), so by Theorem 4

$$U(s) = \frac{1}{1 - \frac{ps}{1-qs}} = \frac{1-qs}{1-qs-ps},$$

but $q + p = 1$ so

$$\begin{aligned} U(s) &= \frac{1-qs}{1-s} = (1-qs)\sum_{n=0}^{\infty} s^n \\ &= 1 + (1-q)s + (1-q)s^2 + (1-q)s^3 + \dots \\ &= 1 + ps + ps^2 + ps^3 + \dots, \end{aligned}$$

giving $u_0 = 1$ (always true, by convention) and $u_n = p$ for $n \geq 1$. (Obvious from definition of process.)

Example 12. *Bernoulli trials with "blocking"*

Again we have a sequence of Bernoulli trials with "success" probability p, and we have a renewal every time there is a success, except now the renewal is "blocked" if there was a renewal the previous trial, so we do not get renewals on consecutive trials.

Here $f_1 = 0$, and for $n \geq 2$ $f_n = q^{n-2}p$ (to get a renewal on the nth trial it does not matter whether the first trial is a success or failure, the next $n-2$ must be failures, and the nth must be a success). Hence

$$F(s) = \sum_{n=2}^{\infty} q^{n-2}ps^n = \frac{ps^2}{1-qs}.$$

Then we can find the mean time to renewal by differentiating:

$$F'(s) = \frac{(1-qs)\cdot 2ps = ps^2(-q)}{(1-qs)^2} = \frac{2ps - pqs^2}{(1-qs)^2},$$

and so $\mu = F'(1) = \frac{2p-pq}{(1-q)^2} = \frac{2-q}{p} = \frac{1+p}{p}$.

By Theorem 4

$$\begin{aligned}
U(s) &= \frac{1}{1-\frac{ps^2}{1-qs}} \\
&= \frac{1-qs}{1-qs-ps^2} \\
&= \frac{1-qs}{(1-s)(1+ps)} \\
&= \frac{1}{1+p}\left(\frac{p}{1-s}+\frac{1}{1+ps}\right) \\
& \quad \text{(partial fractions)} \\
&= \frac{1}{1+p}(p(1+s+s^2+s^3+\ldots)+(1-ps+p^2s^2-p^3s^3+\ldots)).
\end{aligned}$$

So u_n, the coefficient of s^n in $U(s)$, is $\frac{1}{1+p}(p+(-p)^n)$. As $n \to \infty$, $u_n \to \frac{p}{1+p}$. Note that this is $1/\mu$.

Example 13. *Equalisations in rolling a six-sided dice*

Let E_n be the event that after n rolls of an ordinary dice, the total numbers of 1s, 2s, 3s, 4s, 5s and 6s are all equal. Then E_n can occur if and only if n is a multiple of 6, so if we construct a renewal process by saying that a renewal occurs at n if and only if E_n occurs,

$$\{n : f_n > 0\} = \{0, 6, 12, 18, 24, \ldots\}$$

and so the renewal process is periodic with period 6.

Example 14. *Bernoulli trials with blocking revisited*

Consider the Bernoulli trials with blocking in Example 12, but now assume that the process does not behave as if there was a renewal at time 0, so the renewal at time 1 is not blocked. Then the time D until the first renewal occurs has a geometric distribution (as for Bernoulli trials): $b_n = q^{n-1}p$ for $n \geq 1$, with generating function $B(s) = \frac{ps}{1-qs}$. Once the first renewal has occurred the distributions of $T_1, T_2, \ldots$ are as in Example 12, so $F(s) = \frac{ps^2}{1-qs}$.

By Theorem 7,

$$\begin{aligned} V(s) &= \frac{\frac{ps}{1-qs}}{1-\frac{ps^2}{1-qs}} \\ &= \frac{ps}{1-qs-ps^2} \\ &= \frac{p}{1+p}\left(\frac{1}{1-s}-\frac{1}{1+ps}\right) \\ &\quad \text{(partial fractions)} \\ &= \frac{p}{1+p}((1+s+s^2+s^3+\ldots)-(1-ps+p^2s^2-p^3s^3+\ldots)). \end{aligned}$$

So v_n, the coefficient of s^n in $V(s)$, is $\frac{p}{1+p}(1-(-p)^n)$ (e.g. $v_0=0$, $v_1=p$, $v_2=\frac{p}{1+p}(1-p^2)=p(1-p)$). As before, $n\to\infty$, $v_n\to\frac{p}{1+p}$.

Example 15. *Expected time until sequence completed*

Assuming the coin fair, find the expected number of tosses up to and including the first occurrence of (a) HHT; (b) THT.

(a) This can be treated as a non-delayed renewal process, with renewals being completions of HHT. Then $u_0=1$ (convention), $u_1=u_2=0$ (renewals impossible), and $u_n=1/8$ for $n\geq 3$. So

$$U(s)=1+\frac{1}{8}\sum_{n=3}^{\infty}s^n=1+\frac{1}{8}\frac{s^3}{1-s}=\frac{8-8s+s^3}{8-8s}.$$

Then

$$F(s)=1-\frac{8-8s}{8-8s+s^3}=\frac{s^3}{8-8s+s^3}.$$

We want the mean time to a renewal, which is $F'(1)$. Now

$$F'(s)=\frac{(8-8s+s^3)(3s^2)-s^3(-8+3s^2)}{(8-8s+s^3)^2},$$

and evaluating this at $s=1$ gives $\mu=8$.

(b) This has to be treated as a delayed renewal process, and we want the mean time to the first renewal, $B'(1)$. Here the probabilities of completing THT at time n give $v_0 = v_1 = v_2 = 0$ and $v_n = 1/8$ if $n \geq 3$, but $u_0 = 1$ (convention), $u_1 = 0$, $u_2 = 1/4$ (because after a renewal another renewal occurs if the first two tosses are HT), $u_n = 1/8$ for $n \geq 3$.

So $U(s) = 1 + \frac{1}{4}s^2 + \frac{1}{8}\frac{s^3}{1-s}$ and $V(s) = \frac{1}{8}\frac{s^3}{1-s}$. Using $V(s) = B(s)U(s)$,

$$B(s) = \frac{V(s)}{U(s)} = \frac{s^3}{8(1-s)}\frac{8(1-s)}{8-8s+2s^2-s^3} = \frac{s^3}{8-8s+2s^2-s^3},$$

and

$$B'(s) = \frac{(8-8s+2s^2-s^3)(3s^2) - s^3(-8+4s-3s^2)}{(8-8s+2s^2-s^3)^2},$$

giving $B'(1) = 10$.

(Is this counterintuitive? Note that the probability of getting the first THT after 5 tosses is different from the probability of getting the first HHT after 5 tosses: the latter is 1/8 but the former is only 3/32.)

3 Chapter 3

Example 16. *Random walk and coin tossing*

In the simple random walk (Section 2.4) we have already considered the renewal process of returns to zero given that we started there. So each state is null recurrent if $p = \frac{1}{2}$ (i.e. the random walk is symmetric) and transient otherwise. Also, each state is periodic with period 2.

Example 17. *Mean recurrence time*

Consider the wet/dry days Markov chain of Example 1, with

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

If the chain starts in state 1 (wet) then with probability $1-\alpha$ the first return to state 1 happens at time 1, so $f_{11}^{(1)} = 1-\alpha$. For the first return to state 1 to happen at time n, we must move to state 2 at time 1, and then we must remain in state 2 until we move to state 1 at time n, which happens with probability $\alpha(1-\beta)^{n-2}\beta$. So $f_{11}^{(n)} = \alpha(1-\beta)^{n-2}\beta$ for $n \geq 2$. The generating function of this sequence is

$$F(s) = (1-\alpha)s + \sum_{n=2}^{\infty} \alpha(1-\beta)^{n-2}\beta s^n = (1-\alpha)s + \frac{\alpha\beta s^2}{1-(1-\beta)s}.$$

We can check that $F(1) = 1$, so the state is recurrent, and

$$F'(s) = (1-\alpha) + \frac{(1-(1-\beta)s)(2s\alpha\beta) + (1-\beta)\alpha\beta s^2}{(1-(1-\beta)s)^2},$$

so the mean time to the first renewal is

$$F'(1) = (1-\alpha) + \frac{\beta(2\alpha\beta) + (1-\beta)\alpha\beta}{\beta^2} = \frac{\alpha+\beta}{\beta}.$$

So the state is positive recurrent.

Example 18. *Irreducible Markov chains*

The simple random walk and the Ehrenfest model for diffusion (Examples 2.4 and 4 are examples of irreducible Markov chains. So is a random walk on a graph (Example 5) as long as the graph is connected.

Example 19. *Gambler's ruin*

In the gambler's ruin Markov chain (Example 2) the states 0 and N do not commnunicate with any other states, since once entered they cannot be left. The states $1, 2, \ldots, n-1$ do communicate with each other, so the classes are $\{0\}$, $\{N\}$ and $\{1, 2, \ldots, N-1\}$.

Example 20. *Finding classes and recurrence properties*

Find the classes, the closed classes and the periods of the classes of a Markov chain on $\{1,2,3,4,5,6,7\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{7} & 0 & 0 & \frac{6}{7} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(Draw graph showing possible one step transitions)

State 3 cannot be left once entered, so $\{3\}$ is a closed class (absorbing), obviously aperiodic. It is possible to get from 1 to 2, from 2 to 5, and from 5 to 1, and also from 2 to 6 and from 6 to 1, so all these classes communicate. States 4 and 7 communicate with each other but with no other states (as $\{4,7\}$ cannot be left) so $\{4,7\}$ is a closed class, whose period is obviously 2. The remaining class $\{1,2,5,6\}$ is not closed, as it is possible to go from state 6 to state 7, from state 5 to state 4 or from state 1 to state 3, and its period is 3 as it is only possible to return to state 1 after passing through state 2 and then either state 5 or state 6.

Example 21. *Modelling the game of Monopoly*

See separate slides.

Example 22. *A non-irreducible chain*

Let

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Here the classes are $\{1,4\}$, $\{2\}$ and $\{3\}$, of which the first two are closed.

The equations for a stationary distribution are

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_4 \quad (5)$$
$$\pi_2 = \pi_2 + \frac{1}{2}\pi_3 \quad (6)$$
$$\pi_3 = \frac{1}{2}\pi_3 \quad (7)$$
$$\pi_4 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_4 \quad (8)$$

Immediately from (7) $\pi_3 = 0$, and either (8) or (5) gives $\pi_4 = \frac{4}{3}\pi_1$. The remaining equation (6) gives no information. So there is a continuum of stationary distributions, of the form $\begin{pmatrix}\frac{3}{7}(1-\alpha) & \alpha & 0 & \frac{4}{7}(1-\alpha)\end{pmatrix}$, for any $\alpha \in [0,1]$.

Example 23. *A periodic chain*

Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

(e.g. random walk on three vertex path).

This chain is irreducible with period 2. The equations for a stationary distribution are $\pi_1 = \frac{1}{2}\pi_2$, $\pi_2 = \pi_1 + \pi_3$ and $\pi_3 = \frac{1}{2}\pi_2$, with unique solution $\boldsymbol{\pi} = \begin{pmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{pmatrix}$.

However, if $\boldsymbol{\pi}^{(0)} = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}$, then

$$\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)}P = \begin{pmatrix}0 & 1 & 0\end{pmatrix}$$

and

$$\boldsymbol{\pi}^{(2)} = \boldsymbol{\pi}^{(1)}P = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix} = \boldsymbol{\pi}^{(0)}.$$

So for all $n \in \mathbb{N}_0$, $\boldsymbol{\pi}^{(2n)} = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}$ and $\boldsymbol{\pi}^{(2n+1)} = \begin{pmatrix}0 & 1 & 0\end{pmatrix}$. Hence we do not get convergence to the stationary distribution.

Example 24. *Bernoulli trials with blocking*

Consider the Bernoulli trials with blocking renewal process (Example 12). We calculated $\mu = \frac{1+p}{p}$, so the renewal theorem tells us that $u_n \to \frac{p}{1+p}$ as $n \to \infty$. (We already saw this from the direct calculation of u_n in Example 12, but there are some processes where it is not easy to get a general form for u_n.)

4 Chapter 4

Example 26. *Random surfer on a mini-Web*

Use example from admissions talk

Example 27. *PageRank for a mini-Web*

Use example from admissions talk

5 Chapter 5

Example 28. *University course model*

A university course has three levels. Each year, independently, a student passes and progresses (with probability p), quits with probability q, or repeats the year with probability r, with $p, q, r > 0$ and $p + q + r = 1$. What is the probability that the student will graduate?

Markov chain with 5 states: the three levels 1,2,3; "graduated" (4), "left without graduating" (5). The transition matrix is

$$\begin{pmatrix} r & p & 0 & 0 & q \\ 0 & r & p & 0 & q \\ 0 & 0 & r & p & q \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the absorbing classes are $\{4\}$ and $\{5\}$. We are interested in the probability of absorption in $\{4\}$, starting in state 1. We have boundary conditions $q_4 = 1$ and $q_5 = 0$, and equations

$$\begin{aligned} q_1 &= rq_1 + pq_2 + qq_5 = rq_1 + pq_2 \quad (9) \\ q_2 &= rq_2 + pq_3 + qq_5 = rq_2 + pq_3 \quad (10) \\ q_3 &= rq_3 + pq_4 + qq_5 = rq_3 + p \quad (11) \end{aligned}$$

Solving in reverse order, $q_3 = \frac{p}{1-r}$, then $q_2(1-r) = \frac{p^2}{1-r}$ so $q_2 = \frac{p^2}{(1-r)^2}$, then $q_1(1-r) = \frac{p^3}{(1-r)^2}$ so $q_1 = \frac{p^3}{(1-r)^3}$.

Example 29. *Gambler's ruin*

(See Example 2.) States 0 (ruin) and N (target reached) are absorbing states. Let q_i be the probability of reaching N starting in state i. Then

$$q_i = pq_{i+1} + qq_{i-1}$$

for $i = 1, 2, \ldots, N-1$, and the boundary conditions are $q_0 = 0$ and $q_N = 1$.

Using $p+q = 1$, the general equation can be re-written $pq_i+qq_i = pq_{i+1}+qq_{i-1}$ and hence

$$p(q_{i+1} - q_i) = q(q_i - q_{i-1}).$$

So if we write $d_i = q_{i+1} - q_i$ we have $d_i = (q/p)d_{i-1} = (q/p)^i d_0$. Because $q_0 = 0$ we know $d_0 = q_1$, so $d_i = (q/p)^i q_1$.

Now, note that

$$\sum_{i=0}^{k-1} d_i = \sum_{i=0}^{k-1} (q_{i+1} - q_i) = q_k - q_0 = q_k.$$

So

$$q_k = \sum_{i=0}^{k-1} d_i = q_1 \sum_{i=0}^{k-1} (q/p)^i = q_1 \frac{1-(q/p)^k}{1-q/p}$$

(geometric series) unless $q = p$. Hence

$$q_k = q_1 \frac{1-(q/p)^k}{1-q/p} \quad (12)$$

We now use $q_N = 1$. Hence

$$q_1 \frac{1-(q/p)^N}{1-q/p} = 1,$$

and so $q_1 = \frac{1-q/p}{1-(q/p)^N}$. Substituting into (12) we get

$$q_k = \frac{1-(q/p)^k}{1-(q/p)^N}.$$

If $q = p = \frac{1}{2}$, then $d_i = d_0 = q_1$ for all i. Thus

$$q_k = \sum_{i=0}^{k-1} d_i = kq_1,$$

so $q_k = kq_1$). Using $q_N = 1$ gives $q_1 = 1/N$, so we have $q_k = k/N$.

Example 30. *Random walk on cube*

One face has vertices labelled A, B, C, D cyclically, opposite face has E, F, G, H.

What is the probability of reaching $\{E, F, G, H\}$ before C, starting in A? Treat $\{E, F, G, H\}$ and $\{C\}$ as absorbing classes; then we have boundary conditions $q_C = 0$ and $q_E = q_F = q_G = q_H = 1$.

Equations:

$$q_A = \frac{1}{3}(q_B + q_D + q_E) = \frac{1}{3}(q_B + q_D + 1) \quad (13)$$
$$q_B = \frac{1}{3}(q_A + q_C + q_F) = \frac{1}{3}(q_A + 1) \quad (14)$$
$$q_D = \frac{1}{3}(q_A + q_C + q_H) = \frac{1}{3}(q_A + 1) \quad (15)$$

so $q_A = \frac{1}{9}(2q_A + 5)$, giving $q_A = \frac{5}{7}$. (Also $q_B = q_D = \frac{4}{7}$.)

Example 31. *University course model revisited*

Recall the university course model in Example 28. What is the expected number of years spent at university by the student?

Let e_i be the expected time to absorption in state 4 or 5 starting in state i. We have $e_4 = e_5 = 0$, and

$$\begin{aligned} e_1 &= 1 + re_1 + pe_2 + qe_5 = 1 + re_1 + pe_2 &(16)\\ e_2 &= 1 + re_2 + pe_3 + qe_5 = 1 + re_2 + pe_3 &(17)\\ e_3 &= 1 + re_3 + pe_4 + qe_5 = 1 + re_3 &(18)\end{aligned}$$

Solving in reverse order, $e_3 = \frac{1}{1-r}$, then

$$e_2 = \frac{1 + pe_3}{1-r} = \frac{1}{1-r} + \frac{p}{(1-r)^2}$$

and

$$e_1 = \frac{1 + pe_2}{1-r} = \frac{1}{1-r} + \frac{p}{(1-r)^2} + \frac{p^2}{(1-r)^3}.$$

Example 32. *Patterns in coin tossing*

This uses a Markov chain method to solve the same problem as Example 15(b): find the expected number of tosses of a fair coin until the first appearance of THT.

(b) Define states of "partial achievement" of THT by 0 (not started), 1 (last toss was T not preceded by TH), 2 (last two tosses were TH) and 3 (THT completed). Then the sequence of these states is a Markov chain with transition matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

This time

$$e_3 = 0 \tag{19}$$

$$e_2 = 1 + \frac{1}{2}(e_0 + e_3) = 1 + \frac{1}{2}e_0 \tag{20}$$

$$e_1 = 1 + \frac{1}{2}(e_1 + e_2) \tag{21}$$

$$e_0 = 1 + \frac{1}{2}(e_0 + e_1) \tag{22}$$

Substituting (20) into (21) gives $\frac{1}{2}e_1 = 1 + \frac{1}{2} + \frac{1}{4}e_0$ or $e_1 = 3 + \frac{1}{2}e_0$. Into (22): $\frac{1}{2}e_0 = 1 + \frac{3}{2} + \frac{1}{4}e_0$, giving $\frac{1}{4}e_0 = \frac{5}{2}$ hence $e_0 = 10$ (as before). (Also: $e_1 = 8$, $e_2 = 6$.)

6 Chapter 6

Example 33. *Volcanic eruptions*

Eruptions of a volcano occur, on average, once per century. Making suitable assumptions, find the probabilities of the following events:

(a). There are exactly two eruptions in the next 80 years.

(b). The time until the next eruption is at least t years.

We use a Poisson process model with rate 1/100. Then for (a) the number of eruptions is a Poisson random variable with parameter $80 \times 0.01 = 0.8$, and so the probability that it is equal to 2 is $\frac{e^{-0.8} \times 0.8^2}{2} = 0.1438$ (also obtained by `dpois(2,0.8)` in R).

For (b), the event of interest is the same as saying that there are no eruptions in the next t years. Under the Poisson process model, the number of eruptions in the next t years has a $Po(0.01t)$ distribution, so the probability that it is zero is $e^{-0.01t}$.

Example 34. *Email arrivals*

Suppose that emails arise over the course of a day with rate function $\lambda(t) = t(24-t)/12$ for $0 \leq t \leq 24$, with time measured in hours.

Then the number of emails up to time t is Poisson with parameter $\int_0^t (24s - s^2)/12 \; ds = s^2 - s^3/36$. For example the probability that there are no emails between midnight and 2am can be calculated as the probability that a $Po(34/9)$ random variable is zero, which is $\exp(-34/9) = 0.0229$.

Example 35. *University applications*

Modelling applications for Maths at Sheffield. Overall applications Poisson process with variable rate $\lambda(t)$; candidate applying at time t for BSc with probability $p(t)$ (otherwise MMath) independently of others. Marking gives BSc and MMath applications as independent Poisson processes with variable rates $\lambda(t)p(t)$ and $\lambda(t)(1-p(t))$ respectively.

Example 36. *Conditioning on number of events*

Let $\lambda(t) = a + b\cos(2\pi t)$ with $a > b > 0$ (so that $\lambda(t) > 0$ everywhere). Given that there were three occurrences in $(0, 1]$, what is the probability that they all occurred in $(1/4, 3/4]$?

The times of the three occurrences will be a random sample from the distribution with pdf

$$f(u) = \frac{a + b\cos(2\pi u)}{\int_0^1 (a + b\cos(2\pi v)) \; dv} = \frac{a + b\cos(2\pi u)}{a},$$

for $u \in (0, 1)$. So the probability that a given occurrence is in $(1/4, 3/4]$ is

$$\int_{1/4}^{3/4} \left(1 + \frac{b}{a}\cos(2\pi u)\right) \; du = \left[u + \frac{b}{2\pi a}\sin(2\pi u)\right]_{1/4}^{3/4} = \frac{1}{2} - \frac{b}{\pi a},$$

and the probability that all three are will, by independence, be $(\frac{1}{2} - \frac{b}{\pi a})^3$.

Example 37. *Distance to nearest tree*

The positions of trees in an infinite forest are modelled by a two-dimensional Poisson process with density λ.

(a). Given there are 3 trees within 10m of a given point, find the probability none are within 5m;

(b). Find the pdf of the distance R from a chosen point to the nearest tree.

Let A be a disc of radius 10m, and B be a disc of radius 5m, both centred on our given point. Then $N(A) \sim Po(100\pi\lambda)$ and $N(B) \sim Po(25\pi\lambda)$. Given $N(A) = 3$, the distribution of those three points will be that of a random sample of 3 points from the uniform distribution on A. Each is then in B with probability $|B|/|A| = \frac{25\pi\lambda}{100\pi\lambda} = 1/4$. Hence the probability there are none in B is $(3/4)^3 = 27/64$.

The distribution function of R is given for $r \geq 0$ by $P(R \leq r)$, which is the probability that a disc of radius r centred on our point contains at least one tree. The number of trees N_r in this disc is Poisson with parameter $\lambda\pi r^2$, so this probability is $1 - P(N(r) = 0) = 1 - e^{-\lambda\pi r^2}$. Differentiate to get the pdf:

$$f_R(r) = 2\lambda\pi r e^{-\lambda\pi r^2},$$

for $r \geq 0$.