1. a) $v_n = P(\text{Renewal on } n^{th} \text{ arrival})$

$v_1 = 0$ since need at least two arrivals

$v_n = p^2 \quad n \geq 2$ since require

$n-1^{th}$ arrival EU w.p. p

n^{th} " " " p

So $V(s) = \sum_{n=2}^{\infty} p^2 s^n$

$$= \frac{p^2 s^2}{1-s} \quad |s| < 1$$

b) $u_n = P(\text{Renewal on } t+n^{th} \text{ arrival} \mid \text{Renewal on } t^{th})$

$u_0 = 1$ (by convention)

$u_1 = p$ (if $t+1^{th}$ also EU)

$u_n = p^2 \quad n \geq 2$ (as for v_n)

Hence $U(s) = 1 + ps + \sum_{n=2}^{\infty} p^2 s^n$

$$= 1 + ps + \frac{p^2 s^2}{1-s} = \frac{1 + (p-1)s + (p^2-p)s^2}{1-s}$$

c) $B(s) = \dfrac{V(s)}{U(s)} = \dfrac{p^2 s^2}{(1+ps)(1-s) + p^2 s^2}$

$$= \frac{p^2 s^2}{1 + ps - s - ps^2 + p^2 s^2}$$

$$= \frac{p^2 s^2}{1 + s(p-1) + s^2 (p^2 - p)}$$

$$B'(s) = \frac{2p^2 s\left[1 + s(p-1) + s^2(p^2-p)\right] - p^2 s^2 \left[(p-1) + 2s(p^2-p)\right]}{\left(1 + s(p-1) + s^2(p^2-p)\right)^2}$$

$$\Rightarrow B'(1) = \frac{2p^2\left[1 + p - 1 + p^2 - p\right] - p^2\left[p - 1 + 2p^2 - 2p\right]}{\left(1 + p - 1 + p^2 - p\right)^2}$$

$$= \frac{2p^2 \cdot p^2 - p^2\left[2p^2 - p - 1\right]}{(p^2)^2}$$

$$= \frac{1 + p}{p^2}$$

d) $$F(s) = 1 - \frac{1}{U(s)} = 1 - \frac{1-s}{1+(p-1)s+(p^2-p)s^2}$$

$$= \frac{1+(p-1)s+(p^2-p)s^2 - 1 + s}{1+(p-1)s+(p^2-p)s^2}$$

$$= \frac{ps + (p^2-p)s^2}{1+(p-1)s+(p^2-p)s^2}$$

$$F'(s) = \frac{\left\{ \left[p + 2s(p^2-p)\right]\left[1+(p-1)s+(p^2-p)s^2\right] - \left[ps + (p^2-p)s^2\right]\left[(p-1)+2s(p^2-p)\right] \right\}}{\left[1+(p-1)s + (p^2-p)s^2\right]^2}$$

So $\mathbb{E}[T_i] = F'(1) = \dfrac{\left\{ (p+2(p^2-p))(1+p-1+p^2-p) - (p+p^2-p)((p-1)+2(p^2-p)) \right\}}{p^4}$

$$= \frac{\left\{ p^2(2p^2-p) - p^2(2p^2-p-1) \right\}}{p^4}$$

$$= \frac{1}{p^2}$$

$\Rightarrow$ Time until m^{th} renewal $= \mathbb{E}[D] + (m-1)\,\mathbb{E}[T_i]$

$$= \frac{1+p}{p^2} + \frac{(m-1)}{p^2}$$

a) Let S_i be state of chain at beginning of day i. Then transition matrix

$$P_S = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/2 & 0 & 0 & 1/4 \\ 1/4 & 1/4 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$$

b) Let e_i be expected number of days until restocked given in state i

$$e_1 = 1 + 1/2\, e_1$$

$$e_2 = 1 + 1/4\, e_1 + 1/2\, e_2$$

$$e_3 = 1 + 1/4\, e_1 + 1/4\, e_2 + 1/2\, e_3$$

$$e_4 = 1 + 1/4\, e_2 + 1/4\, e_3 + 1/2\, e_4$$

so $e_1 = 2$

$$e_2 = 2\left[1 + \tfrac{1}{4}e_1\right] = 3$$

$$e_3 = 2\left[1 + \tfrac{1}{4}e_1 + \tfrac{1}{4}e_2\right] = \frac{9}{2}$$

$$e_4 = 2\left[1 + \tfrac{1}{4}e_2 + \tfrac{1}{4}e_3\right] = \frac{23}{4}$$

c) Let p_i be prob. in state 1 before state 4 given currently in state i

$$p_1 = 1 \quad (1)$$

$$p_2 = \tfrac{1}{4}p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{4}p_5 \quad (2)$$

$$p_3 = \tfrac{1}{4}p_1 + \tfrac{1}{4}p_2 + \tfrac{1}{2}p_3 \quad (3)$$

$$p_4 = 0 \quad (4)$$

$$p_5 = \tfrac{1}{4}p_3 + \tfrac{1}{4}p_4 + \tfrac{1}{2}p_5 \quad (5)$$

$\Rightarrow$ From (2), (3) + (5)

$$p_2 = \frac{1}{2}\left[1 + p_5\right]$$

$$p_3 = \frac{1}{2}\left[1 + p_2\right]$$

$$p_5 = \frac{1}{2}\left[p_3\right]$$

$$\Rightarrow \quad p_5 = \frac{1}{4}\left[1 + p_2\right]$$

$$p_2 = \frac{1}{2}\left[1 + \frac{1}{4} + p_2/4\right]$$

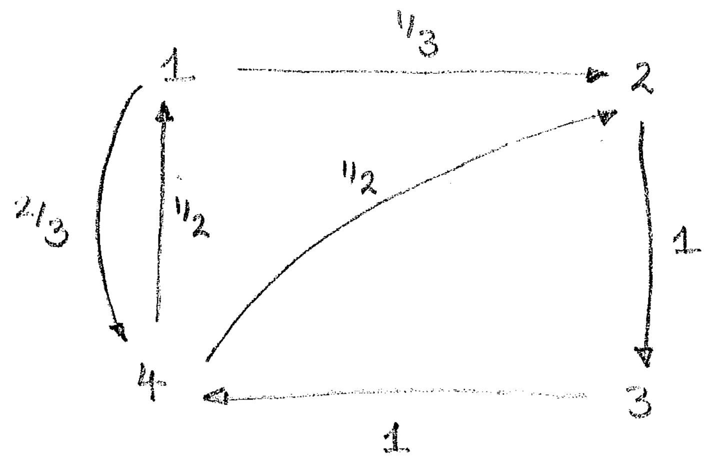
$$= \frac{5}{8} + \frac{p_2}{8}$$

$$\Rightarrow \quad p_2 = 5/7$$

$$p_3 = \frac{1}{2}\left[1 + 5/7\right]$$

$$= 6/7$$

$$p_5 = 3/7$$



a) i) Stationary distribution solves $\pi P = \pi$

$$\pi_1 = \tfrac{1}{2}\pi_4 \qquad (1)$$

$$\pi_2 = \tfrac{1}{3}\pi_1 + \tfrac{1}{2}\pi_4 \qquad (2)$$

$$\pi_3 = \pi_2 \qquad (3)$$

$$\pi_4 = \tfrac{2}{3}\pi_1 + \pi_3 \qquad (4)$$

From (2)

$$\pi_2 = \tfrac{1}{6}\pi_4 + \tfrac{1}{2}\pi_4$$

$$= \tfrac{4}{6}\pi_4$$

So sol.^{n}

$$\underline{\pi} = \left(\frac{1}{2}\pi_4, \frac{4}{6}\pi_4, \frac{4}{6}\pi_4, \pi_4\right)$$

As $\sum_i \pi_i = 1$ we have

$$\pi_4 = \frac{6}{17}$$

$\Rightarrow$ Unique equilibrium is

$$\pi = \left(\frac{3}{17}, \frac{4}{17}, \frac{4}{17}, \frac{6}{17} \right)$$

ii) • Chain irreducible since a possible path is

$$1 \to 2 \to 3 \to 4 \to 1$$

• Aperiodic since two paths are

$$2 \to 3 \to 4 \to 2$$

$$2 \to 3 \to 4 \to 1 \to 2$$

and hcf $(3,4) = 1$ so state 2 is aperiodic and so by solidarity so are all states

• Positive recurrent since any irreducible chain with finite state space is positive recurrent

- As irreducible, aperiodic and positive recurrent,

by Th. 7

$$P(X_n = i) \longrightarrow \pi_i \quad \text{as} \quad n \to \infty$$

and so

$$P(X_n = 1) \longrightarrow 3/17$$

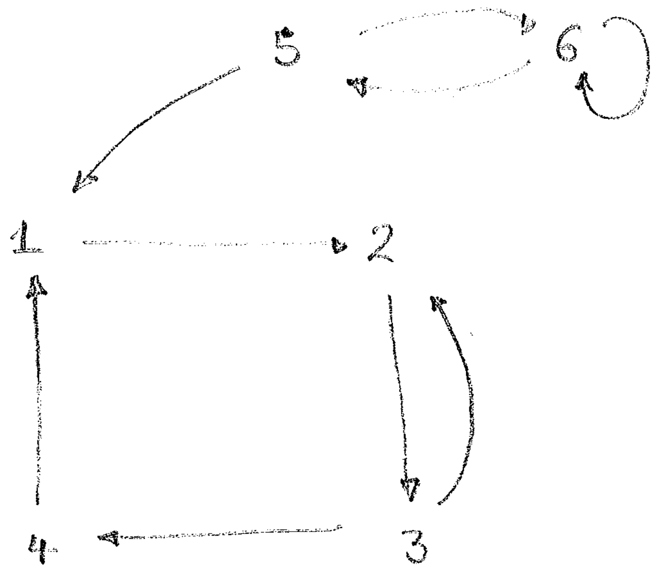
$$P(X_n = 2) \longrightarrow 4/17$$

$$P(X_n = 3) \longrightarrow 4/17$$

$$P(X_n = 4) \longrightarrow 6/17$$

as $n \to \infty$

b)



i) Communicating classes are

$\{1, 2, 3, 4\}$ closed

$\{5, 6\}$ open

ii) • States $\{5, 6\}$ are aperiodic since path is $6 \to 6$ and solidarity

• States $\{1, 2, 3, 4\}$ have period 2 since e.g. returns to 2 are either

$2 \to 3 \to 2$

$2 \to 3 \to 4 \to 1 \to 4$

and solidarity

iii) We will prove by induction

- True for $k=0$ since

$$\pi^{(0)} = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$$

- Suppose true for k i.e.

$$\pi^{(2k)} = (0 \quad p_k \quad 0 \quad 1-p_k \quad 0 \quad 0)$$

Then

$$\pi^{(2k+1)} = (1-p_k \quad 0 \quad p_k \quad 0 \quad 0 \quad 0)$$

$$\pi^{(2k+2)} = \left(0 \quad (1-p_k)+\frac{p_k}{2} \quad 0 \quad \frac{p_k}{2} \quad 0 \quad 0\right)$$

where $p_k = \frac{1}{3}\left[2+\left(-\frac{1}{2}\right)^k\right]$

i.e. $p_{k+1} = 1 - \frac{1}{2}\cdot\frac{1}{3}\left[2+\left(-\frac{1}{2}\right)^k\right]$

$$= \frac{1}{3}\left[3 - \frac{1}{2}\left[2+\left(-\frac{1}{2}\right)^k\right]\right]$$

$$= \frac{1}{3}\left[2+\left(-\frac{1}{2}\right)^{k+1}\right]$$

$\Rightarrow$ True for $k+1$

So by induction true for all k

$\blacksquare$

a) Poisson with parameter

$$\frac{1}{100}\int_0^{12}(72t-6t^2)\ dt = \frac{1}{100}\left[36t^2-2t^3\right]_0^{12}$$

$$= 17.28$$

so $Po(17.28)$

b) The number of claims in 1^{st} two hours is Poisson with parameter

$$\mu = \frac{1}{100}\int_0^{2}(72t-6t^2) = \left[36t-2t^3\right]_0^{2}$$

$$= 1.28$$

Hence

$$P(\text{Four claims in } 1^{st} \text{ two hours}) = \frac{\mu^4 e^{-\mu}}{4!}$$

$$\approx 0.031$$

c) The prob. a claim known to arrive in $(0,2]$ hours arrives in $(0,1]$ is

$$\frac{\int_0^1 \lambda(t)\, dt}{\int_0^2 \lambda(t)\, dt} = \frac{0 \cdot 34}{1 \cdot 28} = \frac{17}{64}$$

Given that two arrive in $(0,2]$, the number arriving in $(0,1]$ has

$$\mathrm{Bin}\left(2, \tfrac{17}{64}\right)$$

So Probability there are none is

$$\left(1 - \frac{17}{64}\right)^2 = \left(\frac{47}{64}\right)^2$$

$$\approx 0 \cdot 54$$

d) i) Number of insurance (lorry) claims in $(0,2]$

is $Po(2/5)$ so prob. of 1 lorry claim

is

$$\frac{2}{5} e^{-2/5} \approx 0.268$$

As processes independent

$$P(4 \text{ car and } 1 \text{ lorry in } (0,2])$$

$$= P(4 \text{ car}) P(1 \text{ lorry})$$

$$\approx 0.0083$$

ii) Let C_t and L_t be number of Car and Lorry claims up to time t.
By independence and additivity of Poisson distributions, $C_t + L_t$ has a Poisson dist. with parameter given by sum of individual parameters.

$$C_t \sim Po\left(\frac{1}{100}(36t^2 - 2t^3)\right)$$

$$L_t \sim Po(t/5)$$

so required dist is

$$Po\left(\frac{1}{100}(36t^2 - 2t^3) + \frac{t}{5}\right)$$