MAS275

2018 Exam Solutions

1. a) $v_1 = v_2 = 0$ since has only worn 1/2 shirts by that point

$v_n = (1/3)^2 \quad n \geq 3$

since needs shirt on day $n-1$ and n to be the same as that at $n-2$

$v_0 = 0$ since by def.^{n} no renewal at time 0

$$\Rightarrow \quad V(s) = 1/9 \sum_{n=3}^{\infty} s^n = \frac{s^3}{9(1-s)}.$$

b) If renewal occurred at time t then another renewal occurs at time $t+1$ iff choose same shirt again

$$\Rightarrow \quad u_1 = 1/3$$

Similarly, renewal at time $t+2$ iff shirt on day $t+1$ and $t+2$ are same as previous three

$$\Rightarrow \quad u_2 = (1/3)^2 = 1/9$$

For $n \geq 3$, renewal occurs at time $t+n$ iff shirt at $t+n$, $t+n-1$ is same as at time $t+n-2$

$$\Rightarrow \quad u_n = \left(1/3\right)^2 = 1/9 \qquad n \geq 3$$

By convention $u_0 = 1$, so

$$U(s) = 1 + 1/3\, s + \left(1/3\right)^2 \sum_{n=2}^{\infty} s^n$$

$$= 1 + 1/3\, s + 1/9 \, \frac{s^2}{1-s}$$

$$= \frac{(9+3s)(1-s) + s^2}{9(1-s)}$$

c) By given result,

$$B(s) = \frac{V(s)}{U(s)} = \frac{s^3}{(9+3s)(1-s)+s^2}$$

$$= \frac{s^3}{9-6s-2s^2}$$

so

$$B'(s) = \frac{3s^2\left[(9-6s-2s^2)\right] - s^3\left[-6-4s\right]}{(9-6s-2s^2)^2}$$

Evaluating at $s = 1$ gives

$$B'(1) = \frac{3 + 10}{1^2} = 13$$

so $\mathbb{E}[\text{Days}] = 13$

d) Use either

$$\lim_{n \to \infty} u_n = 1/9$$

and so by renewal theorem, mean inter-renewal time is 9

<u>Or</u>

$$F(s) = 1 - \frac{1}{U(s)} = 1 - \frac{9(1-s)}{(9+3s)(1-s)+s^2}$$

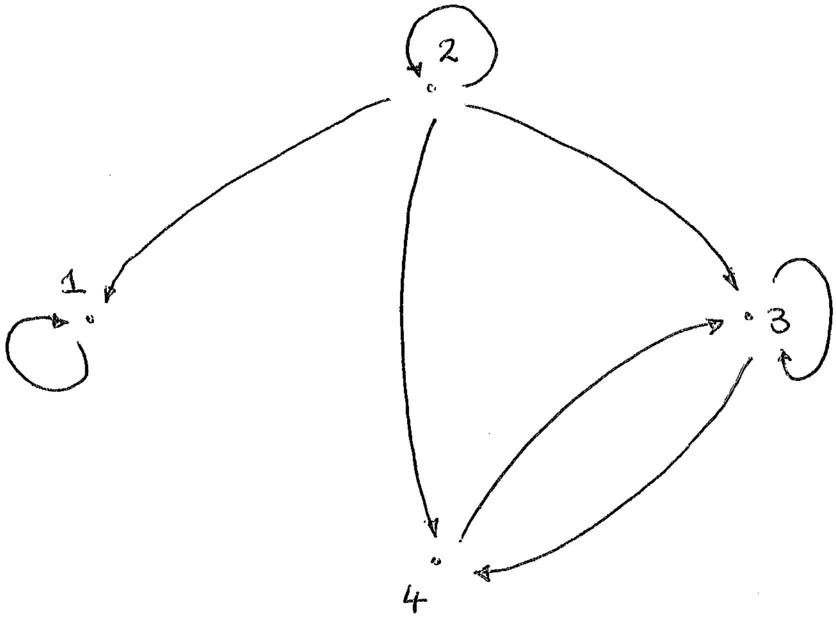
$$F'(s) = -\left[\frac{-9\{(9+3s)(1-s)+s^2\} - 9(1-s)\ldots}{[(9+3s)(1-s)+s^2]^2} \right]$$

$$\Rightarrow \quad F'(1) = 9$$

Hence, expected time until 2^{nd} wash is

$$\mathbb{E}[D + T_1] = \mathbb{E}[D] + \mathbb{E}[T_1] = 13 + 9 = 22 \text{ days}$$

2a) Begin by drawing possible transitions



Clear that communicating classes are

$\{1\}$ - recurrent

$\{2\}$ - transient

$\{3,4\}$ - recurrent

b) Solve $\pi P = \pi$

$$\pi_1 = \pi_1 + \tfrac{1}{4}\pi_2 \quad \Rightarrow \quad \pi_2 = 0, \; \pi_1 = \pi_1$$

$$\pi_2 = \tfrac{1}{4}\pi_2$$

$$\pi_3 = \tfrac{1}{4}\pi_2 + \tfrac{1}{3}\pi_3 + \pi_4 \quad \Rightarrow \quad \pi_4 = \tfrac{2}{3}\pi_3$$

$$\pi_4 = \tfrac{1}{4}\pi_2 + \tfrac{2}{3}\pi_3$$

$\Rightarrow$ Sol.^{n} $\quad (\pi_1, 0, \pi_3, \tfrac{2}{3}\pi_3)$

Require add to 1 hence

$$\pi_1 + \tfrac{5}{3}\pi_3 = 1$$

$$\Rightarrow \quad \pi_1 = 1 - \tfrac{5}{3}\pi_3 \quad \text{or} \quad \pi_3 = \frac{3}{5}(1-\pi_1)$$

so sol.^{n} (can be written as either)

$$\left(1-\tfrac{5}{3}\alpha,\ 0,\ \alpha,\ \tfrac{2}{3}\alpha\right) \quad \underline{\text{or}} \quad \left(\beta,\ 0,\ \tfrac{3}{5}(1-\beta),\ \tfrac{2}{5}(1-\beta)\right)$$

for $\tfrac{5}{3}\alpha \leq 1$ i.e. $0 \leq \alpha \leq \tfrac{3}{5}$ for $0 \leq \beta \leq 1$

c) Assume $\pi_0 = (0 \quad 1 \quad 0 \quad 0)$

Then true $n=0$ i.e.

$$P(Y_0 = 2) = 1 = (\tfrac{1}{4})^0$$

$$P(Y_0 = 1) = 0 = \tfrac{1}{3}\left(1-(\tfrac{1}{4})^0\right)$$

Suppose true $n=k$ then

$$P(Y_{n+1} = 2) = \tfrac{1}{4}P(Y_n = 2) = (\tfrac{1}{4})^{n+1}$$

$$P(Y_{n+1} = 1) = P(Y_n = 1) + \tfrac{1}{4}P(Y_n=2) = \frac{1}{3}\left[1-(\tfrac{1}{4})^n\right] + (\tfrac{1}{4})^{n+1}$$

$$= \frac{1}{3}\left[1-\left(\frac{1}{4}\right)^{n+1}(4-3)\right]$$

$$= \frac{1}{3}\left[1-\left(\frac{1}{4}\right)^{n+1}\right]$$

so true for $n=k+1$

$\blacksquare$

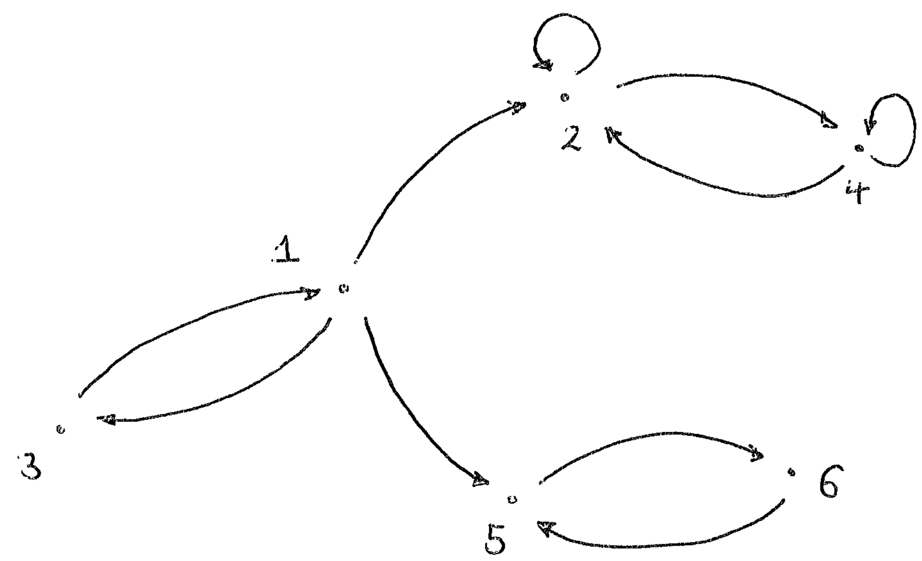
ii) As $n \to \infty$, from part i)

$$P(Y_n = 1) \to 1/3$$

$\Rightarrow$ Stationary dist.

$$\left(1/3,\ 0,\ 2/5,\ 4/15\right)$$

3 a) First draw the possible transitions



Clear that communicating classes

$\{1, 3\}$

$\{2, 4\}$

$\{5, 6\}$

b) $\{1,3\}$ - transient as can leave but not return
period 2 as to return to e.g. 1
only possibility $1 \to 3 \to 1$
(and solidarity means equal period)

$\{2,4\}$ - ~~closed~~ recurrent as can't leave
aperiodic as can go $2 \to 2$ and $4 \to 4$

$\{5,6\}$ - recurrent with period 2 as path $5 \to 6 \to 5$

c) Stationary dist. equations $\pi P = \pi$

$$\pi_1 = \pi_3 \quad ①$$

$$\pi_2 = \tfrac{1}{3}\pi_1 + \tfrac{1}{2}\pi_2 + \tfrac{1}{2}\pi_4 \quad ②$$

$$\pi_3 = \tfrac{1}{3}\pi_1 \quad ③$$

$$\pi_4 = \tfrac{1}{2}\pi_2 + \tfrac{1}{2}\pi_4 \quad ④$$

$$\pi_5 = \tfrac{1}{3}\pi_1 + \pi_6 \quad ⑤$$

$$\pi_6 = \pi_5 \quad ⑥$$

Clear from above $\pi_1 = \tfrac{1}{3}\pi_1 \Rightarrow \pi_1 = \pi_3 = 0$

② $\tfrac{1}{2}\pi_2 = \tfrac{1}{2}\pi_4 \Rightarrow \pi_2 = \pi_4$

⑤ $\pi_5 = \pi_6$

and rest give no more info.

Hence stationary dist

$$\left(0, \frac{\alpha}{2}, 0, \frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right) \qquad 0 \leq \alpha \leq 1$$

4 a) Transition matrix

$$
\begin{array}{c|cccccc}
 & H & A & B & C & D & E \\
\hline
H & 1/2 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 \\
A & 1/3 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\
B & 1/3 & 1/6 & 0 & 1/6 & 1/6 & 1/6 \\
C & 1/3 & 1/6 & 1/6 & 0 & 1/6 & 1/6 \\
D & 1/3 & 1/6 & 1/6 & 1/6 & 0 & 1/6 \\
E & 1/3 & 1/6 & 1/6 & 1/6 & 1/6 & 0
\end{array}
$$

b) Clearly irreducible as possible path is

$$H \to A \to B \to C \to D \to E \to H$$

Aperiodic as can have transition $H \to H$ and solidarity gives all states same period.

c) We ~~have~~ know $\pi_A = \pi_B = \pi_C = \pi_D = \pi_E$ in equilibrium dist. due to symmetry i.e. ~~we~~ could relabel any flower and still have same process

d) Require (using symmetry) - $\pi_A = \pi_B = \dots = \pi_E$

$$\pi_H = \tfrac{1}{2}\pi_H + \tfrac{5}{3}\pi_A$$

$$\pi_A = \tfrac{1}{10}\pi_H + \tfrac{4}{6}\pi_A \qquad (\dagger)$$

All other equations are same as $(\dagger)$. Hence

$$\tfrac{1}{2}\pi_H = \tfrac{5}{3}\pi_A \Rightarrow \pi_H = \frac{10}{3}\pi_A$$

so equilibrium

$$\left(\underset{H}{\frac{10}{3}\pi_A}, \underset{A}{\pi_A}, \underset{B}{\pi_A}, \underset{C}{\pi_A}, \underset{D}{\pi_A}, \underset{E}{\pi_A}\right)$$

and

$$\frac{10 + (3 \times 5)}{3}\pi_A = 1$$

$$\Rightarrow \quad \pi_A = \frac{3}{25}$$

so equilibrium

$$\left(\frac{10}{25}, \frac{3}{25}, \frac{3}{25}, \frac{3}{25}, \frac{3}{25}, \frac{3}{25}\right)$$

e) Let p_i be prob. bee visits hive before flower C given start at i.

Then $p_H = 1$, $p_C = 0$ and by symmetry $p_A = p_B = p_D = p_E$

$$p_A = \tfrac{1}{2} p_H + \tfrac{4}{10} p_A + \tfrac{1}{10} p_C$$

$$\Rightarrow \frac{6}{10} p_A = \tfrac{1}{2}$$

$$\Rightarrow p_A = \frac{5}{6}$$

f) Let e_i be expected number of steps until bee reaches flower C given start at i. Then again by symmetry $e_A = e_B = e_D = e_E$

$$e_H = 1 + \tfrac{1}{2} e_H + \tfrac{4}{10} e_A \Rightarrow \tfrac{1}{2} e_H = 1 + \tfrac{2}{5} e_A$$

$$e_A = 1 + \tfrac{1}{3} e_H + \tfrac{3}{6} e_A \Rightarrow \tfrac{1}{2} e_A = 1 + \tfrac{1}{3} e_H$$

so

$$\tfrac{1}{2} e_H = 1 + \tfrac{2}{5}\left(2 + \tfrac{2}{3} e_H\right)$$

$$\Rightarrow \tfrac{1}{2} e_H = \tfrac{9}{5} + \tfrac{4}{15} e_H$$

$$\Rightarrow \frac{15-8}{30} e_H = \tfrac{9}{5} \Rightarrow \frac{7}{30} e_H = \frac{9}{5}$$

$$\Rightarrow e_H = \frac{9 \times 6}{7} = \tfrac{54}{7} \qquad e_A = 2 + \frac{2}{3} \cdot \frac{9.6}{7} = \frac{14 + 9.4}{7} = \tfrac{50}{7}$$

g) $\frac{25}{3}$ direct from $\pi_c = 1/\mu_c$

or, considering where could make initial move from C
$z_c = 1 + 2/3\, e_A + 1/3\, e_H$