Notation reminder

(X_n) is a Markov chain

$P = (p_{ij})$ is the transition matrix

$p_{ij}^{(n)} = (i,j)$ entry of the matrix P^n

$=$ probability of going from i to j

in n steps.

transient $\iff f_{ii} < 1$

$\iff \sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$

recurrent $\iff f_{ii} = 1$

$\iff \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$

period $= \gcd \{ n \in \mathbb{N} \; p_{ii}^{(n)} > 0 \}$

Example 16 : SRW

$$i-1 \xleftarrow{\;1-p\;} i \xrightarrow{\;p\;} i+1$$

previous work shows that

- if $p = \frac{1}{2}$ then all states are null recurrent
- if $p \neq \frac{1}{2}$ then all states are transient.

Also, each state has period 2.

<u>example 17:</u> wet/dry days

$$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$

- $f_{11}^{(1)} = 1 - \alpha$
- $f_{11}^{(2)} = \alpha \beta \quad (1 \to 2 \to 1)$

$\forall n \geq 2$, $\boxed{f_{11}^{(n)} = \alpha(1-\beta)^{n-2}\beta}$

so the generating function is

$$F(s) = (1-\alpha)s + \sum_{n=2}^{\infty} \alpha(1-\beta)^{n-2}\beta s^n$$

$$= (1-\alpha)s + \frac{\alpha\beta s^2}{1-(1-\beta)s}$$

$F(1) = 1-\alpha + \alpha = 1$

so the state "1" is recurrent

Moreover, $F'(s) = (1-\alpha)$

$$+ \frac{(1-(1-\beta)s)(2s\alpha\beta) + (1-\beta)\alpha\beta^2}{(1-(1-\beta)s)^2}$$

$$F'(1) = \cdots = \frac{\alpha+\beta}{\alpha} < \infty$$

So the state "1" is

positive recurrent

Theorem 8: For a Markov chain on a finite state space, there are some positive recurrent states, and the rest are transient.

proof:

step 1: not every state is transient.

Otherwise we would run out of places to go.

step 2: no null recurrent states.

Let i be a recurrent state, let us show that it is positive recurrent.

Let j be another state.

~~Con~~

Consider V_j = number of visits to j before returning to i.

Want to show that $E[V_j] < \infty$

q_{ij} = probability of visiting before returning to i.

$q_{j,i} = \dots$

- If ~~$q_{j,i} = 0$~~ $q_{ij} = 0$

  then $V_j = 0$ so $E[V_j] = 0$ ✓

- If $q_{ij} > 0$ then $q_{ji} > 0$

  then $P(V_j = 0) = 1 - q_{ij}$

$\forall n \geq 1 \quad P(V_j = n) = q_{ij} (1 - q_{ji})^{n-1} q_{ji}$

So $E[V_j] < \infty$, and so

mean return time to i

$$= \sum_{j \neq i} E[V_j] < \infty.$$

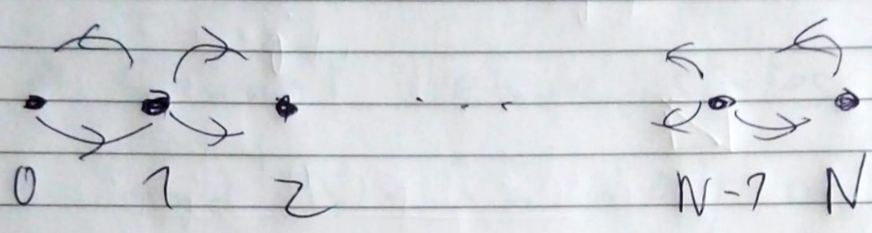
$\boxtimes$

<u>Example 18:</u> Simple random walk.

- Can go from n to $n+1$ and $n-1$ in 1 step.
- Can go from a to b, $\forall a, b$.

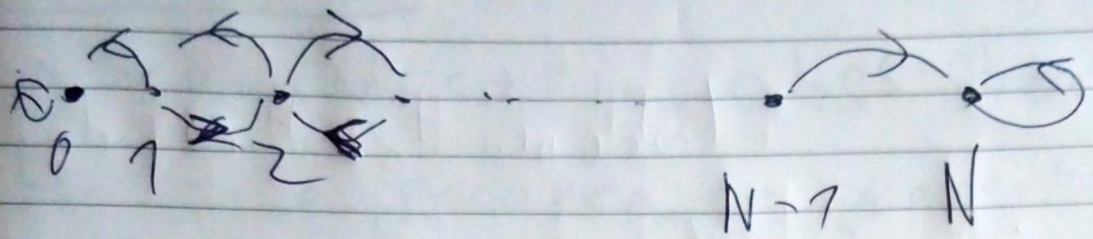
So SRW is irreducible.

· Ehrenfest model



Also irreducible

Example 1·9 : Gambler's ruin



- Cannot leave states 0 and N. So $\{0\}$ and $\{N\}$ are communicating classes.
- All the other states communicate. So, $\{1, 2, 3, \ldots, N-1\}$ is also a class.

Theorem 9: All states within a communicating class share the same recurrence properties.