Periodic XXX spin chains: What is this under the carpet?

Sébastien Leurent

IMB, Dijon

April 13, 2018

Disclaimer

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Discussions and preliminary work with D. Volin
Overlaps [Jiang, Zhang, '17]

XXX-type spin chains

1. Two versions of a Fairy tale
   - Spin chain
   - Coordinate Bethe ansatz
   - Hirota equation and Wronskian determinants
   - A look under the carpet
2. Solving Bethe equations
   - Multiplets
   - Each solution applies to several models
   - Resolution of Bethe equations
3. Counting solutions

"Coordinate Bethe Ansatz" for $XXX_{1/2}$ Heisenberg spin chain

Eigenstates of $H = -\sum_{i=1}^{L} \vec{\sigma_i} \cdot \vec{\sigma_{i+1}}$ $\quad \mathcal{H} = (\mathbb{C}^2)^{\otimes L}; \quad \vec{\sigma}_{L+1} = \vec{\sigma}_1$

$= L - 2\sum_{i=1}^{L} \mathcal{P}_{i,i+1}$

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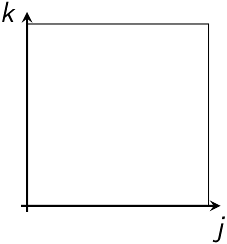

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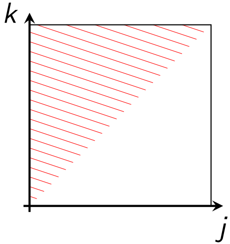

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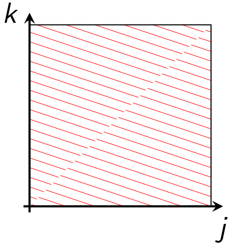

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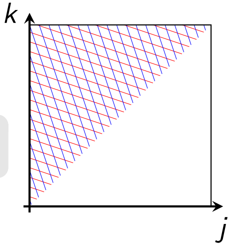

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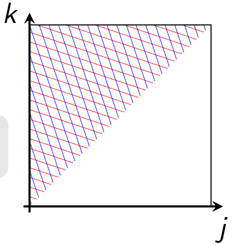

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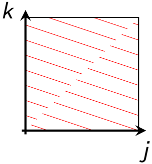
- n excitations:

$$|\psi\rangle = \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant L} \; \sum_{\sigma\in\mathfrak{S}_n} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} j_k} \, |\{j_1, j_2, \ldots, j_n\}\rangle$$

  is an eigenstate if
  - $\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} (1 + e^{i(p_{\sigma(j)}+p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}})$
  - $\forall j, \; e^{iLp_j} = \prod_{k\neq j} S(p_j, p_k)$ where $S(p,p') =$

The corresponding eigenvalue is

$$E = -L + \sum_k 8 \sin^2 \frac{p_k}{2}$$



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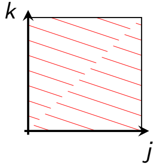
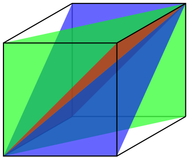
- $\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} \left(1 + e^{i(p_{\sigma(j)}+p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right)$
- $\forall j, e^{i\,L\,p_j} = \prod_{k\neq j} S(p_j, p_k)$ where $S(p,p') \equiv -\frac{1+e^{i(p+p')}-2e^{i\,p}}{1+e^{i(p+p')}-2e^{i\,p'}}$

The corresponding eigenvalue is

$$E = -L + \sum_k 8\sin^2\frac{p_k}{2}$$





"Coordinate Bethe Ansatz" for $XXX_{1/2}$ Heisenberg spin chain

- Two excitations: $|\psi\rangle = \sum_{j<k} (\mathcal{A}\, e^{i(p_1 j + p_2 k)} + \tilde{\mathcal{A}}\, e^{i(p_1 k + p_2 j)}) \,|\{j,k\}\rangle$

  where $e^{i L p_2} = e^{-i L p_1} = S = \frac{\tilde{\mathcal{A}}}{\mathcal{A}}$, with $S = -\frac{1+e^{i(p_1+p_2)}-2e^{ip_2}}{1+e^{i(p_1+p_2)}-2e^{ip_1}}$
- n excitations:

$$|\psi\rangle = \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant L} \; \sum_{\sigma\in\mathfrak{S}_n} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} j_k} \,|\{j_1, j_2, \ldots, j_n\}\rangle$$

  is an eigenstate if
  - $\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} \left(1 + e^{i(p_{\sigma(j)}+p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right)$
  - $\forall j,\, e^{i L p_j} = \prod_{k\neq j} S(p_j, p_k)$ where $S(p,p') \equiv -\frac{1+e^{i(p+p')}-2e^{i\,p}}{1+e^{i(p+p')}-2e^{i\,p'}}$

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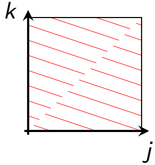

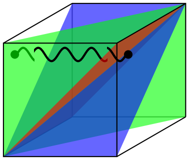

"Coordinate Bethe Ansatz" for $XXX_{1/2}$ Heisenberg spin chain

- Two excitations: $|\psi\rangle = \sum_{j<k}(\mathcal{A}\, e^{i(p_1 j+p_2 k)} + \tilde{\mathcal{A}}\, e^{i(p_1 k+p_2 j)})\, |\{j,k\}\rangle$
  where $e^{i\,L\,p_2} = e^{-i\,L\,p_1} = S = \frac{\tilde{\mathcal{A}}}{\mathcal{A}}$, with $S = -\frac{1+e^{i(p_1+p_2)}-2e^{ip_2}}{1+e^{i(p_1+p_2)}-2e^{ip_1}}$
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The corresponding eigenvalue is

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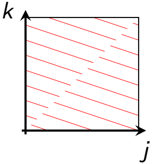

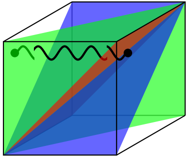

"Coordinate Bethe Ansatz" for $XXX_{1/2}$ Heisenberg spin chain

- Two excitations: $|\psi\rangle = \sum_{j<k} (\mathcal{A}\, e^{i(p_1 j + p_2 k)} + \tilde{\mathcal{A}}\, e^{i(p_1 k + p_2 j)}) \, |\{j, k\}\rangle$

  where $e^{i\,L\,p_2} = e^{-i\,L\,p_1} = S = \frac{\tilde{\mathcal{A}}}{\mathcal{A}}$, with $S = -\frac{1+e^{i(p_1+p_2)}-2e^{ip_2}}{1+e^{i(p_1+p_2)}-2e^{ip_1}}$
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The corresponding eigenvalue is

$$E = -L + \sum_k 8\sin^2 \frac{p_k}{2}$$

Bethe equations $\rightsquigarrow$ spectrum

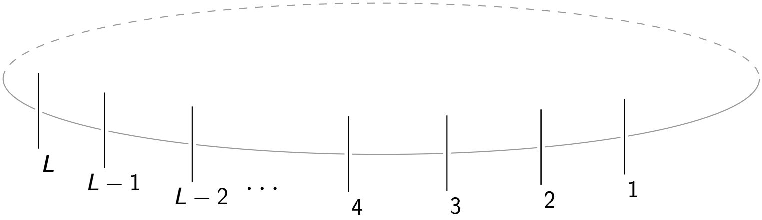
1. Two versions of a Fairy tale
   - Spin chain
   - Coordinate Bethe ansatz
   - Hirota equation and Wronskian determinants
   - A look under the carpet
2. Solving Bethe equations
   - Multiplets
   - Each solution applies to several models
   - Resolution of Bethe equations
3. Counting solutions

T-operators for $XXX_{1/2}$ Heisenberg spin chain

$H = -\sum_i \mathcal{P}_{i,i+1} = -\frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$

$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$

operator on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



partial trace: $\forall x, y \in \mathcal{H}_p$, $\langle y|\,\mathrm{tr}_a M\,|x\rangle = \sum_{z\in\mathcal{B}_a} (\langle y| \otimes \langle z|)\ M\ (|x\rangle \otimes |z\rangle)$
where $M \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_a)$, $\mathrm{tr}_a(M) \in \mathcal{L}(\mathcal{H}_p)$, $\mathcal{B}_a$=orthonorm. bas. of $\mathcal{H}_a$.

- $((u-v)\mathbb{I} + \mathcal{P}_{i,j})(u\mathbb{I} + \mathcal{P}_{i,k})(v\mathbb{I} + \mathcal{P}_{j,k})$
$= (v\mathbb{I} + \mathcal{P}_{j,k})(u\mathbb{I} + \mathcal{P}_{i,k})((u-v)\mathbb{I} + \mathcal{P}_{i,j})$

T-operators for $XXX_{1/2}$ Heisenberg spin chain

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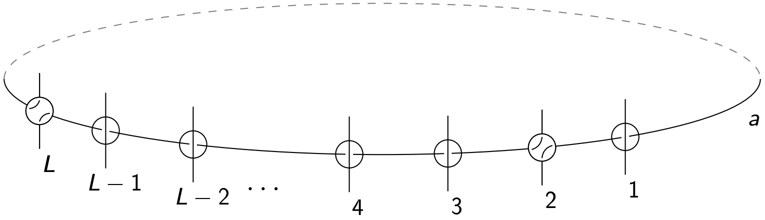
$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$$

operators on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



partial trace: $\forall x, y \in \mathcal{H}_p$, $\langle y|\, \mathrm{tr}_a M \,|x\rangle = \sum_{z\in\mathcal{B}_a} (\langle y| \otimes \langle z|)\; M\; (|x\rangle \otimes |z\rangle)$
where $M \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_a)$, $\mathrm{tr}_a(M) \in \mathcal{L}(\mathcal{H}_p)$, $\mathcal{B}_a$=orthonorm. bas. of $\mathcal{H}_a$.

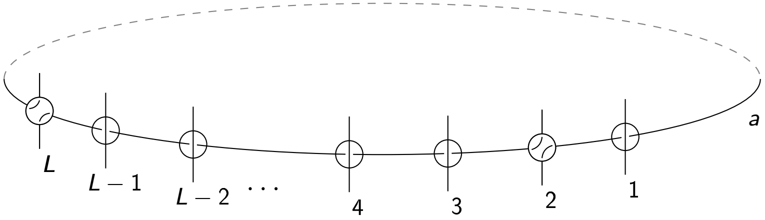
- $((u-v)\mathbb{I} + \mathcal{P}_{i,j})(u\mathbb{I} + \mathcal{P}_{i,k})(v\mathbb{I} + \mathcal{P}_{j,k})$
  $= (v\mathbb{I} + \mathcal{P}_{j,k})(u\mathbb{I} + \mathcal{P}_{i,k})((u-v)\mathbb{I} + \mathcal{P}_{i,j})$

T-operators for $XXX_{1/2}$ Heisenberg spin chain

$$H = -\sum_i \mathcal{P}_{i,i+1} = -\frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$$

$$T(u) = \mathrm{tr}_a\left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a})\right)$$

operators on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



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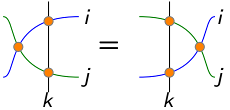
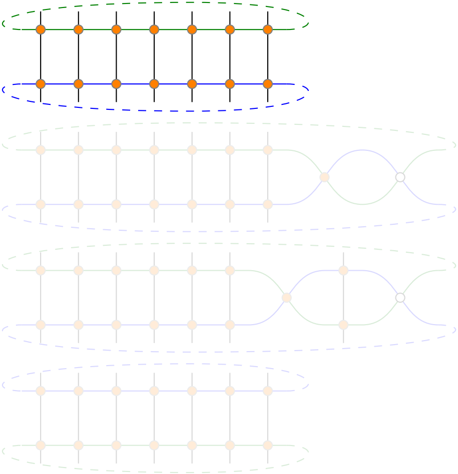


Commutation of T-operators

$T(v)T(u) =$



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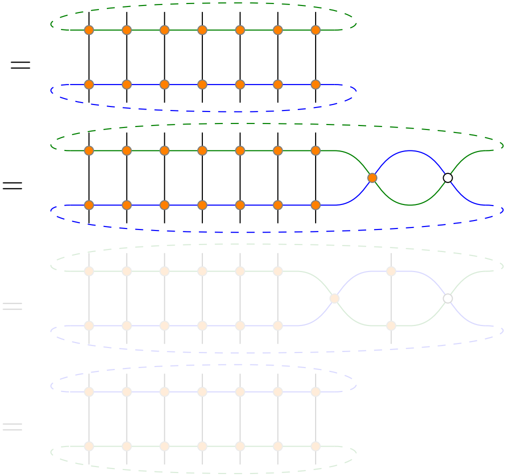
$= T(u)T(v)$

Commutation of T-operators

$T(v)T(u) =$



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Commutation of T-operators

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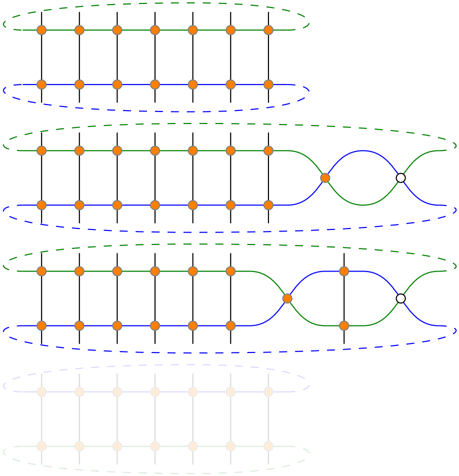

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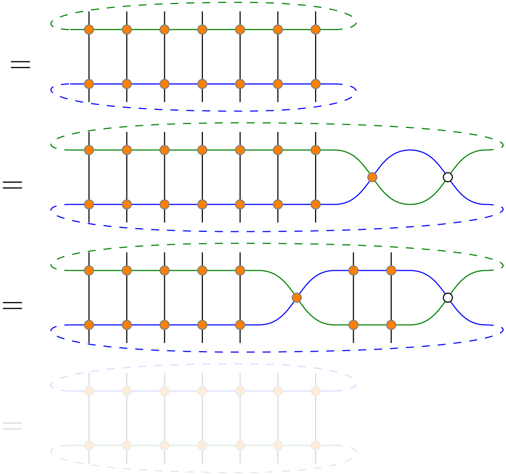

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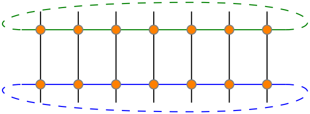

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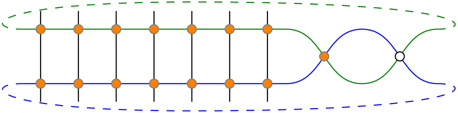

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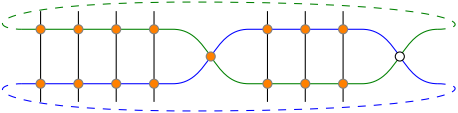

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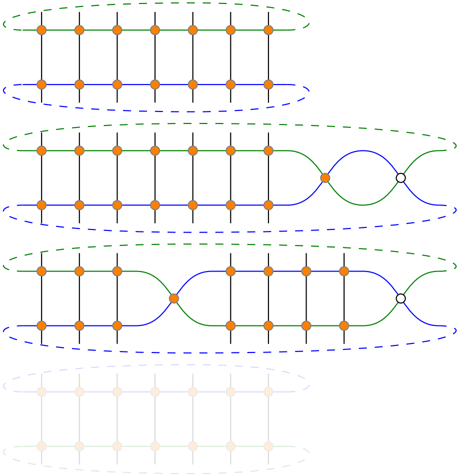
$= T(u)T(v)$

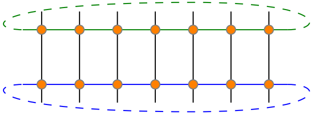
Commutation of T-operators

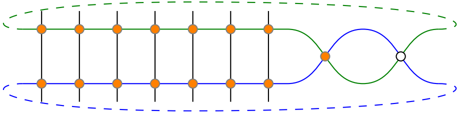
$$T(v)T(u) =$$

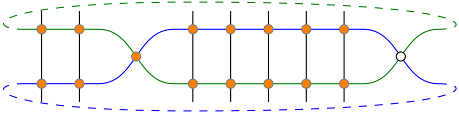


$$= \ldots = \ldots = \ldots = T(u)T(v)$$

Commutation of T-operators

$T(v)T(u) =$ 

$=$ 

$=$ 

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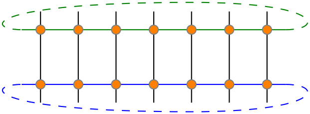
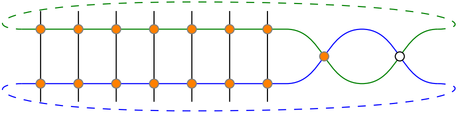
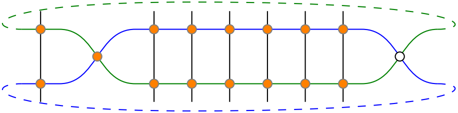
$T(v)T(u) =$ 

$=$ 

$=$ 

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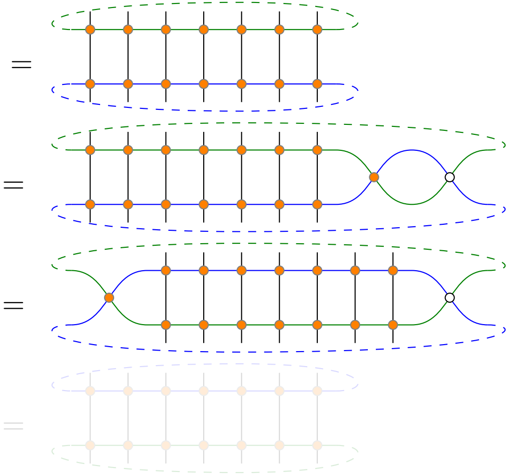

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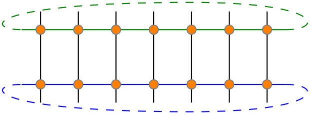
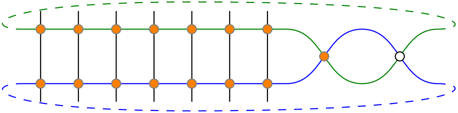
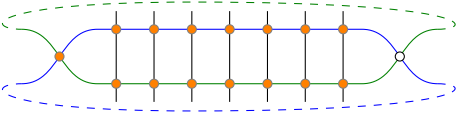
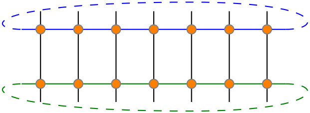
$T(v)T(u) =$ 

$=$ 

$=$ 

$=$ 

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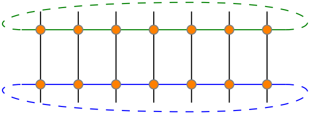
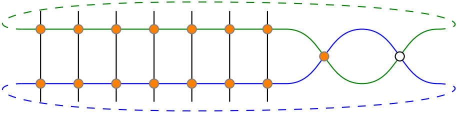
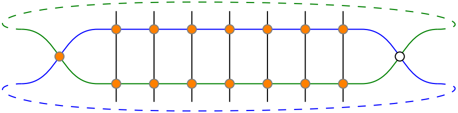
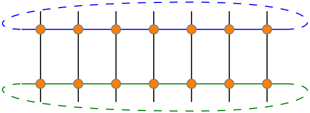
Commutation of T-operators

$T(v)T(u) =$ 

$=$ 

$=$ 

$=$ 

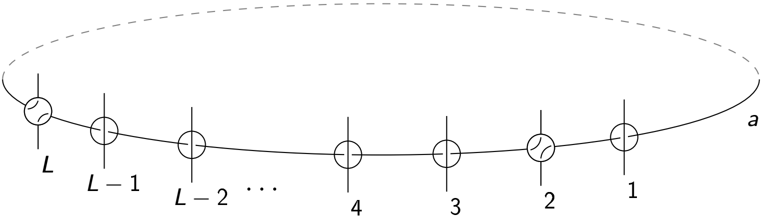
$= T(u)T(v)$

T-operators for $XXX_{1/2}$ spin chain

$$H = -\sum_i \mathcal{P}_{i,i+1} = -\frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$$

$$T(u) = \mathrm{tr}_a\left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a})\right)$$

operators on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



partial trace: $\forall x, y \in \mathcal{H}_p$, $\langle y|\,\mathrm{tr}_a M\,|x\rangle = \sum_{z\in\mathcal{B}_a}(\langle y|\otimes\langle z|)\; M\; (|x\rangle\otimes|z\rangle)$
where $M \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_a)$, $\mathrm{tr}_a(M) \in \mathcal{L}(\mathcal{H}_p)$, $\mathcal{B}_a$=orthonorm. bas. of $\mathcal{H}_a$.

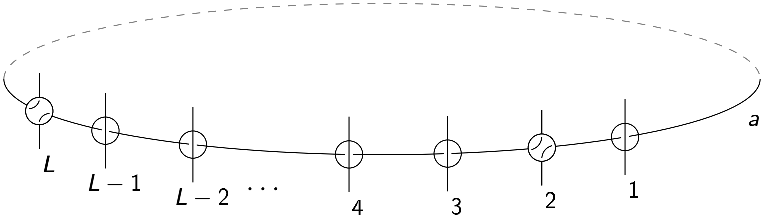
- $\left[T\ (u), T\ (v)\right] = 0$

T-operators for XXX-type spin chains

$$H = -\sum_i \mathcal{P}_{i,i+1} = -\frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$$

$$T(u) = \mathrm{tr}_a\left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a})\right)$$

operators on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



partial trace: $\forall x, y \in \mathcal{H}_p$, $\langle y|\,\mathrm{tr}_a M\,|x\rangle = \sum_{z\in\mathcal{B}_a}(\langle y| \otimes \langle z|)\ M\ (|x\rangle \otimes |z\rangle)$
where $M \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_a)$, $\mathrm{tr}_a(M) \in \mathcal{L}(\mathcal{H}_p)$, $\mathcal{B}_a$=orthonorm. bas. of $\mathcal{H}_a$.

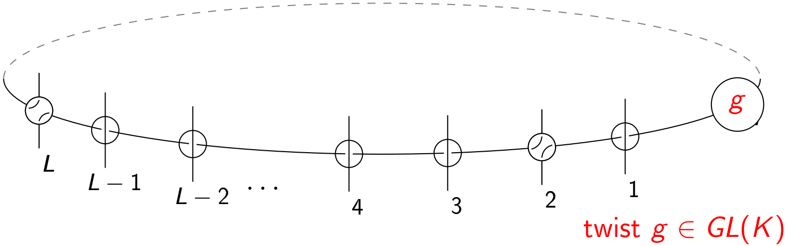
- $\left[T\ (u), T\ (v)\right] = 0$

T-operators for XXX-type spin chains

$$H = « - \sum_i \mathcal{P}_{i,i+1} » = - \frac{\mathrm{d}}{\mathrm{du}} \log T(u)\big|_{u=0}$$

$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \cdot g \right)$$

operators on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



partial trace: $\forall x, y \in \mathcal{H}_p$, $\langle y | \, \mathrm{tr}_a M \, | x \rangle = \sum_{z \in \mathcal{B}_a} (\langle y | \otimes \langle z |) \; M \; (| x \rangle \otimes | z \rangle)$
where $M \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_a)$, $\mathrm{tr}_a(M) \in \mathcal{L}(\mathcal{H}_p)$, $\mathcal{B}_a$=orthonorm. bas. of $\mathcal{H}_a$.

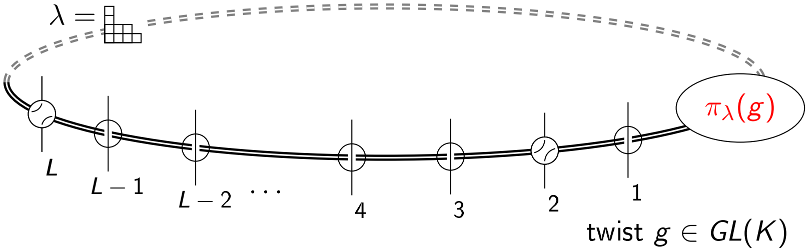
- $\left[T \;\; (u), T \;\; (v) \right] = 0$

T-operators for XXX-type spin chains

$$H = « - \sum_i \mathcal{P}_{i,i+1} » = - \frac{\mathrm{d}}{\mathrm{du}} \log T^{\square}(u)\big|_{u=0}$$

$$T^{\lambda}(u) = \mathrm{tr}_a \left((u\, \mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\, \mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\, \mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_\lambda(g) \right)$$

operators on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



Generalized permutation operator: $\mathcal{P}_{i,j} = \sum_{\alpha,\beta} e^{(i)}_{\alpha,\beta} \otimes \pi_\lambda(e^{(j)}_{\beta,\alpha})$

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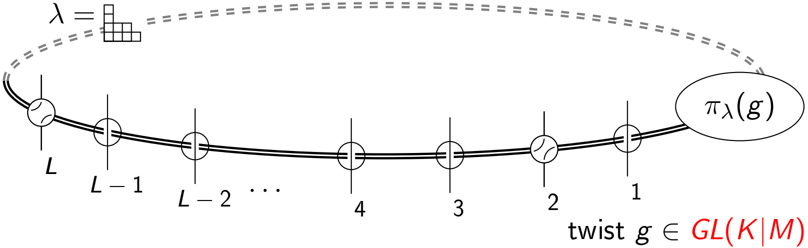

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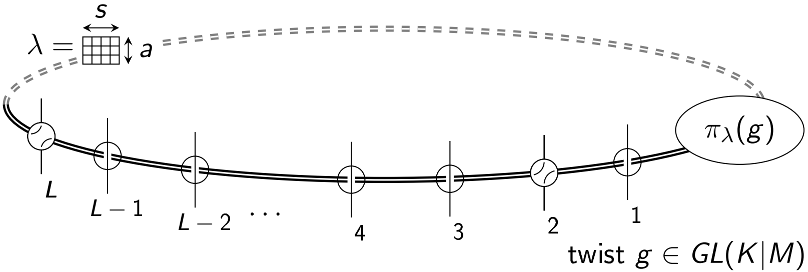

- $\left[T^{\lambda}(u), T^{\mu}(v) \right] = 0$
- For rectangular Young diagrams,

$$T^{a,s}(u+1) \cdot T^{a,s}(u) = T^{a+1,s}(u+1) \cdot T^{a-1,s}(u) + T^{a,s-1}(u+1) \cdot T^{a,s+1}(u)$$

Wronskian determinant

Generic solution of Hirota equation

("bosonic" case)

$$T^{\lambda}(u) = \frac{\det\left(x_j^{\lambda_k - k + 1} \mathcal{Q}_j(u + \lambda_k - k + 1)\right)_{1 \leqslant j,k \leqslant N}}{\Delta(x_1, \cdots, x_N)}$$

where $g = \mathrm{diag}(x_1, \cdots x_N)$; $\quad \Delta(x_1, \cdots, x_N) = \det\left(x_j^{1-k}\right)_{1 \leqslant j,k \leqslant N}$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$, ... commute among themselves and with T, and are polynomial in u.

For spin chains, their explicit expression is known.

Wronskian determinant

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For generic solutions of Hirota equations, $\mathcal{Q}$'s are solution of

$$\begin{vmatrix} \mathcal{Q}_i & x_i\mathcal{Q}_i(u+1) & x_i^2\mathcal{Q}_i(u+2) & \dots & x_i^N\mathcal{Q}_i(u+N) \\ T^{1,0}(u) & T^{1,1}(u) & T^{1,2}(u) & \dots & T^{1,N}(u) \\ T^{1,1}(u-1) & T^{1,2}(u-1) & T^{1,3}(u-1) & \dots & T^{1,N+1}(u-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{1,N-1}(u-N+1) & T^{1,N}(u-N+1) & T^{1,N+1}(u-N+1) & \dots & T^{1,2N-1}(u-N+1) \end{vmatrix} = 0$$

Q-system

$SU(N)$ case

- $\mathcal{Q}_\emptyset = 1$
- $\mathcal{Q}_{i_1,i_2,\dots,i_m} = \dfrac{\det\left(x_{i_k}^{1-l}\mathcal{Q}_{i_k}(u+1-l)\right)_{1\leqslant k,l\leqslant m}}{\Delta(x_1,\cdots,x_N)}$
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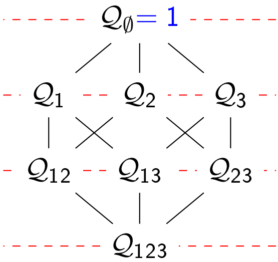
Energy

$$E = E_0 + \sum_{\substack{u \\ Q_{23\dots N}(u)=0}} \frac{-1}{u(u+1)}$$



"$\mathcal{Q}\mathcal{Q}$-relations"

- $\mathcal{Q}_{123}(u)\mathcal{Q}_1(u-1) = \dfrac{\begin{vmatrix} x_2\mathcal{Q}_{12}(u) & x_3\mathcal{Q}_{13}(u) \\ \mathcal{Q}_{12}(u-1) & \mathcal{Q}_{13}(u-1)\end{vmatrix}}{x_2 - x_3}$
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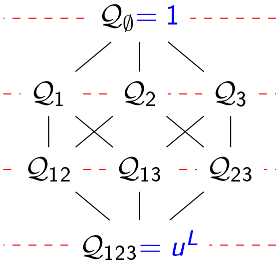

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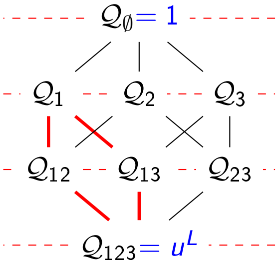

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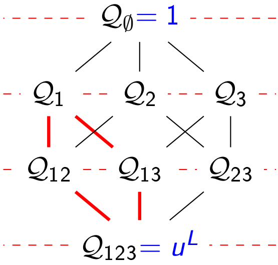

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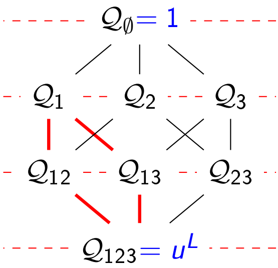

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Q-system in $SU(K|M)$ case

Pictures are for $K = 2$ and $M = 1$

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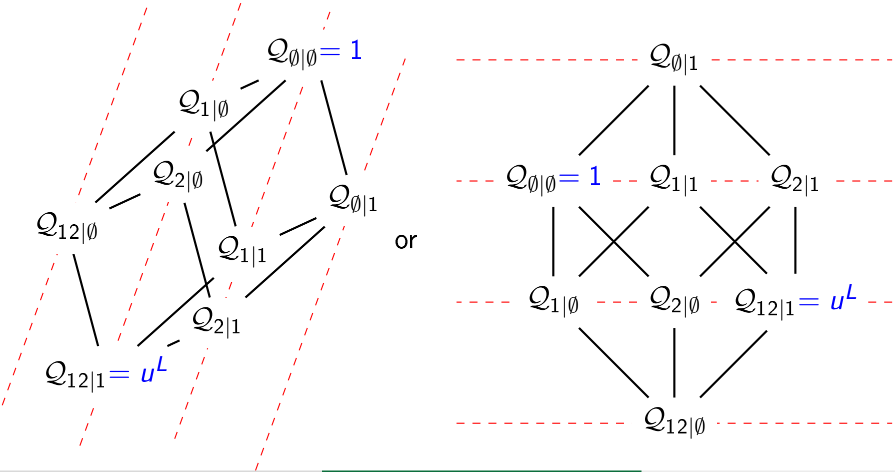

Bethe equation

- Restrict to eigenspace: operators become complexed-valued functions of u.

Bethe equations $\Leftrightarrow$ Existence of Polynomial $\mathcal{Q}$-functions such that

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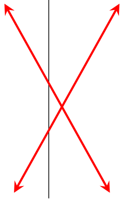
Importance of counting solutions

Coordinate Bethe Ansatz

- "Each solution" of Bethe equations provides an eigenstate
- The number of states produced this way is the dimension of the Hilbert space
- All eigenstates are produced this way



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Physicalness of solutions to Bethe equation

Reminder: Bethe equations

$$|\psi\rangle = \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant L} \; \sum_{\sigma\in\mathfrak{S}_n} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} j_k} \left|\{j_1, j_2, \ldots, j_n\}\right\rangle$$

$$\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} \left(1 + e^{i(p_{\sigma(j)}+p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right)$$

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What about the (completely symmetric) state $\sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant L} \left|\{j_1, j_2, \ldots, j_n\}\right\rangle$?

- Corresponds to $p_1 = p_2 = \cdots = 0$, and arbitrary $\mathcal{A}_\sigma$.

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$$|\psi\rangle = \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant L} \sum_{\sigma\in\mathfrak{S}_n} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} j_k} |\{j_1, j_2, \ldots, j_n\}\rangle$$

$$\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} \left(1 + e^{i(p_{\sigma(j)}+p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right)$$

$$\forall j, e^{i L p_j} = \prod_{k\neq j} S(p_j, p_k) \text{ where } S(p, p') \equiv -\frac{1+e^{i(p+p')}-2e^{i p}}{1+e^{i(p+p')}-2e^{i p'}}$$

$$E = -L + \sum_k 8 \sin^2 \frac{p_k}{2}$$

What about the (completely symmetric) state $\sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant L} \quad |\{j_1, j_2, \ldots, j_n\}\rangle$?

- Corresponds to $p_1 = p_2 = \cdots = 0$, and arbitrary $\mathcal{A}_\sigma$.

$\rightsquigarrow$ Same energy as "vacuum" $|\downarrow\downarrow\downarrow\cdots\downarrow\rangle$.

- Should it be counted as well ?

$\rightsquigarrow$ do Bethe equations actually hold, as $S(p, p') = -\frac{0}{0}$?

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Importance of counting solutions

Coordinate Bethe Ansatz

- "Each solution" of Bethe equations provides an eigenstate
- The number of states produced this way is the dimension of the Hilbert space
- Hence all eigenstates are produced this way

Q-system

- Each eigenstate provides a consistent $\mathcal{Q}$-system
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Periodic limit

- $\mathcal{Q}$-system has an unambiguous construction for generic twist
- $Q_{i_1,\dots,i_n} = \Delta(x_{i_1},\dots,x_{i_n})\prod_k x_{i_k}^u \mathcal{Q}_{i_1,\dots,i_n}(u)$ obey

$$Q_{Ijk}(u)Q_I(u-1) = \begin{vmatrix} Q_{Ij}(u) & Q_{Ik}(u) \\ Q_{Ij}(u-1) & Q_{Ik}(u-1) \end{vmatrix}$$

- Example of $SU(2)$ state: $|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}}\,|\uparrow\downarrow\rangle$

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Wronskian determinant

Generic solution of Hirota equation

("bosonic" case)

$$T^{\lambda}(u) = \frac{\det\left(x_j^{\lambda_k - k+1}\mathcal{Q}_j(u+\lambda_k - k+1)\right)_{1\leqslant j,k\leqslant N}}{\Delta(x_1,\cdots,x_N)}$$

For generic solutions of Hirota equations, $\mathcal{Q}$'s are solution of

$$\begin{vmatrix} \mathcal{Q}_i & x_i\mathcal{Q}_i(u+1) & x_i^2\mathcal{Q}_i(u+2) & \dots & x_i^N\mathcal{Q}_i(u+N) \\ T^{1,0}(u) & T^{1,1}(u) & T^{1,2}(u) & \dots & T^{1,N}(u) \\ T^{1,1}(u-1) & T^{1,2}(u-1) & T^{1,3}(u-1) & \dots & T^{1,N+1}(u-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{1,N-1}(u-N+1) & T^{1,N}(u-N+1) & T^{1,N+1}(u-N+1) & \dots & T^{1,2N-1}(u-N+1) \end{vmatrix} = 0$$

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Rotational degrees of freedom

Determinant $\rightsquigarrow$ possibility to take linear combinations of columns: $Q'_i = \sum_j H_{i,j} Q_j$ (where $H_{i,j}(u+1) = H_{i,j}(u)$)

- Q' obeys the same QQ-relation as Q
- Q' is equivalent to Q: gives the same T-functions
- This transformation preserves polynomiality only in the periodic limit ($x_j \to 1$)

- Example of $SU(2)$ state: $|\downarrow\uparrow\rangle + x_1 |\uparrow\downarrow\rangle$:
  $\lim Q_\emptyset = 1,\qquad \lim(1-x_1)Q_1 = 1,\qquad \lim(x_2-x_1)Q_{12} = u^2,$
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And periodic limit of twisted spin chain

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Multiplets
in the periodic limit

Symmetries

In the periodic limit, the Hamiltonian commutes with

- Permutation group: $[H, \mathcal{P}_{...}] = 0$
- $GL(K|M)$ action $[H, h \otimes h \otimes h \cdots] = 0$.

Hence, solutions form multiplets having the same Q-system

Example of $SU(2)$ spin chain at $L = 2$:

For $|\downarrow\downarrow\rangle$, $|\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle$+$|\uparrow\downarrow\rangle$,

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- twist: partial degeneracy lift
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Ex $Q_{[0|0]}$ o $Q_{[1|0]}$) s $Q_{[2|0]}$ in at $L = 2$:

For $|\downarrow\downarrow\rangle$, $|\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle$,

$Q_\emptyset = Q_1 = 1,\ Q_{12} = u^2$,unambiguous

$Q_2 = \frac{2}{3}u^3 + u^2 + \frac{u}{3}$ ambiguous

For $|\downarrow\uparrow\rangle$-$|\uparrow\downarrow\rangle$,

$Q_\emptyset = 1,\ Q_1 = u + \frac{1}{2},\ Q_{12} = u^2$

$Q_2 = u^2 + \frac{u}{2} - \frac{1}{4}$

- twist: partial degeneracy lift
- $|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}}\,|\uparrow\downarrow\rangle$ interpolates between two multiplets

Rank does not matter

$|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}}\,|\uparrow\downarrow\rangle$ is an eigenstate of $SU(2)$ spin chain, and of $SU(K|M)$ spin chains if $K \geq 2$:

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$$Q_{[K|M]} = u^L \;\text{———}\; Q_{[K-1|M]} = u + \tfrac{1}{2} \;\text{———}\; Q_{[K-2|M]} = 1$$

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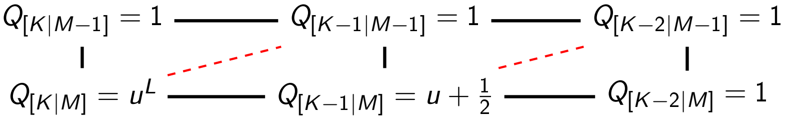

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$Q_{[K|M]} = u^L$ — $Q_{[K-1|M]} = u + \frac{1}{2}$ — $Q_{[K-2|M]} = 1$

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$$
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| & & | & & | \\
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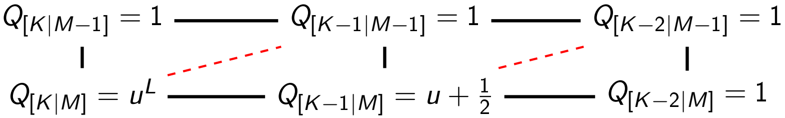

Resolution of Bethe equation

Solving Bethe equation for a specific multiplet:

- write down its Young Diagram, associate a Q-function to each node
- $Q = u^L$ at the corner
- $Q = 1$ at the opposite boundary
- enforce QQ-relations on each facet

- Each multiplet corresponds to a solution of QQ-equations
- *Conjecture*: No "unphysical solution"

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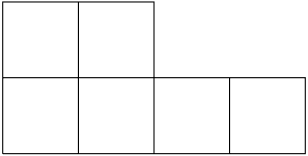

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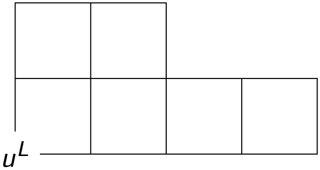

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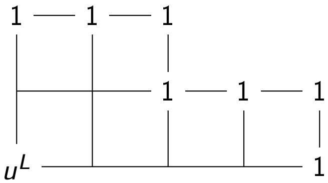

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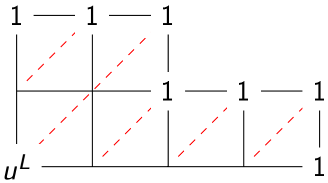

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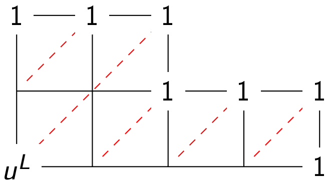

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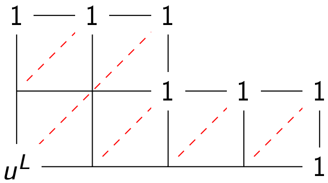

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Outlook

- Completeness is a conceptual issue in the “nice Fairy Tale”
- Q-system is well defined in the presence of a twist
- Periodic case involves a non-straightforward limit, and multiplets
- Resolution method [Marboe, Volin, ’17] suitable for efficient analytic resolution via algebraic-geomtric methods implemented in Mathematica
- Algebraic-geometric resolution methods explained in [Jiang, Zhang, ’18]
- At the level of completeness, no need to actually solve equations but only to count solutions (use BKK theorem).

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