Periodic XXX spin chains: What is this under the carpet?

Sébastien Leurent

IMB, Dijon

April 13, 2018

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XXX-type spin chains

April 13, 2018 1 / 27

Disclaimer

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Discussions and preliminary work with D. Volin Overlaps [Jiang, Zhang, '17]

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XXX-type spin chains

April 13, 2018 2 / 27

XXX-type spin chains

Two versions of a Fairy tale

- Spin chain
- Coordinate Bethe ansatz
- Hirota equation and Wronskian determinants
- A look under the carpet
- 2 Solving Bethe equations
 - Multiplets
 - Each solution applies to several models
 - Resolution of Bethe equations
- Counting solutions

Solving Bethe equations

Counting solutions

 $\vec{\sigma}_{L+1} = \vec{\sigma}_1$ $|\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow\downarrow\rangle$

"Coordinate Bethe Ansatz"

for $XXX_{1/2}$ Heisenberg spin chain

 $\mathcal{H} = (\mathbb{C}^2)^{\otimes L};$

Eigenstates of $H = -\sum_{i=1}^{L} \vec{\sigma_i} \cdot \vec{\sigma_{i+1}}$ = $L - 2\sum_{i=1}^{L} \mathcal{P}_{i,i+1}$

• "Vacuum": $|\downarrow\downarrow\cdots\downarrow
angle$

Single "excitation":

 $\ket{\psi} \propto \sum_k e^{i\,k\,p} \ket{\{k\}}$ where $e^{2i\,p\,L} = 1$

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XXX-type spin chains

April 13, 2018 4 / 27

Solving Bethe equations

Counting solutions

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Two	versions	ofa	Fairy	tale
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$$\rightsquigarrow \Psi(k) = \alpha x^k + \beta y^k$$
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$$\rightsquigarrow \Psi(k+L) = \Psi(k)$$

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for $XXX_{1/2}$ Heisenberg spin chain

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Solving Bethe equations

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$$\begin{array}{ll} |\psi\rangle \ = \ \mathcal{A}\sum_{j\leq k} e^{i(p_1j+p_2k)} \left|\{j,k\}\right\rangle \\ + \tilde{\mathcal{A}}\sum_{j>k} e^{i(p_1j+p_2k)} \left|\{j,k\}\right\rangle \end{array}$$

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for $XXX_{1/2}$ Heisenberg spin chain

 $(m^2) \otimes I$

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• Two excitations:

• Single "excitation"

$$\begin{split} \psi \rangle &= \sum_{j < k} (\mathcal{A} \, e^{i(p_1 j + p_2 k)} + \tilde{\mathcal{A}} \, e^{i(p_1 k + p_2 j)}) \, |\{j, k\}\rangle \\ \text{where } e^{i \, L \, p_2} &= e^{-i \, L \, p_1} = S = \frac{\tilde{\mathcal{A}}}{\mathcal{A}}, \text{ with} \\ S &= -\frac{1 + e^{i(p_1 + p_2)} - 2e^{ip_2}}{1 + e^{i(p_1 + p_2)} - 2e^{ip_1}} \end{split}$$



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- is an eigenstate i
 - $\sim A_{\sigma} \propto (-1)^{\sigma} \prod_{i \leq k} (1 + e^{i(k_{i}a_{i}+k_{i}a_{i})} 2e^{ik_{i}a_{i}})$
 - $V_{i}, e^{iL,p} = \prod_{k \in I} S(p_i, p_k)$ where $S(p_i, p') = -\frac{1+e^{ip_k p_k}-2e^{ip_k}}{1+e^{ip_k p_k}-2e^{ip_k}}$
- The corresponding eigenvalue is

$$E = -L + \sum_{k} 8\sin^2\frac{p_k}{2}$$



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XXX-type spin chains

April 13, 2018 7 / 27

Two versions of a Fairy tale	Solving Bethe equations	Counting solutions
"Coordinate Bethe Ansatz	II for XXX.	Heisenherg spin chain

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- n excitations:

$$|\psi\rangle = \sum_{1 \leq j_1 < j_2 < \dots < j_n \leq L} \sum_{\sigma \in \mathfrak{S}_n} \mathcal{A}_{\sigma} e^{j \sum_k p_{\sigma(k)} j_k} |\{j_1, j_2, \dots, j_n\}\rangle$$

- is an eigenstate if
 - $\mathcal{A}_{\sigma} \propto (-1)^{\sigma} \prod_{i \leq k} \left(1 + e^{i(p_{\sigma(i)} + p_{\sigma(k)})} 2e^{ip_{\sigma(k)}} \right)$
 - $\forall j, e^{i L p_j} = \prod_{k \neq j} S(p_j, p_k)$ where $S(p, p') \equiv -\frac{1 + e^{i(p+p')} 2e^{ip'}}{1 + e^{i(p+p')} 2e^{ip'}}$

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Two versions of a Fairy tale ○CO●○○○○○○○○○○○○○○	Solving Bethe equations	Counting solutions
"Coordinate Bethe Ansatz	II for $XXX_{1/2}$	Heisenberg spin chain

- Two excitations: $|\psi\rangle = \sum_{j < k} (\mathcal{A} e^{i(p_1 j + p_2 k)} + \tilde{\mathcal{A}} e^{i(p_1 k + p_2 j)}) |\{j, k\}\rangle$ where $e^{i \, L \, p_2} = e^{-i \, L \, p_1} = S = \frac{\tilde{\mathcal{A}}}{\mathcal{A}}$, with $S = -\frac{1 + e^{i(p_1 + p_2)} 2e^{ip_2}}{1 + e^{i(p_1 + p_2)} 2e^{ip_1}}$
- n excitations:

$$|\psi\rangle = \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq L} \sum_{\sigma \in \mathfrak{S}_n} \mathcal{A}_{\sigma} e^{j \sum_k p_{\sigma(k)} j_k} |\{j_1, j_2, \dots, j_n\}\rangle$$

is an eigenstate if

•
$$\mathcal{A}_{\sigma} \propto (-1)^{\sigma} \prod_{j < k} \left(1 + e^{i(p_{\sigma(j)} + p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}} \right)$$

•
$$\forall j, e^{j L p_j} = \prod_{k \neq j} S(p_j, p_k)$$
 where $S(p, p') \equiv -\frac{1 + e^{i(p+p')} - 2e^{ip}}{1 + e^{i(p+p')} - 2e^{ip'}}$

$$E = -L + \sum_{k} 8\sin^2\frac{p_k}{2}$$

Two versions of a Fairy tale ○CO●○○○○○○○○○○○○○	Solving Bethe equations	Counting solutions
"Coordinate Bethe Ansatz	II for XXX_1	/2 Heisenberg spin chain

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The corresponding eigenvalue is

$$E = -L + \sum_{k} 8\sin^2\frac{p_k}{2}$$

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April 13, 2018

7 / 27

Two versions of a Fairy tale	Solving Bethe equations 000	Counting solutions
"Coordinate Bethe Ansatz	for $XXX_{1/2}$	Heisenberg spin chain

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• $\forall j, e^{j L p_j} = \prod_{k \neq j} S(p_j, p_k)$ where $S(p, p') \equiv -\frac{1 + e^{i(p+p')} - 2e^{ip}}{1 + e^{i(p+p')} - 2e^{ip'}}$

The corresponding eigenvalue is

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XXX-type spin chains

Bethe equations \rightsquigarrow spectrum

April 13, 2018 7 / 27

- Spin chain
- Coordinate Bethe ansatz
- Hirota equation and Wronskian determinants
- A look under the carpet
- 2 Solving Bethe equations
 - Multiplets
 - Each solution applies to several models
 - Resolution of Bethe equations

3 Counting solutions

Counting solutions

T-operators for $XXX_{1/2}$ Heisenberg spin chain



partial trace: $\forall x, y \in \mathcal{H}_p$, $\langle y | \operatorname{tr}_a M | x \rangle = \sum_{z \in \mathcal{B}_a} (\langle y | \otimes \langle z |) M (|x\rangle \otimes |z\rangle)$ where $M \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_a)$, $\operatorname{tr}_a(M) \in \mathcal{L}(\mathcal{H}_p)$, \mathcal{B}_a =orthonorm. bas. of \mathcal{H}_a .

• $((u-v)\mathbb{I} + \mathcal{P}_{i,j})(u\mathbb{I} + \mathcal{P}_{i,k})(v\mathbb{I} + \mathcal{P}_{j,k})$

Solving Bethe equations

Counting solutions

T-operators for $XXX_{1/2}$ Heisenberg spin chain



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• $((u-v)\mathbb{I} + \mathcal{P}_{i,j})(u\mathbb{I} + \mathcal{P}_{i,k})(v\mathbb{I} + \mathcal{P}_{j,k})$ = $(v\mathbb{I} + \mathcal{P}_{j,k})(u\mathbb{I} + \mathcal{P}_{i,k})((u-v)\mathbb{I} + \mathcal{P}_{i,j})$

Solving Bethe equations

Counting solutions

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•
$$((u - v)\mathbb{I} + \mathcal{P}_{i,j})(u\mathbb{I} + \mathcal{P}_{i,k})(v\mathbb{I} + \mathcal{P}_{j,k})$$

= $(v\mathbb{I} + \mathcal{P}_{j,k})(u\mathbb{I} + \mathcal{P}_{i,k})((u - v)\mathbb{I} + \mathcal{P}_{i,j})$

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April 13, 2018 9 / 27

Solving Bethe equations

Counting solutions

Commutation of *T*-operators



Solving Bethe equations

Counting solutions

Commutation of *T*-operators



Solving Bethe equations

Counting solutions

Commutation of *T*-operators



Solving Bethe equations

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Solving Bethe equations

Counting solutions

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Solving Bethe equations

Counting solutions

T-operators for $XXX_{1/2}$ spin chain



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•
$$[T(u), T(v)] = 0$$

Solving Bethe equations

Counting solutions

T-operators for XXX-type spin chains



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Solving Bethe equations

Counting solutions

T-operators for XXX-type spin chains



Generalized permutation operator:

$$\mathcal{P}_{i,j} = \sum_{lpha,eta} e^{(i)}_{lpha,eta} \otimes \pi_{\lambda}(e^{(j)}_{eta,lpha})$$

•
$$[T^{\lambda}(u), T^{\mu}(v)] = 0$$

Solving Bethe equations

Counting solutions

T-operators for XXX-type spin chains



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Solving Bethe equations

Counting solutions

T-operators for *XXX*-type spin chains



•
$$[T^{\lambda}(u), T^{\mu}(v)] = 0$$

• For rectangular Young diagrams,
 $T^{a,s}(u+1) \cdot T^{a,s}(u) = T^{a+1,s}(u+1) \cdot T^{a-1,s}(u) + T^{a,s-1}(u+1) \cdot T^{a,s+1}(u)$
Sébastien Leurent, *IMB*, Dijon XXX-type spin chains April 13, 2018 11/27

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XXX-type spin chains

Wronskian determinant

Generic solution of Hirota equation $\mathcal{T}^{\lambda}(u) = \frac{\det \left(x_{j}^{\lambda_{k}-k+1} \mathcal{Q}_{j}(u+\lambda_{k}-k+1) \right)_{1 \leq j,k \leq N}}{\Delta(x_{1}, \cdots, x_{N})}$ where $g = \operatorname{diag}(x_{1}, \cdots x_{N}); \quad \Delta(x_{1}, \cdots, x_{N}) = \det \left(x_{j}^{1-k} \right)_{1 \leq j,k \leq N}$

where Q_1 , Q_2 , ... commute among themselves and with T, and are polynomial in u. For spin chains, their explicit expression is known.

Wronskian determinant

Generic solution of Hirota equation (``bosonic'' case) $T^{\lambda}(u) = \frac{\det \left(x_{j}^{\lambda_{k}-k+1} \mathcal{Q}_{j}(u+\lambda_{k}-k+1) \right)_{1 \leq j,k \leq N}}{\Delta(x_{1}, \cdots, x_{N})}$

For generic solutions of Hirota equations, \mathcal{Q} 's are solution of

$$\begin{vmatrix} Q_i & x_i Q_i(u+1) & x_i^2 Q_i(u+2) & \dots & x_i^N Q_i(u+N) \\ T^{1,0}(u) & T^{1,1}(u) & T^{1,2}(u) & \dots & T^{1,N}(u) \\ T^{1,1}(u-1) & T^{1,2}(u-1) & T^{1,3}(u-1) & \dots & T^{1,N+1}(u-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{1,N-1}(u-N+1) & T^{1,N}(u-N+1) & T^{1,N+1}(u-N+1) & \dots & T^{1,2N-1}(u-N+1) \end{vmatrix} = 0$$

Solving Bethe equations

Counting solutions

Q-system SU(N) case

•
$$\mathcal{Q}_{\emptyset} = 1$$

• $\mathcal{Q}_{i_1, i_2, \dots, i_m} = \frac{\det \left(x_{i_k}^{1-l} \mathcal{Q}_{i_k}(u+1-l) \right)_{1 \leq k, l \leq m}}{\Delta(x_1, \dots, x_N)}$
• Hence $\mathcal{Q}_{\overline{\emptyset}} \equiv \mathcal{Q}_{1, 2, \dots, N} = \mathcal{T}^{0, 0}(u) = u$

Energy

$$E = E_0 + \sum_{\substack{u \ Q_{23...N}(u)=0}} \frac{-1}{u(u+1)}$$

"QQ-relations"

$$(-1) = \frac{\begin{vmatrix} x_2 Q_{12}(u) & x_3 Q_{13}(u) \\ Q_{12}(u-1) & Q_{13}(u-1) \end{vmatrix}}{x_2 - x_3}$$

$$\frac{\left|\begin{array}{c}x_{j}\mathcal{Q}_{lj}(u) & x_{k}\mathcal{Q}_{lk}(u)\\\mathcal{Q}_{lj}(u-1) & \mathcal{Q}_{lk}(u-1)\end{array}\right|}{x_{i}-x_{k}}$$

• $\mathcal{Q}_{lik}(u)\mathcal{Q}_l(u-1) =$

Solving Bethe equations

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$$\begin{array}{c} & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ &$$

Solving Bethe equations

Counting solutions

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Solving Bethe equations

Counting solutions

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Solving Bethe equations

Counting solutions

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Solving Bethe equations

Counting solutions

Q-system in SU(K|M) case Pictures are for K = 2 and M = 1

2^{K+M} Q-operators. Each is labeled by a subset of {1,2,..., K} and a subset of {1,2,..., M}.





Solving Bethe equations

Counting solutions

Q-system in SU(K|M) case Pictures are for K = 2 and M = 1

• 2^{K+M} Q-operators. Each is labeled by a subset of $\{1, 2, \dots, K\}$ and a subset of $\{1, 2, \dots, M\}$.



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XXX-type spin chains

Two versions of a Fairy tale	Solving Bethe equations	Counting solutions
Bethe equation		
 Restrict to eigenspace: functions of u. 	operators become com	plexed-valued

Existence of Polynomial \mathcal{Q} -functions such that

Bethe equations

Example of SU(2)

 $\begin{aligned} &(x_1 - x_2) u^L = x_1 Q_1(u) Q_2(u-1) - x_2 Q_2(u) Q_1(u-1) \\ &(x_1 - x_2) (u+1)^L = x_1 Q_1(u+1) Q_2(u) - x_2 Q_2(u+1) Q_1(u) \\ &\text{Hence } \frac{u^L}{(u+1)^L} = -\frac{x_2}{x_1} \frac{Q_1(u-1)}{Q_1(u+1)}. \end{aligned}$ $\text{Matches Bethe equation up to } u = \frac{e^{ip}}{\sqrt{u}}. \end{aligned}$

Two versions of a Fairy tale	Solving Bethe equations 000	Counting solutions
Bethe equation		

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 and $\mathcal{Q}_{\overline{\emptyset}} = u^{L}$.

Example of SU(2)



Two versions of a Fairy tale	Solving Bethe equations 000	Counting solutions
Bethe equation		

Existence of Polynomial $\mathcal{Q}\text{-}\mathsf{functions}$ such that

Bethe equations \Leftrightarrow

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Two versions of a Fairy tale ○○○○○○○○○○●○○○○○○○	Solving Bethe equations 000	Counting solutions
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Example of SU(2)

$$\begin{array}{l} \text{if } u \text{ is a root of } \mathcal{Q}_1, \\ (x_1 - x_2) \, u^L = x_1 \mathcal{Q}_1(u) \mathcal{Q}_2(u-1) - x_2 \mathcal{Q}_2(u) \mathcal{Q}_1(u-1) \\ (x_1 - x_2) \, (u+1)^L = x_1 \mathcal{Q}_1(u+1) \mathcal{Q}_2(u) - x_2 \mathcal{Q}_2(u+1) \mathcal{Q}_1(u) \\ \text{Hence } \frac{u^L}{(u+1)^L} = -\frac{x_2}{x_1} \frac{\mathcal{Q}_1(u-1)}{\mathcal{Q}_1(u+1)}. \\ \text{Matches Bethe equation up to } u = \frac{e^{i\rho}}{1 - e^{i\rho}}. \end{array}$$

Two versions of a Fairy tale ○○○○○○○○○○●○○○○○○○	Solving Bethe equations 000	Counting solutions
Bethe equation		

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Example of SU(2)

if *u* is a root of
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 $(x_1 - x_2) (u+1)^L = x_1 Q_1(u+1) Q_2(u) - x_2 Q_2(u+1) Q_1(u)$
Hence $\frac{u^L}{(u+1)^L} = -\frac{x_2}{x_1} \frac{Q_1(u-1)}{Q_1(u+1)}$.
Matches Bethe equation up to $u = \frac{e^{ip}}{1 - e^{ip}}$.

Two versions of a Fairy tale	Solving Bethe equations 000	Counting solutions
Bethe equation		

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Example of SU(2)

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Hence $\frac{u^L}{(u+1)^L} = -\frac{x_2}{x_1} \frac{Q_1(u-1)}{Q_1(u+1)}$.
Matches Bethe equation up to $u = \frac{e^{ip}}{1 - e^{ip}}$.

- Spin chain
- Coordinate Bethe ansatz
- Hirota equation and Wronskian determinants
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3 Counting solutions

Solving Bethe equations

Counting solutions

Importance of counting solutions

Coordinate Bethe Ansatz

- "Each solution" of Bethe equations provides an eigenstate
- The number of states produced this way is the dimension of the Hilbert space
- All eigenstates are produced this way



• Each eigenstate provides a consistent *Q*-system

- The number of consistent *Q*-systems is the dimension of the Hilbert space
- All consistent *Q*-systems correspond to eigenstates

Solving Bethe equations

Counting solutions

Importance of counting solutions

Coordinate Bethe Ansatz

- "Each solution" of Bethe equations provides an eigenstate
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- Hence all eigenstates are produced this way

Q-system

- Each eigenstate provides a consistent Q-system
- The number of consistent *Q*-systems is the dimension of the Hilbert space
- Hence all consistent *Q*-systems correspond to eigenstates

Physicalness of solutions to Bethe equation

Reminder: Bethe equations

$$\begin{aligned} |\psi\rangle &= \sum_{1 \leqslant j_1 < j_2 < \dots < j_n \leqslant L} \sum_{\sigma \in \mathfrak{S}_n} \mathcal{A}_{\sigma} e^{i \sum_k p_{\sigma(k)} j_k} |\{j_1, j_2, \dots, j_n\}\rangle \\ \mathcal{A}_{\sigma} \propto (-1)^{\sigma} \prod_{j < k} \left(1 + e^{i(p_{\sigma(j)} + p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right) \\ \forall j, e^{i L p_j} &= \prod_{k \neq j} S(p_j, p_k) \text{ where } S(p, p') \equiv -\frac{1 + e^{i(p + p')} - 2e^{ip}}{1 + e^{i(p + p')} - 2e^{ip'}} \\ E &= -L + \sum_k 8 \sin^2 \frac{p_k}{2} \end{aligned}$$

What about the (completely symmetric) state $\sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq L} |\{j_1, j_2, \dots, j_n\}\rangle$?

- Corresponds to $p_1 = p_2 = \cdots = 0$, and arbitrary \mathcal{A}_{σ} .
- \rightarrow Same energy as "vacuum" $|\downarrow\downarrow\downarrow\downarrow\cdots\downarrow\rangle$.

ightarrow do Bethe equations actually hold, as $S(
ho,
ho')=-rac{0}{n}$?

Physicalness of solutions to Bethe equation

Reminder: Bethe equations

$$\begin{aligned} |\psi\rangle &= \sum_{1 \leqslant j_1 < j_2 < \dots < j_n \leqslant L} \sum_{\sigma \in \mathfrak{S}_n} \mathcal{A}_{\sigma} e^{i \sum_k p_{\sigma(k)} j_k} |\{j_1, j_2, \dots, j_n\}\rangle \\ \mathcal{A}_{\sigma} \propto (-1)^{\sigma} \prod_{j < k} \left(1 + e^{i(p_{\sigma(j)} + p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right) \\ \forall j, e^{i L p_j} &= \prod_{k \neq j} S(p_j, p_k) \text{ where } S(p, p') \equiv -\frac{1 + e^{i(p + p')} - 2e^{ip}}{1 + e^{i(p + p')} - 2e^{ip'}} \\ E &= -L + \sum_k 8 \sin^2 \frac{p_k}{2} \end{aligned}$$

What about the (completely symmetric) state $\sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq L} |\{j_1, j_2, \dots, j_n\}\rangle$?

• Corresponds to $p_1 = p_2 = \cdots = 0$, and arbitrary \mathcal{A}_{σ} .

 \rightsquigarrow Same energy as "vacuum" $|\downarrow\downarrow\downarrow\downarrow\cdots\downarrow\rangle$.

Should it be counted as well 1

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Solving Bethe equations

Counting solutions

Importance of counting solutions

Coordinate Bethe Ansatz

- "Each solution" of Bethe equations provides an eigenstate
- The number of states produced this way is the dimension of the Hilbert space
- Hence all eigenstates are produced this way

Q-system

- Each eigenstate provides a consistent Q-system
- The number of consistent *Q*-systems is the dimension of the Hilbert space
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| Two versions of a Fairy tale | Solving Bethe equations | Counting solutions |
|------------------------------|--------------------------------|--------------------|
| Periodic limit | | |
| • $\mathcal Q$ -system has a | n unambiguous construction for | generic twist |

• $Q_{i_1,\ldots,i_n} = \Delta(x_{i_1},\ldots,x_{i_n}) \prod_k x_{i_k}^u \mathcal{Q}_{i_1,\ldots,i_n}(u)$ obey

• Example of SU(2) state: $|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}}|\uparrow\downarrow\rangle$

Counting solutions

Wronskian determinant

Generic solution of Hirota equation (``bosonic'' case) $T^{\lambda}(u) = \frac{\det \left(x_{j}^{\lambda_{k}-k+1} \mathcal{Q}_{j}(u+\lambda_{k}-k+1) \right)_{1 \leq j,k \leq N}}{\Delta(x_{1}, \cdots, x_{N})}$

For generic solutions of Hirota equations, \mathcal{Q} 's are solution of

$$\begin{vmatrix} Q_i & x_i Q_i(u+1) & x_i^2 Q_i(u+2) & \dots & x_i^N Q_i(u+N) \\ T^{1,0}(u) & T^{1,1}(u) & T^{1,2}(u) & \dots & T^{1,N}(u) \\ T^{1,1}(u-1) & T^{1,2}(u-1) & T^{1,3}(u-1) & \dots & T^{1,N+1}(u-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{1,N-1}(u-N+1) & T^{1,N}(u-N+1) & T^{1,N+1}(u-N+1) & \dots & T^{1,2N-1}(u-N+1) \end{vmatrix} = 0$$

Two versions of a Fairy tale	Solving Bethe equations	Counting solutions
Periodic limit		
• Q -system has an unam • $Q_{i_1,,i_n} = \Delta(x_{i_1},,x_{i_n})$ $Q_{ljk}(u)Q_l(u-1) = \Big _Q$	biguous construction for $a_n \prod_k x_{i_k}^u \mathcal{Q}_{i_1,,i_n}(u)$ obey $Q_{lj}(u) \qquad Q_{lk}(u) \mid Q_{lk}(u-1) \mid Q_{lk}(u-1)$	generic twist

• Example of SU(2) state: $|\downarrow\uparrow\rangle + \sqrt{\frac{\chi_1}{\chi_2}} |\uparrow\downarrow\rangle$



Iwo versions of a Fairy tale	Solving Bethe equations	Counting solutions
Periodic limit		

 $\circ \mathcal{Q}$ -system has an unambiguous construction for generic twist

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$$Q_{i_1,...,i_n} = \Delta(x_{i_1},...,x_{i_n}) \prod_k x_{i_k}^u Q_{i_1,...,i_n}(u)$$
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Two versions of a Fairy tale	Solving Bethe equations 000	Counting solutions
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Two versions of a Fairy tale	Solving Bethe equations	Counting solutions
Periodic limit		
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• Example of <i>SU</i> (2) s	tate: $\left \downarrow\uparrow\right\rangle+\sqrt{rac{x_1}{x_2}}\left \uparrow\downarrow\right\rangle$	
$Q_{\emptyset}=\mathcal{Q}_{\emptyset}=1;$	$Q_1(u)$	$) = u + \frac{\frac{1+1}{\sqrt{\frac{x_{1}}{x_{2}}}}}{1-\frac{x_{2}}{x_{1}}}$
$\mathcal{Q}_2(u)=u+rac{1+\sqrt{rac{x_1}{x_2}}}{1-rac{x_1}{x_2}};$	$Q_{\overline{\emptyset}}$.	$\propto \mathcal{Q}_{12}(u) = u^2$

• when $x_i \rightarrow 1$, $\lim Q_1 = \lim Q_2 \propto \begin{cases} 1 & \text{if } \sqrt{\frac{x_1}{x_2}} = x_1 \rightarrow 1 \\ u + \frac{1}{2} & \text{if } \sqrt{\frac{x_1}{x_2}} = -x_1 \rightarrow -1 \end{cases}$

Two versions of a Fairy tale

Solving Bethe equations

Counting solutions

Rotational degrees of freedom

Determinant \rightarrow possibility to take linear combinations of columns:

- $Q_i' = \sum_j H_{i,j} Q_j$ (where $H_{i,j}(u+1) = H_{i,j}(u)$)
 - Q' obeys the same QQ-relation as Q
 - Q' is equivalent to Q: gives the same T-functions
 - This transformation preserves polynomiality only in the periodic limit $(x_j
 ightarrow 1)$
 - Example of SU(2) state: $|\downarrow\uparrow\rangle + x_1 |\uparrow\downarrow\rangle$: $\lim Q_{\emptyset} = 1$, $\lim (1 - x_1)Q_1 = 1$, $\lim (x_2 - x_1)Q_{12} = u^2$, $\lim \frac{4}{(x_1 - x_2)^2}(Q_1 + Q_2 - (1 - x_1)Q_1) = \frac{2}{3}u^3 + u^2 + \frac{u}{3}$
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Counting solutions

Rotational degrees of freedom And periodic limit of twisted spin chain

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Two	versi	ons	of	а	Fairy	tale	
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Counting solutions

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Sébastien Leurent, IMB, Dijon

Two	versi	ons	of	а	Fairy	tale	
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Counting solutions

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Two versions of a Fairy tale	Solving Bethe equations ●○○	Counting solution:	
Multiplets in the periodic limit			

In the periodic limit, the Hamiltonian commutes with

- Permutation group: $[H, \mathcal{P}_{...}] = 0$
- GL(K|M) action $[H, h \otimes h \otimes h \cdots] = 0.$

Hence, solutions form multiplets having the same Q-system.

Example of SU(2) spin chain at L = 2:

For
$$|\downarrow\downarrow\rangle$$
, $|\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle$,

$$Q_{\emptyset} = Q_1 = 1, \ Q_{12} = u^2$$
,
 $Q_2 = \frac{2}{2}u^3 + u^2 + \frac{u}{2}$

For
$$|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle$$
,
 $Q_{\emptyset} = 1, \ Q_1 = u + \frac{1}{2}, \ Q_{12} = u^2$
 $Q_2 = u^2 + \frac{u}{2} - \frac{1}{4}$

twist: partial degeneracy lift

• $|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}}|\uparrow\downarrow\rangle$ interpolates between two multiplets

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Multiplets			

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Two versions of a Fairy tale	Solving Bethe equations ●○○	Counting solutions
Multiplets		

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Example of
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 spin chain at $L = 2$:

For
$$|\downarrow\downarrow\rangle$$
, $|\uparrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle$,
 $Q_{\emptyset} = Q_1 = 1$, $Q_{12} = u^2$, unambiguous
 $Q_2 = \frac{2}{3}u^3 + u^2 + \frac{u}{3}$ ambiguous
• twist: partial degeneracy lift
• $|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}} |\uparrow\downarrow\rangle$ interpolates between two multiplets
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- GL(K|M) action $[H, h \otimes h \otimes h \cdots] = 0.$

Hence, solutions form multiplets having the same Q-system

$$\begin{array}{c} \mathsf{Ex} \ Q_{[0|0]} \ P_{[1|0]} \ s \ Q_{[2|0]} \ \text{in at } L = 2: \\ \mbox{For } |\downarrow\downarrow\rangle, \ |\downarrow\uparrow\rangle \ \text{and } |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle, \\ Q_{\emptyset} = Q_1 = 1, \ Q_{12} = u^2, \ \text{unambiguous} \ Q_{\emptyset} = 1, \ Q_1 = u + \frac{1}{2}, \ Q_{12} = u^2 \\ Q_2 = \frac{2}{3}u^3 + u^2 + \frac{u}{3} \ \text{ambiguous} \ Q_2 = u^2 + \frac{u}{2} - \frac{1}{4} \\ \mbox{etails: partial degeneracy lift} \\ \mbox{etails: partial degeneracy lift} \ |\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}} |\uparrow\downarrow\rangle \ \text{interpolates between two multiplets} \end{array}$$

$|\downarrow\uparrow\rangle + \sqrt{\frac{x_1}{x_2}} |\uparrow\downarrow\rangle$ is an eigenstate of SU(2) spin chain, and of SU(K|M) spin chains if $K \ge 2$:

- SU(4): $Q_{[0|0]} = Q_{[1|0]} = Q_{[2|0]} = 1$, $Q_{[3|0]} = u + \frac{1}{2}$, $Q_{[4|0]} = u^2$.
- SU(2|1): $Q_{[0|0]} = Q_{[0|1]} = Q_{[1|0]} = Q_{[2|0]} = 1$, $Q_{[1|1]} = u + \frac{1}{2}$, $Q_{[2|1]} = u^2$
- SU(2|2): $Q_{[0|0]} = Q_{[2|0]} = Q_{[0|1]} = Q_{[1|1]} = Q_{[2|1]} = Q_{[2|2]} = 1,$ $Q_{[1|2]} = u + \frac{1}{2}, \quad Q_{[2|2]} = 0$

but there is no $Q_{[1|0]}$: both $Q_{1|0}$ and $Q_{2|0}$ are ambiguous in the periodic limit.

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Resolution of Bethe equation

Solving Bethe equation for a specific multiplet:

- write down its Young Diagram, associate a Q-function to each node
- $Q = u^L$ at the corner
- Q = 1 at the opposite boundary
- enforce QQ-relations on each facet



Each multiplet corresponds to a solution of QQ-equations
 Conjecture: No "unphysical solution"

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XXX-type spin chains

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Outlook

- Completeness is a conceptual issue in the "nice Fairy Tale"
- Q-system is well defined in the presence of a twist
- Periodic case involves a non-straightforward limit, and multiplets
- Resolution method [Marboe, Volin, '17] suitable for efficient analytic resolution via algebraic-geomtric methods implemented in Mathematica
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