Hirota equation and the spectrum of (some) quantum integrable models

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Semestre thématique: *Correspondance AdS/CFT, holographie, intégrabilité*

CRM, Montréal

Outline

1. Integrability and Bethe equations
   - Coordinate Bethe ansatz
   - Nested Bethe equations for rational spin chains
2. Coderivative approach to rational spin chains
   - Coderivative formalism
   - Hirota equation $\leftrightarrow$ Wronskian determinants
   - Non-twisted limit
   - Hirota equation $\leftrightarrow$ spectrum
3. Finite size spectrum of sigma models
   - Thermodynamic Bethe Ansatz
   - “Quantum Spectral Curve” for AdS/CFT

"Coordinate Bethe Ansatz" for $XXX_{1/2}$ Heisenberg spin chain

Eigenstates of $H = -\sum_{i=1}^{L} \vec{\sigma}_i \cdot \vec{\sigma_{i+1}}$ $\quad \mathcal{H} = (\mathbb{C}^2)^{\otimes L}; \quad \vec{\sigma}_{L+1} = \vec{\sigma}_1$

$= L - 2\sum_{i=1}^{L} \mathcal{P}_{i,i+1}$

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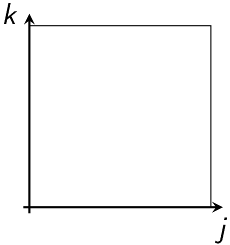

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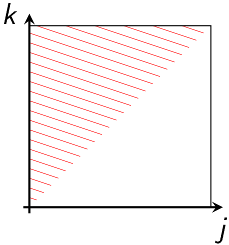

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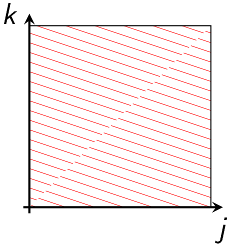

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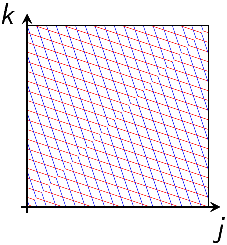

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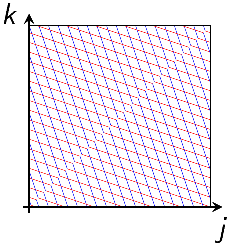

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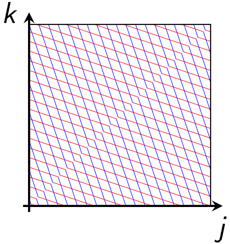
1+1 D *Integrability* field theories / spin chains

Bethe Ansatz of the form $\psi(n_1, n_2, \cdots, n_M) \equiv \sum_{\sigma \in \mathcal{S}^M} \mathcal{A}_\sigma e^{i \sum_k p_{\sigma(k)} n_k}$

$\rightsquigarrow$ wave-function of the eigenstates of several theories such that

1. The space is one-dimensional and there are periodic boundary conditions.
2. The interactions are local.
3. A factorization formula holds
4. There are infinitely many conserved charges

- Conditions (1,2) are necessary for this ansatz
- Conditions (1,2,4) are sufficient for this ansatz [Zamolodchikov Zamolodchikov 79]



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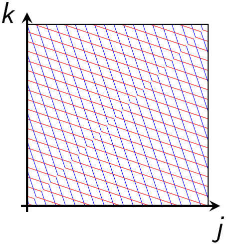

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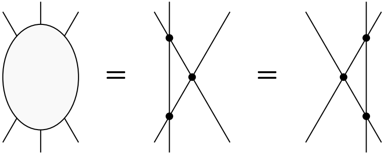

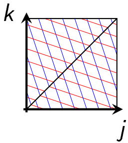

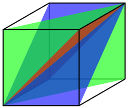

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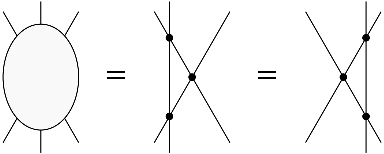

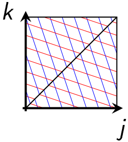

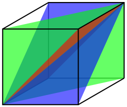

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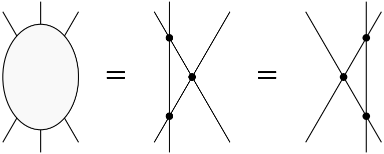

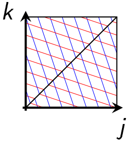

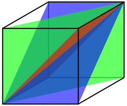

Spectrum

$E = \sum_i \mathcal{E}(p_i) \qquad e^{i L p_j} = \prod_{k \neq j} \mathcal{S}(p_j, p_k)$

$\mathcal{E}$ and $\mathcal{S}$ are model-dependent functions

Nested Bethe Ansatz

unexpected simplicity of the equations for higher rank rational spin chain

SU(2) spin chain:

$\forall j, e^{i L p_j} = \prod_{k \neq j} S(p_j, p_k)$

$S(p, p') \equiv -\frac{1+e^{i(p+p')}-2e^{i p}}{1+e^{i(p+p')}-2e^{i p'}}$

$E = -L + \sum_k (4 - 4 \cos p_k)$

$$\xleftrightarrow[Q(u)=\prod_k (u-\theta_k)]{\tan \frac{p_k}{2}=-\frac{1}{2\theta_k}}$$

$\forall j, \left(\frac{\theta_j - i/2}{\theta_j + i/2}\right)^L = -\frac{Q(\theta_j - i)}{Q(\theta_j + i)}$

$E = -L + 2 \sum_k \frac{1}{\theta_k^2 + 1/4}$

SU(N) spin chain:

polynomials $Q_0, Q_1, Q_2, \ldots, Q_N$, with $Q_0 = 1$, $Q_N(u) = u^L$

$$E = -L + 2 \sum_{\theta_k : Q_{N-1}(\theta_k)=0} \frac{1}{\theta_k^2 + 1/4}$$

$\frac{Q_{i-1}(\theta+i/2) Q_i(\theta-i) Q_{i+1}(\theta+i/2)}{Q_{i-1}(\theta-i/2) Q_i(\theta+i) Q_{i+1}(\theta-i/2)} = -1$ when $Q_i(\theta) = 0$

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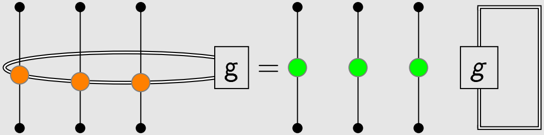
Contents

Group-derivative

Advertisement

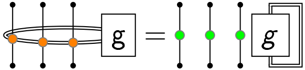
$T^{\lambda}(u) =$



picture for $L = 3$ spin chain,
operators on $\mathcal{H} = (\mathbb{C}^N)^{\otimes 3}$

Group-derivative

let $\boxed{g} \in GL(\mathbb{C}^N)$, $\underbrace{\boxed{f(g)}}_{L} \in \mathcal{L}\left((\mathbb{C}^N)^{\otimes L}\right)$



derivative of f w.r.t. $\log(g)$:

$$\underbrace{\boxed{\hat{D} \otimes f(g)}}_{L+1} \equiv \boxed{\partial_{\phi^t}}\boxed{f(e^{\phi} g)}\Big|_{\phi \to 0}$$

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Group-derivative

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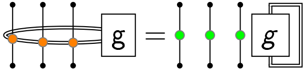
well-behaved under change of representation



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derivation exchanges in-going and outgoing indices:

$\partial_i = \partial_{x^i}$, $\partial^i = \partial_{x_i}$

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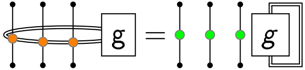

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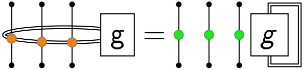

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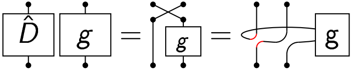

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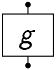
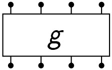
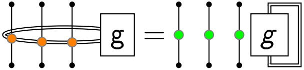
Group-derivative

let  $\in GL(\mathbb{C}^N)$,  $\in \mathcal{L}\left((\mathbb{C}^N)^{\otimes L}\right)$



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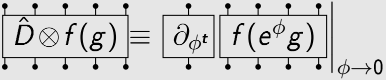

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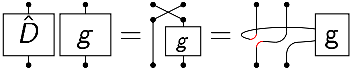

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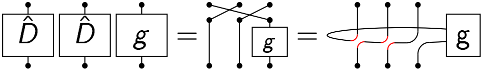

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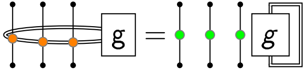

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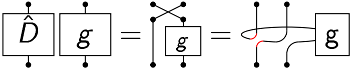

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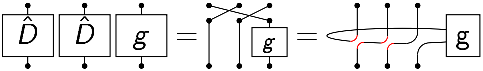

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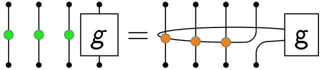

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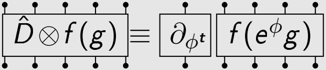
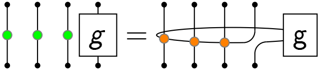
Group-derivative and transfer matrices

derivative of f w.r.t. $\log(g)$:





where $\color{orange}{\bullet} = u\,\mathbb{I} + i\mathcal{P}$, $\color{green}{\bullet} = u\,\mathbb{I} + i\hat{D}$

Transfer matrix $T(u) =$ $= (u + i\hat{D})^{\otimes L}\,\mathrm{Tr}\, g$

$$[T(u), T(v)] = 0 \qquad H = L - 2i\partial_u \log T(u)\big|_{u=0, g=1}$$

Arbitrary irrep (in auxiliary space):

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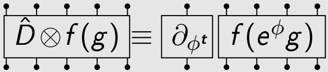

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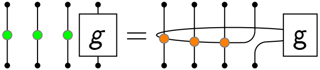

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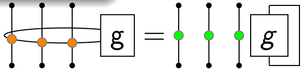

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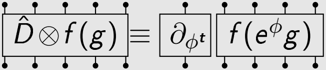

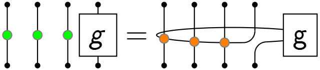

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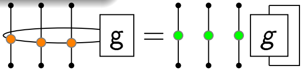

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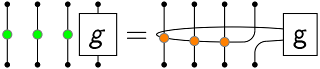

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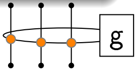
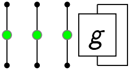
Transfer matrix $T(u) =$  $=$  $= (u + i\hat{D})^{\otimes L}\, \mathrm{Tr}\, g$

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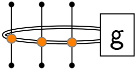
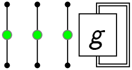
$T^{\lambda}(u) =$  $=$  $= (u + i\hat{D})^{\otimes L} \chi_{\lambda}(g)$

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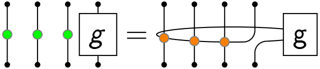

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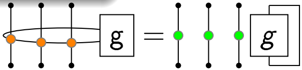

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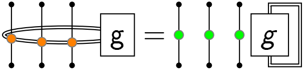


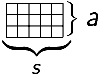
$= (u + i\hat{D})^{\otimes L} \chi_\lambda(g)$

$$[T^{\lambda}(u), T^{\mu}(v)] = 0$$

$$H = L - 2i\partial_u \log T^{\square}(u)\big|_{u=0, g=1}$$

Rectangular irrep: $T_{a,s}(u) = T^{\lambda_{a,s}}(u + i\frac{a-s}{2})$

where $\lambda_{a,s} =$



Hirota equation

Hirota equation

$$T_{a,s}(u+i/2)\,T_{a,s}(u-i/2) = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s-1}$$

Some arguments of the proof (combinatorial): [Kazakov Vieira 08]

$$\sum_{s\geq 0} z^s\, T_{1,s}(u+i\tfrac{s-1}{2}) = (u+\hat{D})^{\otimes L} \underbrace{\sum_{s\geq 0} z^s \chi_{1,s}(g)}_{w(z)}$$

$\hat{D}^{\otimes 3} w(z) =$ $w(z)$

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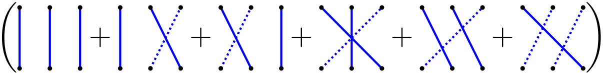
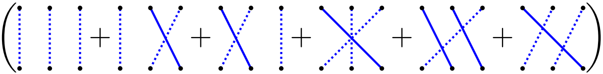
$\hat{D}^{\otimes 3} w(z) =$  $w(z)$

$(1+\hat{D})^{\otimes 3} w(z) =$  $w(z)$

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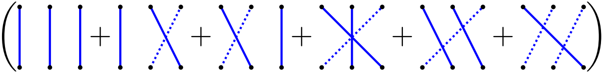
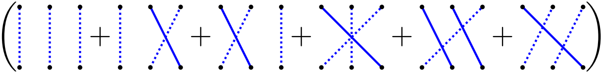
$\hat{D}^{\otimes 3} w(z) =$  $w(z)$

$(1+\hat{D})^{\otimes 3} w(z) =$  $w(z)$

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Characteristic property

$$z_2 \tau_{n+1} \left(\mathbf{t} - [z_2^{-1}]\right) \tau_n \left(\mathbf{t} - [z_1^{-1}]\right) - z_1 \tau_{n+1} \left(\mathbf{t} - [z_1^{-1}]\right) \tau_n \left(\mathbf{t} - [z_2^{-1}]\right) + (z_1 - z_2)\tau_{n+1}(\mathbf{t})\tau_n \left(\mathbf{t} - [z_1^{-1}] - [z_2^{-1}]\right) = 0.$$

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Classical ↔ quantum integrability

T-operators form a τ-function:

- Set of times $\mathbf{t} \leftrightsquigarrow$ representations λ :

$$\tau(u,\mathbf{t}) = \sum_{\lambda} \underbrace{s_\lambda(\mathbf{t})}_{\text{Schur polynomial}} \tau(u,\lambda) \qquad s_\lambda(\mathbf{t}) = \det\left(h_{\lambda_i - i + j}(\mathbf{t})\right)_{1\leq i,j\leq|\lambda|}$$

where $e^{\xi(\mathbf{t},z)} = \sum_{k\geq 0} h_k(\mathbf{t}) z^k, \qquad \xi(\mathbf{t},z) = \sum_{k\geq 1} t_k z^k$

If $\tau(u,\lambda) = T^\lambda(u) = (u + i\hat{D})^{\otimes L}\ \chi^\lambda(g)$, we get

$$\tau(u,\mathbf{t}) = (u + i\hat{D})^{\otimes L}\ e^{\sum_{k\geq 1} t_k \operatorname{tr}(g^k)}$$

- Then $\tau(u,\mathbf{t}+[z]) = (u + i\hat{D})^{\otimes L}\ w(z) e^{\sum_{k\geq 1} t_k \operatorname{tr}(g^k)}$

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Wronskian determinant (quantum integrability)

Generic solution of Hirota equation

[Krichever Lipan Wiegmann Zabrodin 97]

$$T^{\lambda}(u) = \frac{\det \left(x_j^{\lambda_k - k+1} \mathcal{Q}_j(u + i(\lambda_k - k + 1)) \right)_{1 \leq j,k \leq N}}{\Delta(x_1, \cdots, x_N)}$$

where $g = \mathrm{diag}(x_1, \cdots x_N)$; $\quad \Delta(x_1, \cdots, x_N) = \det \left(x_j^{1-k} \right)_{1 \leq j,k \leq N}$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$, ... commute among themselves and with T.

$$T = \begin{vmatrix} Q_1(\ldots) & Q_2(\ldots) & \ldots & Q_N(\ldots) \\ Q_1(\ldots) & Q_2(\ldots) & \ldots & Q_N(\ldots) \\ \vdots & \vdots & \ddots & \vdots \\ Q_1(\ldots) & Q_2(\ldots) & \ldots & Q_N(\ldots) \end{vmatrix} \qquad T_{a,s} = Q_{(a)}^{[+s]} \wedge Q_{(N-a)}^{[-s]}$$

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$\mathcal{Q}$'s are solution of

$$\begin{vmatrix} \mathcal{Q}_i & x_i \mathcal{Q}_i^{[+2]} & x_i^2 \mathcal{Q}_i^{[+4]} & \dots & x_i^N \mathcal{Q}_i^{[2N]} \\ T_{1,0} & T_{1,1}^{[-1]} & T_{1,2}^{[-2]} & \dots & T_{1,N}^{[-N]} \\ T_{1,1}^{[-1]} & T_{1,2}^{[-2]} & T_{1,3}^{[-3]} & \dots & T_{1,N+1}^{[-N-1]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{1,N-1}^{[-N+1]} & T_{1,N}^{[-N]} & T_{1,N+1}^{[-N-1]} & \dots & T_{1,2N-1}^{[-2N+1]} \end{vmatrix} = 0$$

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General rational τ-function [Krichever 78]

Polynomial τ-functions of this MKP hierarchy

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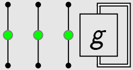
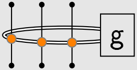
$$T^{\lambda}(u) = (u + i\hat{D})^{\otimes L} \chi^{\lambda}(g) =$$  $$=$$ 

Quantum integrability

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Bäcklund flow

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[Kazakov, SL, Tsuboi 12]

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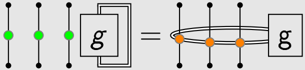

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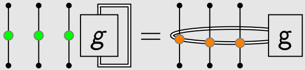

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- Equivalently, T-operator for a complicated irrep in auxiliary space

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Wronskian determinants in the $g \to \mathbb{I}$ limit

Twisted case:

- under the redefinition $Q_j = \mathcal{Q}_j \, x_j^{-i\,u}$

$$T^\lambda(u) = \frac{\det\left(x_j^{\lambda_k - k+1} \mathcal{Q}_j(u+i(\lambda_k-k+1))\right)_{1\leq j,k\leq N}}{\Delta(x_1,\cdots,x_N)} \rightsquigarrow \frac{\det\left(Q_j(u+i(\lambda_k-k+1))\right)_{1\leq j,k\leq N}}{\Delta(x_1,\cdots,x_N)\det g^{i\,u}}$$

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$\rightsquigarrow$ All lines of the determinant become equal

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$g \to \mathbb{I}$ **limit :**

"Rotational symmetry"

Wronskian determinant invariant under

$$Q_j \rightsquigarrow H_j{}^k \, Q_k$$

where the coefficients $H_j{}^k$ are i-periodic functions of u

(up to the normalization $\det H$)

- One should consider the limits of Q_1, $Q_1 - Q_2$, ...

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Hirota equation $\leftrightarrow$ Spectrum

Three conditions fix the spectrum

- Q_j is polynomial in u (for $1 \leq j \leq N$)
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The same works for $N > 2$

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Hirota equation ↔ Spectrum

Three conditions fix the twisted spectrum

- $x_j^{i\,u} Q_j$ is polynomial in u (for $1 \le j \le N$)
- $T^{\emptyset}(u) = u^L \det g^{i\,u}$
- $H = L - 2i \, \frac{\mathrm{d}}{\mathrm{du}} \log T^{\square}(u)\big|_{u=0}$

- Example for periodic $SU(2)$ spin chain

$$Q_1(u) = x_j^{-i\,u} \prod_j (u - \theta_j)$$

$$\det g^{-\frac{1}{2}-i\,u}(\theta_j - \tfrac{i}{2})^L = \cancel{Q_1(\theta_j)}\, Q_2(\theta_j - i) - Q_2(\theta_j) Q_1(\theta_j - i)$$

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- Twist: Q-functions multiplied by $x_i^{-i\,u}$

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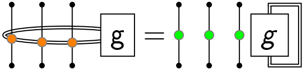
- 

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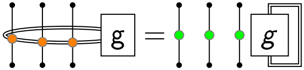
- 

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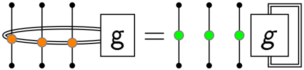
- 
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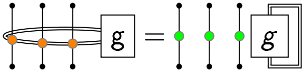
- 

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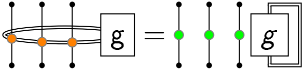
- 

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Thanks for your attention

Appendices

Disclaimer : The following slides are additional material, not necessarily part of the presentation

4. Master identity
5. Integral definition of τ-functions
6. Nested Bethe equations
7. Finite size spectrum of sigma models
   - Thermodynamic Bethe Ansatz
   - "Quantum Spectral Curve" for AdS/CFT

Master Identity

Combinatorics of coderivatives

"Master Identity" [Kazakov, S.L, Tsuboi 10]

when $\Pi = \prod_j w(t_j)$,

$$(t-z)\left[(u+1+\hat{D})^{\otimes L}\ w(z)w(t)\Pi\right]\cdot\left[(u+\hat{D})^{\otimes L}\ \Pi\right]$$
$$= t\left[(u+\hat{D})^{\otimes L}\ w(z)\Pi\right]\cdot\left[(u+1+\hat{D})^{\otimes L}\ w(t)\Pi\right]$$
$$- z\left[(u+1+\hat{D})^{\otimes L}\ w(z)\Pi\right]\cdot\left[(u+\hat{D})^{\otimes L}\ w(t)\Pi\right]$$

where $w(z) = \det \frac{1}{1-zg} = \sum_{s=0}^{\infty} z^s\ \chi_s(g)$

Integral definition of τ-functions

τ-functions are often defined as the functions such that $\forall n \geq n'$, $\forall \mathbf{t}, \mathbf{t}'$

Definition of τ-functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t} - [z^{-1}]) \tau_{n'}(\mathbf{t}' + [z^{-1}]) dz = 0$$

where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots \right)$, and $\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k$, and $\mathcal{C}$ encircles the singularities of $\tau_n(\mathbf{t} - [z^{-1}])\tau_{n'}(\mathbf{t}' + [z^{-1}])$ (typically finite), but not the singularities of $e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'}$ (typically at infinity).

- Baker-Akhiezer Ψ-functions:

$$\Psi_n(\mathbf{t}, z) = z^n e^{\xi(\mathbf{t},z)} \frac{\tau_n(\mathbf{t}-[z^{-1}])}{\tau_n(\mathbf{t})}$$

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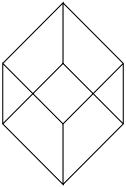
Definition of τ-functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t} - [z^{-1}]) \tau_{n'}(\mathbf{t}' + [z^{-1}]) dz = 0$$

where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$, and $\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k$, and $\mathcal{C}$ encircles the singularities of $\tau_n(\mathbf{t} - [z^{-1}])\tau_{n'}(\mathbf{t}' + [z^{-1}])$ (typically finite), but not the singularities of $e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'}$ (typically at infinity).

- Baker-Akhiezer Ψ-functions:
$$\Psi_n(\mathbf{t}, z) = z^n e^{\xi(\mathbf{t},z)} \frac{\tau_n(\mathbf{t}-[z^{-1}])}{\tau_n(\mathbf{t})}$$

Hirota equation $\leftrightarrow$ Spectrum (nested Bethe equations)



Three conditions fix the spectrum

- Q_j is polynomial in u (for $1 \leq j \leq N$)
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$$Q_I = Q_{i_1 i_2 \dots i_k} = \begin{vmatrix} Q_{i_1}(u+i\frac{k-1}{2}) & \dots & Q_{i_k}(u+i\frac{k-1}{2}) \\ \vdots & \ddots & \vdots \\ Q_{i_1}(u-i\frac{k-1}{2}) & \dots & Q_{i_k}(u-i\frac{k-1}{2}) \end{vmatrix} = \det\left(Q_j(u+\tfrac{i}{2}(|I|+1-2k))\right)_{\substack{j\in I \\ 1\leq k\leq |I|}}$$

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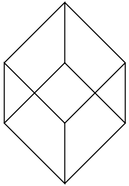

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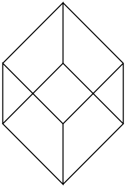

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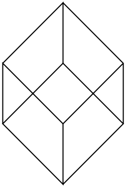

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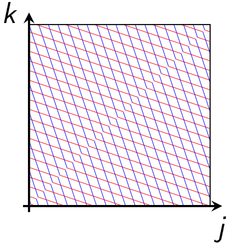
Integrable field theories

Bethe Ansatz of the form $\psi(n_1, n_2, \cdots, n_M) \equiv \sum_{\sigma \in \mathcal{S}^M} \mathcal{A}_\sigma e^{i \sum_k p_{\sigma(k)} n_k}$

$\rightsquigarrow$ wave-function of the eigenstates of several theories such that

- The space is one-dimensional and there are periodic boundary conditions.
- The interactions are local.
- A factorization formula holds



One can argue that it is sufficient to have infinitely many conserved charges

[Zamolodchikov Zamolodchikov 79]

- "Locality" requires a large spatial period

$\rightsquigarrow$ Question of the finite size effects

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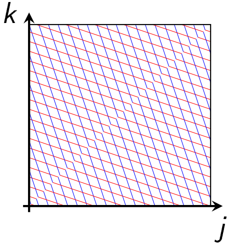

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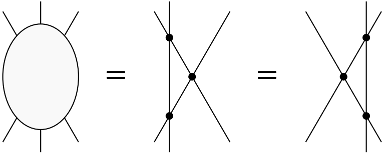

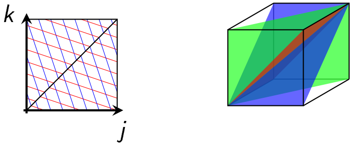

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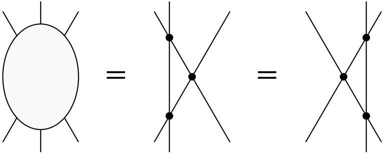

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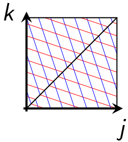

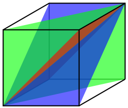

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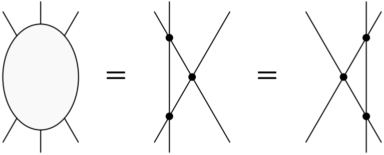

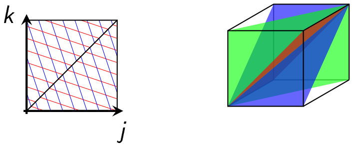

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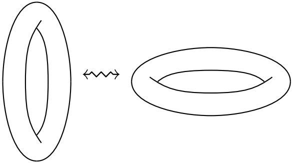
Thermodynamic Bethe Ansatz

- *Matsubara Trick*: "double Wick Rotation"

  finite size $\leftrightsquigarrow$ finite temperature



- At finite temperature, the Bethe equations give rise to several different types of bound states

  $\rightsquigarrow$ introduce one density of excitations (as a function of the rapidity) for each type of bound state.

- For vacuum, densities given by T-functions $T_{a,s}(u)$ obeying the Hirota equation

  $T_{a,s}(u+i/2)T_{a,s}(u-i/2) = T_{a+1,s}(u)T_{a-1,s}(u) + T_{a,s+1}(u)T_{a,s-1}(u)$

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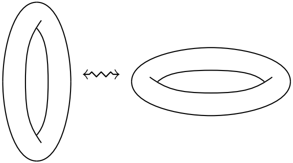

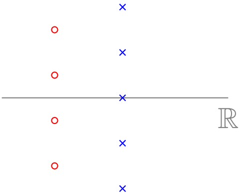

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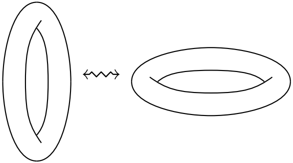

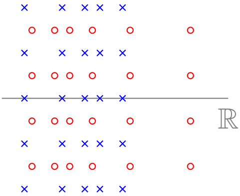

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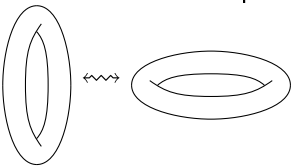

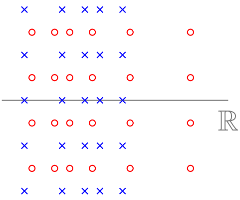

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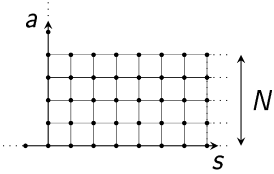
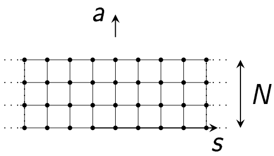
Boundary conditions for Hirota equation

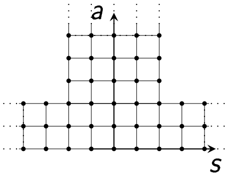
- **Symmetry Group $\leftrightsquigarrow$ boundary condition**



$SU(N)$ spin chain



$SU(N) \times SU(N)$
principal chiral model



AdS_5/CFT_4
$PSU(2,2|4)$

$\rightsquigarrow$ $T^{a,s}(u) = \det(Q_j(u+\dots))_{1\leq j,k\leq \dots}$

- Additional information: analyticity properties

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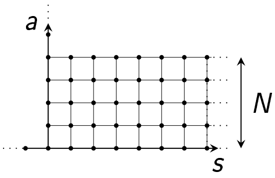

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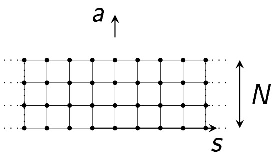

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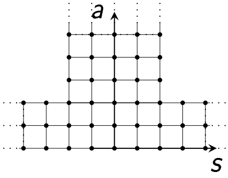

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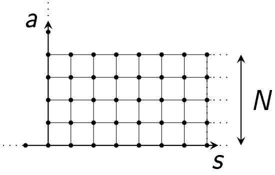

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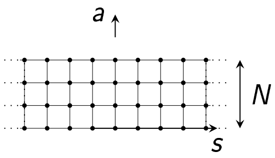

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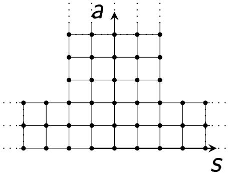

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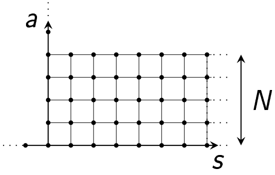

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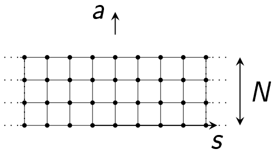

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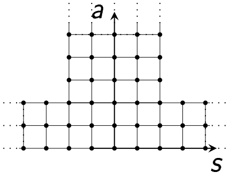

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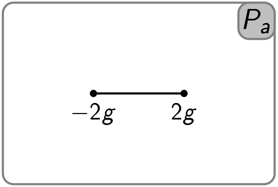
Analyticity requirements

Quantum Spectral Curve

- Functions P_a and P^a holomorphic on $\mathbb{C} \setminus [-2g, 2g]$ $\quad (1 \leq a \leq 4)$
  where $g = \frac{\sqrt{\lambda}}{4\pi}$, $\lambda = g_{YM} N_c^2$.



- Denote by tilde the analytic continuation around $\pm 2g$.

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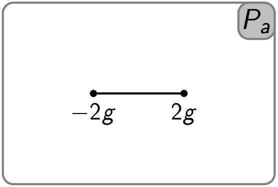

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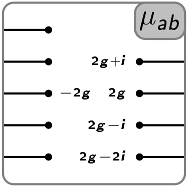

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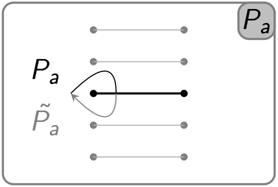

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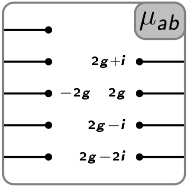

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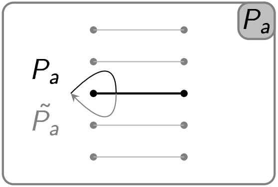

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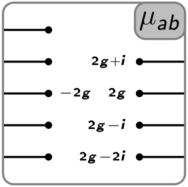

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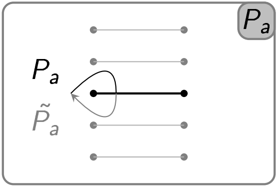

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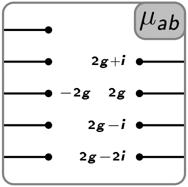

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Non-exhaustive list

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Interpretation of the QSC

Ambiguity in shifts

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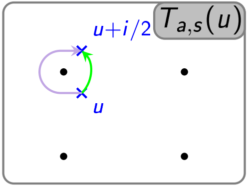

Symmetries

$$T^{a,s}(u) = \det(Q_j(u+\dots))_{1\leq j,k\leq \dots}$$

Invariant under

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$$\tilde{P}_a = \mu_{ab} P^b$$

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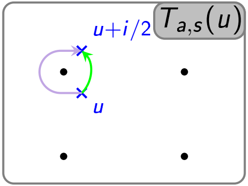

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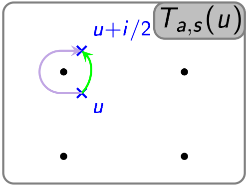

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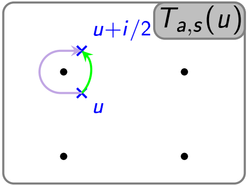

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