Classical and quantum integrability : from spin chains to the AdS/CFT duality.

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[arXiv:1112.3310] A. Alexandrov, V. Kazakov, SL, Z.Tsuboi, A. Zabrodin

[arXiv:1110.0562] N.Gromov, V. Kazakov, SL, D. Volin

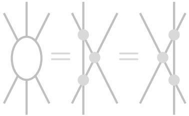
[arXiv:1302.1135] SL, D. Volin

Laboratoire de Physique, *ENS Lyon*, February 14, 2013

Quantum Integrability

Quantum integrability is a property of very specific models (spin chains or quantum field theories), which usually have

- 1+1 dimensions
- local interactions
- many conserved charges
- specific boundary conditions (eg periodic)



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Properties of integrable models

- n-points interactions factorize into 2-points interactions
- the exact diagonalization of the Hamiltonian reduces to solving the Bethe Equation(s).

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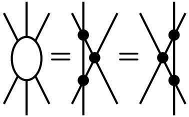

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1+1 D integrable field theories

Bethe Ansatz : wavefunction for a large volume

- planar waves when particles are far from each other
- an *S-matrix* describes 2-points interactions

⇒ Bethe equations

This ansatz works if the space is peridic with a very large period

Aim in this talk

The last part of this talk will address the question of the spectrum of such a theory in finite size, when the above asumptions cannot be used.

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AdS/CFT duality (conjecture)

AdS	**CFT**
"Type IIB" string theory on $AdS_5 \times S^5$	3+1 dimensional conformal field theory "$\mathcal{N} = 4$ super Yang-Mills"

strong coupling regime ↔ weak coupling regime

weak coupling regime ↔ strong coupling regime

Energy of string excitations ↔ Conformal dimensions of operators

Integrability

Many integrable features were understood in the *planar limit* of this duality, in particular with respect to the spectrum.

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Energy of string excitations	↔	Conformal dimensions of operators

(Strong coupling regime on the AdS side ↔ weak coupling regime on the CFT side; weak coupling regime on the AdS side ↔ strong coupling regime on the CFT side.)

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Outline

0. Motivation

1. Solving $GL(K|M)$ spin chains
   - Hirota equation for T-operators
   - Solution of Hirota equation through Q-operators
   - Classical integrability of this quantum system

2. Finite size effects in integrable field theories
   - Y-system $\rightsquigarrow$ spectrum
   - Analyticity properties of Q-functions

3. Weak coupling expansion in AdS/CFT

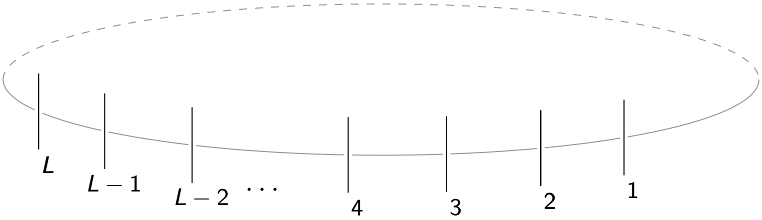
Heisenberg "XXX" spin chain

Construction of T-operators

$$H = -\sum_i \vec{\sigma_i} \cdot \vec{\sigma_{i+1}} = L - 2 \left. \frac{\mathrm{d}}{\mathrm{du}} \log T(u) \right|_{u=0}$$

$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$$

operator on the Hilbert space $(\mathbb{C}^2)^{\otimes L}$



permutation operator: $\mathcal{P}_{1,2}|\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle = |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$

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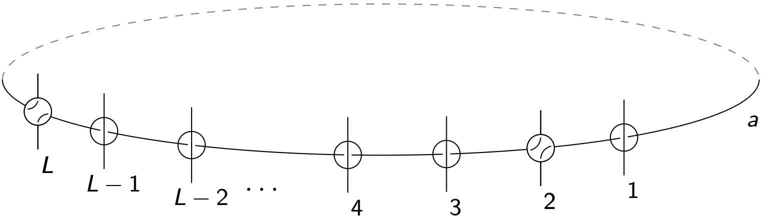

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$GL(K)$ spin chains

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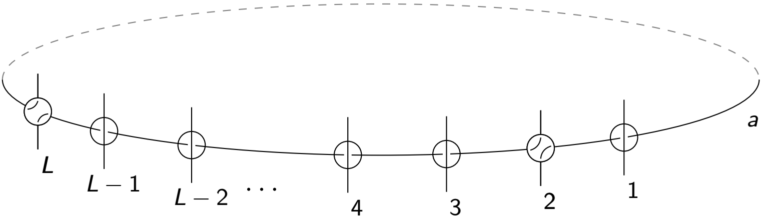

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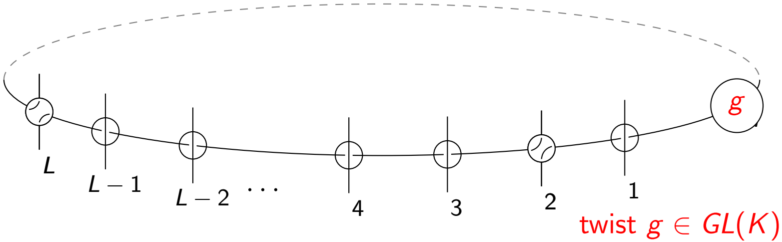

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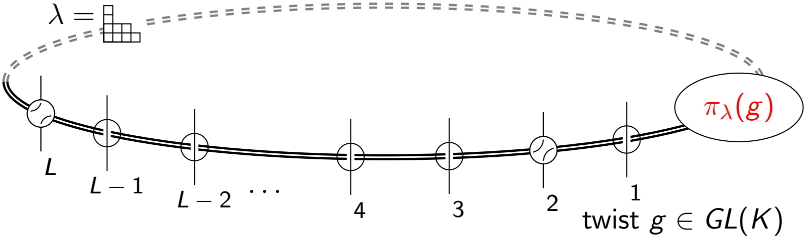

generalized permutation operator:

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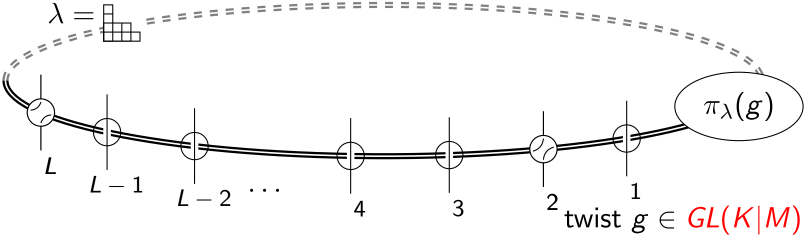

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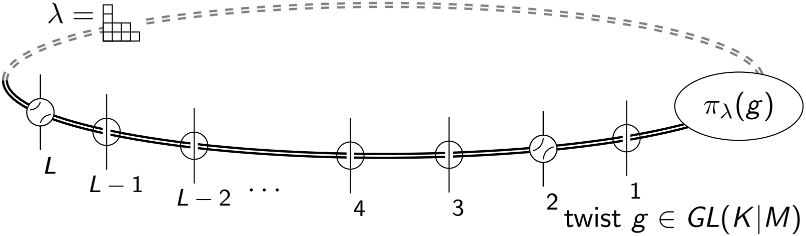

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T-operators $\leftrightsquigarrow$ characters

+ Cherednik-Bazhanov-Reshetikhin formula

$$T^\lambda(u) = \mathrm{tr}_a\left(((u-\xi_L)\mathbb{I}+\mathcal{P}_{L,a})\cdots((u-\xi_1)\mathbb{I}+\mathcal{P}_{1,a})\cdot\pi_\lambda(g)\right)$$

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- In general $\qquad u_i \equiv u - \xi_i$

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Rectangular representation : $a, s \leftrightarrow \lambda =$

CBR Determinant identity

[Cherednik 86] [Bazhanov-Reshetikhin 90] [Kazakov Vieira 08]

$$\chi^\lambda(g) = \det\left(\chi^{1,\lambda_i+j-i}(g)\right)_{1\le i,j\le|\lambda|}$$

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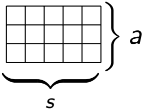
Rectangular representation : $a, s \leftrightarrow \lambda =$ 

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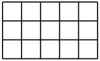
Rectangular representation : $a, s \leftrightarrow \lambda =$  $\Big\}\, a$

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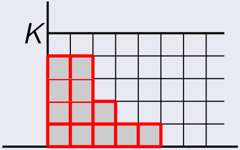
- "equivalent" to the Hirota equation :

$$T^{a,s}(u+1)\cdot T^{a,s}(u) =$$
$$T^{a+1,s}(u+1)\cdot T^{a-1,s}(u) + T^{a,s-1}(u+1)\cdot T^{a,s+1}(u)$$

"Fat hooks" and "Bäcklund Flow"

Authorised Young diagrams for a given symmetry group

GL(K) symmetry



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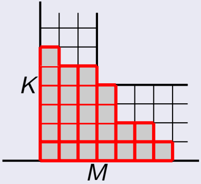

Hirota equation solved by gradually reducing the size of the "fat hook" [Krichever, Lipan, Wiegmann & Zabrodin 97] [Kazakov Sorin Zabrodin 08]

$$\underbrace{\chi_\lambda\begin{pmatrix}x_1&0&0\\0&x_2&0\\0&0&x_3\end{pmatrix}}_{\in GL(2|1)} \rightsquigarrow \underbrace{\chi_\lambda\begin{pmatrix}x_2&0\\0&x_3\end{pmatrix}}_{\in GL(1|1)} \rightsquigarrow \underbrace{\chi_\lambda\begin{pmatrix}x_2\end{pmatrix}}_{\in GL(1)} \rightsquigarrow \underbrace{\chi_\lambda\begin{pmatrix}\ \end{pmatrix}}_{\in\{1\}}$$

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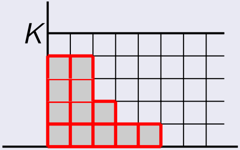

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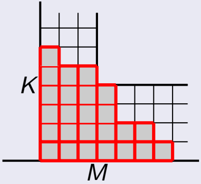

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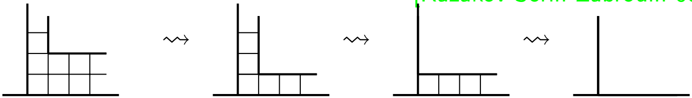

$$\chi_\lambda \underbrace{\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}}_{\in GL(2|1)} \rightsquigarrow \chi_\lambda \underbrace{\begin{pmatrix} x_2 & 0 \\ 0 & x_3 \end{pmatrix}}_{\in GL(1|1)} \rightsquigarrow \chi_\lambda \underbrace{\begin{pmatrix} x_2 \end{pmatrix}}_{\in GL(1)} \rightsquigarrow \chi_\lambda \underbrace{\begin{pmatrix} \quad \end{pmatrix}}_{\in \{1\}}$$

Bäcklund Transformations

linear system

Bäcklund Transformations

if $T^{a,s}(u)$ is a solution of Hirota equation and

$$\begin{cases} T^{a+1,s}(u)F^{a,s}(u) - T^{a,s}(u)F^{a+1,s}(u) \\ \qquad = \underbrace{x_j}_{\text{eigenvalue of } g} T^{a+1,s-1}(u+1)F^{a,s+1}(u-1), \\ T^{a,s+1}(u)F^{a,s}(u) - T^{a,s}(u)F^{a,s+1}(u) \\ \qquad = x_j T^{a+1,s}(u+1)F^{a-1,s+1}(u-1). \end{cases}$$

Then $F^{a,s}(u)$ is a solution of Hirota equation.

Moreover, if $T^{a,s}(u) = 0$, outside the $(K|M)$ “fat hook”, one can choose $F^{a,s}(u) = 0$ outside the $(K-1|M)$ “fat hook”.

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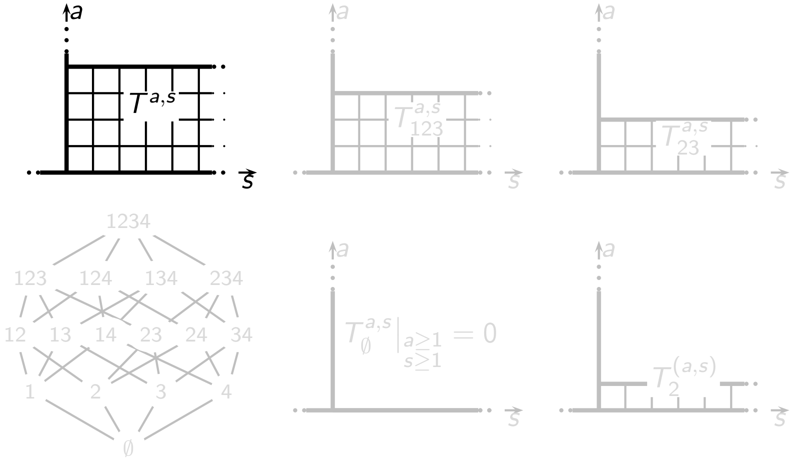
Hasse Diagram

example of $GL(4)$ Bäcklund flow



⇝ Defines 2^4 Q-operators, lying on the nodes of this *Hasse Diagram*

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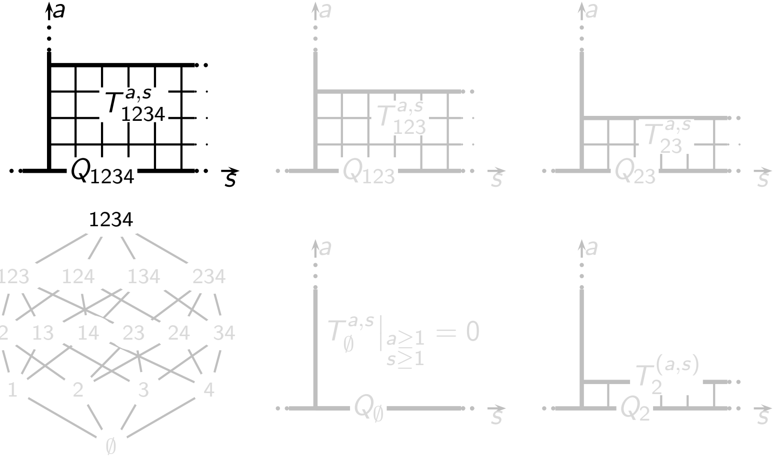

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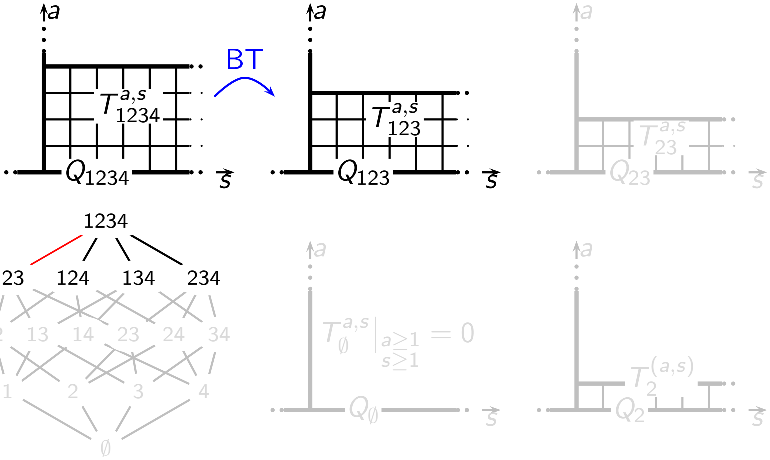

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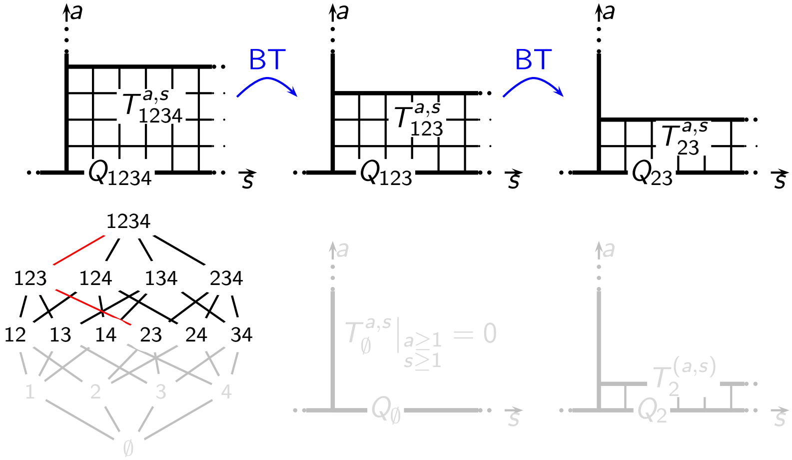

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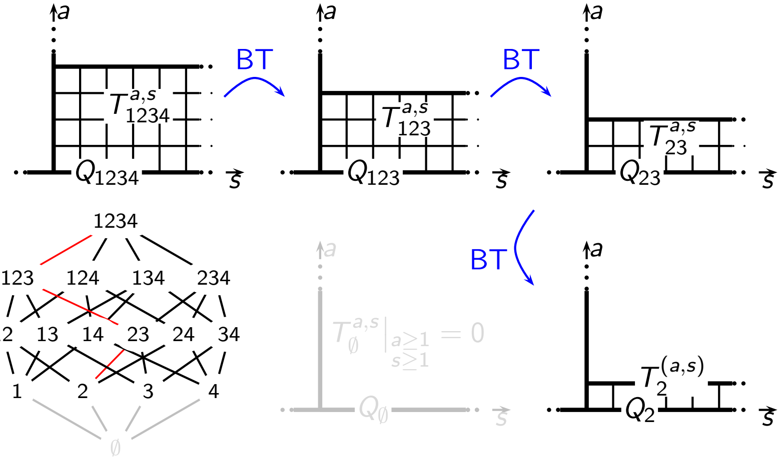

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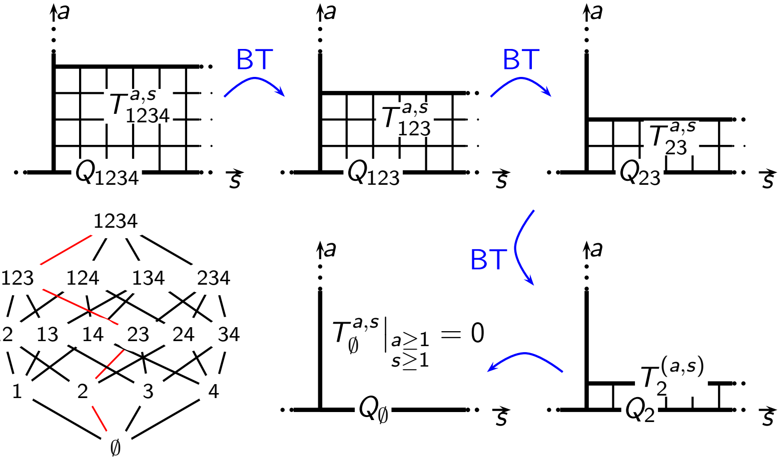

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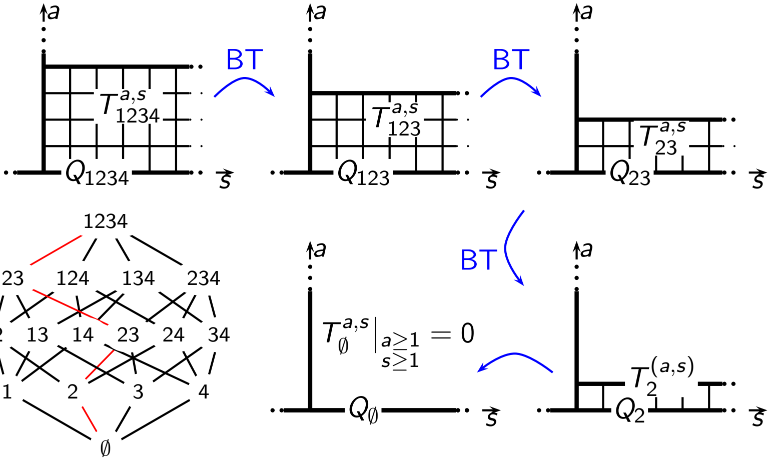

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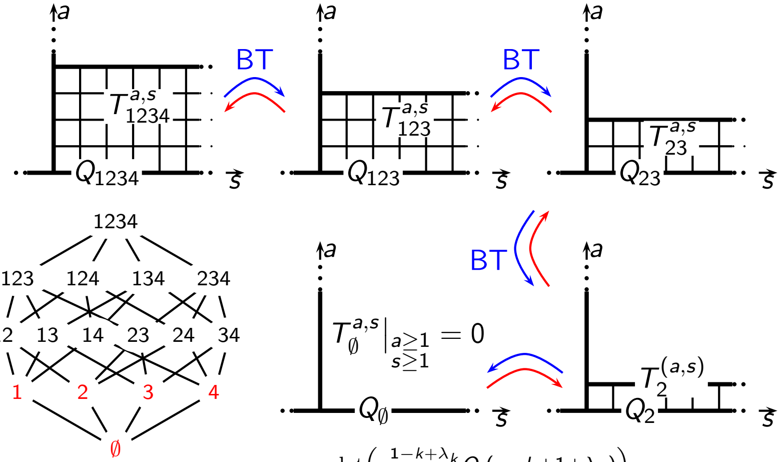


$$\rightsquigarrow \quad T^{\lambda}(u) = Q_{\emptyset}(u-K)\cdot \frac{\det\left(x_j^{1-k+\lambda_k}\, Q_j(u-k+1+\lambda_k)\right)_{1\leq j,k\leq K}}{\Delta(x_1,\cdots,x_K)\prod_{k=1}^{K} Q_{\emptyset}(u-k+\lambda_k)}$$

where $\Delta(x_1,\cdots,x_K) = \det\left(x_j^{1-k}\right)_{1\leq j,k\leq K}$

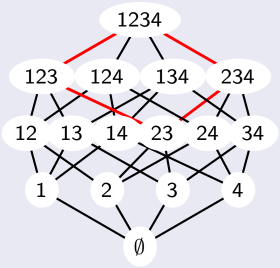
QQ-relations and Bethe Equations

The consistency of the construction imposes the QQ-relations

$$(x_i - x_j)\, Q_{\mathrm{I}}(u-1) Q_{\mathrm{I},i,j}(u) = x_i\, Q_{\mathrm{I},j}(u-1) Q_{\mathrm{I},i}(u) - x_j\, Q_{\mathrm{I},j}(u) Q_{\mathrm{I},i}(u-1)$$

example : $\mathrm{I} = \{23\}, i = 1, j = 4$

$$(x_1 - x_4)\, Q_{23}(u-2) Q_{1234}(u) = x_1\, Q_{234}(u-1) Q_{123}(u) - x_4\, Q_{234}(u) Q_{123}(u-1)$$



The relation involves Q-operators lying on the same facet of the Hasse diagram

Bäcklund flow for this spin chain [Krichever, Lipan, Wiegmann, Zabrodin 97] [Kazakov Sorin Zabrodin 08]

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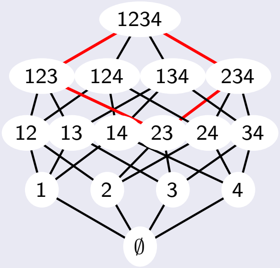
Consequences: diagonalization of the Q-operators



- Q-operators are polynomial $\Rightarrow$ parameterized by their roots
- Zeroes of the left hand side have to be zereoes of the right hand side $\rightsquigarrow$ Bethe equations

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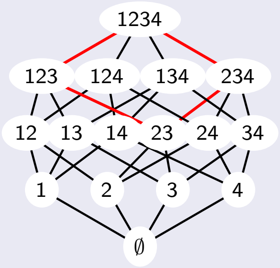

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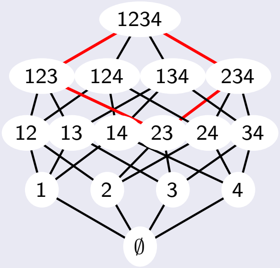

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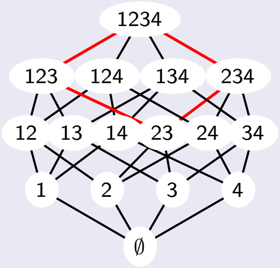

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Outline

τ-functions of the MKP hierarchy

- A τ-function of the *MKP hierarchy* is a function of a variable n and an infinite set $\mathbf{t} = (t_1, t_2, \cdots)$ of "times", such that $\forall n \geq n'$, $\forall \mathbf{t}, \mathbf{t}'$

Definition of τ-functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}', z)} z^{n-n'} \tau_n(\mathbf{t} - [z^{-1}]) \tau_{n'}(\mathbf{t}' + [z^{-1}]) dz = 0$$

where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$, $\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k$, and $\mathcal{C}$ encircles the singularities of $\tau_n(\mathbf{t} - [z^{-1}])\tau_{n'}(\mathbf{t}' + [z^{-1}])$ (typically finite), but not the singularities of $e^{\xi(\mathbf{t}-\mathbf{t}', z)} z^{n-n'}$ (typically at infinity).

- An example of such τ-function is the expectation value

$$\tau_n(\mathbf{t}) = \langle n | e^{J_+(\mathbf{t})} G | n \rangle$$

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$$\tau_n(\mathbf{t}) = \langle n| e^{J_+(\mathbf{t})} G |n\rangle$$

over an infinite set of fermionic oscillators ($\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$), where $G = \exp\left(\sum_{i,k\in\mathbb{Z}} A_{ik} \psi_i^\dagger \psi_k\right)$ and $J_+ = \sum_{k\geq 1} t_k J_k$, where $J_k = \sum_{j\in\mathbb{Z}} \psi_j \psi_{j+k}^\dagger$. (and $\psi_n |n\rangle = |n+1\rangle$)

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Characteristic property

τ-functions are characterised by

$$z_2 \tau_{n+1}\left(\mathbf{t} - [z_2^{-1}]\right) \tau_n\left(\mathbf{t} - [z_1^{-1}]\right) - z_1 \tau_{n+1}\left(\mathbf{t} - [z_1^{-1}]\right) \tau_n\left(\mathbf{t} - [z_2^{-1}]\right) + (z_1 - z_2)\tau_{n+1}(\mathbf{t})\tau_n\left(\mathbf{t} - [z_1^{-1}] - [z_2^{-1}]\right) = 0.$$

(obtained from $n' = n - 1$ and $\mathbf{t}' = \mathbf{t} - [z_1{}^{-1}] - [z_1{}^{-1}]$)

Spin-chains ⇜⇝ MKP hierarchy

T-operators are τ-functions

- Set of times $\mathbf{t}$ ⇜⇝ representations λ :

$$\tau(u,\mathbf{t}) = \sum_\lambda \underbrace{s_\lambda(\mathbf{t})}_{\text{Schur polynomial}} \tau(u,\lambda) \qquad s_\lambda(\mathbf{t}) = \det\left(h_{\lambda_i - i + j}(\mathbf{t})\right)_{1\le i,j\le|\lambda|}$$

where $e^{\xi(\mathbf{t},z)} = \sum_{k\ge 0} h_k(\mathbf{t}) z^k$

If $\tau(u,\lambda) = T^\lambda(u) = \bigotimes_{i=1}^L (u_i + \hat{D})\ \chi^\lambda(g)$, we get

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- The function
$\tau(u, \mathbf{t}) = \bigotimes_{i=1}^{L}(u_i + \hat{D})\ e^{\sum_{k\geq 1} t_k \mathrm{tr}(g^k)}$
is a τ-function of the MKP hierarchy.
- The Bäcklund transformation is the transformation
$\tau(u, \mathbf{t}) \rightsquigarrow \mathrm{Res}_{z=x_i}\tau(u+1, \mathbf{t}+[z^{-1}])$.
- Fermionic realisation of this τ-function :

$$\tau(n, \mathbf{t}) = \langle n|\, e^{J_+(\mathbf{t})}\Psi_1 \dots \Psi_K \,|\, n-K\rangle \ ,$$

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[Alexandrov, Kazakov, S.L., Tsuboi, Zabrodin 11]

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1+1 D integrable field theories

Bethe Ansatz: wavefunction for a large volume

- planar waves when particles are far from each other
- an *S-matrix* describes 2-points interactions

$\Rightarrow$ Bethe equations

- "Thermodynamic Bethe Ansatz" for finite size effects : "double Wick Rotation"

finite size $\leftrightsquigarrow$ finite temperature

- At finite temperature, the Bethe equations give rise to several different types of bound states
$\rightsquigarrow$ introduce one density of excitations (as a function of the rapidity) for each type of bound state.

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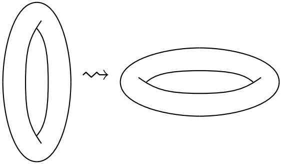

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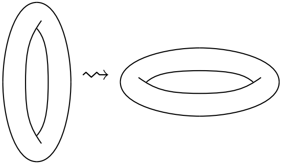

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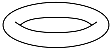
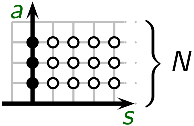
Y- and T-systems

case of the SU(N) "Gross Neveu" 1+1 dimensional field theory

TBA : 

bound states labelled by $a \in \{1, \ldots, N-1\}$ and $s \in \mathbb{N}$



their densities obey

$$\begin{cases} \frac{Y_{a,s}{}^{+}\, Y_{a,s}{}^{-}}{Y_{a+1,s}\, Y_{a+1,s}} = \frac{1+Y_{a,s+1}}{1+Y_{a+1,s}} \frac{1+Y_{a,s-1}}{1+Y_{a-1,s}} \\ \text{analyticity constraints} \end{cases}$$

where $Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$

Lattice regularization

Representation labelled by a and s

$$\Leftrightarrow T_{a,s}{}^{+} T_{a,s}{}^{-} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$

$$\text{if } Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}.$$

[Gromov Kazakov Vieira 09] [Bombardelli Fioravanti Tateo 09]

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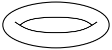
TBA : 

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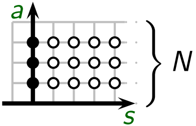

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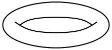
TBA : 

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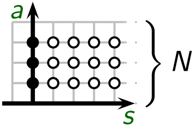

their densities obey

$$\begin{cases} \dfrac{Y_{a,s}{}^{+}\, Y_{a,s}{}^{-}}{Y_{a+1,s}\, Y_{a+1,s}} = \dfrac{1+Y_{a,s+1}}{1+Y_{a+1,s}} \dfrac{1+Y_{a,s-1}}{1+Y_{a-1,s}} \\ \text{analyticity constraints} \end{cases}$$

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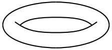
[Gromov Kazakov Vieira 09] [Bombardelli Fioravanti Tateo 09]

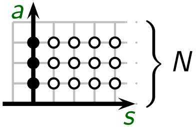
[Autyunov Frolov 09]

Y- and T-systems

case of the SU(N) "Gross Neveu" 1+1 dimensional field theory

TBA : 

bound states labelled by $a \in \{1, \dots, N-1\}$ and $s \in \mathbb{N}$



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Lattice regularization

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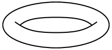
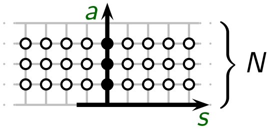
Y- and T-systems

case of the SU(N)×SU(N) Principal chiral model

TBA : 

bound states labelled by $a \in \{1, \dots, N-1\}$ and $s \in \mathbb{Z}$



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[Gromov Kazakov Vieira 09] [Bombardelli Fioravanti Tateo 09]

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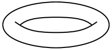
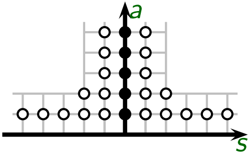
Y- and T-systems

case of the planar limit of the AdS/CFT duality

TBA : 

bound states labelled by integers a and s



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Analyticity properties of Q-functions

$\rightsquigarrow$ simple equations

- Typical solution of Hirota equation :
  $T_{a,s} = \det_{k,l}\Big(Q_k(u + f(a,s,l))\Big)$, where T and Q are the eigenvalues of T and Q operators.
- Some Q functions are holomorphic functions of u in the upper half plane $\mathrm{Im}(u) > 0$ (others are holomorphic in the lower half plane).

$\Rightarrow$ Each Q-function reduces to a real function on the real axis.

- $\rightsquigarrow$ Additional analyticity conditions (typically at $u \to \infty$) give rise to a finite set of non-linear integral equations (FiNLIE)[S.L. Kazakov 10] [Gromov Kazakov S.L. Volin 11]

Statement

The outcome of these works is that the (previously conjectured) Thermodynamic Bethe Ansatz is proven to be equivalent to analyticity conditions on the Q-functions.

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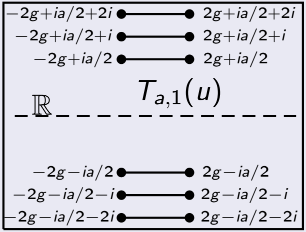
Analyticity properties for AdS/CFT

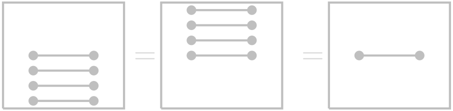
Branch points

The Y-, T- and Q-functions have square-root-types branch points at positions $\pm 2g + n\mathrm{i}$ or $\pm 2g + (n+\frac{1}{2})\mathrm{i}$, where $n \in \mathbb{Z}$.



- New symmetries identified, expressed very simply in terms of Q-functions :

For instance, there exists a Q-function Q_1 such that $Q_1 = -\bar{Q}_1$. Then



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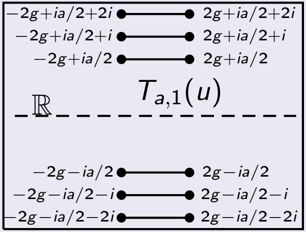

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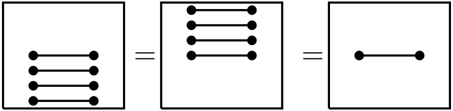
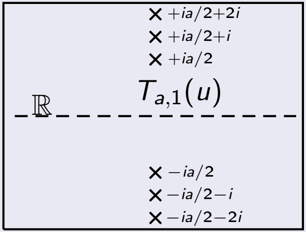


Weak coupling expansion in AdS/CFT [S.L. Volin Serban 12]

Weak coupling

When $g \ll 1$, the branch points collide to give rise to ladders of poles.



Q-functions can then be expressed analytically in terms of sums of the type

$$\sum_{0 \leq n_1 < n_2 < \ldots < n_k < \infty} \frac{1}{(u + \mathrm{i}\, n_1)^{m_1} (u + \mathrm{i}\, n_2)^{m_2} \ldots (u + \mathrm{i}\, n_k)^{m_k}} .$$

Conformal dimension of the Konishi operator

$$\Delta_{\text{Konishi}} = 4 + 12\, g^2 - 48\, g^4 + 336\, g^6 + 96\, g^8(-26 + 6\,\zeta_3 - 15\,\zeta_5) - 96\, g^{10}(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7)$$

[Bajnok Egedüs Janik Łukowski 09]

[Eden Heslop Korchemsky Smirnov Sokatchev 12]

Conformal dimension of the Konishi operator

$$\begin{aligned}\Delta_{\mathrm{Konishi}} =& 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + 96\,g^8(-26 + 6\,\zeta_3 - 15\,\zeta_5)\\ &- 96\,g^{10}(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7)\\ &- 48\,g^{12}(160 + 5472\,\zeta_3 - 3240\,\zeta_3\,\zeta_5 + 432\,\zeta_3^2\\ &- 2340\,\zeta_5 - 1575\,\zeta_7 + 10206\,\zeta_9)\end{aligned}$$

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[SL Volin Serban 12]

$$\begin{aligned}& + 48\, g^{14}(-44480 + 108960\,\zeta_3 + 8568\,\zeta_3\,\zeta_5\\ & \quad - 40320\,\zeta_3\,\zeta_7 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 - 4776\,\zeta_5\\ & \quad - 20700\,\zeta_5^2 - 26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11})\end{aligned}$$

[Bajnok Janik 12]

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& + 48\, g^{16}(1133504 + 263736\,\zeta_2\,\zeta_9 - 1739520\,\zeta_3 \\
& \qquad - 90720\,\zeta_3\,\zeta_5 - 129780\,\zeta_3\,\zeta_7 + 78408\,\zeta_3\,\zeta_8 \\
& \qquad + 483840\,\zeta_3\,\zeta_9 + 165312\,\zeta_3^2 - 82080\,\zeta_3^2\,\zeta_5 \\
& \qquad + 41472\,\zeta_3^3 + 178200\,\zeta_4\,\zeta_7 - 409968\,\zeta_5 \\
& \qquad + 121176\,\zeta_5\,\zeta_6 + 463680\,\zeta_5\,\zeta_7 + 49680\,\zeta_5^2 \\
& \qquad + 455598\,\zeta_7 + 194328\,\zeta_9 - 555291\,\zeta_{11} \\
& \qquad - 2208492\,\zeta_{13} - 14256\,\zeta_{1,2,8}) + \mathcal{O}(g^{18})
\end{aligned}
$$

[SL Volin 13]

Conclusion

- Rational spin chains (very well understood)
  - Bäcklund Flow to gradually simplify the system
  - Bethe Equations
  - Expression of the Hamiltonian from T and Q-functions
- For these rational spin chains, the classical integrability of τ-functions sheds light on the whole constriction, and helps for generalizations.
- For finite-size effects in integrable field theories, gives a guideline to write FiNLIE
  - Simple parameterization
  - Clearer analyticity properties
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How general is This procedure ?

- Open question for these finite-size effects is : Can we prove these analyticity properties ?

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Thank you !

- S. Leurent and D. Volin, "Multiple zeta functions and double wrapping in planar N=4 SYM," arXiv : 1302.1135.
- S. Leurent, D. Serban, and D. Volin, "Six-loop Konishi anomalous dimension from the Y-system," *Phys.Rev.Lett.* **109** (2012) 241601, arXiv : 1209.0749.
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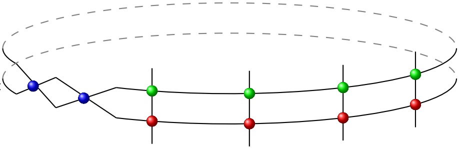
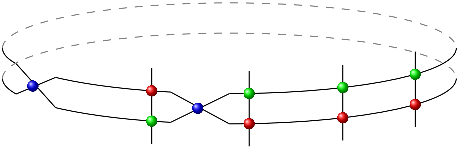
- N. Gromov, V. Kazakov, S. Leurent, and Z. Tsuboi, "Wronskian Solution for AdS/CFT Y-system," *JHEP* **1101** (2011) 155, arXiv : 1010.2720.
- V. Kazakov and S. Leurent, "Finite Size Spectrum of SU(N) Principal Chiral Field from Discrete Hirota Dynamics," arXiv : 1007.1770.

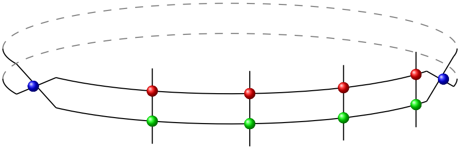
Appendices

Disclaimer : The following slides are additional material, not necessarily part of the presentation

Commutation of T-operators

$T(u) \cdot T(v) =$ 

$=$ 

$= \cdots =$ 

$= T(v) \cdot T(u)$

back

Expression of T through co-derivative

- $\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi} g)\Big|_{\phi=0}$ $\qquad \phi \in GL(K)$
- If $f(g)$ acts on $\mathcal{H}$, then $\hat{D} \otimes f$ acts on $\tilde{\mathcal{H}} = \mathbb{C}^K \otimes \mathcal{H}$
- $\hat{D} \otimes g = \mathcal{P}(1 \otimes g)$ and Leibnitz rule :
  $\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot \left[\hat{D} \otimes \tilde{f}\right] + \left[\hat{D} \otimes f\right] \cdot \left[\mathbb{I} \otimes \tilde{f}\right]$
  $\rightsquigarrow$ compute any $\hat{D} \otimes f(g)$
- $\hat{D} \otimes \pi_\lambda(g) = \left[\sum_{\alpha,\beta} \underbrace{e_{\beta\alpha}}_{generator} \otimes \underbrace{\pi_\lambda(e_{\alpha\beta})}_{generator}\right] \cdot \mathbb{I} \otimes \pi_\lambda(e_{\alpha\beta})$

hence

$$((u-\xi_L)\mathbb{I} + \mathcal{P}_{L,a}) \cdots ((u-\xi_1)\mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_\lambda(g) = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D}) \pi_\lambda(g)$$

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back

Thermodynamic Bethe Ansatz

⇝ Equations of the form
$$Y_{a,s}(u) = -L\, E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right)$$
$$+\langle Source\ \ Terms\rangle$$

- Vacuum energy
$$E_0 = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$$
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- Extra assumption : Excited states obey the same equations.
Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles
- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

⇒ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

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Parameterzation of Q-functions

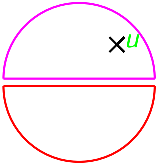
A simple Riemann-Hilbert Problem

Form the Cauchy theorem, we get

If $Q(u)$ is an analytic function on the upper half plane (when $\mathrm{Im}(u) > 0$), and $Q(u) \ll 1/u$ in the vicinity of ∞, then

$$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{Q(v)-\bar{Q}(v)}{v-u}\mathrm{d}v = \begin{cases} Q(u) & \text{if } \mathrm{Im}(u) > 0 \\ \bar{Q}(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$

where $\bar{Q}(u)$ is the complex-conjugate of $Q(\bar{u})$.



Indeed, if $\mathrm{Im}(u) > 0$, then

$\frac{1}{2i\pi}\int_{\text{upwards}} \frac{Q(v)}{v-u}\mathrm{d}v = Q(u)$ and

$\frac{1}{2i\pi}\int_{\text{downwards}} \frac{\bar{Q}(v)}{v-u}\mathrm{d}v = 0$