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Integrable field theorie

Y-systems Analyticity properties o

Weak coupling AdS/CFT

Integrability and analyticity of the conserved charges of quantum spin chains : a guideline for the AdS/CFT duality.

Sébastien Leurent postdoc - Imperial College

[arXiv:1112.3310]

A. Alexandrov, V. Kazakov, SL, Z.Tsuboi, A. Zabrodin

[arXiv:1110.0562]

N.Gromov, V. Kazakov, SL, D. Volin

[arXiv:1302.soon]

SL, D. Volin

Imperial College, February 5, 2013

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Quantum Integrability

Quantum integrability is a property of very specific models (spin chains or quantum field theories), which have



- 1+1 dimensions
- local interactions
- and many conserved charges.

Then, they have the following properties

Properties of integrable models

- n-points interactions factorize into 2-points interactions
- the exact diagonalization of the Hamiltonian reduces to solving the Bethe Equation(s).

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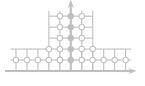
One method of resolution

through a Bäcklund flow and Q-operators

Solving rational spin chains

- Introduce conserved charges (T-operators) by changing the "auxiliary space"
- Reduce the model to a simpler and simpler system (Bäcklund flow → Q-operators)
- Express the original Hamiltonian through Q-operators
- Analyticity properties of these conserved charges give constraint such as the "Bethe equations"

 \rightarrow Surprisingly this method gives a guideline for more complicated models, including a 3+1 dimensional field theory: the $\mathcal{N}=4$ Super-Yang-Mills (of AdS/CFT duality)



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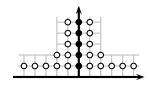
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- 4 Weak coupling expansion in AdS/CFT

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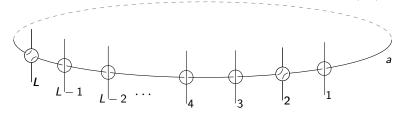
Analyticity properties of Q-functions

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Heisenberg "XXX" spin chain

Construction of T-operators

$$T(u) = \operatorname{tr}_{a} ((u \, \mathbb{I} + \mathcal{P}_{L,a}) \cdot (u \, \mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u \, \mathbb{I} + \mathcal{P}_{1,a}))$$
 operator on the Hilbert space $(\mathbb{C}^{2})^{\otimes L}$



permutation operator :
$$\mathcal{P}_{1,2} | \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow \downarrow \cdots \rangle = | \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow \downarrow \cdots \rangle$$

 $\mathcal{P}_{1,2} | \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \cdots \rangle = | \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \downarrow \cdots \rangle$

•
$$[T(u), T(v)] = 0$$

(proved from relations like $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}$)
• $H = -\sum_{i} \vec{\sigma_{i}} \cdot \vec{\sigma_{i+1}} = L - 2 \frac{d}{dv} \log T(u)|_{u=0}$

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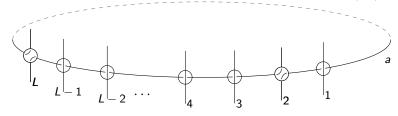
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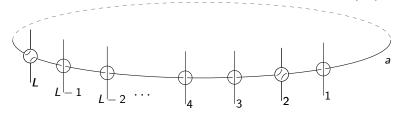
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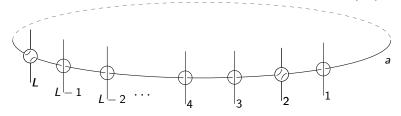
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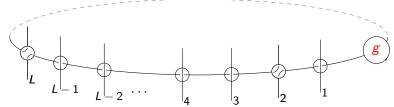
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twist $g \in GL(K)$

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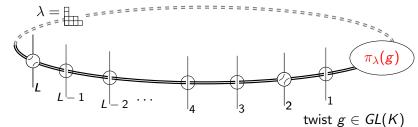
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$$T^{\lambda}(u) = \operatorname{tr}_{a}\left(\left((u - \xi_{L})\mathbb{I} + \mathcal{P}_{L,a}\right) \cdots \left((u - \xi_{1})\mathbb{I} + \mathcal{P}_{1,a}\right) \cdot \pi_{\lambda}(g)\right)$$
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generalized permutation operator:

$$\mathcal{P}_{i,j} = \sum_{lpha,eta}^{\cdot} e_{lpha,eta}^{(i)} \otimes \pi_{\lambda}(e_{eta,lpha}^{(j)})$$

- $[T^{\lambda}(u), T^{\mu}(v)] = 0$ (proved from relations like $\mathcal{P}_{i,i}\mathcal{P}_{i,k} = \mathcal{P}_{i,k}\mathcal{P}_{i,k}$)
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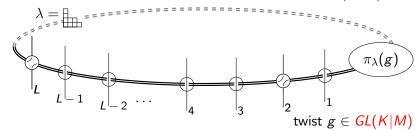
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- In general $u_i \equiv u T^{\lambda}(u) = \left(u_1 + \hat{D}\right) \otimes \left(u_2 + \hat{D}\right) \otimes \cdots \otimes \left(u_L + \hat{D}\right) \chi^{\lambda}(g)$

Rectangular representation : $a, s \leftrightarrow \lambda =$

CBR Determinant identity

[Cherednik 86] [Bazhanov-Reshetikhin 90] [Kazakov Vieira 08

$$\chi^{\lambda}(g) = \det\left(\chi^{1,\lambda_i+j-i}(g)\right)_{1 \leq i,j \leq |\lambda|}$$

$$\Rightarrow \qquad T^{\lambda}(u) = \frac{\det\left(T^{1,\lambda_i+j-i}(u+1-j)\right)_{1 \leq i,j \leq |\lambda|}}{\prod_{k=1}^{|\lambda|-1} T^{0,0}(u-k)}$$

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• In general
$$u_i \equiv u - \xi_i$$
 $T^{\lambda}(u) = \left(u_1 + \hat{D}\right) \otimes \left(u_2 + \hat{D}\right) \otimes \cdots \otimes \left(u_L + \hat{D}\right) \chi^{\lambda}(g)$

Rectangular representation : $a, s \leftrightarrow \lambda =$

CBR Determinant identity

[Cherednik 86] [Bazhanov-Reshetikhin 90] [Kazakov Vieira 08]

• "equivalent" to the Hirota equation :

$$T^{a,s}(u+1) \cdot T^{a,s}(u) = T^{a+1,s}(u+1) \cdot T^{a-1,s}(u) + T^{a,s-1}(u+1) \cdot T^{a,s+1}(u)$$

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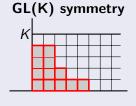
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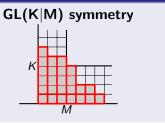
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"Fat hooks" and "Bäcklund Flow"

Authorised Young diagrams for a given symmetry group





Hirota equation solved by gradually reducing the size of the "fat hook" [Krichever, Lipan, Wiegmann & Zabrodin



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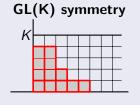
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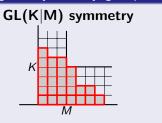
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$$\chi_{\lambda} \underbrace{\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}}_{\in GL(2|1)} \rightsquigarrow \chi_{\lambda} \underbrace{\begin{pmatrix} x_2 & 0 \\ 0 & x_3 \end{pmatrix}}_{\in GL(1|1)} \rightsquigarrow \chi_{\lambda} \underbrace{\begin{pmatrix} x_2 \\ 0 & x_3 \end{pmatrix}}_{\in GL(1)} \rightsquigarrow \chi_{\lambda} \underbrace{\begin{pmatrix} x_2 \\ 0 & x_3 \end{pmatrix}}_{\in GL(1)}$$

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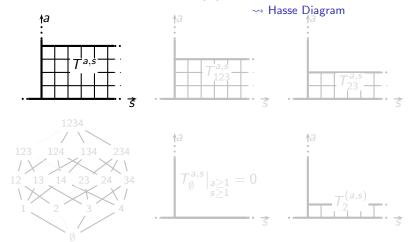
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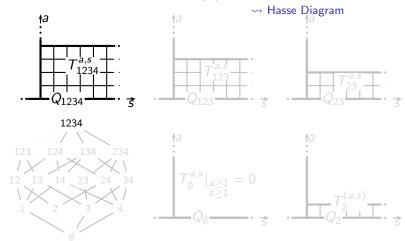
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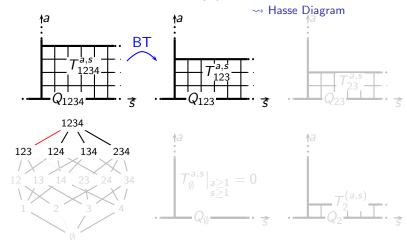
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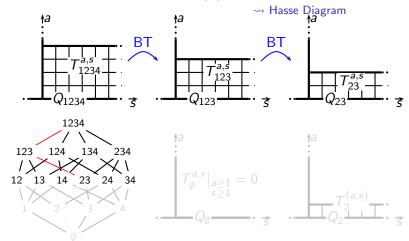
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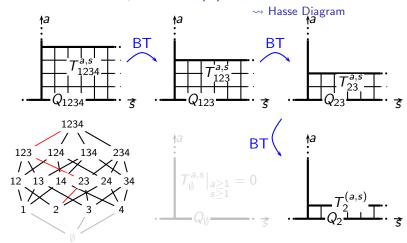
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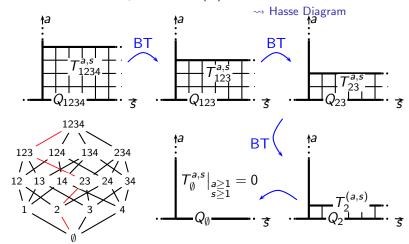
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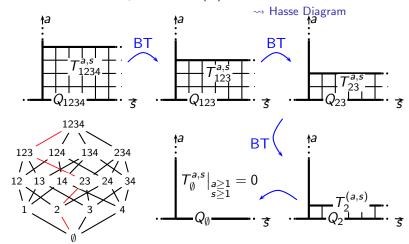
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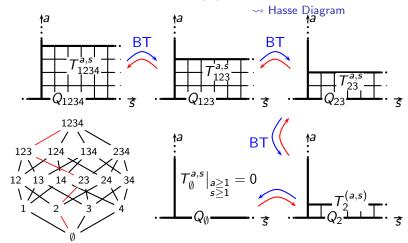
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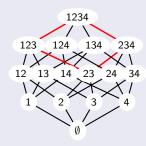
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QQ-relations and Bethe Equations

The consistency of the construction imposes the QQ-relations $(x_i - x_j) Q_{\rm I}(u-1) Q_{{\rm I},i,j}(u) = x_i Q_{{\rm I},j}(u-1) Q_{{\rm I},i}(u) - x_j Q_{{\rm I},j}(u) Q_{{\rm I},i}(u-1)$

example : $I = \{23\}, i = 1, j = 4$

$$(x_1 - x_4) Q_{23}(u - 2) Q_{1234}(u) = x_1 Q_{234}(u - 1) Q_{123}(u) - x_4 Q_{234}(u) Q_{123}(u - 1)$$



The relation involves
Q-operators lying on the same
facet of the Hasse diagram

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Motivat

GL(K|M) spin

T-operators Bäcklund flow

Integrable

field theorie

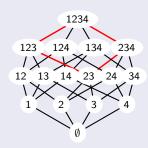
Y-systems Analyticity properties o

Weak coupling AdS/CFT

QQ-relations and Bethe Equations

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Consequences



- Q-operators are polynomial
 - ⇒ parameterized by their roots
- Zeroes of the left hand side have to be zereoes of the right hand side
 Bethe equations

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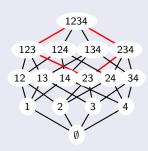
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Expression of T^{λ}

 T-operators are reconstructed from Q-operators as [Krichever, Lipan, Wiegmann & Zabrodin 97]

$$T^{\lambda}(u) = Q_{\emptyset}(u - K) \cdot rac{\det\left(x_{j}^{1-k+\lambda_{k}}Q_{j}(u - k + 1 + \lambda_{k})
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 where $\Delta(x_{1},\cdots,x_{K}) = \det\left(x_{j}^{1-k}
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outcome

- T-operators expressed through Q-operators
 - Hamiltonian : $H = 2\frac{L}{K} 2\frac{d}{du} \log T^{1,1}(u)|_{u=0}$
 - Q-operators constrained by the Bethe equation
 - Requires the existence and polynomiality of the
 - Q-operators

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Some results for these spin chains

 can explicitely construct the Q-operators by combinatoric methods, and prove their polynomiality, their degree, their commutations, etc.

[Kazakov, S.L., Tsuboi 10]

ullet This construction turns out to be deeply related to classical integrability. The Hirota equation is the statement that the T-operators form a au-function, whose residues are the Q-operators

[Alexandrov, Kazakov, S.L., Tsuboi, Zabrodin 11]

• Stated in terms of classical integrability, the construction is generalizable to other models

[Zabrodin 12]

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Outline

- Motivation
 - 2 Solving GL(K|M) spin chains
 - T-operators
 - Bäcklund flow
 - Some results
 - 3 Finite size effects in integrable field theories
 - Y-systems
 - Analyticity properties of Q-functions
 - Weak coupling expansion in AdS/CFT

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Integrable field theories

1+1 D integrable field theories

Wavefunction for a large volume

- planar waves when particles are far from each other
- an S-matrix describes 2-points interactions
- ⇒ Bethe equations
 - "Thermodynamic Bethe Ansatz" for finite size effects :

- At finite temperature, the Bethe equations give rise to

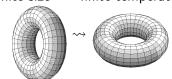
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- planar waves when particles are far from each other
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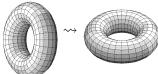
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finite size w finite temperature



- At finite temperature, the Bethe equations give rise to several different types of bound states
 - → introduce one density of particles (as a function of the rapidity) for each type of bound state.

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Y-systems

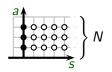
Y-systems

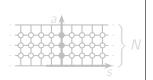
Classification of Bound states

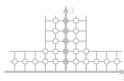
up to a change of variables

 $Y_{(a,s)}(u)$ = density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

SU(N)Gross-Neveu







 $\bullet \ \ Y_{a,s}{}^+ \ Y_{a,s}{}^- = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}} \ \text{``Y-system equation'}$

$$\Leftrightarrow T_{a,s}^{+}T_{a,s}^{-} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$
if $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$

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Y-systems

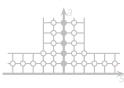
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$$\begin{array}{c|c} SU(N) & SU(N) \times SU(N) \\ \hline \text{Gross-Neveu} & \text{Principal Chiral Model} \\ \hline \\ \bullet & \bullet & \bullet & \bullet \\ \hline \\ \bullet & \bullet & \bullet & \bullet \\ \hline \\ S & & & & & \\ \hline \\ \bullet &$$



•
$$Y_{a,s}^+ Y_{a,s}^- = \frac{1 + Y_{a,s+1}}{1 + (Y_{a+1,s})^{-1}} \frac{1 + Y_{a,s-1}}{1 + (Y_{a-1,s})^{-1}}$$
 "Y-system equation" $Y_{a,s}^{\pm} \equiv Y_{a,s}(u \pm i/2)$

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Y-systems

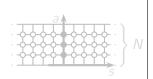
Y-systems → Energy spectrum

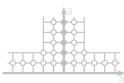
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 "Y-system equation"
 \Leftrightarrow Hirota equation

$$E = -\sum_{a,s} \int E_{a,s}(u) \log(1 + Y_{a,s}(u)) du$$

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Y-systems

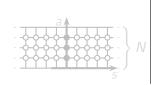
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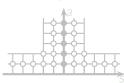
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+ analyticity condition

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Y-systems

Y-systems → Energy spectrum

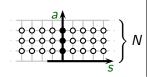
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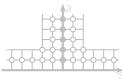
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$$SU(N) \times SU(N)$$

Principal Chiral Model



planar limit of
$$AdS_5/CFT_4$$



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Motivation

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Y-systems → Energy spectrum

Classification of Bound states

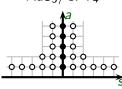
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$$SU(N)$$
 $SU(N) \times SU(N)$ Principal Chiral Model

 N

planar limit of AdS_5/CFT_4



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⇔ Hirota equation

[Gromov Kazakov Vieira 09]

[Bombardelli Fioravanti Tateo 09][Autyunov Frolov 09]
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Integral form : TBA equation

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Motivation

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Analyticity properties of Q-functions

→ simple equations

• Typical solution of Hirota equation :

$$T_{a,s} = \det_{k,l} \left(Q_k(u + f(a, s, l)) \right)$$
, where T and Q are the eigenvalues of T and Q operators.

- Q functions are holomorphic functions of u in the upper half plane Im(u) > 0.
- \Rightarrow Each Q-function reduces to a real function on the real axis
- \leadsto Additional analyticity conditions (typically at $u \to \infty$) give rise to a finite set of non-linear integral equations (FiNLIE)[Gromov Kazakov Vieira 08] [S.L. Kazakov 10] [Gromov Kazakov S.L. Volin 11]

Statement

The outcome of these works is that the (previously conjectured) Thermodynamic Bethe Ansatz is proven to be equivalent to analyticity conditions on the Q-functions.

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Analyticity properties of Q-functions

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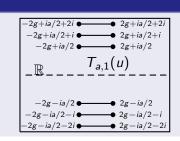
Y-systems Analyticity properties of Q-functions

Weak coupling AdS/CFT

Analyticity properties for AdS/CFT

Branch points

The Y-, T- and Q-functions have square-root-types branch points at positions $\pm 2g + ni$ or $\pm 2g + (n + \frac{1}{2})i$, where $n \in \mathbb{Z}$.



 New symmetries identified, expressed very simply in terms of Q-functions :

For instance, there exists a Q-function Q_1 such that $Q_1 = -\bar{Q}_1$. The



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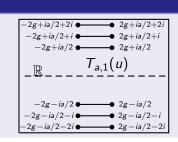
Weak coupling AdS/CFT

Analyticity properties for AdS/CFT

[Gromov Kazakov S.L. Volin 11]

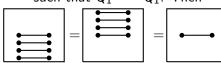
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Y-systems Analyticity properties o Q-functions

Weak coupling AdS/CFT Weak coupling expansion in AdS/CFT [S.L. Volin Serban 12]

Weak coupling

When $g \ll 1$, the branch points collide to give rise to ladders of poles.

Q-functions can then be expressed analytically in terms of sums of the type

$$\sum_{0 \le n_1 < n_2 < \ldots < n_k < \infty} \frac{1}{(u + i n_1)^{m_1} (u + i n_2)^{m_2} \ldots (u + i n_k)^{m_k}}.$$

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Weak coupling AdS/CFT

Conformal dimension of the Konishi operator

$$\Delta_{\text{Konishi}} = 4 + 12 g^2 - 48 g^4 + 336 g^6 + 96 g^8 (-26 + 6 \zeta_3 - 15 \zeta_5)$$
$$-96 g^{10} (-158 - 72 \zeta_3 + 54 \zeta_3^2 + 90 \zeta_5 - 315 \zeta_7)$$
[Paired Faulia Ladia Labarahi 00]

[Bajnok Egedüs Janik Łukowski 09]

[Eden Heslop Korchemsky Smirnov Sokatchev 12]

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$$- 2340 \zeta_5 - 1575 \zeta_7 + 10206 \zeta_9)$$
[SL Volin Serban 12]

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$$+ 48 g^{14} (-44480 + 108960 \zeta_3 + 8568 \zeta_3 \zeta_5$$

$$- 40320 \zeta_3 \zeta_7 - 8784 \zeta_3^2 + 2592 \zeta_3^3 - 4776 \zeta_5$$

$$- 20700 \zeta_5^2 - 26145 \zeta_7 - 17406 \zeta_9 + 152460 \zeta_{11})$$
[Bajnok Janik 12]

analyticity of conserved charges S. Leurent

Integrability:

Weak coupling

AdS/CFT

 $-2208492(_{13}-14256(_{128})+\mathcal{O}(g^{18})$

[SL Volin 13]

 $+41472 \zeta_3^3 + 178200 \zeta_4 \zeta_7 - 409968 \zeta_5$

 $-90720 \zeta_3 \zeta_5 - 129780 \zeta_3 \zeta_7 + 78408 \zeta_3 \zeta_8$ $+483840 (_3 (_9 + 165312 (_3^2 - 82080 (_3^2 (_5 + 165312 (_3 + 16540 (_3 + 16540 (_3 + 166$

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 $-96 g^{10} (-158 - 72 \zeta_3 + 54 \zeta_3^2 + 90 \zeta_5 - 315 \zeta_7)$

 $+455598 \zeta_7 + 194328 \zeta_9 - 555291 \zeta_{11}$

 $-40320 \zeta_3 \zeta_7 - 8784 \zeta_3^2 + 2592 \zeta_3^3 - 4776 \zeta_5$ $-20700 \, \zeta_5^2 - 26145 \, \zeta_7 - 17406 \, \zeta_9 + 152460 \, \zeta_{11}$ $+48 g^{16} (1133504 + 263736 \zeta_2 \zeta_9 - 1739520 \zeta_3$

 $+ 121176 \zeta_5 \zeta_6 + 463680 \zeta_5 \zeta_7 + 49680 \zeta_5^2$

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Motivation

GL(K|M) sp

T-operators Bäcklund flo Some results

Integrable field theorie

Y-systems Analyticity properties of Q-functions

Weak coupling AdS/CFT

Conclusion

- Rational spin chains (very well understood)
 - Bäcklund Flow to gradually simplify the system
 - Bethe Equations
 - ullet Expression of the Hamiltonian from T and Q-functions
- ullet For these rational spin chains, the classical integrability of au-functions sheds light on the whole constriction, and helps for generalizations.
- For finite-size effects in integrable field theories, gives a guideline to write FiNLIE
 - Simple parameterization
 - Clearer analyticity properties
 - New symmetries
 - → Perturbative expansion
- One big open question for these finite-size effects is : Can we prove these analyticity properties ?

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Undressing procedure

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Thermodynam Bethe Ansatz

Riemann Hilbert

Appendices

Disclaimer: The following slides are additional material, not necessarily part of the presentation

- 5 Commutation of *T*-operators
 - 6 Co-derivatives
 - Explicit expression of Q-operators
 - Classical integrability of the MKP-hierarchy
 - τ-functions
 - General rational solution
 - Undressing procedure
 - Construction of Q-operators
 - GL(k) spin chain
- Thermodynamic Bethe Ansatz
- Riemann-Hilbert

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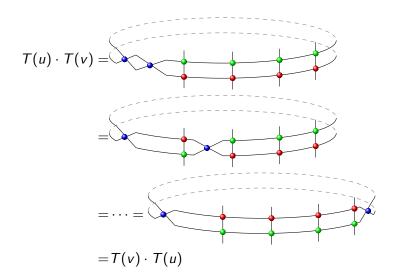
Q-operator:

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Bethe Ansatz

Riemann Hilbert

Commutation of T-operators



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Expression of T through co-derivative

$$\bullet \hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi}g) \bigg|_{\phi=0} \qquad \qquad \phi \in GL(K)$$

- If f(g) acts on \mathcal{H} , then $\hat{D}\otimes f$ acts on $\tilde{\mathcal{H}}=\mathbb{C}^K\otimes\mathcal{H}$
- $\hat{D} \otimes g = \mathcal{P} (1 \otimes g)$ and Leibnitz rule : $\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot [\hat{D} \otimes \tilde{f}] + [\hat{D} \otimes f] \cdot [\mathbb{I} \otimes \tilde{f}]$ \rightsquigarrow compute any $\hat{D} \otimes f(g)$

$$\hat{D}\otimes\pi_{\lambda}(g)=\left[\sum_{\alpha,\beta}\underbrace{e_{\beta\alpha}}_{\text{generator}}\otimes\underbrace{\pi_{\lambda}(e_{\alpha\beta})}_{\text{generator}}\right]\cdot\mathbb{I}\otimes\pi_{\lambda}(e_{\alpha\beta})$$

Therefore
$$((u - \xi_L)\mathbb{I} + \mathcal{P}_{L,a}) \cdots ((u - \xi_1)\mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_{\lambda}(g) \\ = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D}) \pi_{\lambda}(g)$$
 and
$$T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D}) \chi_{\lambda}(g)$$
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nence
$$((u - \xi_L)\mathbb{I} + \mathcal{P}_{L,a}) \cdots ((u - \xi_1)\mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_{\lambda}(g) \\ = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D})\pi_{\lambda}(g)$$
 and
$$T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D})\chi_{\lambda}(g)$$

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$$\hat{D}\otimes\pi_{\lambda}(g)=\left[\sum_{lpha,eta}\underbrace{e_{etalpha}}_{generator}\otimes\underbrace{\pi_{\lambda}(e_{lphaeta})}_{generator}
ight]\cdot\mathbb{I}\otimes\pi_{\lambda}(e_{lphaeta})$$

$$\begin{array}{c} \text{Hence} \\ ((u-\xi_L)\mathbb{I}+\mathcal{P}_{L,a})\cdots((u-\xi_1)\mathbb{I}+\mathcal{P}_{1,a})\cdot\pi_{\lambda}(g) \\ &= \bigotimes_{i=1}^{N}(u-\xi_i+\hat{D})\pi_{\lambda}(g) \\ \text{and} \ T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N}(\underline{u-\xi_i}+\hat{D})\chi_{\lambda}(g) \end{array}$$

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Bäcklund Transformations

if $T^{a,s}(u)$ is a solution of Hirota equation and

$$\begin{cases} T^{a+1,s}(u)F^{a,s}(u) - T^{a,s}(u)F^{a+1,s}(u) \\ = \underbrace{x_j} T^{a+1,s-1}(u+1)F^{a,s+1}(u-1), \\ \text{eigenvalue of } g \\ T^{a,s+1}(u)F^{a,s}(u) - T^{a,s}(u)F^{a,s+1}(u) \\ = x_j T^{a+1,s}(u+1)F^{a-1,s+1}(u-1). \end{cases}$$

Then $F^{a,s}(u)$ is a solution of Hirota equation.

Moreover, if $T^{a,s}(u) = 0$, outside the (K|M) "fat hook", one can choose $F^{a,s}(u) = 0$ outside the (K-1|M) "fat hook".



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Bäcklund Transformations

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Explicit expression of nested T and Q-operators

Generating series of T-operators:

$$\begin{array}{l} \text{let } w(z) \equiv \sum_{s \geq 0} \chi^{1,s} z^s = \det \frac{1}{1-g\,z}, \\ \text{then } \sum_{s \geq 0} T^{1,s} z^s = \bigotimes_{i=1}^L (u_i + \hat{D}) \ w(z) \end{array}$$

Explicit solution of this linear system

$$T_{\mathtt{I}}^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{\mathtt{I}}}} B_{\mathtt{I}} \cdot \left[\bigotimes_{i=1}^{L} (u_i + \hat{D} + |\overline{\mathtt{I}}|) \ \chi_{\lambda}(g_{\mathtt{I}}) \Pi_{\mathtt{I}} \right]$$

$$\Pi_{\mathtt{I}} = \prod_{i=1}^{n} w(t_i)$$

$$B_{\mathtt{I}} = \prod_{i \in \overline{\mathtt{I}}} (1 - \mathsf{x}_{j} \ t_{j}) \cdot (1 - \mathsf{g} \ t_{j})^{\otimes}$$

 $Q_{\mathrm{I}} = T_{\mathrm{I}}^{(\mathbf{0},\mathbf{s})}$ $g_{\{j_1,j_2,\cdots,j_k\}} = \mathrm{diag}(x_{j_1},x_{j_2},\cdots,x_{j_k})$

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$\begin{array}{c} \mathsf{Explicit} \ \mathsf{expression} \ \mathsf{of} \ \mathsf{nested} \\ \mathsf{T} \ \mathsf{and} \ \mathsf{Q}\text{-}\mathsf{operators} \end{array}$

Generating series of T-operators:

let
$$w(z) \equiv \sum_{s\geq 0} \chi^{1,s} z^s = \det \frac{1}{1-gz}$$
,
then $\sum_{s\geq 0} T^{1,s} z^s = \bigotimes_{i=1}^L (u_i + \hat{D}) w(z)$

Explicit solution of this linear system

$$T_{\mathbf{I}}^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \overline{\mathbf{I}}}} B_{\mathbf{I}} \cdot \left[\bigotimes_{i=1}^{L} (u_i + \hat{D} + |\overline{\mathbf{I}}|) \chi_{\lambda}(g_{\mathbf{I}}) \Pi_{\mathbf{I}} \right],$$

$$\overline{\Pi_{\mathbf{I}} = \prod_{j \in \overline{\mathbf{I}}} w(t_j)} \qquad B_{\mathbf{I}} = \prod_{j \in \overline{\mathbf{I}}} (1 - x_j \ t_j) \cdot (1 - g \ t_j)^{\otimes N}}$$

$$Q_{\mathbf{I}} = T_{\mathbf{I}}^{(0,s)} \qquad g_{\{j_1,j_2,\cdots,j_k\}} = \operatorname{diag}(x_{j_1}, x_{j_2}, \cdots, x_{j_k})$$

 x_i =eigenvalue of g

[Kazakov, S.L, Tsuboi 10]

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(Hints of) combinatorial proof

 Determinant identities (similar to Jacobi-Trudi) reduce the proofs to a few bilinear identities proven from the following

"Master Identity"

when $\Pi = \prod_{j} w(t_j)$,

$$(t-z) \left[\bigotimes (u_i + 1 + \hat{D}) \ w(z)w(t)\Pi \right] \cdot \left[\bigotimes (u_i + \hat{D}) \ \Pi \right]$$

$$= t \left[\bigotimes (u_i + \hat{D}) \ w(z)\Pi \right] \cdot \left[\bigotimes (u_i + 1 + \hat{D}) \ w(t)\Pi \right]$$

$$- z \left[\bigotimes (u_i + 1 + \hat{D}) \ w(z)\Pi \right] \cdot \left[\bigotimes (u_i + \hat{D}) \ w(t)\Pi \right]$$
where $w(z) = \det \frac{1}{1-zg} = \sum_{s=0}^{\infty} z^s \chi_s(g)$

Master identity proven by combinatorial arguments

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Master identity proven by combinatorial arguments

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• Master identity proven by combinatorial arguments

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Combinatorics of coderivatives

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"Master Identity"

[Kazakov, S.L, Tsuboi 10]

when $\Pi = \prod_{j} w(t_{j})$, $(t-z) \left[\bigotimes (u_{i}+1+\hat{D}) \ w(z)w(t)\Pi \right] \cdot \left[\bigotimes (u_{i}+\hat{D}) \ \Pi \right]$ $= t \left[\bigotimes (u_{i}+\hat{D}) \ w(z)\Pi \right] \cdot \left[\bigotimes (u_{i}+1+\hat{D}) \ w(t)\Pi \right]$ $-z \left[\bigotimes (u_{i}+1+\hat{D}) \ w(z)\Pi \right] \cdot \left[\bigotimes (u_{i}+\hat{D}) \ w(t)\Pi \right]$

• Master identity proven by combinatorial arguments

where $w(z) = \det \frac{1}{1-z\sigma} = \sum_{s=0}^{\infty} z^s \chi_s(g)$

For instance,

[Kazakov Vieira 07]

$$\hat{D} \otimes \hat{D} \otimes \hat{D} \ w(x) = \left(\prod \left[+ \prod \left[+ \prod \right] + \prod \right] + \prod \left[+ \prod \left[+ \prod \right] \right] \right) w(x)$$

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au-functions of the MKP hierarchy

• A τ -function of the *MKP hierarchy* is a function of a variable n and an infinite set $\mathbf{t} = (t_1, t_2, \cdots)$ of "times", such that $\forall n \geq n', \ \forall \mathbf{t}, \mathbf{t}'$

Definition of τ -functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t}-[z^{-1}]) \tau_{n'}(\mathbf{t}'+[z^{-1}]) dz = 0$$

where
$$\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$$
, $\xi(\mathbf{t}, z) = \sum_{k \geq 1} t_k z^k$, and \mathcal{C} encircles the singularities of $\tau_n(\mathbf{t} - [z^{-1}])\tau_{n'}(\mathbf{t}' + [z^{-1}])$ (typically finite), but not the singularities of $e^{\xi(\mathbf{t} - \mathbf{t}', z)}z^{n-n'}$ (typically at infinity).

ullet An example of such au-function is the expectation value

$$\tau_n(\mathbf{t}) = \langle n | e^{J_+(\mathbf{t})} G | n \rangle$$

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au-functions of the MKP hierarchy

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Definition of τ -functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t}-[z^{-1}]) \tau_{n'}(\mathbf{t}'+[z^{-1}]) dz = 0$$

where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$, and $\xi(\mathbf{t}, z) = \sum_{k \ge 1} t_k z^k$.

• An example of such τ -function is the expectation value

$$\tau_n(\mathbf{t}) = \langle n | e^{J_+(\mathbf{t})} G | n \rangle$$

over an infinite set of fermionic oscillators $(\{\psi_i, \psi_j^{\dagger}\} = \delta_{ij})$, where $G = \exp\left(\sum_{i,k \in \mathbb{Z}} A_{ik} \psi_i^{\dagger} \psi_k\right)$ and $J_+ = \sum_{k \geq 1} t_k J_k$, where $J_k = \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+k}^{\dagger}$. (and $\psi_n | n \rangle = | n + 1 \rangle$)

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τ -functions of the MKP hierarchy

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$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t}-[z^{-1}]) \tau_{n'}(\mathbf{t}'+[z^{-1}]) dz = 0$$

where
$$\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$$
,

Characteristic property

au-functions are characterised by

$$\begin{split} z_2 \tau_{n+1} \left(\mathbf{t} - [z_2^{-1}] \right) \tau_n \left(\mathbf{t} - [z_1^{-1}] \right) \\ &- z_1 \tau_{n+1} \left(\mathbf{t} - [z_1^{-1}] \right) \tau_n \left(\mathbf{t} - [z_2^{-1}] \right) \\ &+ \left(z_1 - z_2 \right) \tau_{n+1} (\mathbf{t}) \tau_n \left(\mathbf{t} - [z_1^{-1}] - [z_2^{-1}] \right) = 0. \end{split}$$
 (obtained from $n' = n - 1$ and $\mathbf{t}' = \mathbf{t} - [z_1^{-1}] - [z_1^{-1}]$)

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au-functions of the MKP hierarchy

• A τ -function is a function of a variable n and an infinite set $\mathbf{t} = (t_1, t_2, \cdots)$ of "times", such that $\forall n \geq n'$, $\forall \mathbf{t}, \mathbf{t}'$

Definition of τ -functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t}-[z^{-1}]) \tau_{n'}(\mathbf{t}'+[z^{-1}]) dz = 0$$

where
$$\mathbf{t} \pm [z^{-1}] = (t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots),$$

Characteristic property "Master identity"

au-functions are characterised by

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General rational au-function

[Krichever 78]

If $\tau_n(\mathbf{t})$ is polynomial in n, we substitute $n \rightsquigarrow u$, where $u \in \mathbb{C}$.

Polynomial τ -functions of this MKP hierarchy

$$\tau(u, \mathbf{t}) = \det (A_i(u - j, \mathbf{t}))_{1 \le i, j \le N}$$
where $A_i(u, \mathbf{t}) = \sum_{m=0}^{d_i} a_{i,m} \partial_z^m \left(z^u e^{\xi(\mathbf{t}, z)} \right) \bigg|_{z = p_i}$

parameterized by : the integer $N \ge 0$, the numbers $\{p_i\}$ and d_i , and the coefficients $\{a_{i,m}\}$.

 Analogous to the spin-chain's "undressing procedure": restricting to smaller and smaller minors of the determinant

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- $A_i(u, \mathbf{t} + [z^{-1}]) = \sum_{m=0}^{d_i} a_{i,m} \partial_x^m \left(x^u e^{\xi(\mathbf{t}, x)} \frac{1}{1 x/z} \right) \Big|_{x = p_i}$ has a pole at $z = p_i$.
- One can show that $A_k(u, \mathbf{t} + [z^{-1}]) = A_k(u, \mathbf{t}) + z^{-1}A_k(u+1, \mathbf{t} + [z^{-1}])$, hence

$$\tau_{u}(\mathbf{t} + [z^{-1}]) = \begin{vmatrix} A_{1}(u-1,\mathbf{t}+[z^{-1}]) & A_{1}(u-2,\mathbf{t}) & \dots & A_{1}(u-N,\mathbf{t}) \\ A_{2}(u-1,\mathbf{t}+[z^{-1}]) & A_{2}(u-2,\mathbf{t}) & \dots & A_{2}(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ A_{N}(u-1,\mathbf{t}+[z^{-1}]) & A_{N}(u-2,\mathbf{t}) & \dots & A_{N}(u-N,\mathbf{t}). \end{vmatrix}$$

Undressing procedure for rational τ -functions

- First step : $\operatorname{Res}_{z=p_i} \tau(u+1,\mathbf{t}+[z^{-1}])$
- Second step : $\operatorname{Res}_{z_1=p_i}\operatorname{Res}_{z_2=p_i}\tau(u+2,t+[z_1^{-1}]+[z_2^{-1}])$
- Et cetera

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Undressing the rational au-functions

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T-operators are au-functions

• Set of times $\mathbf{t} \iff$ representations λ :

$$au(u,\mathbf{t}) = \sum_{\lambda} \underbrace{s_{\lambda}(\mathbf{t})}_{ ext{Schur polynomial}} au(u,\lambda) \qquad s_{\lambda}(\mathbf{t}) = \det\left(h_{\lambda_i-i+j}(\mathbf{t})\right)_{1 \leq i,j \leq |\lambda|}$$

where
$$e^{\xi(\mathbf{t},z)} = \sum_{k\geq 0} h_k(\mathbf{t}) z^k$$

If
$$\tau(u,\lambda) = T^{\lambda}(u) = \bigotimes_{i=1}^{L} (u_i + \hat{D}) \chi^{\lambda}(g)$$
, we get

$$\tau(u,\mathbf{t}) = \bigotimes_{i=1}^{L} (u_i + \hat{D}) e^{\sum_{k \ge 1} t_k \operatorname{tr}(g^k)}$$

• Then
$$\tau(u, \mathbf{t} + [z^{-1}]) = \bigotimes_{i=1}^{L} (u_i + \hat{D}) \ w(1/z) e^{\sum_{k \ge 1} t_k \operatorname{tr}(g^k)}$$

where $w(1/z) \equiv \sum_{s \ge 0} \chi^{1,s} z^{-s} = \det \frac{1}{1-g/z}$

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Spin-chains <>>> MKP hierarchy

T-operators are au-functions

 The master identity coincides with the characteristic property of the MKP hierarchy

$$\Rightarrow \tau(u, \mathbf{t}) = \bigotimes_{i=1}^{L} (u_i + \hat{D}) e^{\sum_{k \geq 1} t_k \operatorname{tr}(g^k)}$$
 is a τ -function of the MKP hierarchy.

- The undressing procedure for τ -functions (ie $\operatorname{Res}_{z=p_i}\tau(u+1,\mathbf{t}+[z^{-1}])$) explains the explicit expression found from the combinatorics of coderivatives.
- Fermionic realisation of this τ -function :

$$\tau_n(\mathbf{t}) = \langle n | e^{J_+(\mathbf{t})} \Psi_1 \dots \Psi_N | n - N \rangle$$

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$$\Psi_i = \sum_{m \geq 0} a_{im} \, \partial_z^m \sum_{k \in \mathbb{Z}} \psi_k z^k \Big|_{z=p_i}$$

[Alexandrov, Kazakov, S.L., Tsuboi, Zabrodin 11]

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→ Equations of the form

$$Y_{a,s}(u) = -L E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log \left(1 + Y_{a',s'}(u)^{\pm 1}\right) + \langle Source Terms \rangle$$

Vacuum energy

$$E_0 = -\sum_{a,s} \int E_{a,s}(u) \log(1 + Y_{a,s}(u)) du$$

- Extra assumption : Excited states obey the same equations.
 - Each state corresponds to a different solution of Y-system characterized by its zeroes and poles
- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root
- ⇒ TBA-equations contain analyticity information under a form which is hard to decode (infinite sum:

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 Back to the presentation
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Parameterzation of Q-functions

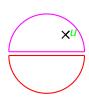
A simple Riemann-Hilbert Problem

Form the Cauchy theorem, we get

If Q(u) is an analytic function on the upper half plane (when ${\rm Im}(u)>0$), and $Q(u)\ll 1/u$ in the vicinity of ∞ , then

$$\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{Q(v) - \bar{Q}(v)}{v - u} dv = \begin{cases} Q(u) & \text{if } \operatorname{Im}(u) > 0 \\ \bar{Q}(u) & \text{if } \operatorname{Im}(u) < 0 \end{cases}$$

where $\bar{Q}(u)$ is the complex-conjugate of $Q(\bar{u})$.



Indeed, if
$$\operatorname{Im}(u) > 0$$
, then $\frac{1}{2i\pi} \int_{\operatorname{upwards}} \frac{Q(v)}{v-u} \mathrm{d}v = Q(u)$ and $\frac{1}{2i\pi} \int_{\operatorname{downwards}} \frac{\overline{Q}(v)}{v-u} \mathrm{d}v = 0$