Integrability and analyticity of the conserved charges of quantum spin chains : a guideline for the AdS/CFT duality.

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postdoc - Imperial College

[arXiv:1112.3310] A. Alexandrov, V. Kazakov, SL, Z.Tsuboi, A. Zabrodin

[arXiv:1110.0562] N.Gromov, V. Kazakov, SL, D. Volin

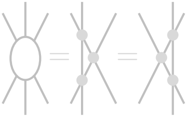
[arXiv:1302.soon] SL, D. Volin

Imperial College, February 5, 2013

Quantum Integrability

Quantum integrability is a property of very specific models (spin chains or quantum field theories), which have

- 1+1 dimensions
- local interactions
- and many conserved charges.



Then, they have the following properties

Properties of integrable models

- n-points interactions factorize into 2-points interactions
- the exact diagonalization of the Hamiltonian reduces to solving the Bethe Equation(s).

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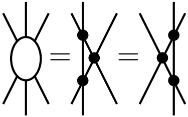

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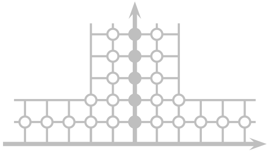
One method of resolution

through a Bäcklund flow and Q-operators

Solving rational spin chains

- Introduce conserved charges (T-operators) by changing the "auxiliary space"
- Reduce the model to a simpler and simpler system (Bäcklund flow$\rightsquigarrow$ Q-operators)
- Express the original Hamiltonian through Q-operators
- Analyticity properties of these conserved charges give constraint such as the "Bethe equations"

$\rightarrow$ Surprisingly this method gives a guideline for more complicated models, including a 3+1 dimensional field theory: the $\mathcal{N}=4$ Super-Yang-Mills (of AdS/CFT duality)



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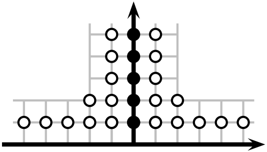


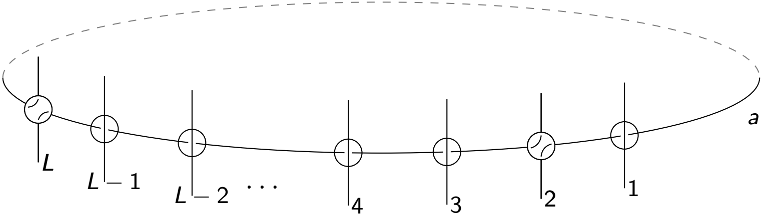
Outline

Heisenberg “XXX” spin chain

Construction of T-operators

$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$$

operator on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



permutation operator :
$$\mathcal{P}_{1,2} \,|\, \downarrow\downarrow \uparrow\downarrow\uparrow\downarrow\downarrow \cdots \rangle = |\, \downarrow\downarrow \uparrow\downarrow\uparrow\downarrow\downarrow \cdots \rangle$$
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- $\left[T\ \ (u), T\ \ (v)\right] = 0$

  (proved from relations like $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}$)
- $H = -\sum_i \vec{\sigma_i} \cdot \vec{\sigma_{i+1}} = \ \ L - 2\, \frac{\mathrm{d}}{\mathrm{d}u} \log T(u)\Big|_{u=0}$

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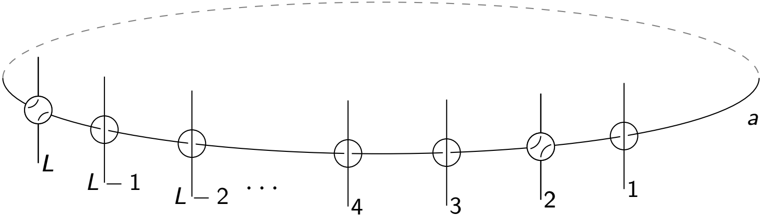

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$GL(K)$ spin chains

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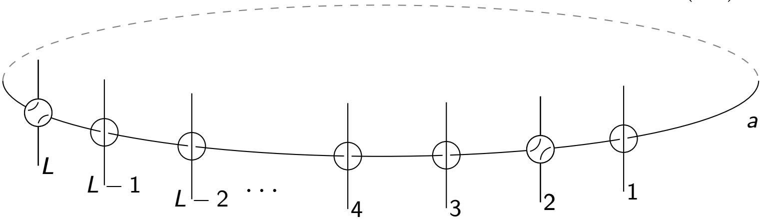

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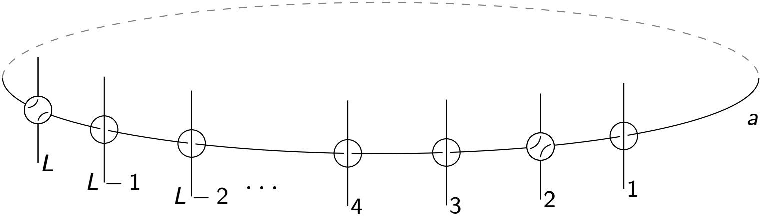

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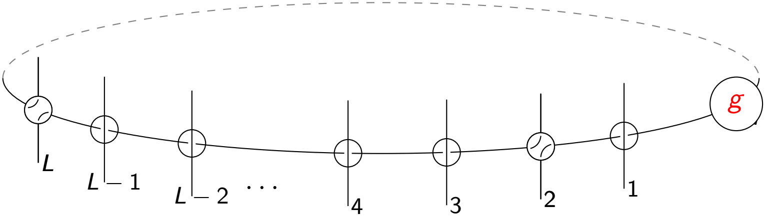

twist $g \in GL(K)$

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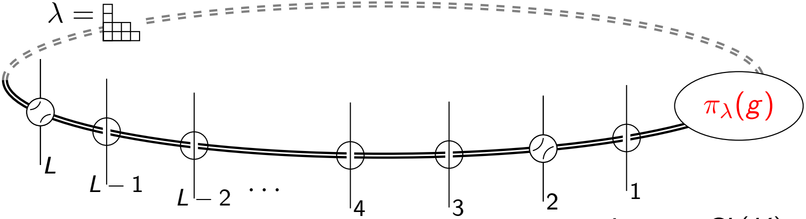

twist $g \in GL(K)$

generalized permutation operator :

$$\mathcal{P}_{i,j} = \sum_{\alpha,\beta} e^{(i)}_{\alpha,\beta} \otimes \pi_\lambda(e^{(j)}_{\beta,\alpha})$$

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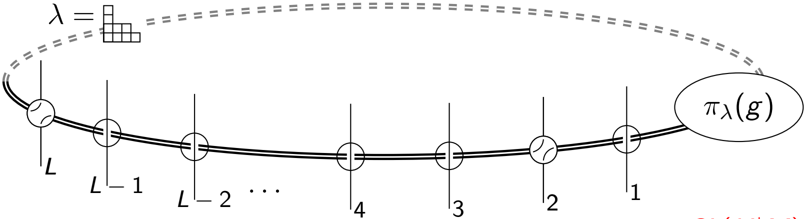
operator on the Hilbert space $\left(\mathbb{C}^{K|M}\right)^{\otimes L}$



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T-operators $\leftrightsquigarrow$ characters

+ Cherednik-Bazhanov-Reshetikhin formula

$$T^\lambda(u) = \mathrm{tr}_a\left(((u-\xi_L)\mathbb{I} + \mathcal{P}_{L,a})\cdots((u-\xi_1)\mathbb{I} + \mathcal{P}_{1,a})\cdot\pi_\lambda(g)\right)$$

- At $L=0$, $T^\lambda(u) = \chi^\lambda(g) \equiv \mathrm{tr}\,\pi_\lambda(g)$
- In general $\qquad u_i \equiv u - \xi_i$

$$T^\lambda(u) = \left(u_1 + \hat{D}\right) \otimes \left(u_2 + \hat{D}\right) \otimes \cdots \otimes \left(u_L + \hat{D}\right) \chi^\lambda(g)$$

Rectangular representation : $a, s \leftrightarrow \lambda = \underbrace{\square\square\square\square\square}_{s}\Big\}\, a$

CBR Determinant identity

[Cherednik 86] [Bazhanov-Reshetikhin 90] [Kazakov Vieira 08]

$$\chi^\lambda(g) = \det\left(\chi^{1,\lambda_i+j-i}(g)\right)_{1\le i,j\le|\lambda|}$$

$$\leadsto \qquad T^\lambda(u) = \frac{\det\left(T^{1,\lambda_i+j-i}(u+1-j)\right)_{1\le i,j\le|\lambda|}}{\prod_{k=1}^{|\lambda|-1} T^{0,0}(u-k)}$$

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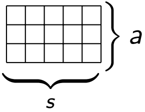

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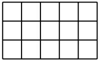
Rectangular representation : $a, s \leftrightarrow \lambda =$  $\Big\}\, a$

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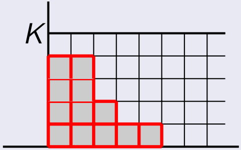
- "equivalent" to the Hirota equation :
$$T^{a,s}(u+1)\cdot T^{a,s}(u) =$$
$$T^{a+1,s}(u+1)\cdot T^{a-1,s}(u) + T^{a,s-1}(u+1)\cdot T^{a,s+1}(u)$$

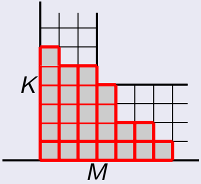
"Fat hooks" and "Bäcklund Flow"

Authorised Young diagrams for a given symmetry group

GL(K) symmetry



GL(K|M) symmetry



Hirota equation solved by gradually reducing the size of the "fat hook" [Krichever, Lipan, Wiegmann & Zabrodin 97] [Kazakov Sorin Zabrodin 08]

$$\chi_\lambda \underbrace{\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}}_{\in GL(2|1)} \rightsquigarrow \chi_\lambda \underbrace{\begin{pmatrix} x_2 & 0 \\ 0 & x_3 \end{pmatrix}}_{\in GL(1|1)} \rightsquigarrow \chi_\lambda \underbrace{\begin{pmatrix} x_2 \end{pmatrix}}_{\in GL(1)} \rightsquigarrow \chi_\lambda \underbrace{\begin{pmatrix} \end{pmatrix}}_{\in \{1\}}$$

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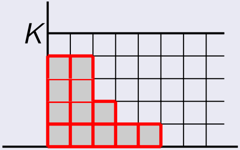

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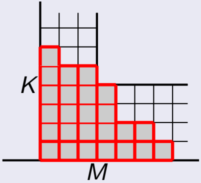

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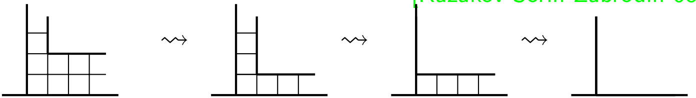

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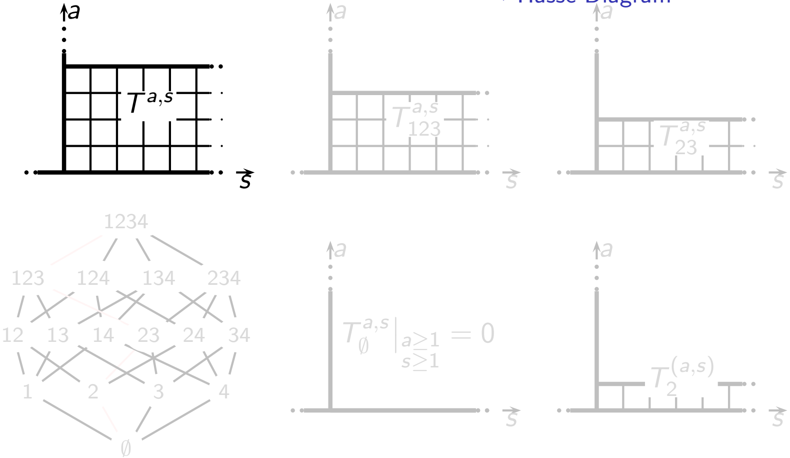
example of $GL(4)$ Bäcklund flow

$\rightsquigarrow$ Hasse Diagram



$\rightsquigarrow$ Defines 2^4 Q-operators, lying on the nodes of this *Hasse Diagram*

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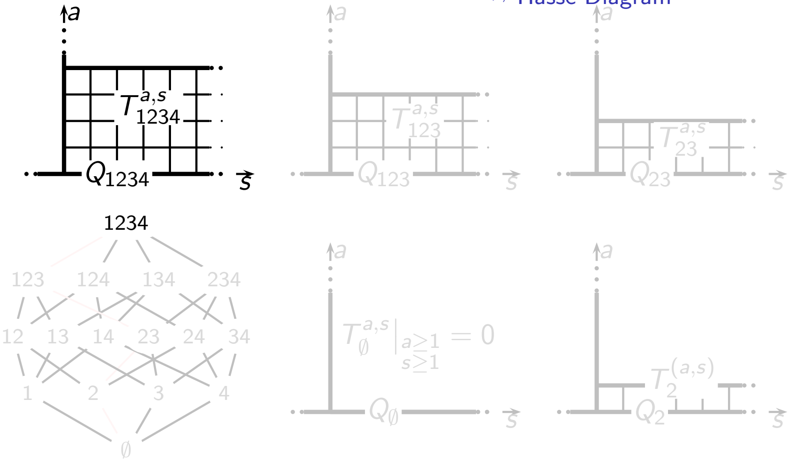

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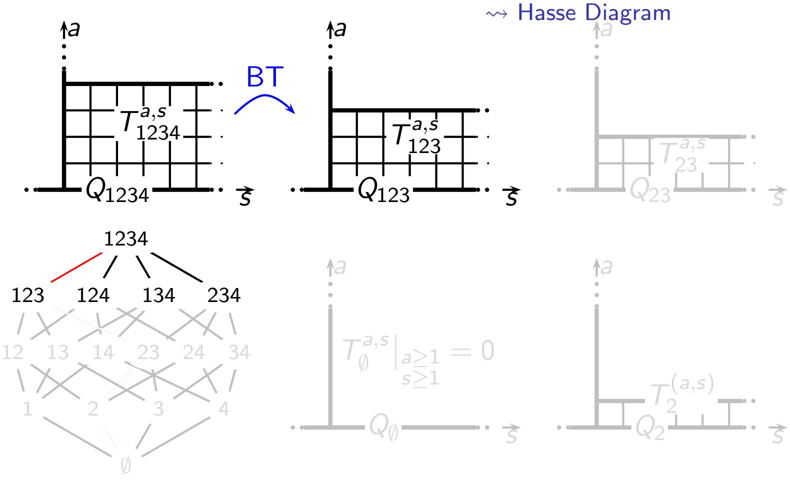

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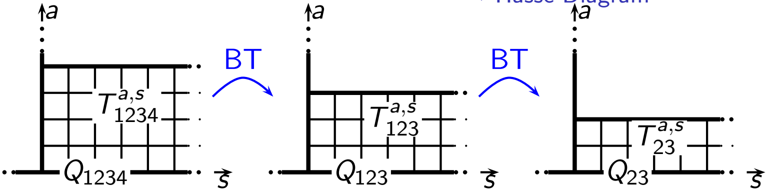

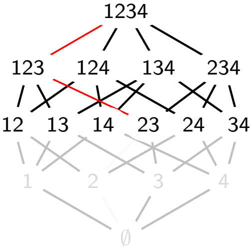

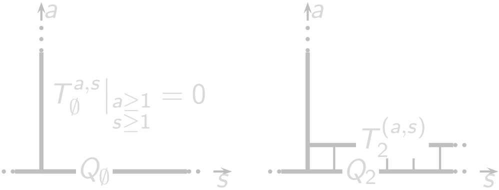

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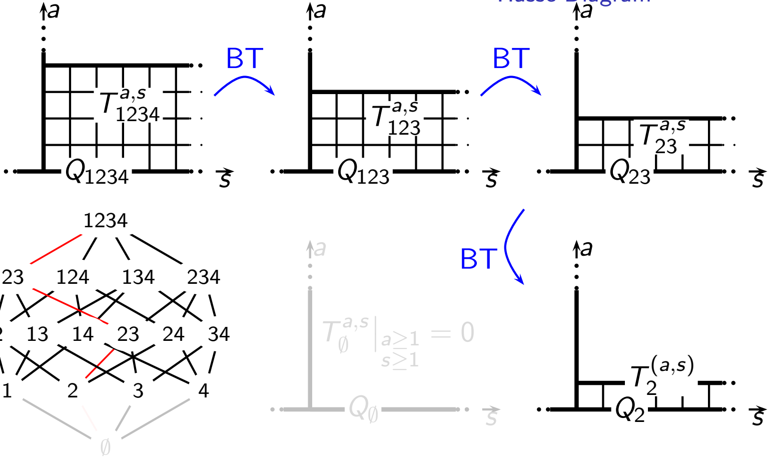

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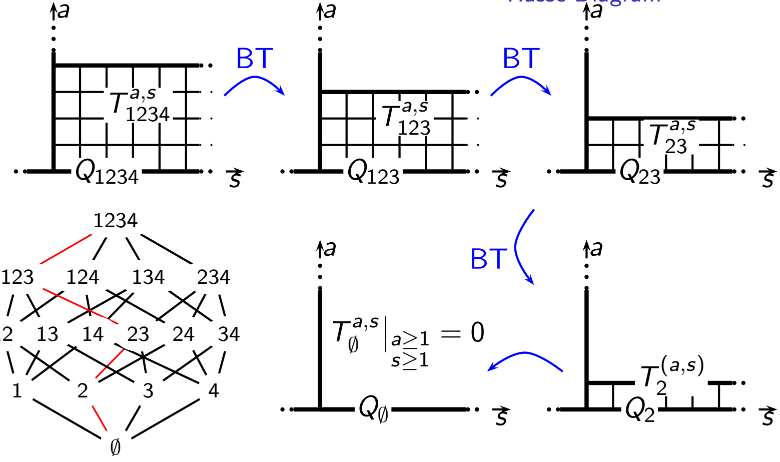

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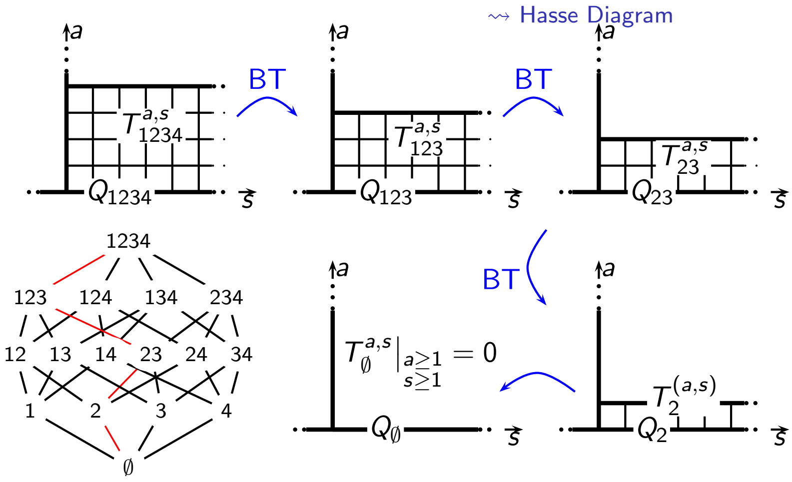

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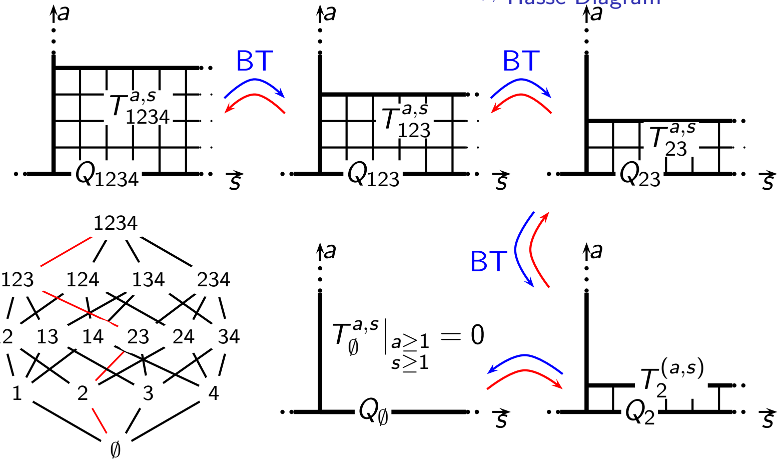

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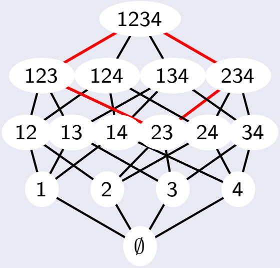
QQ-relations and Bethe Equations

The consistency of the construction imposes the QQ-relations

$$(x_i - x_j)\, Q_{\mathrm{I}}(u-1) Q_{\mathrm{I},i,j}(u) = x_i Q_{\mathrm{I},j}(u-1) Q_{\mathrm{I},i}(u) - x_j Q_{\mathrm{I},j}(u) Q_{\mathrm{I},i}(u-1)$$

example : $\mathrm{I} = \{23\}, i = 1, j = 4$

$$(x_1 - x_4)\, Q_{23}(u-2) Q_{1234}(u) = x_1 Q_{234}(u-1) Q_{123}(u) - x_4 Q_{234}(u) Q_{123}(u-1)$$



The relation involves Q-operators lying on the same facet of the Hasse diagram

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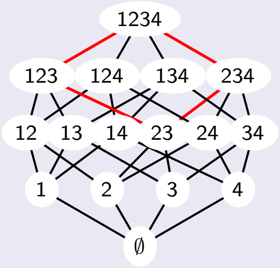

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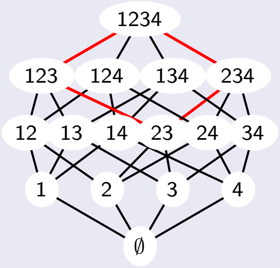

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Expression of T^λ

- T-operators are reconstructed from Q-operators as [Krichever, Lipan, Wiegmann & Zabrodin 97]

$$T^\lambda(u) = Q_\emptyset(u-K) \cdot \frac{\det\left(x_j^{1-k+\lambda_k} Q_j(u-k+1+\lambda_k)\right)_{1\le j,k\le K}}{\Delta(x_1,\cdots,x_K)\prod_{k=1}^K Q_\emptyset(u-k+\lambda_k)}$$

where $\Delta(x_1,\cdots,x_K) = \det\left(x_j^{1-k}\right)_{1\le j,k\le K}$

outcome

- T-operators expressed through Q-operators
- Hamiltonian : $H = 2\frac{L}{K} - 2\frac{\mathrm{d}}{\mathrm{d}u}\log T^{1,1}(u)\Big|_{u=0}$
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Some results for these spin chains

- can explicitely construct the Q-operators by combinatoric methods, and prove their polynomiality, their degree, their commutations, etc.

  [Kazakov, S.L., Tsuboi 10]

- This construction turns out to be deeply related to classical integrability. The Hirota equation is the statement that the T-operators form a τ-function, whose residues are the Q-operators

  [Alexandrov, Kazakov, S.L., Tsuboi, Zabrodin 11]

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Outline

1+1 D integrable field theories

Wavefunction for a large volume

- planar waves when particles are far from each other
- an *S-matrix* describes 2-points interactions

$\Rightarrow$ Bethe equations

- "Thermodynamic Bethe Ansatz" for finite size effects : "double Wick Rotation"

finite size $\leftrightsquigarrow$ finite temperature

$\rightsquigarrow$

- At finite temperature, the Bethe equations give rise to several different types of bound states
$\rightsquigarrow$ introduce one density of particles (as a function of the rapidity) for each type of bound state.

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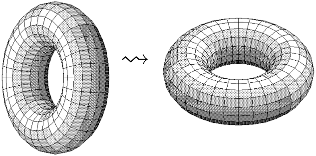

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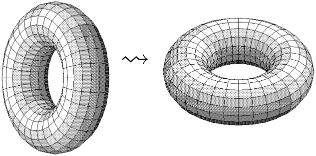

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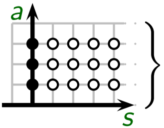
Y-systems

Classification of Bound states

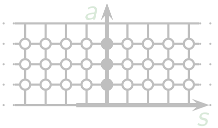
up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a, s) and rapidity $u \in \mathbb{C}$.

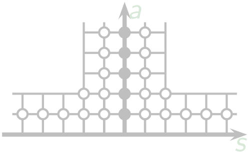
$SU(N)$ Gross-Neveu



$SU(N) \times SU(N)$ Principal Chiral Model



planar limit of AdS_5/CFT_4



- $Y_{a,s}{}^{+} Y_{a,s}{}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$$

$\Leftrightarrow \quad T^{+}_{a,s} T^{-}_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$

if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$

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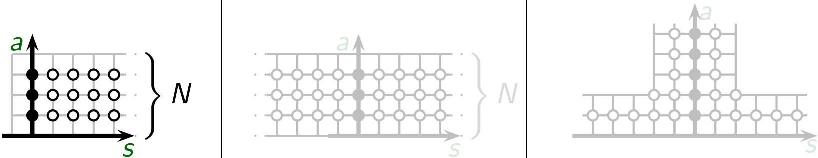

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Y-systems ⇝ Energy spectrum

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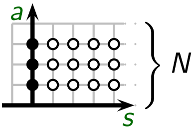

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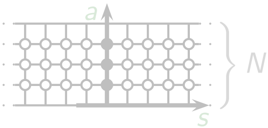

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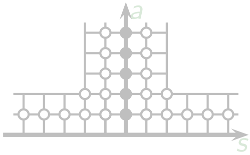

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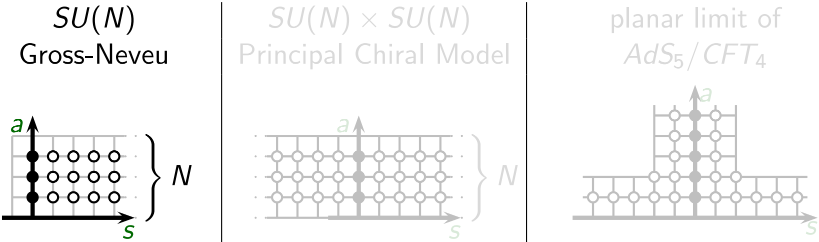

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▸ Integral form : TBA equation

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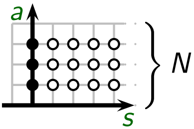	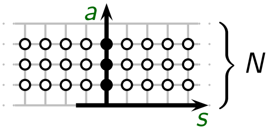	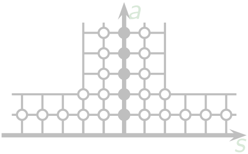

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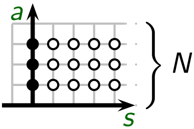

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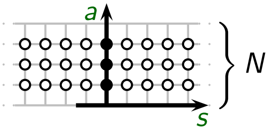

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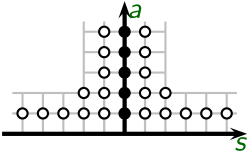

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  [Bombardelli Fioravanti Tateo 09][Autyunov Frolov 09]

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Analyticity properties of Q-functions

⇝ simple equations

- Typical solution of Hirota equation : $T_{a,s} = \det_{k,l}\left(Q_k(u + f(a,s,l))\right)$, where T and Q are the eigenvalues of T and Q operators.
- Q functions are holomorphic functions of u in the upper half plane $\mathrm{Im}(u) > 0$.

⇒ Each Q-function reduces to a real function on the real axis.

- ⇝ Additional analyticity conditions (typically at $u \to \infty$) give rise to a finite set of non-linear integral equations (FiNLIE)[Gromov Kazakov Vieira 08] [S.L. Kazakov 10] [Gromov Kazakov S.L. Volin 11]

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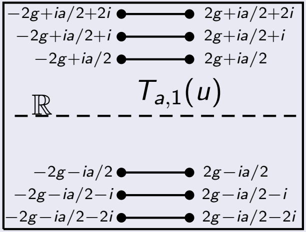
Analyticity properties for AdS/CFT

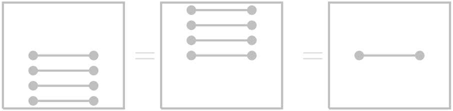
Branch points

The Y-, T- and Q-functions have square-root-types branch points at positions $\pm 2g + n\mathrm{i}$ or $\pm 2g + (n+\frac{1}{2})\mathrm{i}$, where $n \in \mathbb{Z}$.



- New symmetries identified, expressed very simply in terms of Q-functions :

For instance, there exists a Q-function Q_1 such that $Q_1 = -\bar{Q}_1$. Then



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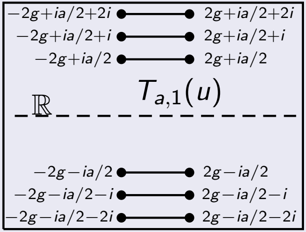

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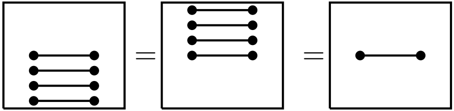
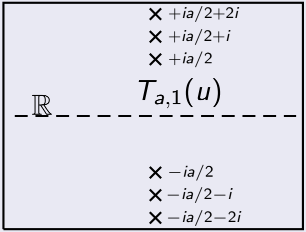


Weak coupling expansion in AdS/CFT [S.L. Volin Serban 12]

Weak coupling

When $g \ll 1$, the branch points collide to give rise to ladders of poles.



Q-functions can then be expressed analytically in terms of sums of the type

$$\sum_{0\le n_1<n_2<\ldots<n_k<\infty} \frac{1}{(u+\mathrm{i}\,n_1)^{m_1}(u+\mathrm{i}\,n_2)^{m_2}\ldots(u+\mathrm{i}\,n_k)^{m_k}}\,.$$

Conformal dimension of the Konishi operator

$$\Delta_{\text{Konishi}} = 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + 96\,g^8(-26 + 6\,\zeta_3 - 15\,\zeta_5) - 96\,g^{10}(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7)$$

[Bajnok Egedüs Janik Łukowski 09]

[Eden Heslop Korchemsky Smirnov Sokatchev 12]

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[SL Volin Serban 12]

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& - 48\, g^{12}(160 + 5472\, \zeta_3 - 3240\, \zeta_3\, \zeta_5 + 432\, \zeta_3^2 \\
& \qquad - 2340\, \zeta_5 - 1575\, \zeta_7 + 10206\, \zeta_9)
\end{aligned}
$$

[SL Volin Serban 12]

$$
\begin{aligned}
& + 48\, g^{14}(-44480 + 108960\, \zeta_3 + 8568\, \zeta_3\, \zeta_5 \\
& \qquad - 40320\, \zeta_3\, \zeta_7 - 8784\, \zeta_3^2 + 2592\, \zeta_3^3 - 4776\, \zeta_5 \\
& \qquad - 20700\, \zeta_5^2 - 26145\, \zeta_7 - 17406\, \zeta_9 + 152460\, \zeta_{11})
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[Bajnok Janik 12]

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& \qquad - 90720\,\zeta_3\,\zeta_5 - 129780\,\zeta_3\,\zeta_7 + 78408\,\zeta_3\,\zeta_8 \\
& \qquad + 483840\,\zeta_3\,\zeta_9 + 165312\,\zeta_3^2 - 82080\,\zeta_3^2\,\zeta_5 \\
& \qquad + 41472\,\zeta_3^3 + 178200\,\zeta_4\,\zeta_7 - 409968\,\zeta_5 \\
& \qquad + 121176\,\zeta_5\,\zeta_6 + 463680\,\zeta_5\,\zeta_7 + 49680\,\zeta_5^2 \\
& \qquad + 455598\,\zeta_7 + 194328\,\zeta_9 - 555291\,\zeta_{11} \\
& \qquad - 2208492\,\zeta_{13} - 14256\,\zeta_{1,2,8}) + \mathcal{O}(g^{18})
\end{aligned}
$$

[SL Volin 13]

Conclusion

- Rational spin chains (very well understood)
  - Bäcklund Flow to gradually simplify the system
  - Bethe Equations
  - Expression of the Hamiltonian from T and Q-functions
- For these rational spin chains, the classical integrability of τ-functions sheds light on the whole constriction, and helps for generalizations.
- For finite-size effects in integrable field theories, gives a guideline to write FiNLIE
  - Simple parameterization
  - Clearer analyticity properties
  - New symmetries
  - $\rightsquigarrow$ Perturbative expansion
- One big open question for these finite-size effects is : Can we prove these analyticity properties ?

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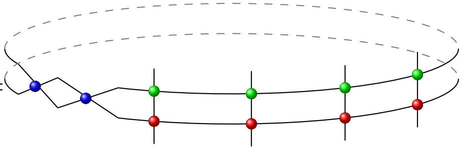
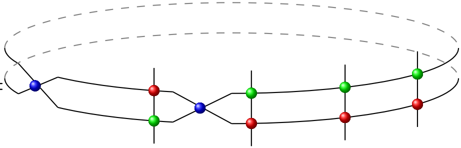
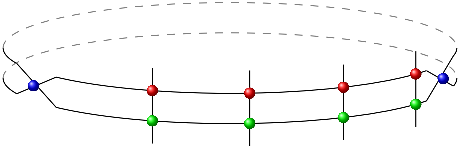
Appendices

Disclaimer : The following slides are additional material, not necessarily part of the presentation

Commutation of T-operators

$T(u) \cdot T(v) =$ 

$=$ 

$= \cdots =$ 

$= T(v) \cdot T(u)$

back

Expression of T through co-derivative

- $\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi} g)\Big|_{\phi=0}$ $\qquad \phi \in GL(K)$
- If $f(g)$ acts on $\mathcal{H}$, then $\hat{D} \otimes f$ acts on $\tilde{\mathcal{H}} = \mathbb{C}^K \otimes \mathcal{H}$
- $\hat{D} \otimes g = \mathcal{P}(1 \otimes g)$ and Leibnitz rule :
  $\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot \left[\hat{D} \otimes \tilde{f}\right] + \left[\hat{D} \otimes f\right] \cdot \left[\mathbb{I} \otimes \tilde{f}\right]$
  $\rightsquigarrow$ compute any $\hat{D} \otimes f(g)$
- $\hat{D} \otimes \pi_\lambda(g) = \left[\sum_{\alpha,\beta} \underbrace{e_{\beta\alpha}}_{generator} \otimes \underbrace{\pi_\lambda(e_{\alpha\beta})}_{generator}\right] \cdot \mathbb{I} \otimes \pi_\lambda(e_{\alpha\beta})$

hence

$$((u-\xi_L)\mathbb{I} + \mathcal{P}_{L,a}) \cdots ((u-\xi_1)\mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_\lambda(g) = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D}) \pi_\lambda(g)$$

and $T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N} (\underbrace{u - \xi_i}_{u_i} + \hat{D}) \chi_\lambda(g)$

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Bäcklund Transformations

linear system

Bäcklund Transformations

if $T^{a,s}(u)$ is a solution of Hirota equation and

$$\begin{cases} T^{a+1,s}(u)F^{a,s}(u) - T^{a,s}(u)F^{a+1,s}(u) \\ \qquad = \underbrace{x_j}_{\text{eigenvalue of } g} T^{a+1,s-1}(u+1)F^{a,s+1}(u-1), \\ T^{a,s+1}(u)F^{a,s}(u) - T^{a,s}(u)F^{a,s+1}(u) \\ \qquad = x_j T^{a+1,s}(u+1)F^{a-1,s+1}(u-1). \end{cases}$$

Then $F^{a,s}(u)$ is a solution of Hirota equation.

Moreover, if $T^{a,s}(u) = 0$, outside the $(K|M)$ "fat hook", one can choose $F^{a,s}(u) = 0$ outside the $(K-1|M)$ "fat hook".

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Explicit expression of nested T and Q-operators

- Generating series of T-operators:

let $w(z) \equiv \sum_{s\geq 0} \chi^{1,s} z^s = \det \frac{1}{1-g\,z}$,
then $\sum_{s\geq 0} T^{1,s} z^s = \bigotimes_{i=1}^{L} (u_i + \hat{D})\; w(z)$

Explicit solution of this linear system

$$T_{\mathrm{I}}^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \bar{\mathrm{I}}}} B_{\mathrm{I}} \cdot \left[\bigotimes_{i=1}^{L} (u_i + \hat{D} + |\bar{\mathrm{I}}|)\; \chi_\lambda(g_{\mathrm{I}}) \Pi_{\mathrm{I}} \right],$$

$$\Pi_{\mathrm{I}} = \prod_{j \in \bar{\mathrm{I}}} w(t_j) \qquad B_{\mathrm{I}} = \prod_{j \in \bar{\mathrm{I}}} (1 - x_j\, t_j) \cdot (1 - g\, t_j)^{\otimes N}$$

$$Q_{\mathrm{I}} = T_{\mathrm{I}}^{(0,s)} \qquad g_{\{j_1, j_2, \cdots, j_k\}} = \mathrm{diag}(x_{j_1}, x_{j_2}, \cdots, x_{j_k})$$

x_j=eigenvalue of g [Kazakov, S.L, Tsuboi 10]

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(Hints of) combinatorial proof

- Determinant identities (similar to Jacobi-Trudi) reduce the proofs to a few bilinear identities proven from the following

"Master Identity" [Kazakov, S.L, Tsuboi 10]

when $\Pi = \prod_j w(t_j)$,

$$(t-z)\left[\bigotimes(u_i+1+\hat{D})\ w(z)w(t)\Pi\right]\cdot\left[\bigotimes(u_i+\hat{D})\ \Pi\right]$$
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Combinatorics of coderivatives

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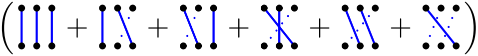
- For instance, [Kazakov Vieira 07]

$\hat{D}\otimes\hat{D}\otimes\hat{D}\ w(x) =$  $w(x)$

τ-functions of the MKP hierarchy

- A τ-function of the *MKP hierarchy* is a function of a variable n and an infinite set $\mathbf{t} = (t_1, t_2, \cdots)$ of "times", such that $\forall n \geq n'$, $\forall \mathbf{t}, \mathbf{t}'$

Definition of τ-functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t} - [z^{-1}]) \tau_{n'}(\mathbf{t}' + [z^{-1}]) dz = 0$$

where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$, $\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k$, and $\mathcal{C}$ encircles the singularities of $\tau_n(\mathbf{t} - [z^{-1}])\tau_{n'}(\mathbf{t}' + [z^{-1}])$ (typically finite), but not the singularities of $e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'}$ (typically at infinity).

- An example of such τ-function is the expectation value

$$\tau_n(\mathbf{t}) = \langle n| e^{J_+(\mathbf{t})} G |n\rangle$$

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where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$, and $\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k$.

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$$\tau_n(\mathbf{t}) = \langle n| \, e^{J_+(\mathbf{t})} G \, |n\rangle$$

over an infinite set of fermionic oscillators ($\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$), where $G = \exp\left(\sum_{i,k\in\mathbb{Z}} A_{ik} \psi_i^\dagger \psi_k\right)$ and $J_+ = \sum_{k\geq 1} t_k J_k$, where $J_k = \sum_{j\in\mathbb{Z}} \psi_j \psi_{j+k}^\dagger$. (and $\psi_n \, |n\rangle = |n+1\rangle$)

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Characteristic property

τ-functions are characterised by

$$\begin{aligned} z_2\tau_{n+1}\left(\mathbf{t} - [z_2^{-1}]\right)\tau_n\left(\mathbf{t} - [z_1^{-1}]\right) & \\ - z_1\tau_{n+1}\left(\mathbf{t} - [z_1^{-1}]\right)\tau_n\left(\mathbf{t} - [z_2^{-1}]\right) & \\ + (z_1 - z_2)\tau_{n+1}(\mathbf{t})\tau_n\left(\mathbf{t} - [z_1^{-1}] - [z_2^{-1}]\right) & = 0. \end{aligned}$$

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(obtained from $n' = n - 1$ and $\mathbf{t}' = \mathbf{t} - [z_1{}^{-1}] - [z_1{}^{-1}]$)

General rational τ-function

[Krichever 78]

If $\tau_n(\mathbf{t})$ is polynomial in n, we substitute $n \rightsquigarrow u$, where $u \in \mathbb{C}$.

Polynomial τ-functions of this MKP hierarchy

$$\tau(u, \mathbf{t}) = \det \left(A_i(u - j, \mathbf{t}) \right)_{1 \leq i,j \leq N}$$

$$\text{where } \; A_i(u, \mathbf{t}) = \sum_{m=0}^{d_i} a_{i,m} \partial_z^m \left(z^u e^{\xi(\mathbf{t},z)} \right) \Big|_{z=p_i}$$

parameterized by : the integer $N \geq 0$, the numbers $\{p_i\}$ and d_i, and the coefficients $\{a_{i,m}\}$.

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Undressing the rational τ-functions

- $A_i(u, \mathbf{t} + [z^{-1}]) = \sum_{m=0}^{d_i} a_{i,m} \partial_x^m \left(x^u e^{\xi(\mathbf{t},x)} \frac{1}{1-x/z} \right)\Big|_{x=p_i}$ has a pole at $z = p_i$.
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$$\tau_u(\mathbf{t} + [z^{-1}]) = \begin{vmatrix} A_1\big(u-1,\mathbf{t}+[z^{-1}]\big) & A_1(u-2,\mathbf{t}) & \dots & A_1(u-N,\mathbf{t}) \\ A_2\big(u-1,\mathbf{t}+[z^{-1}]\big) & A_2(u-2,\mathbf{t}) & \dots & A_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ A_N\big(u-1,\mathbf{t}+[z^{-1}]\big) & A_N(u-2,\mathbf{t}) & \dots & A_N(u-N,\mathbf{t}). \end{vmatrix}$$

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T-operators are τ-functions

- Set of times $\mathbf{t} \leftrightsquigarrow$ representations λ :

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- The undressing procedure for τ-functions (ie $\mathrm{Res}_{z=p_i}\tau(u+1,\mathbf{t}+[z^{-1}])$) explains the explicit expression found from the combinatorics of coderivatives.
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[Alexandrov, Kazakov, S.L., Tsuboi, Zabrodin 11]

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Thermodynamic Bethe Ansatz

$\rightsquigarrow$ Equations of the form

$$Y_{a,s}(u) = -L\, E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right) + \langle \textit{Source Terms} \rangle$$

- Vacuum energy

$$E_0 = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$$

Back to the presentation

- Extra assumption : Excited states obey the same equations.
  Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles
- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

$\Rightarrow$ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

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Parameterzation of Q-functions

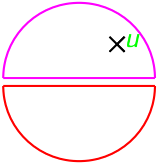
A simple Riemann-Hilbert Problem

Form the Cauchy theorem, we get

If $Q(u)$ is an analytic function on the upper half plane (when $\mathrm{Im}(u) > 0$), and $Q(u) \ll 1/u$ in the vicinity of ∞, then

$$\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{Q(v)-\bar{Q}(v)}{v-u}\mathrm{d}v = \begin{cases} Q(u) & \text{if } \mathrm{Im}(u) > 0 \\ \bar{Q}(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$

where $\bar{Q}(u)$ is the complex-conjugate of $Q(\bar{u})$.



Indeed, if $\mathrm{Im}(u) > 0$, then

$\frac{1}{2i\pi}\int_{\text{upwards}}\frac{Q(v)}{v-u}\mathrm{d}v = Q(u)$ and

$\frac{1}{2i\pi}\int_{\text{downwards}}\frac{\bar{Q}(v)}{v-u}\mathrm{d}v = 0$