Q-functions in the spectral problem of integrable quantum field theories

Sébastien Leurent, Burgundy University, DIJON–**FRANCE**

UNIVERSITÉ DE BOURGOGNE

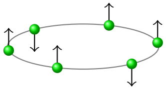
Talk based on collaborations with
N. Gromov, V. Kazakov, Z. Tsuboi, D. Volin:

ArXiv: 1007.1770, 1010.4022, 1110.0562, 1302.1135, 1305.1939

Outline

1. Quantum Integrability
   - Coordinate Bethe Ansatz (for XXX Spin chain)
   - Generalisations
   - Hirota Equation
2. Q-functions & spectral problems
   - Bäcklund flow
   - Bethe Equations
   - “FiNLIE” for field theories
3. P-μ system for AdS/CFT
   - Discontinuity relations
   - Quantum charges
   - Weak coupling

Bethe Ansatz for spin chains



Heisenberg spin chains: periodic set of L sites in a superposition of 2 states $\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L \text{ times}}$

Interactions: $\mathrm{H} = -\sum_i \vec{\sigma_i} \cdot \vec{\sigma_{i+1}}$

Identification of eigen-states:

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- 1 excitation: $|\Psi\rangle \propto \sum_k e^{i\,k\,p} |\underbrace{\downarrow\downarrow\cdots\downarrow}_{k-1}\uparrow\underbrace{\downarrow\downarrow\cdots\downarrow}_{L-k}\rangle$ with $e^{2i\,k\,L} = 1$
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$$|\Psi\rangle \propto \sum_{j<k}\left(e^{i(p_1 j+p_2 k)} + S\, e^{i(p_1 k+p_2 j)}\right)|\underbrace{\downarrow\cdots\downarrow}_{j-1}\uparrow\underbrace{\downarrow\cdots\downarrow}_{k-j-1}\uparrow\underbrace{\downarrow\cdots\downarrow}_{L-k}\rangle$$

with $S = -\frac{1+e^{i(p_1+p_2)}-2e^{ip_2}}{1+e^{i(p_1+p_2)}-2e^{ip_1}}$ and $e^{i\,L\,p_2} = S = e^{-i\,L\,p_1}$

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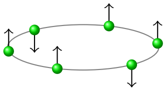

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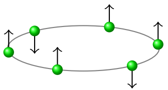

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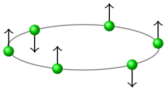

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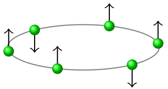

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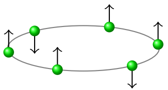

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$$|\Psi\rangle = \sum_{1 \le j_1 < j_2 < \cdots < j_n} \sum_{\sigma \in \mathcal{S}_n} \mathcal{A}_\sigma e^{i \sum_k p_{\sigma(k)} j_k} |\underbrace{\downarrow\downarrow \cdots \downarrow}_{j_1 - 1} \uparrow \underbrace{\downarrow\downarrow \cdots \downarrow}_{j_2 - j_1 - 1} \uparrow \downarrow \cdots\rangle$$

It is an eigenstate if

- $\mathcal{A}_\sigma \propto (-1)^\sigma \prod_{j<k} \left(1 + e^{i(p_{\sigma(j)} + p_{\sigma(k)})} - 2e^{ip_{\sigma(k)}}\right)$
- $\forall j, e^{i L p_j} = \prod_{k \neq j} S(p_j, p_k)$ where $S(p, p') \equiv -\frac{1 + e^{i(p+p')} - 2e^{ip}}{1 + e^{i(p+p')} - 2e^{ip'}}$

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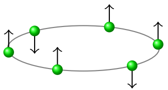

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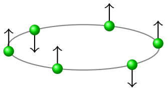

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Generalizations to other spin chains

There exists other integrable spin chains, with e.g.

- spins in a superposition of more than two values
- modified periodicity conditions
- different interactions (eg $H = \sum_k (X\sigma_i^x\sigma_{i+1}^x + Y\sigma_i^y\sigma_{i+1}^y + Z\sigma_i^z\sigma_{i+1}^z)$ instead of $\sum_k \vec{\sigma_i} \cdot \vec{\sigma_{i+1}}$)

The equations for their spectrum have the same form as for the XXX Heisenberg spin chain:

Bethe equations and spectrum

$$\forall j, e^{i\,L\,p_j} = \prod_{k\neq j} S(p_j, p_k)$$

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Bethe Ansatz in field theory

The same Ansatz $\psi(n_1, n_2, \cdots, n_M) \equiv \sum_{\sigma \in \mathcal{S}^M} \mathcal{A}_\sigma e^{i \sum_k p_{\sigma(k)} n_k}$ gives the wave-function of the eigenstates of several theories such that

- The space is one-dimensional and there are periodic boundary conditions.
- The interactions are local.
- A factorization formula holds :

One can argue that it is sufficient to have infinitely many conserved charges [Zamolodchikov & Zamolodchikov 79]

- "Locality" requires a large spatial period

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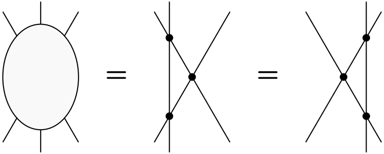

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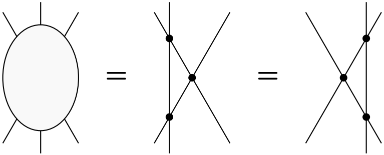

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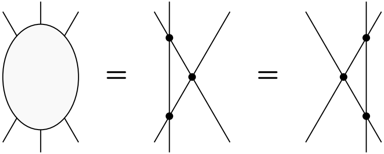

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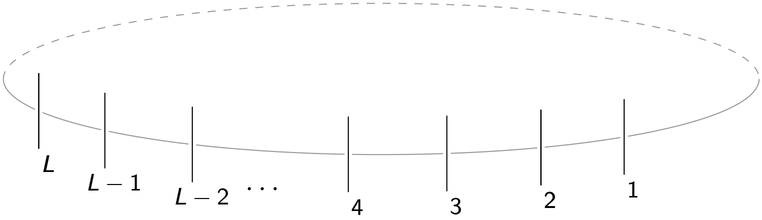
T-operators for Heisenberg XXX spin chain

$$H = -\sum_i \vec{\sigma_i} \cdot \vec{\sigma_{i+1}} = L - 2 \frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$$

$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$$

operator on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



permutation operator: $\mathcal{P}_{2,4}\,|\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle = |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$

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- $((u-v)\mathbb{I} + \mathcal{P}_{i,j})(u\mathbb{I} + \mathcal{P}_{i,k})(v\mathbb{I} + \mathcal{P}_{i,k})$
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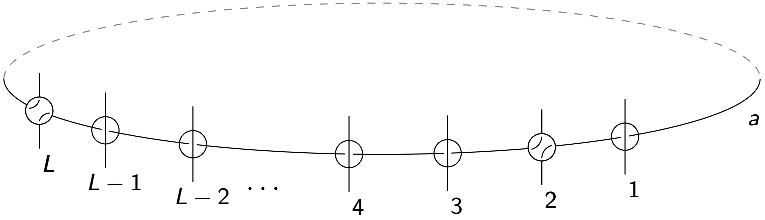

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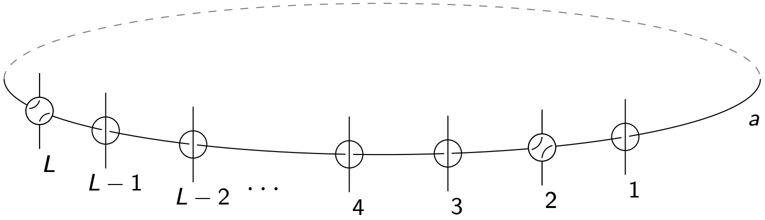

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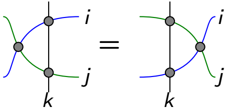

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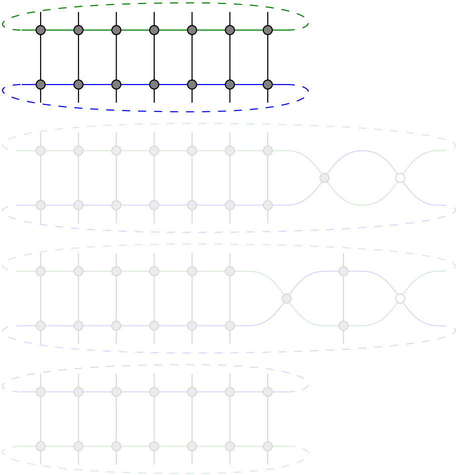

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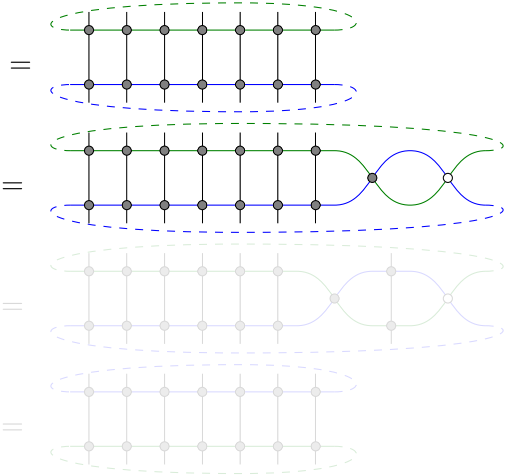

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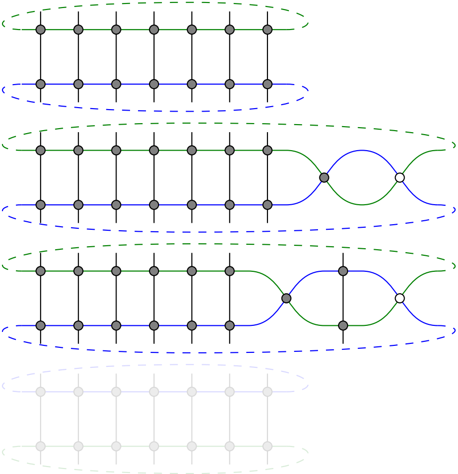

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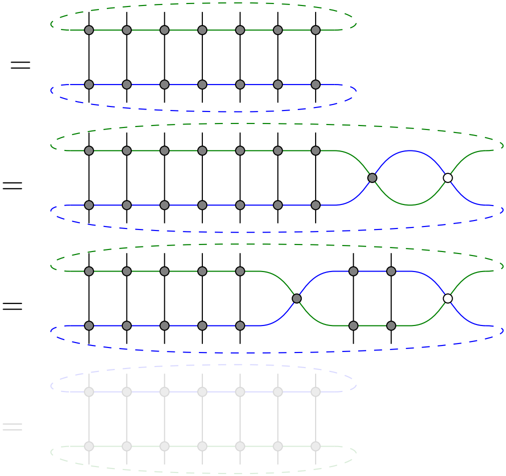

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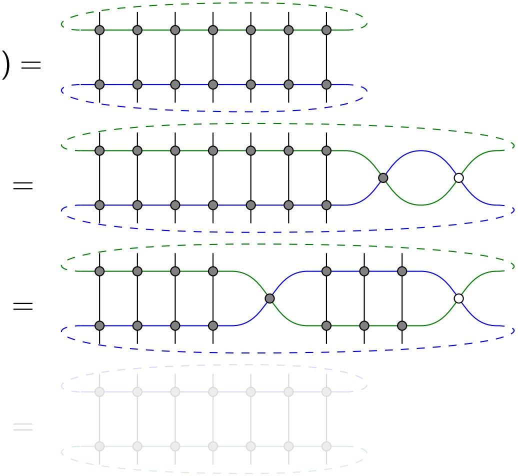

$= T(v)T(u)$

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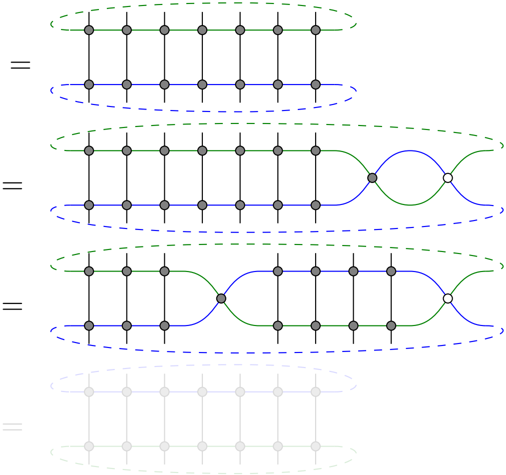

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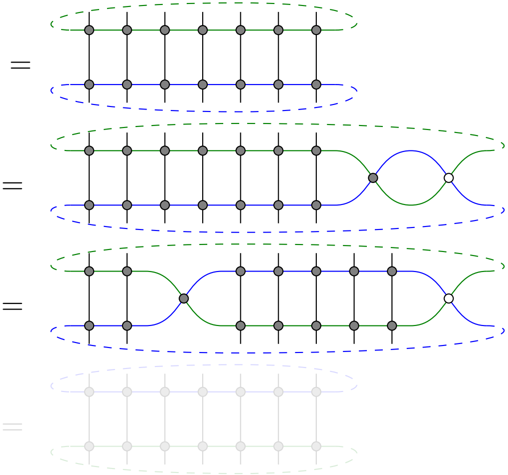

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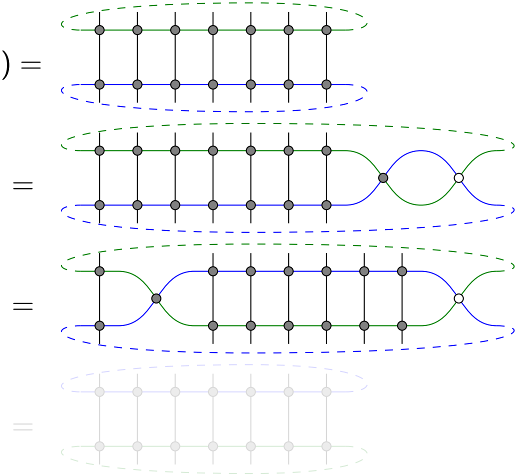

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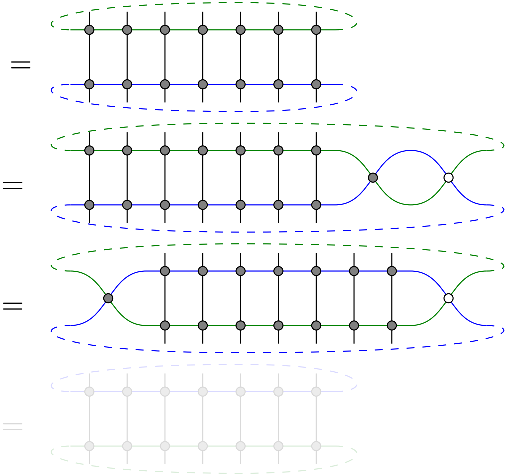

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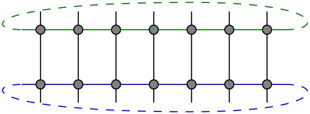
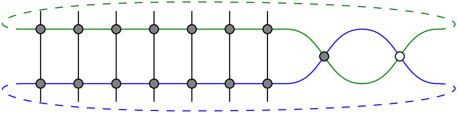
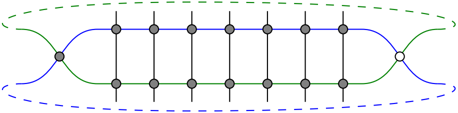
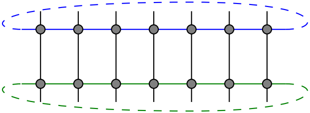
$T(u)T(v) =$ 

$=$ 

$=$ 

$=$ 

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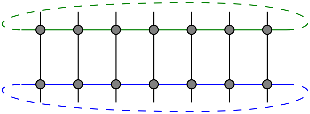

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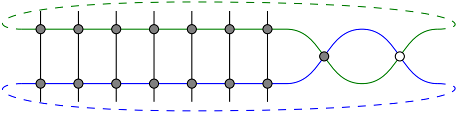

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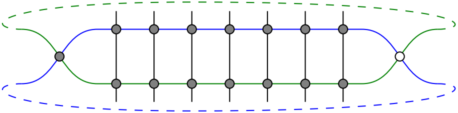

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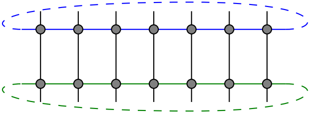


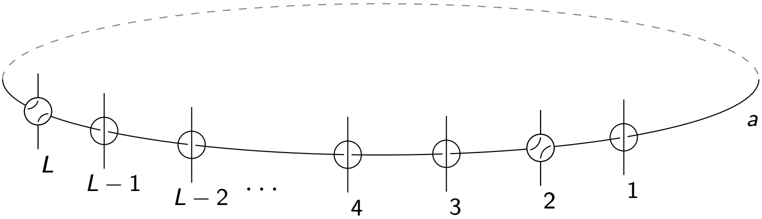
$= T(v)T(u)$

T-operators for Heisenberg XXX spin chain

$$H = -\sum_i \vec{\sigma_i}\cdot\vec{\sigma_{i+1}} = \quad L - 2\,\frac{\mathrm{d}}{\mathrm{du}}\log T(u)\Big|_{u=0}$$

$$T(u) = \mathrm{tr}_a\left(\left(u\,\mathbb{I} + \mathcal{P}_{L,a}\right)\cdot\left(u\,\mathbb{I} + \mathcal{P}_{L-1,a}\right)\cdots\left(u\,\mathbb{I} + \mathcal{P}_{1,a}\right)\right)$$

operators on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



permutation operator:

$$\mathcal{P}_{2,4}\,|\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\;\cdots\rangle = |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\;\cdots\rangle$$
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- $\left[T\ (u), T\ (v)\right] = 0$

T-operators for spin chains generalizing the Heisenberg XXX spin chain

$$H = -\sum_i \vec{\lambda}_i \cdot \vec{\lambda}_{i+1} = \frac{2}{K} L - 2 \frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$$

$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$$

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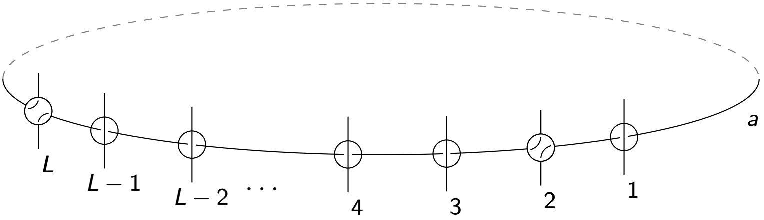

permutation operator:

$$\mathcal{P}_{2,4} \,|bcacdbc \cdots\rangle = |bcacdbc \cdots\rangle$$
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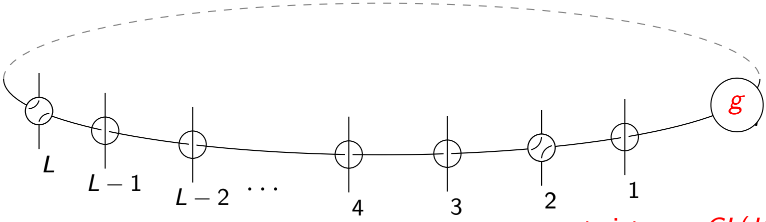
operators on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



twist $g \in GL(K)$

permutation operator:

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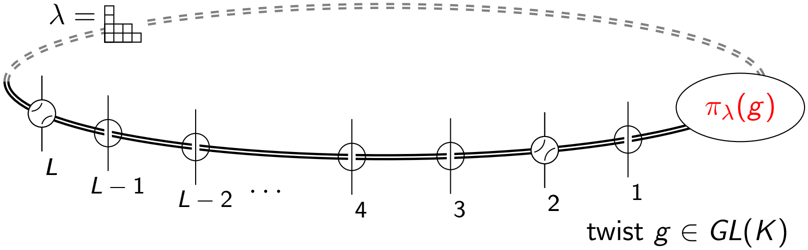
operators on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



generalized permutation operator:

$$\mathcal{P}_{i,j} = \sum_{\alpha,\beta} e^{(i)}_{\alpha,\beta} \otimes \pi_\lambda(e^{(j)}_{\beta,\alpha})$$

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T-operators for spin chains generalizing the Heisenberg XXX spin chain

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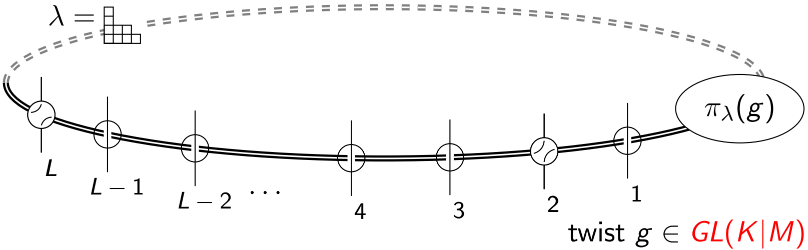
operators on the Hilbert space $\left(\mathbb{C}^{K|M}\right)^{\otimes L}$



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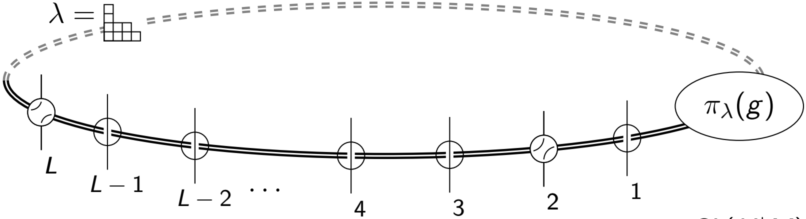

twist $g \in GL(K|M)$

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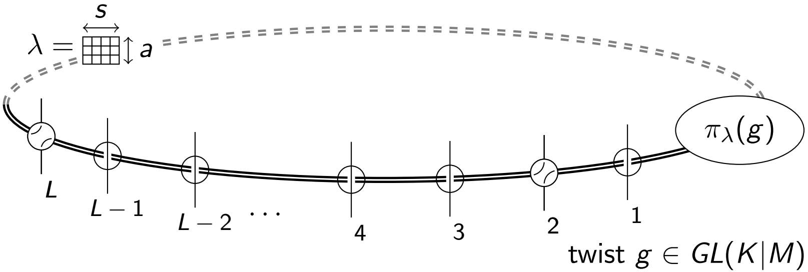

- $\left[T^{\lambda}(u), T^{\mu}(v)\right] = 0$
- for rectangular young diagrams,

$$T^{a,s}(u+1) \cdot T^{a,s}(u) = T^{a+1,s}(u+1) \cdot T^{a-1,s}(u) + T^{a,s-1}(u+1) \cdot T^{a,s+1}(u)$$

1+1 D integrable field theories

Bethe Ansatz: wavefunction for a large volume

- planar waves when particles are far from each other
- an *S-matrix* describes 2-points interactions

$\Rightarrow$ Bethe equations

- "Thermodynamic Bethe Ansatz": "double Wick Rotation"

finite size $\leftrightsquigarrow$ finite temperature

- At finite temperature, the Bethe equations give rise to several different types of bound states
$\rightsquigarrow$ introduce one density of excitations (as a function of the rapidity) for each type of bound state.

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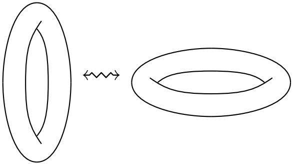

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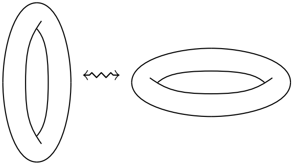

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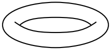
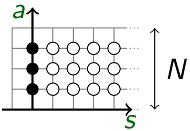
Y- and T-systems

case of the SU(N) "Gross Neveu" 1+1 dimensional field theory

TBA : 

bound states labelled by
$a \in \{1, \ldots, N-1\}$ and $s \in \mathbb{N}$



their densities obey

$$\begin{cases} \frac{Y_{a,s}{}^{+} Y_{a,s}{}^{-}}{Y_{a+1,s} Y_{a+1,s}} = \frac{1+Y_{a,s+1}}{1+Y_{a+1,s}} \frac{1+Y_{a,s-1}}{1+Y_{a-1,s}} \\ \text{analyticity constraints} \end{cases}$$

where $Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$

Lattice regularization

Representation labelled by a and s

$$\Leftrightarrow T_{a,s}{}^{+} T_{a,s}{}^{-} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$

$$\text{if } Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}},$$

[Gromov Kazakov Vieira 09] [Bombardelli Fioravanti Tateo 09][Autyunov Frolov 09]

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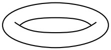
TBA : 

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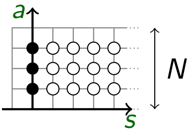

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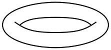
TBA : 

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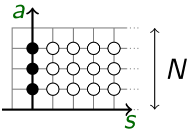

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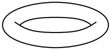
TBA : 

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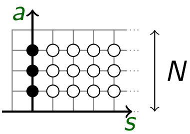

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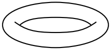
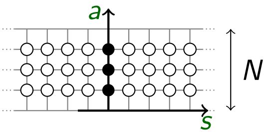
Y- and T-systems

case of the SU(N)×SU(N) Principal chiral model

TBA : 

bound states labelled by
$a \in \{1, \ldots, N-1\}$ and $s \in \mathbb{Z}$



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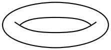
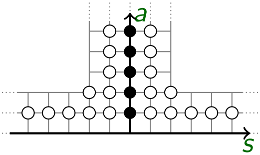
Y- and T-systems

case of the planar limit of the AdS/CFT duality

TBA : 

bound states labelled by integers a and s



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$$\text{if } Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}.$$

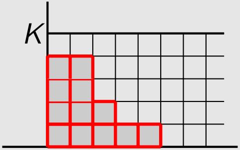
[Gromov Kazakov Vieira 09] [Bombardelli Fioravanti Tateo 09][Autyunov Frolov 09]

outline

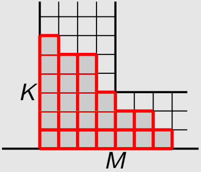
"Fat hooks" and "Bäcklund Flow"

Possible Young diagrams for a given symmetry group

GL(K) symmetry



GL(K|M) symmetry



Hirota equation solved by gradually reducing the size of the "fat hook" [Krichever, Lipan, Wiegmann & Zabrodin 97][Kazakov Sorin Zabrodin 08]

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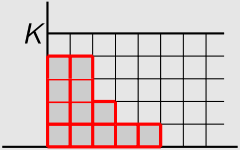

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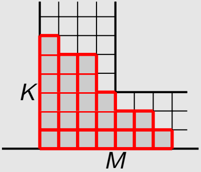

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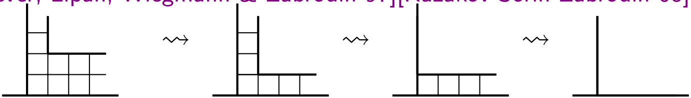

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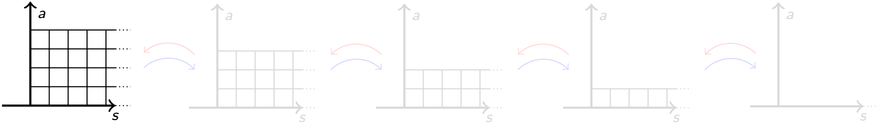
Q-functions and Hasse diagram

example of the SU(4) (a,s)-lattice

- "Undressing" procedure (Bäcklund Transformation)

su(4) $\supset$ su(3) $\supset$ su(2) $\supset$ su(1)



$Q_{1234} \equiv T_{0,0}$ $\quad Q_{123}$ $\quad Q_{23}$ $\quad Q_2$ $\quad Q_\emptyset = 1$

- 2^n conserved charges Q_I related by QQ-relations : $Q_{123} Q_1 = \begin{vmatrix} Q_{12}^+ & Q_{13}^+ \\ Q_{12}^- & Q_{13}^- \end{vmatrix}$
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see for instance [Kazakov, Leurent, Tsuboi 12]

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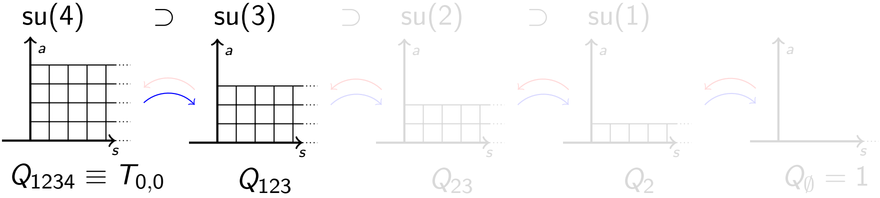

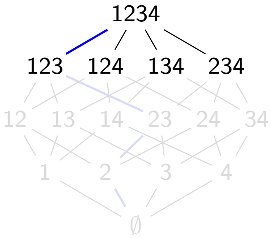

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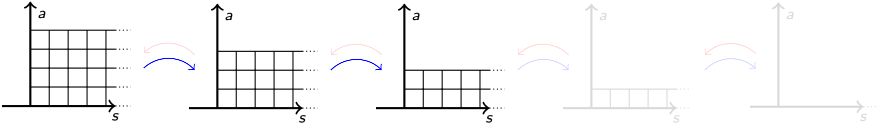

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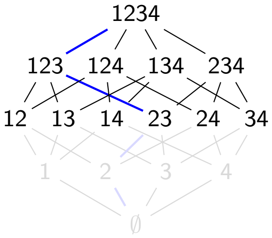

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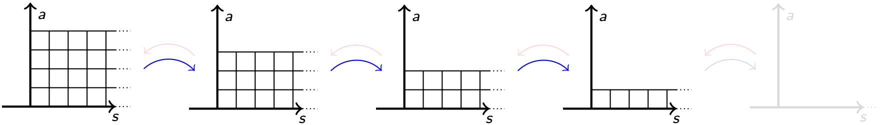

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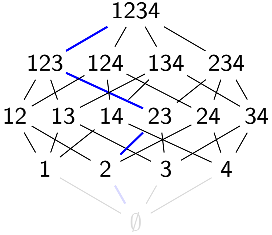

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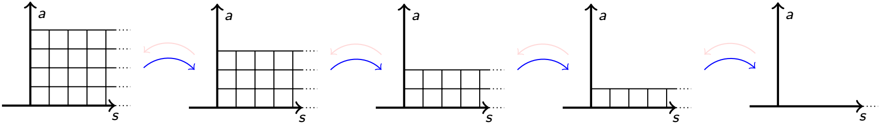

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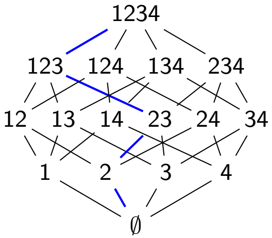

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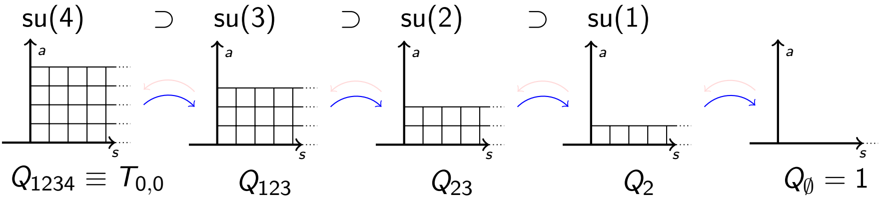

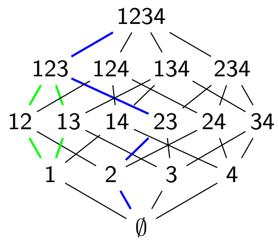

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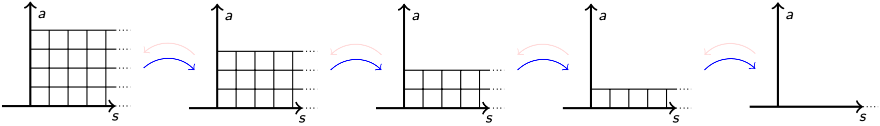

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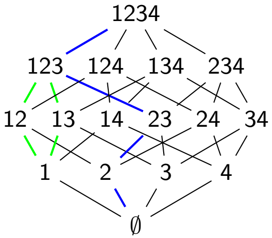

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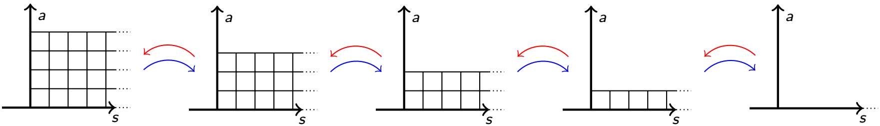

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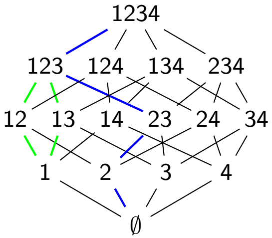

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outline

Q-operators $\leftrightsquigarrow$ spectral problem

Expression of T (for SU(N) representations)

$$T_{a,s}(u) = \frac{\epsilon^{i_1,i_2,i_3,\ldots,i_N}}{a!(N-a)!} Q_{i_1,i_2,\ldots,i_a}\left(u + i\frac{s}{2}\right) Q_{i_{a+1},i_{a+2},\ldots,i_N}\left(u - i\frac{s}{2}\right)$$

- $\rightsquigarrow$ by diagonalising Q's, one diagonalizes $H = \partial \log(T)|_{u=0}$ as well
  - $\rightsquigarrow$ $E = E_0 + \sum_k E(u_k)$ where u_k are the zeroes of the polynomial $Q_{1,2,3,\ldots,N-1}$.

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Spectrum for these rational spin chains

Hence, the spectral problem for these spin chains reduces to finding Q-functions (eigenvalues of the Q-operators), subject to a few constraints

- Polynomiality
- $Q_{1,2,3,\ldots,N} = \prod_i (u - \zeta_i)$
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Thermodynamic Bethe Ansatz: infinitely many Y-functions

- Solution of Hirota written though Q-functions

$\rightsquigarrow$ Finite number of Q-functions, finite number of equations

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- For AdS/CFT, $Y_{a,s}(u)$ is meromorphic in the domain $|\mathrm{Im}(u)| < |a - |s||$.
- For the principal chiral model, the domain $|\mathrm{Im}(u)| \leq \frac{|s|}{2} + \frac{N}{4}$ is where $Y_{a,s}(u)$ has a well defined and analytic behaviour at both $u \to \infty$ and $L \to \infty$.

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Analyticity of Y $\rightsquigarrow$ Analyticity of T $\rightsquigarrow$ Analyticity of Q

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The transformation Y$\rightsquigarrow$T$\rightsquigarrow$Q is complicated and ambiguous

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where $g_{(\pm,\pm)}$ are four arbitrary functions.

⇝ Does there exist a specific gauge where T has nice analyticity properties ?

Principal Chiral Model [Gromov Kazakov Vieira 09] [Kazakov Leurent 10]

- ∃ gauges where $T_{a,s}(u)$ analytic for $|\mathrm{Im}(u)| < \frac{N}{4} + \frac{s}{2}$.
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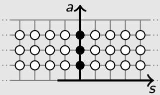

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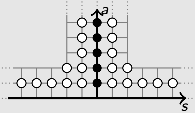
AdS/CFT [Gromov Kazakov Leurent Volin 12]



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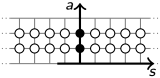
Ambiguities in T⇝QExample of $SU(3)$ principal chiral model :



- T-functions given by "wronskian" determinants like

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$$Q_i \rightsquigarrow \sum_j c_{i,j} Q_j$$

where the matrix $(c_{i,j})$ is i-periodic, has determinant 1, and (in the present case) is real.

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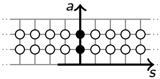
Ambiguities in T$\rightsquigarrow$QExample of $SU(3)$ principal chiral model :



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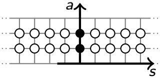

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Q-functions for the spectrum of the principal chiral model

One can chose $Q_i(u)$ such that

- $Q_j(u)$ is analytic with polynomial behavior at $u \to \infty$ (or $L \to \infty$) when $\mathrm{Im}(u) \geq -iN/4$
- Provides an analyticity strip for $T_{a,s}$ with $s \geq 0$.
- The polynomial behaviour of $Q_j(u - iN/4)$ is real and explicit

Parameterisation of these Q-functions [Kazakov Leurent 10]

for $\mathrm{Im}(u) > 0$,

$$Q_j(u - iN/4) = P_j(u) + \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{Q_j(v - iN/4) - \bar{Q}_j(v + iN/4)}{u - v} \mathrm{d}v$$

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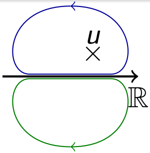

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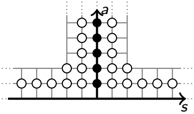

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New symmetries and analytic properties found

four Q-functions have one cut on $[-2g, 2g]$

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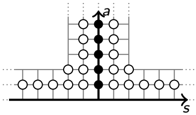

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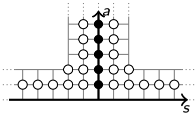

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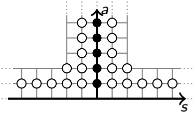

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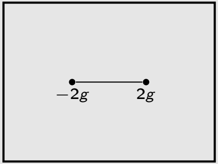

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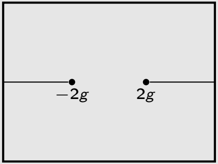

outline

P-μ system: discontinuity relations

P-μ system

$$\tilde{\mathbf{P}}_i = \mathbf{P}^j \, \mu_{ji} \qquad \tilde{\mathbf{P}}^i = \mathbf{P}_j \, \mu^{ji}$$

$$\tilde{\mu}_{ij} - \mu_{ij} = \mathbf{P}_i \tilde{\mathbf{P}}_j - \mathbf{P}_j \tilde{\mathbf{P}}_i$$

$$\tilde{\mu}^{ij} - \mu^{ij} = \mathbf{P}^i \tilde{\mathbf{P}}^j - \mathbf{P}^j \tilde{\mathbf{P}}^i$$

- $i, j \in \{1, 2, 3, 4\}$
- P_i and P^i are analytic on $\mathbb{C} \setminus [-2g, 2g]$
- $\mu_{ji} = -\mu_{ij}$
- $\mu_{12}\,\mu_{34} - \mu_{13}\,\mu_{24} + \mu_{14}\,\mu_{23} = 1$
- μ is i-periodic and analytic on $\mathbb{C} \setminus (]-\infty, -2g] \cup [2g, +\infty[)$

Where $\tilde{f}$ denotes the analytic continuation of f around the branch point at $u = \pm 2g$

Quantum charges $\leftrightsquigarrow$ large u behaviour

Example of the $\mathfrak{sl}_2$ sector

At large u, all functions scale as powers of u. For instance for spin-S twist-L states $\mathrm{Tr}\ Z\nabla_+^S Z^{L-1}$ ($\mathfrak{sl}_2$ sector), one has

$$\mathbf{P}^i = -\chi^{ij}\mathbf{P}_j\,, \qquad \chi^{ij} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu_{12} \sim u^{\Delta-L}, \qquad \mu_{13} \sim u^{\Delta+1},$$

$$\mu_{14} = \mu_{23} \sim u^{\Delta}, \qquad \mu_{24} \sim u^{\Delta-1}, \qquad \mu_{34} \sim u^{\Delta+L}.$$

$$\mathbf{P}_1 \simeq A_1 u^{-\frac{L}{2}}, \quad \mathbf{P}_2 \simeq A_2 u^{-\frac{L+2}{2}}, \quad \mathbf{P}_3 \simeq A_3 u^{\frac{L}{2}}, \quad \mathbf{P}_4 \simeq A_4 u^{\frac{L-2}{2}},$$

$$\text{where } A_2A_3 = \frac{[(L-S+2)^2-\Delta^2][(L+S)^2-\Delta^2)]}{16iL(L+1)},$$

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Application: weak coupling dimension of the Konishi operator

$$\Delta_{\text{Konishi}} = 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + 96\,g^8(-26 + 6\,\zeta_3 - 15\,\zeta_5) - 96\,g^{10}(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7)$$

[Bajnok Egedüs Janik Łukowski 09]

[Eden Heslop Korchemsky Smirnov Sokatchev 12]

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[SL Volin Serban 12]

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$$\begin{aligned}&+ 48\,g^{14}(-44480 + 108960\,\zeta_3 + 8568\,\zeta_3\,\zeta_5\\ &\qquad - 40320\,\zeta_3\,\zeta_7 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 - 4776\,\zeta_5\\ &\qquad - 20700\,\zeta_5^2 - 26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11})\end{aligned}$$

[Bajnok Janik 12]

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&+ 48\,g^{16}(1133504 + 263736\,\zeta_2\,\zeta_9 - 1739520\,\zeta_3\\
&\qquad - 90720\,\zeta_3\,\zeta_5 - 129780\,\zeta_3\,\zeta_7 + 78408\,\zeta_3\,\zeta_8\\
&\qquad + 483840\,\zeta_3\,\zeta_9 + 165312\,\zeta_3^2 - 82080\,\zeta_3^2\,\zeta_5\\
&\qquad + 41472\,\zeta_3^3 + 178200\,\zeta_4\,\zeta_7 - 409968\,\zeta_5\\
&\qquad + 121176\,\zeta_5\,\zeta_6 + 463680\,\zeta_5\,\zeta_7 + 49680\,\zeta_5^2\\
&\qquad + 455598\,\zeta_7 + 194328\,\zeta_9 - 555291\,\zeta_{11}\\
&\qquad - 2208492\,\zeta_{13} - 14256\,\zeta_{1,2,8}) + \mathcal{O}(g^{18})
\end{aligned}
$$

[SL Volin 13]

9-loop result obtained recently

$$\cdots - 96\, g^{18}\Big(10568224 - 11884608\,\zeta_3 + 148896\,\zeta_3\,\zeta_5 - 177768\,\zeta_3\,\zeta_5^2 - 354384\,\zeta_3\,\zeta_7 - 1244484\,\zeta_3\,\zeta_9 + 2901096\,\zeta_{11}\,\zeta_3 + 533952\,\zeta_3^2 + 284904\,\zeta_3^2\,\zeta_5 - 229824\,\zeta_3^2\,\zeta_7 + 209952\,\zeta_3^3 - 5993280\,\zeta_5 + 963954\,\zeta_5\,\zeta_7 + 2553120\,\zeta_5\,\zeta_9 - 576000\,\zeta_5^2 + 2324196\,\zeta_7 + 1184274\,\zeta_7^2 + 2573892\,\zeta_9 + 355266\,\zeta_{11} + 2644434\,\zeta_{13} - 15810795\,\zeta_{15} + 163296\,\frac{\zeta_{11} - \zeta_3\,\zeta_{3,5} + \zeta_{3,5,3}}{5} - 13608(\zeta_3\,\zeta_{3,7} - \zeta_{3,7,3} + \zeta_3^2\,\zeta_5 - \zeta_5\,\zeta_{5,3} + \zeta_{5,3,5})\Big) + \mathcal{O}\left(g^{20}\right)$$

[Volin 13]

Conclusion

- Rational spin chains (very well understood)
  - Bäcklund flow
  - Bethe equations $\leftrightarrow$ polynomiality of Q-operators
  - Expression of the Hamiltonian from T and Q-functions
- For finite-size effects in integrable field theories, gives a guideline to write FiNLIE
  - Simple parameterisation
  - Clearer analyticity properties
  - New symmetries
  - $\rightsquigarrow$ Perturbative expansion
  - $\rightsquigarrow$ Wilson loops

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- Fundamental open questions for these finite-size effects:
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Thank you for your attention

Thermodynamic Bethe Ansatz

⇝ Equations of the form
$Y_{a,s}(u) = -L\ E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right)$
$+\langle Source\ \ Terms\rangle$

- Vacuum energy $E_0 = -\sum_{a,s} \int E_{a,s}(u)\ \ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$

▸ Back to the presentation

- Extra assumption : Excited states obey the same equations. Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles
- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

⇒ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

Thermodynamic Bethe Ansatz

⇝ Equations of the form

$$Y_{a,s}(u) = -L\ E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right) + \langle \textit{Source Terms} \rangle$$

- Vacuum energy $E = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$

- Extra assumption : Excited states obey the same equations. Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles

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- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

⇒ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

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