Integrability of rational spin chains, MKP hierarchy, and Q-operators.

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[arXiv:1010.4022] V. Kazakov, SL, Z. Tsuboi

[arXiv:1112.3310] A. Alexandrov, V. Kazakov, SL, Z.Tsuboi, A. Zabrodin

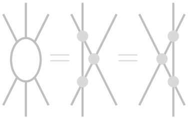
Saclay, November 22, 2012

Quantum Integrability

Very specific models (spin chains or quantum field theories), which have :

- 1+1 dimensions
- local interactions
- and many conserved charges

are *integrable*.



Then, they have the following properties

Properties of integrable models

- n-points interactions factorize into 2-points interactions
- the exact diagonalization of the Hamiltonian reduces to solving the Bethe Equation(s).

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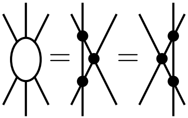

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One method of resolution

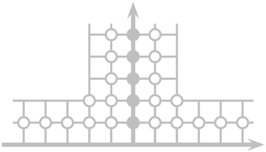
through a Bäcklund flow and Q-operators

Solving integrable models through Q-operators

- Reduce the model to a simpler and simpler system (Bäcklund flow $\rightsquigarrow$ Q-operators)
- Express the original Hamiltonian through Q-operators
- The Bethe equations arise naturally as a consistency constraint, required by Q-operators' analyticity properties

→ A simple example is $GL(K|M)$ spin chains.

- This procedure proved fruitful for many models, including the 3+1-dimensional $\mathcal{N} = 4$ Super-Yang-Mills (of AdS/CFT duality)



[Gromov, Kazakov, SL, Volin 11]
[Balog, Hegedüs 12]

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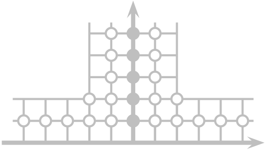

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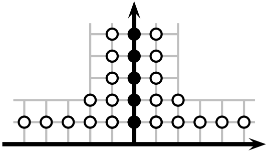

[Gromov, Kazakov, SL, Volin 11]

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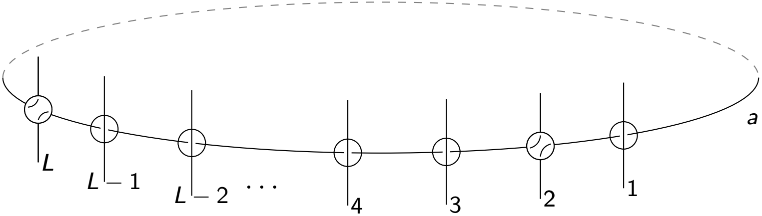
Outline

Heisenberg “XXX” spin chain

Construction of T-operators

$$T(u) = \mathrm{tr}_a \left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a}) \right)$$

operator on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



permutation operator : $\mathcal{P}_{1,2}\,|\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle = |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$

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- $[T\;(u), T\;(v)] = 0$

  (proved from relations like $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}$)

- $H = -\sum_i \vec{\sigma_i} \cdot \vec{\sigma_{i+1}} = \quad L - 2\,\frac{\mathrm{d}}{\mathrm{du}} \log T(u)\Big|_{u=0}$

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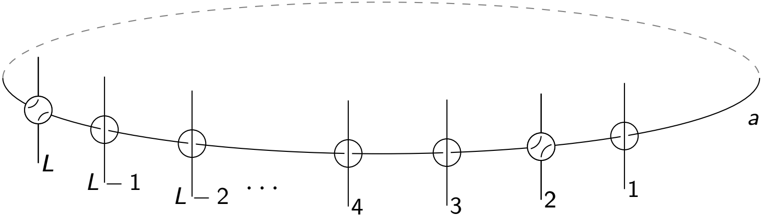

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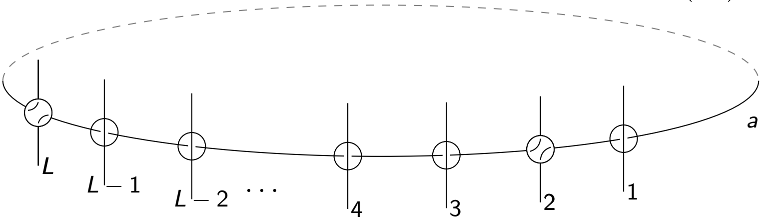

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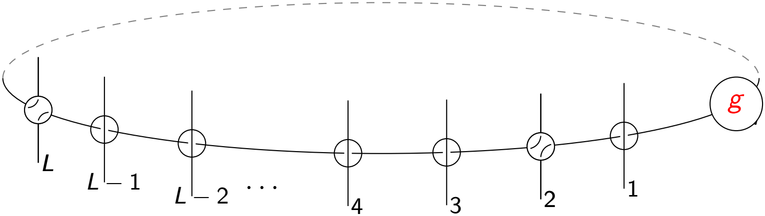

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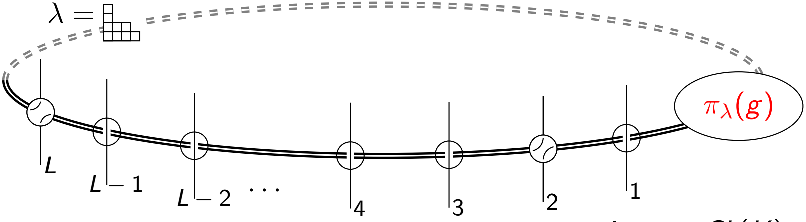

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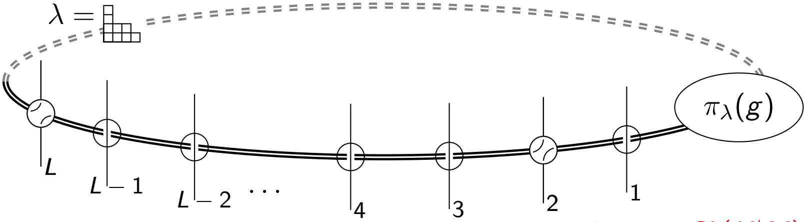

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T-operators $\leftrightsquigarrow$ characters

+ Cherednik-Bazhanov-Reshetikhin formula

$$T^\lambda(u) = \mathrm{tr}_a\left(((u-\xi_L)\mathbb{I} + \mathcal{P}_{L,a})\cdots((u-\xi_1)\mathbb{I} + \mathcal{P}_{1,a})\cdot \pi_\lambda(g)\right)$$

- At $L=0$, $T^\lambda(u) = \chi^\lambda(g) \equiv \mathrm{tr}\,\pi_\lambda(g)$
- In general $\qquad u_i \equiv u - \xi_i$

$$T^\lambda(u) = \left(u_1 + \hat{D}\right) \otimes \left(u_2 + \hat{D}\right) \otimes \cdots \otimes \left(u_L + \hat{D}\right) \chi^\lambda(g)$$

Rectangular representation : $a, s \leftrightarrow \lambda =$ a, s

Determinant identity

[Cherednik 86] [Bazhanov-Reshetikhin 90] [Kazakov Vieira 08]

$$\chi^\lambda(g) = \det\left(\chi^{1,\lambda_i+j-i}(g)\right)_{1\le i,j\le|\lambda|}$$

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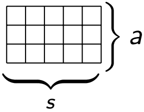
Rectangular representation : $a, s \leftrightarrow \lambda =$ 

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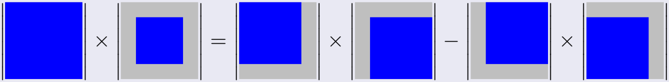
Hirota equation

from Jacobi-Trudi identity

Jacobi-Trudi identity : for an arbitrary determinant



- From the CBR determinant expression
$$T^{a,s}(u) = \frac{\det\left(T^{1,s+j-i}(u+1-j)\right)_{1\le i,j\le a}}{\prod_{k=1}^{a-1} T^{0,0}(u-k)},$$
this identity gives the "Hirota equation":
$$T^{a,s}(u+1)\cdot T^{a,s}(u) = T^{a+1,s}(u+1)\cdot T^{a-1,s}(u) + T^{a,s-1}(u+1)\cdot T^{a,s+1}(u)$$
- Conversely, the Jacobi-Trudy identity allows to show that this Hirota equation implies the CBR determinant expression.

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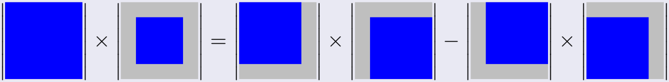

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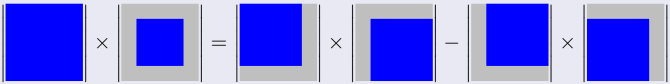

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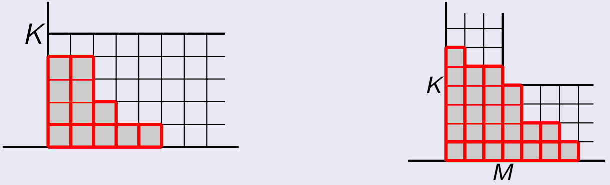
"Fat hooks" and "Bäcklund Flow"

Authorised Young diagrams for a given symmetry group



Hirota equation solved by gradually reducing the size of the "fat hook" [Krichever, Lipan, Wiegmann & Zabrodin 97] [Kazakov Sorin Zabrodin 08]

using inclusions like GL(2|1)⊃ GL(1|1)⊃ GL(1|0)⊃ {1}

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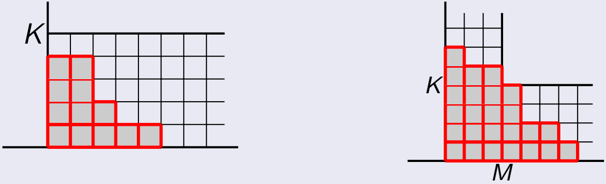
GL(K) symmetry **GL(K|M) symmetry**



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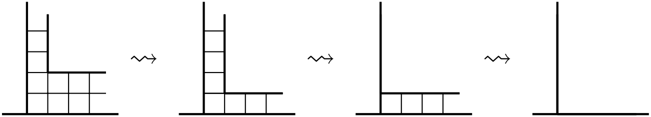

Bäcklund Transformations

linear system

Bäcklund Transformations

if $T^{a,s}(u)$ is a solution of Hirota equation and

$$\begin{cases} T^{a+1,s}(u)F^{a,s}(u) - T^{a,s}(u)F^{a+1,s}(u) \\ \qquad = \underbrace{x_j}_{\text{eigenvalue of } g} T^{a+1,s-1}(u+1)F^{a,s+1}(u-1), \\ T^{a,s+1}(u)F^{a,s}(u) - T^{a,s}(u)F^{a,s+1}(u) \\ \qquad = x_j T^{a+1,s}(u+1)F^{a-1,s+1}(u-1). \end{cases}$$

Then $F^{a,s}(u)$ is a solution of Hirota equation.

Moreover, if $T^{a,s}(u) = 0$, outside the $(K|M)$ "fat hook", one can choose $F^{a,s}(u) = 0$ outside the $(K-1|M)$ "fat hook".

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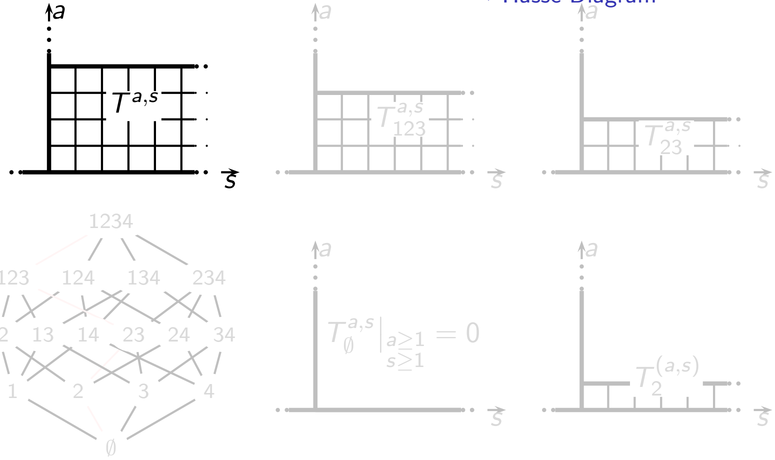
example of $GL(4)$ Bäcklund flow

$\rightsquigarrow$ Hasse Diagram



$\rightsquigarrow$ Defines 2^4 Q-operators, lying on the nodes of this *Hasse Diagram*

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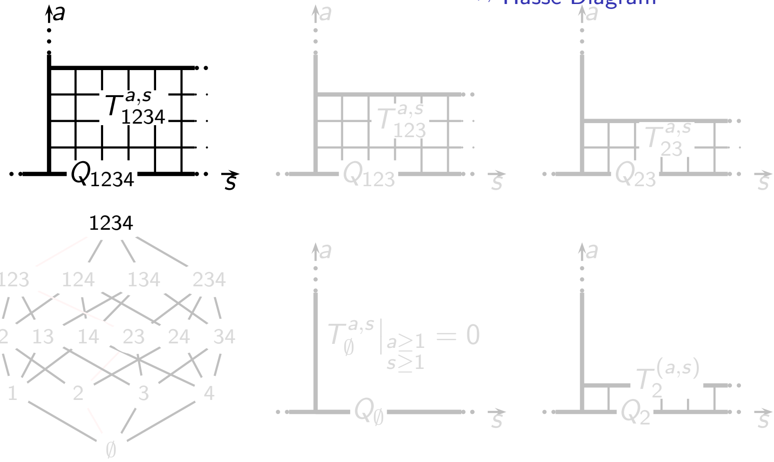

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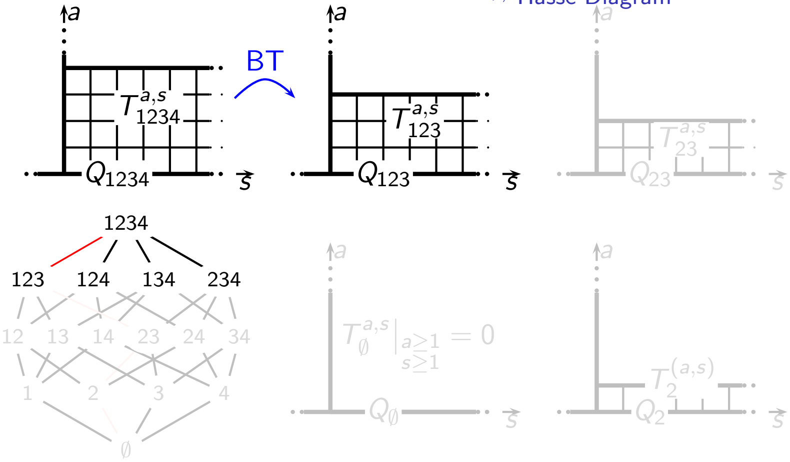
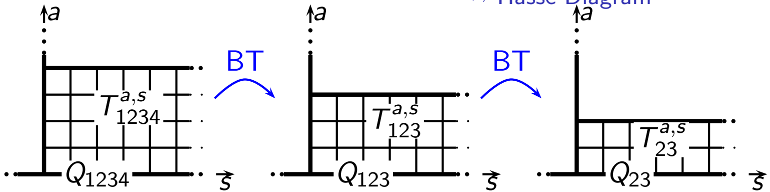
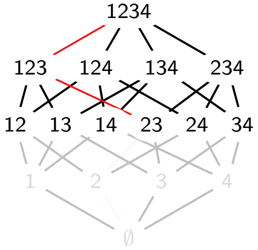


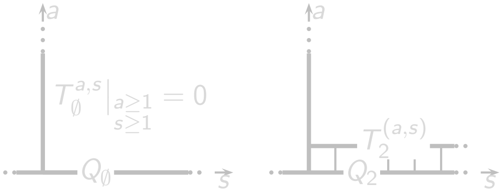
$\rightsquigarrow$ Defines 2^4 Q-operators, lying on the nodes of this *Hasse Diagram*

example of $GL(4)$ Bäcklund flow

⇝ Hasse Diagram







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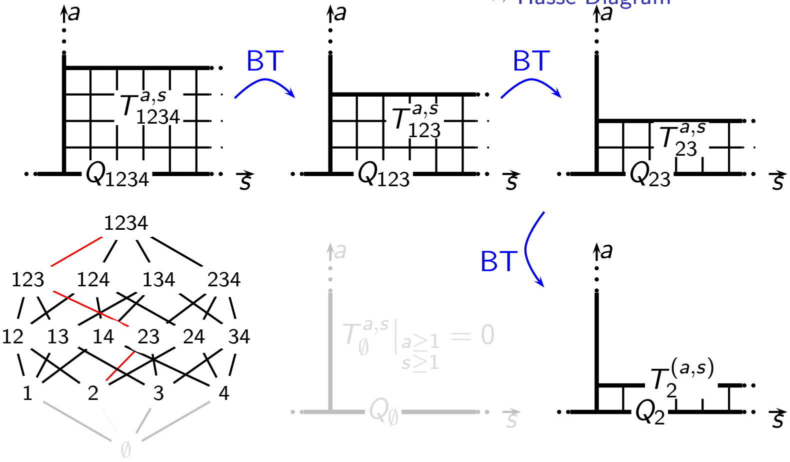

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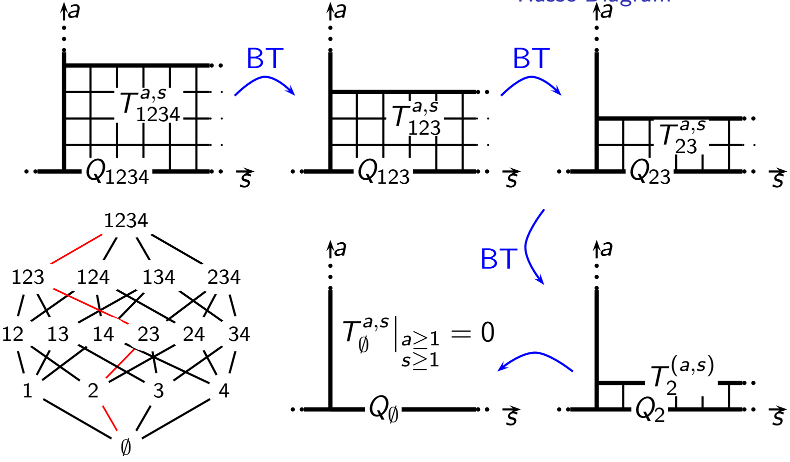

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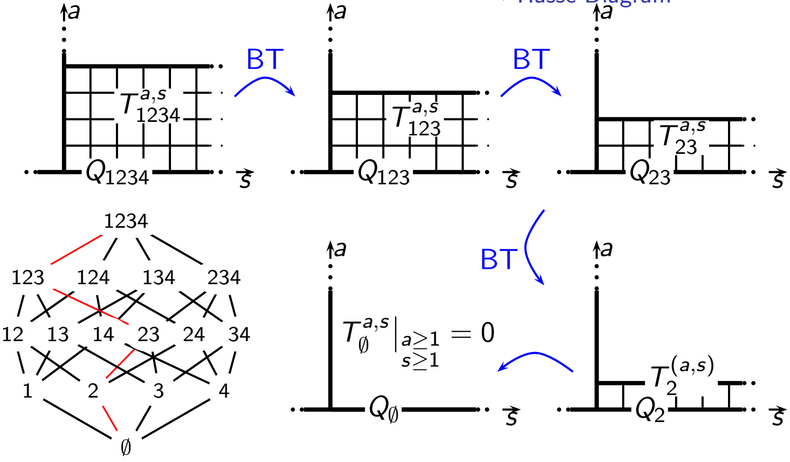

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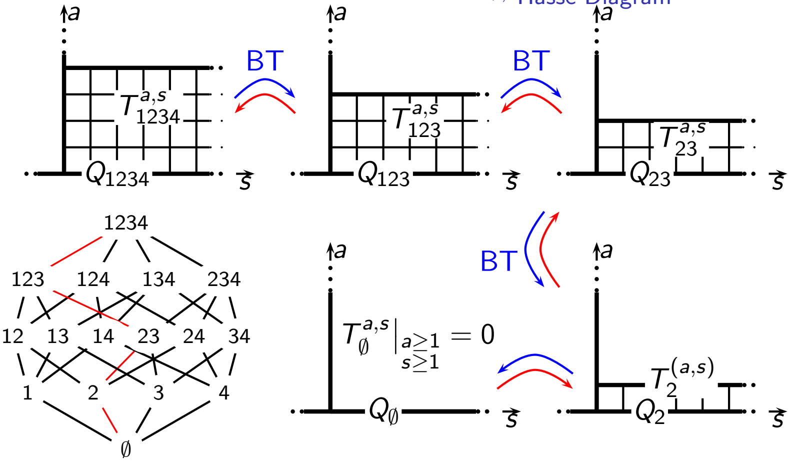


⇝ Defines 2^4 Q-operators, lying on the nodes of this *Hasse Diagram*

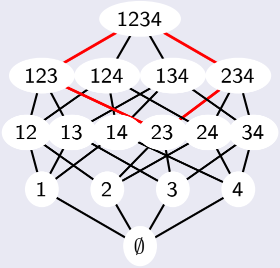
QQ-relations and Bethe Equations

At the level of operators, the QQ-relations

$$(x_i - x_j)\, Q_{\mathrm{I}}(u-1) Q_{\mathrm{I},i,j}(u) = x_i Q_{\mathrm{I},j}(u-1) Q_{\mathrm{I},i}(u) - x_j Q_{\mathrm{I},j}(u) Q_{\mathrm{I},i}(u-1)$$

example : $\mathrm{I} = \{23\}, i = 1, j = 4$

$$(x_1 - x_4)\, Q_{23}(u-2) Q_{1234}(u) = x_1 Q_{234}(u-1) Q_{123}(u) - x_4 Q_{234}(u) Q_{123}(u-1)$$



The relation involves Q-operators lying on the same facet of the Hasse diagram

QQ-relations and Bethe Equations

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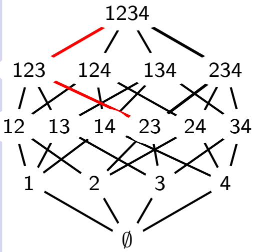
imply

$$Q_{\mathrm{I},i}(u) \mid x_i Q_{\mathrm{I}}(u-1) Q_{\mathrm{I},i,j}(u) Q_{\mathrm{I},i}(u+1) + x_j Q_{\mathrm{I}}(u) Q_{\mathrm{I},i,j}(u+1) Q_{\mathrm{I},i}(u-1).$$



for instance

$$Q_{123}(u) \mid x_1 Q_{23}(u-1) Q_{1234}(u) Q_{123}(u+1) + x_4 Q_{23}(u) Q_{1234}(u+1) Q_{123}(u-1).$$

The relation involves three consecutive Q-operators lying on the same nesting path.

QQ-relations and Bethe Equations

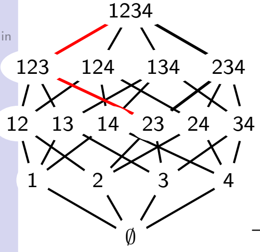
At the level of operators, the QQ-relations

$$(x_i - x_j)\, Q_{\mathtt{I}}(u-1) Q_{\mathtt{I},i,j}(u) = x_i Q_{\mathtt{I},j}(u-1) Q_{\mathtt{I},i}(u) - x_j Q_{\mathtt{I},j}(u) Q_{\mathtt{I},i}(u-1)$$

imply

$$Q_{\mathtt{I},i}(u) \mid x_i Q_{\mathtt{I}}(u-1) Q_{\mathtt{I},i,j}(u) Q_{\mathtt{I},i}(u+1) + x_j Q_{\mathtt{I}}(u) Q_{\mathtt{I},i,j}(u+1) Q_{\mathtt{I},i}(u-1).$$

On a given eigenstate,



$$\mathrm{Q}_{\mathtt{I}}(u) = c_{\mathtt{I}} \prod_{k=1}^{K_{\mathtt{I}}} (u - u_k^{(\mathtt{I})}),$$

we get the Bethe equation

$$-1 = \frac{x_i}{x_j} \frac{Q_{\mathtt{I}}(u_k^{(\mathtt{I},i)} - 1) Q_{\mathtt{I},i}(u_k^{(\mathtt{I},i)} + 1) Q_{\mathtt{I},i,j}(u_k^{(\mathtt{I},i)})}{Q_{\mathtt{I}}(u_k^{(\mathtt{I},i)}) Q_{\mathtt{I},i}(u_k^{(\mathtt{I},i)} - 1) Q_{\mathtt{I},i,j}(u_k^{(\mathtt{I},i)} + 1)}$$

Expression of $T^{a,s}$

- Relations similar to the Jacobi-Trudi Identity give [Krichever, Lipan, Wiegmann & Zabrodin 97]

$$T^{\lambda}(u) = Q_{\emptyset}(u-K) \cdot \frac{\det\left(x_j^{1-k+\lambda_k} Q_j(u-k+1+\lambda_k)\right)_{1\leq j,k\leq K}}{\Delta(x_1,\cdots,x_K)\prod_{k=1}^{K} Q_{\emptyset}(u-k+\lambda_k)}$$

where $\Delta(x_1,\cdots,x_K) = \det\left(x_j^{1-k}\right)_{1\leq j,k\leq K}$

Expression of ${T_{\mathtt{I}}}^{a,s}$

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where $\Delta(\{x_j\}_{j\in\mathtt{I}}) = \det \left(x_j^{1-k} \right)_{j\in\mathtt{I}, k\le|\mathtt{I}|}$

outcome

- T-operators expressed through Q-operators
- Hamiltonian : $H = 2\frac{L}{K} - 2\frac{d}{du}\log T^{1,1}(u)\big|_{u=0}$
- Q-operators constrained by the Bethe equation.

- Requires the existence and polynomiality of the Q-operators

Summary

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Explicit expression of nested T and Q-operators

- Generating series of T-operators :

let $w(z) \equiv \sum_{s\geq 0} \chi^{1,s} z^s = \det \frac{1}{1-g\,z}$,
then $\sum_{s\geq 0} T^{1,s} z^s = \bigotimes_{i=1}^{L} (u_i + \hat{D})\ w(z)$

Explicit solution of this linear system

$$T_{\mathrm{I}}^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j\in\bar{\mathrm{I}}}} B_{\mathrm{I}} \cdot \left[\bigotimes_{i=1}^{L} (u_i + \hat{D} + |\bar{\mathrm{I}}|)\ \chi_\lambda(g_{\mathrm{I}}) \Pi_{\mathrm{I}} \right],$$

$$\Pi_{\mathrm{I}} = \prod_{j\in\bar{\mathrm{I}}} w(t_j) \qquad B_{\mathrm{I}} = \prod_{j\in\bar{\mathrm{I}}} (1 - x_j\ t_j) \cdot (1 - g\ t_j)^{\otimes N}$$

$$Q_{\mathrm{I}} = T_{\mathrm{I}}^{(0,s)} \qquad g_{\{j_1,j_2,\cdots,j_k\}} = \mathrm{diag}(x_{j_1}, x_{j_2}, \cdots, x_{j_k})$$

x_j=eigenvalue of g [Kazakov, S.L, Tsuboi 10]

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x_j=eigenvalue of g [Kazakov, S.L, Tsuboi 10]

(Hints of) combinatorial proof

- Determinant identities (similar to Jacobi-Trudi) reduce the proofs to a few bilinear identities proven from the following

"Master Identity" [Kazakov, S.L, Tsuboi 10]

when $\Pi = \prod_j w(t_j)$,

$$(t-z)\left[\bigotimes(u_i+1+\hat{D})\ w(z)w(t)\Pi\right]\cdot\left[\bigotimes(u_i+\hat{D})\ \Pi\right]$$
$$= t\left[\bigotimes(u_i+\hat{D})\ w(z)\Pi\right]\cdot\left[\bigotimes(u_i+1+\hat{D})\ w(t)\Pi\right]$$
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Combinatorics of coderivatives

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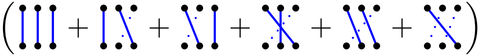
- For instance, [Kazakov Vieira 07]

$\hat{D}\otimes\hat{D}\otimes\hat{D}\ w(x) =$  $w(x)$

Outline

τ-functions of the MKP hierarchy

- A τ-function of the *MKP hierarchy* is a function of a variable n and an infinite set $\mathbf{t} = (t_1, t_2, \cdots)$ of "times", such that $\forall n \geq n'$, $\forall \mathbf{t}, \mathbf{t}'$

Definition of τ-functions.

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'} \tau_n(\mathbf{t} - [z^{-1}]) \tau_{n'}(\mathbf{t}' + [z^{-1}]) dz = 0$$

where $\mathbf{t} \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{z^{-2}}{2}, t_3 \pm \frac{z^{-3}}{3}, \cdots\right)$, $\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k$, and $\mathcal{C}$ encircles the singularities of $\tau_n(\mathbf{t} - [z^{-1}])\tau_{n'}(\mathbf{t}' + [z^{-1}])$ (typically finite), but not the singularities of $e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{n-n'}$ (typically at infinity).

- An example of such τ-function is the expectation value

$$\tau_n(\mathbf{t}) = \langle n | e^{J_+(\mathbf{t})} G | n \rangle$$

τ-functions of the MKP hierarchy

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- An example of such τ-function is the expectation value

$$\tau_n(\mathbf{t}) = \langle n| \, e^{J_+(\mathbf{t})} G \, |n\rangle$$

over an infinite set of fermionic oscillators ($\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$), where $G = \exp\left(\sum_{i,k\in\mathbb{Z}} A_{ik} \psi_i^\dagger \psi_k\right)$ and $J_+ = \sum_{k\geq 1} t_k J_k$, where $J_k = \sum_{j\in\mathbb{Z}} \psi_j \psi_{j+k}^\dagger$. (and $\psi_n \, |n\rangle = |n+1\rangle$)

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Characteristic property

τ-functions are characterised by

$$\begin{aligned} z_2 \tau_{n+1} \left(\mathbf{t} - [z_2^{-1}]\right) \tau_n \left(\mathbf{t} - [z_1^{-1}]\right) & \\ - z_1 \tau_{n+1} \left(\mathbf{t} - [z_1^{-1}]\right) \tau_n \left(\mathbf{t} - [z_2^{-1}]\right) & \\ + (z_1 - z_2) \tau_{n+1}(\mathbf{t}) \tau_n \left(\mathbf{t} - [z_1^{-1}] - [z_2^{-1}]\right) & = 0. \end{aligned}$$

(obtained from $n' = n - 1$ and $\mathbf{t}' = \mathbf{t} - [{z_1}^{-1}] - [{z_1}^{-1}]$)

τ-functions of the MKP hierarchy

- A τ-function is a function of a variable n and an infinite set $\mathbf{t} = (t_1, t_2, \cdots)$ of "times", such that $\forall n \geq n'$, $\forall \mathbf{t}, \mathbf{t}'$

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(obtained from $n' = n - 1$ and $\mathbf{t}' = \mathbf{t} - [z_1{}^{-1}] - [z_1{}^{-1}]$)

General rational τ-function

[Krichever 78]

If $\tau_n(\mathbf{t})$ is polynomial in n, we substitute $n \rightsquigarrow u$, where $u \in \mathbb{C}$.

Polynomial τ-functions of this MKP hierarchy

$$\tau(u, \mathbf{t}) = \det \left(A_i(u-j, \mathbf{t})\right)_{1 \le i,j \le N}$$

$$\text{where } \; A_i(u, \mathbf{t}) = \sum_{m=0}^{d_i} a_{i,m} \partial_z^m \left(z^u e^{\xi(\mathbf{t},z)} \right) \Big|_{z=p_i}$$

parameterized by : the integer $N \ge 0$, the numbers $\{p_i\}$ and d_i, and the coefficients $\{a_{i,m}\}$.

- Analogous to the spin-chain's "undressing procedure" : restricting to smaller and smaller minors of the determinant

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- Analogous to the spin-chain's “undressing procedure” : restricting to smaller and smaller minors of the determinant

Undressing the rational τ-functions

- $A_i(u, \mathbf{t} + [z^{-1}]) = \sum_{m=0}^{d_i} a_{i,m} \partial_x^m \left(x^u e^{\xi(\mathbf{t},x)} \frac{1}{1-x/z} \right) \Big|_{x=p_i}$ has a pole at $z = p_i$.
- One can show that $A_k(u, \mathbf{t} + [z^{-1}]) = A_k(u, \mathbf{t}) + z^{-1} A_k(u+1, \mathbf{t} + [z^{-1}])$, hence

$$\tau_u(\mathbf{t} + [z^{-1}]) = \begin{vmatrix} A_1\big(u-1,\mathbf{t}+[z^{-1}]\big) & A_1(u-2,\mathbf{t}) & \dots & A_1(u-N,\mathbf{t}) \\ A_2\big(u-1,\mathbf{t}+[z^{-1}]\big) & A_2(u-2,\mathbf{t}) & \dots & A_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ A_N\big(u-1,\mathbf{t}+[z^{-1}]\big) & A_N(u-2,\mathbf{t}) & \dots & A_N(u-N,\mathbf{t}). \end{vmatrix}$$

Undressing procedure for rational τ-functions

- First step : $\mathrm{Res}_{z=p_i} \tau(u+1, \mathbf{t} + [z^{-1}])$
- Second step : $\mathrm{Res}_{z_1=p_j} \mathrm{Res}_{z_2=p_i} \tau(u+2, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}])$
- Et cetera

Undressing the rational τ-functions

- $A_i(u, \mathbf{t} + [z^{-1}]) = \sum_{m=0}^{d_i} a_{i,m} \partial_x^m \left(x^u e^{\xi(\mathbf{t},x)} \frac{1}{1-x/z} \right)\Big|_{x=p_i}$ has a pole at $z = p_i$.
- One can show that $A_k(u, \mathbf{t} + [z^{-1}]) = A_k(u, \mathbf{t}) + z^{-1} A_k(u+1, \mathbf{t} + [z^{-1}])$, hence

$$\tau_u(\mathbf{t} + [z^{-1}]) = \begin{vmatrix} A_1\left(u-1, \mathbf{t}+[z^{-1}]\right) & A_1(u-2,\mathbf{t}) & \dots & A_1(u-N,\mathbf{t}) \\ A_2\left(u-1, \mathbf{t}+[z^{-1}]\right) & A_2(u-2,\mathbf{t}) & \dots & A_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ A_N\left(u-1, \mathbf{t}+[z^{-1}]\right) & A_N(u-2,\mathbf{t}) & \dots & A_N(u-N,\mathbf{t}). \end{vmatrix}$$

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Outline

Spin-chains $\leftrightsquigarrow$ MKP hierarchy

T-operators are τ-functions

- Set of times $\mathbf{t} \leftrightsquigarrow$ representations λ :

$$\tau(u,\mathbf{t}) = \sum_{\lambda} \underbrace{s_\lambda(\mathbf{t})}_{\text{Schur polynomial}} \tau(u,\lambda) \qquad s_\lambda(\mathbf{t}) = \det\left(h_{\lambda_i - i + j}(\mathbf{t})\right)_{1\le i,j\le|\lambda|}$$

where $e^{\xi(\mathbf{t},z)} = \sum_{k\ge0} h_k(\mathbf{t}) z^k$

If $\tau(u,\lambda) = T^\lambda(u) = \bigotimes_{i=1}^L (u_i + \hat{D})\ \chi^\lambda(g)$, we get

$$\tau(u,\mathbf{t}) = \bigotimes_{i=1}^{L} (u_i + \hat{D})\ e^{\sum_{k\ge1} t_k \mathrm{tr}(g^k)}$$

- Then $\tau(u,\mathbf{t}+[z^{-1}]) = \bigotimes_{i=1}^L (u_i + \hat{D})\ w(1/z) e^{\sum_{k\ge1} t_k \mathrm{tr}(g^k)}$

where $w(1/z) \equiv \sum_{s\ge0} \chi^{1,s} z^{-s} = \det \frac{1}{1-g/z}$

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Spin-chains $\leftrightsquigarrow$ MKP hierarchy

T-operators are τ-functions

- The master identity coincides with the characteristic property of the MKP hierarchy

$\Rightarrow$ $\tau(u, \mathbf{t}) = \bigotimes_{i=1}^{L}(u_i + \hat{D})\; e^{\sum_{k\geq 1} t_k \mathrm{tr}(g^k)}$ is a τ-function of the MKP hierarchy.

- The undressing procedure for τ-functions (ie $\mathrm{Res}_{z=p_i}\tau(u+1, \mathbf{t}+[z^{-1}])$) explains the explicit expression found from the combinatorics of coderivatives.
- Fermionic realisation of this τ-function :

$$\tau_n(\mathbf{t}) = \langle n|\, e^{J_+(\mathbf{t})}\Psi_1 \ldots \Psi_N \,|\, n-N\rangle \;,$$

where $\Psi_i = \sum_{m\geq 0} a_{im}\, \partial_z^m \sum_{k\in\mathbb{Z}} \psi_k z^k \Big|_{z=p_i}$

[Alexandrov, Kazakov, S.L., Tsuboi, Zabrodin 11]

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1+1 D integrable field theories

Wavefunction for a large volume

- planar waves when particules are far from each other
- an *S-matrix* describes 2-points interactions

$\Rightarrow$ Bethe equations

- Finite size effects : "double Wick Rotation"
  finite size $\leftrightsquigarrow$ finite temperature

$\rightsquigarrow$

- Thermodynamic Bethe Ansatz equations give rise to T-functions which obey the same Hirota equation as spin chains' T-operators

$\Rightarrow$ Some Q-functions must exist

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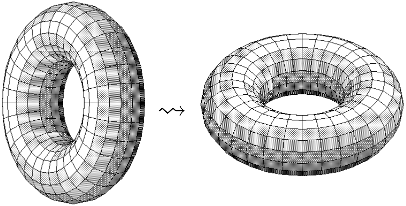

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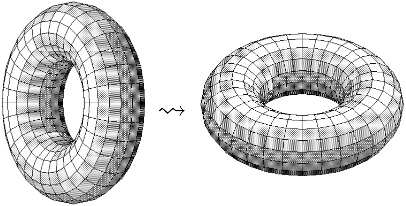

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Example of AdS/CFT

AdS/CFT from Q-functions [Gromov, Kazakov, SL, Volin 11]

- The complicated (and infinite) set of TBA equations can be reduced to some analyticity properties (analyticity strips, continuation around branch points, ...) of the Q-functions.
- Q-functions (or Bäcklund flow) allow to reduce these equations to a finite set of nonlinear integral equations

see also [Balog, Hegedüs, 12]

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Conclusions

- Rational spin chains (very well understood)
  - Bäcklund Flow to gradually simplify the system
  - Bethe Equations
  - Expression of the Hamiltonian from T and Q-functions
- For these rational spin chains, the classical integrability of τ-functions sheds light on the whole constriction
- Generalizes to trigonometric spin chains [Zabrodin 12]
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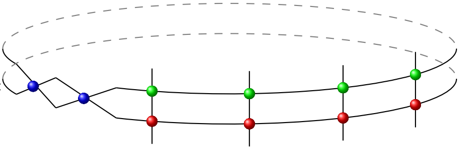
Finally

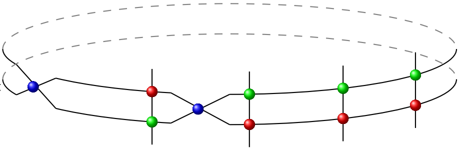
Thank you !

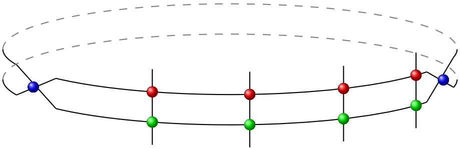
Appendices

Disclaimer : The following slides are additional material, not necessarily part of the presentation

Commutation of T-operators

$T(u) \cdot T(v) =$ 

$=$ 

$= \cdots =$ 

$= T(v) \cdot T(u)$

back

Expression of T through co-derivative

- $\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi} g)\Big|_{\phi=0}$ $\qquad \phi \in GL(K)$
- If $f(g)$ acts on $\mathcal{H}$, then $\hat{D} \otimes f$ acts on $\tilde{\mathcal{H}} = \mathbb{C}^K \otimes \mathcal{H}$
- $\hat{D} \otimes g = \mathcal{P}(1 \otimes g)$ and Leibniz rule :
  $\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot \left[\hat{D} \otimes \tilde{f}\right] + \left[\hat{D} \otimes f\right] \cdot \left[\mathbb{I} \otimes \tilde{f}\right]$
  $\rightsquigarrow$ compute any $\hat{D} \otimes f(g)$
- $\hat{D} \otimes \pi_\lambda(g) = \left[\sum_{\alpha,\beta} \underbrace{e_{\beta\alpha}}_{generator} \otimes \underbrace{\pi_\lambda(e_{\alpha\beta})}_{generator}\right] \cdot \mathbb{I} \otimes \pi_\lambda(e_{\alpha\beta})$

hence

$$((u-\xi_L)\mathbb{I} + \mathcal{P}_{L,a}) \cdots ((u-\xi_1)\mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_\lambda(g) = \bigotimes_{i=1}^{N} (u - \xi_i + \hat{D}) \pi_\lambda(g)$$

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