Integrable systems and AdS/CFT duality.

Sébastien Leurent
LPT-ENS (Paris)

[arXiv:1007.1770]	V. Kazakov & SL
[arXiv:1010.2720]	N. Gromov, V.Kazakov, SL & Z.Tsuboi
[arXiv:1010.4022]	V. Kazakov, SL & Z.Tsuboi
[arXiv:1110.0562]	N. Gromov, V. Kazakov, SL & D. Volin
[arXiv:1112.3310]	A. Alexandrov, V. Kazakov, SL, Z. Tsuboi & A. Zabrodin

PhD defense, LPT-ENS, June 20 2012 [arXiv:1206.4061]

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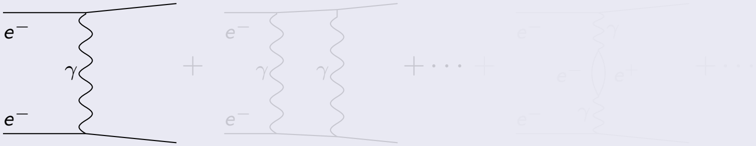
Perturbative quantum field theory (QFT)

Success and challenges

Quantum field theories :

particles interact by exchanging "gauge bosons"

example : electromagnetic repulsion of two electrons



The probability that electrons are deviated in a given direction comes from the sum of contributions from (infinitely many) different processes

→ Many interactions and particles are described this way : for instance the "Standard Model" contains *electromagnetic*, *weak* and *strong* interactions

- Gravity is not described by the "Standard Model".

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Success and challenges

Intégrability in AdS/CFT.

S. Leurent

Introduction
Perturbative quantum field theory (QFT)
AdS/CFT duality
Spin chains' integrability

Spin chains
T-operators
Bäcklund Flow
Explicit Q-operators

TBA⇝finite size
Thermodynamic Bethe Ansatz
The principal chiral model
AdS/CFT

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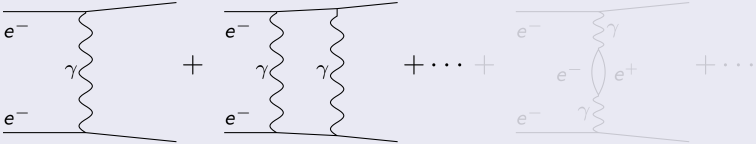

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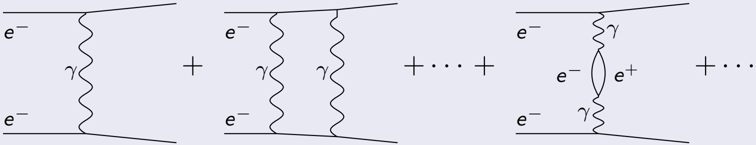

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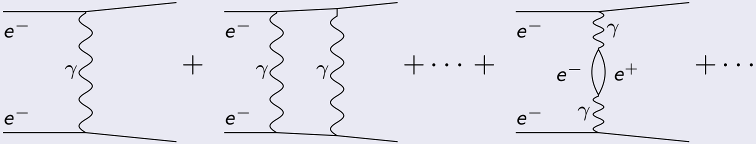

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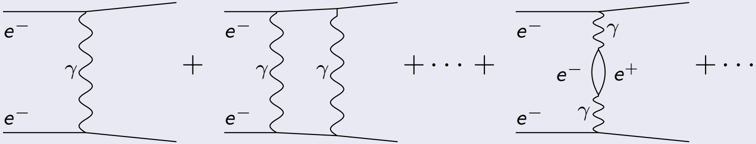

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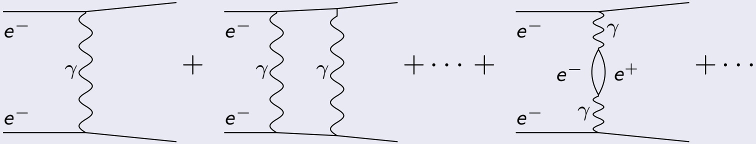

Perturbative description

The (truncated) sum provides a good approximation only in a specific energy domain (the perturbative domain)

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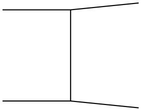
String Theory and AdS/CFT duality

Quantum field theory

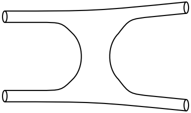
Point-like particles



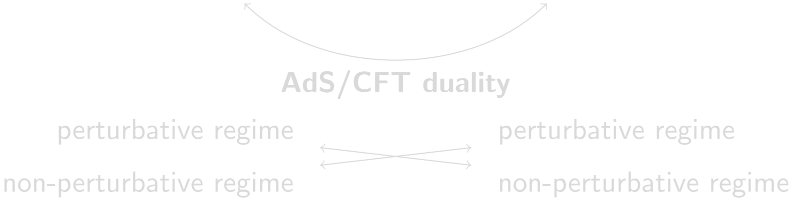
- Standard model

String theory

particles ⇝ strings



- Quantum gravity on a 10-dimensional space-time $AdS_5 \times S^5$



AdS/CFT duality

perturbative regime

non-perturbative regime

perturbative regime

non-perturbative regime

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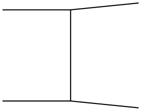

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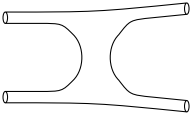

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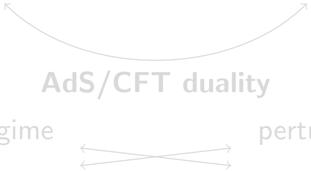

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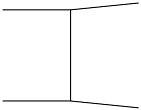

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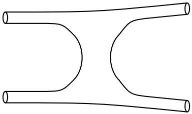

- $\mathcal{N} = 4$ Super-Yang Mills Conformal gauge theory in 4 dimensions

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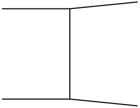

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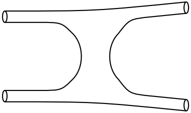

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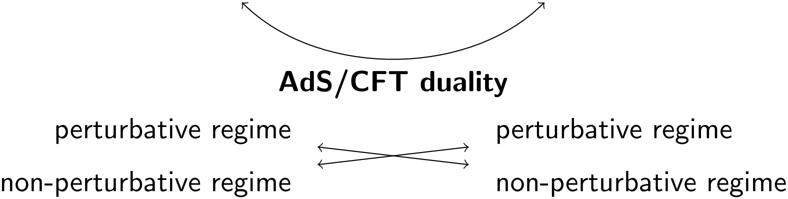


AdS/CFT duality

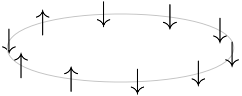
perturbative regime ⟷ non-perturbative regime

non-perturbative regime ⟷ perturbative regime

Spin chains' integrability

Bethe Ansatz

Spin chain :



- Ising model : "state" = a value (↑ or ↓) at each site
- Heisenberg spin chain : "state" = quantum superposition:
$$|\Psi\rangle = \sum_{1\leq n_1\leq n_2\leq\cdots\leq n_M\leq L} \psi(n_1, n_2, \cdots, n_M)\,|n_1, n_2, \cdots, n_M\rangle$$
where $|3,5,\cdots\rangle = |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$.
Quantum evolution : $H = -\sum_i \vec{\sigma}_i \cdot \vec{\sigma}_i$.
- Bethe Ansatz : $\psi(n_1, n_2, \cdots, n_M) \equiv \sum_{\sigma\in\mathcal{S}^M} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} n_k}$
- Bethe equation : $\forall 1 \leq k \leq M, \quad e^{i\,L\,p_k} = \prod_{j\neq k} S(p_k, p_j)$.
(condition to obtain an eigenstate)

Integrability : the diagonalization of the Hamiltonian reduces to solving the Bethe equation.

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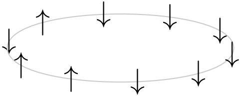

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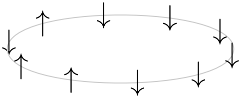

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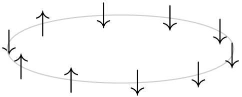

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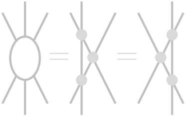
Integrability

Generalization to field theory

The "Bethe Ansatz" $\psi(n_1, n_2, \cdots, n_M) \equiv \sum_{\sigma\in\mathcal{S}^M} \mathcal{A}_\sigma e^{i\sum_k p_{\sigma(k)} n_k}$

gives the eigenstates of several theories such that

- The space is one-dimensional and there are periodic boundary conditions.
- The interactions are local.
- A factorization formula holds :



One can argue that it is sufficient to have infinitely many conserved charges [Zamolodchikov & Zamolodchikov 79]

- "Locality" requires a large spatial period

⇝ Question of the Finite size effects

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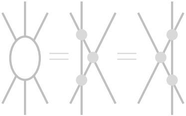
- A factorization formula holds : 

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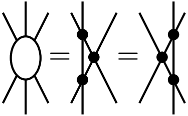

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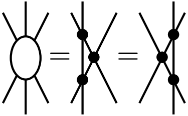

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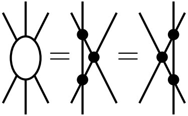

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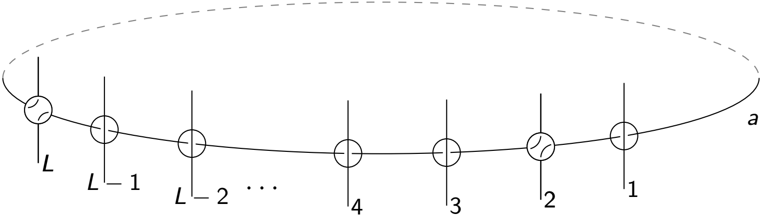
[arXiv:1206.4061]

T-operators for spin chains

family of commuting operators

$$T(u) = \mathrm{tr}_a\left((u\,\mathbb{I} + \mathcal{P}_{L,a}) \cdot (u\,\mathbb{I} + \mathcal{P}_{L-1,a}) \cdots (u\,\mathbb{I} + \mathcal{P}_{1,a})\right)$$

operator on the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$



permutation operator :

$$\mathcal{P}_{1,2}\,|{\downarrow\downarrow}\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle = |{\downarrow\downarrow}\uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$$

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$[T(u), T(v)] = 0$

(proved from relations like $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}$)

$H = \quad L - 2\frac{\mathrm{d}}{\mathrm{du}} \log T(u)$

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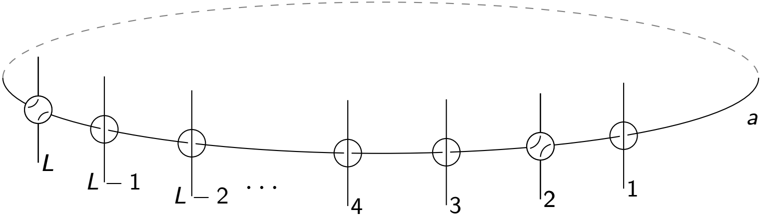

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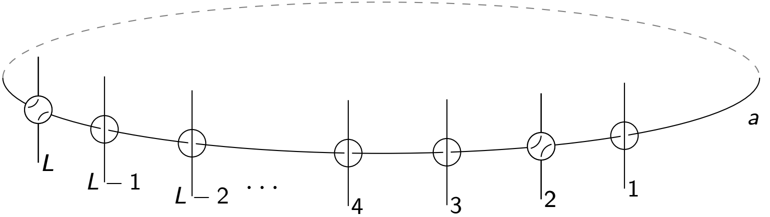

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$$H = \frac{2}{K} L - 2 \frac{\mathrm{d}}{\mathrm{du}} \log T(u)$$

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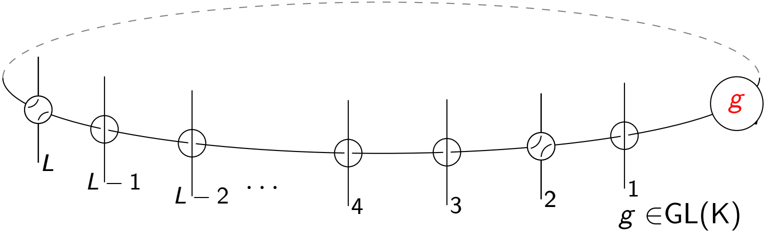
operator on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



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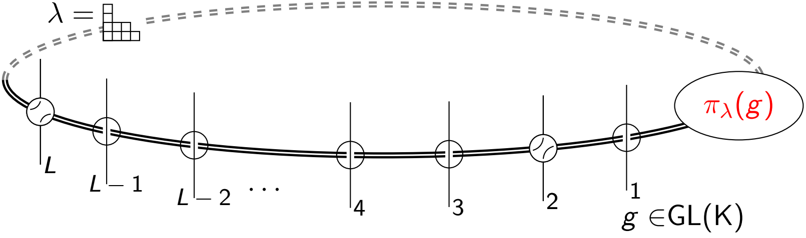
operator on the Hilbert space $\left(\mathbb{C}^K\right)^{\otimes L}$



permutation operator : $\mathcal{P}_{1,2}\,|\downarrow\downarrow\ \uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle = |\downarrow\downarrow\ \uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$

$\mathcal{P}_{1,2}\,|\downarrow\uparrow\ \uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle = |\uparrow\downarrow\ \uparrow\downarrow\uparrow\downarrow\downarrow\cdots\rangle$

$$\left[T^{\lambda}(u), T^{\mu}(v)\right] = 0$$

(proved from relations like $\mathcal{P}_{i,j}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}$)

$$H = \frac{2}{K}L - 2\frac{\mathrm{d}}{\mathrm{du}}\log T^{\square}(u)$$

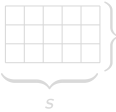
Relation between T-operators and characters

Cherednik-Bazhanov-Reshetikhin formula for T-operators

$$T^{\lambda}(u) = \mathrm{tr}_a\left(((u-\xi_L)\mathbb{I} + \mathcal{P}_{L,a}) \cdots ((u-\xi_1)\mathbb{I} + \mathcal{P}_{1,a}) \cdot \pi_{\lambda}(g)\right)$$

- At $L = 0$, $T^{\lambda}(u) = \chi^{\lambda}(g) \equiv \mathrm{tr}\, \pi_{\lambda}(g)$
- In general $u_i \equiv u - \xi_i$
  $$T^{\lambda}(u) = \left(u_1 + \hat{D}\right) \otimes \left(u_2 + \hat{D}\right) \otimes \cdots \otimes \left(u_L + \hat{D}\right) \chi^{\lambda}(g)$$

Rectangular representation : $(a, s) \leftrightarrow \lambda =$



Determinant identity

[Cherednik 86] [Bazhanov-Reshetikhin 90] [Kazakov Vieira 08]

$$\chi^{\lambda}(g) = \det\left(\chi^{1,\lambda_i+j-i}(g)\right)_{1\leq i,j\leq|\lambda|}$$

$$\rightsquigarrow \qquad T^{\lambda}(u) = \frac{\det\left(T^{1,\lambda_i+j-i}(u+1-j)\right)_{1\leq i,j\leq|\lambda|}}{\prod_{k=1}^{|\lambda|-1} T^{0,0}(u-k)}$$

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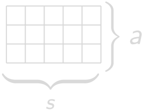

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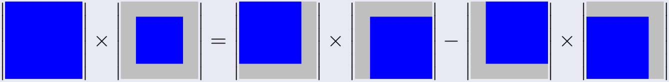
Hirota equation

from Jacobi-Trudi identity

Jacobi-Trudi identity : for an arbitrary determinant



- From the CBR determinant expression $T^{a,s}(u) = \frac{\det\left(T^{1,s+j-i}(u+1-j)\right)_{1\leq i,j\leq a}}{\prod_{k=1}^{a-1} T^{0,0}(u-k)}$, we deduce the "Hirota equation":

$$T^{a,s}(u+1)\cdot T^{a,s}(u) = T^{a+1,s}(u+1)\cdot T^{a-1,s}(u) + T^{a,s-1}(u+1)\cdot T^{a,s+1}(u)$$

- Conversely, the Jacobi-Trudy identity allows to show that this Hirota equation implies the CBR determinant expression. for "typical" solutions

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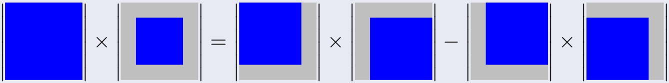

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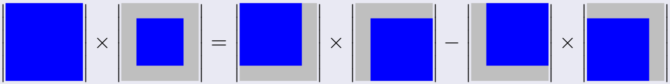

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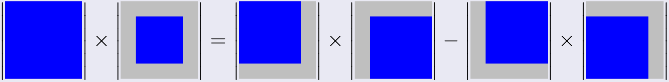

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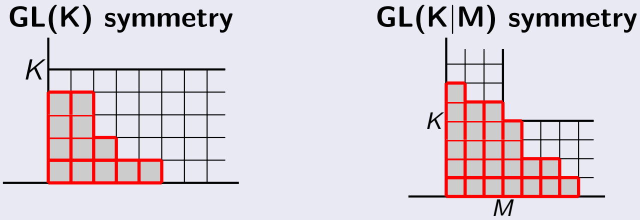
"Fat hooks" and "Bäcklund Flow"

Authorized Young diagrams for a given symmetry group



Hirota equation solved by gradually reducing the size of the "fat hook" [Krichever, Lipan, Wiegmann & Zabrodin 97] [Kazakov Sorin Zabrodin 08]

using inclusions like GL(2|1)⊃ GL(1|1)⊃ GL(1|0)⊃ {1}

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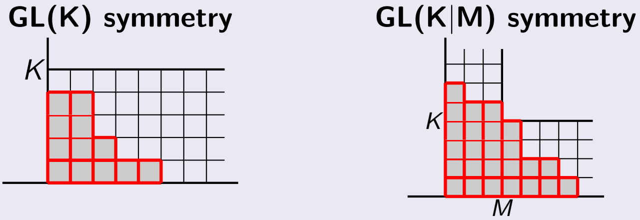
GL(K) symmetry

GL(K|M) symmetry

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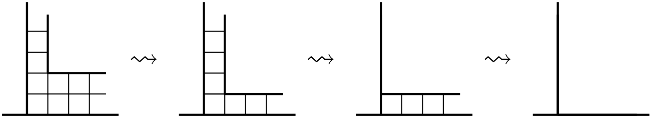

T-operators from Q-operators

For "typical solutions" of Hirota, each step in the Bäcklund flow allows to define a Q-operator, and we obtain (for GL(K)) [Krichever, Lipan, Wiegmann & Zabrodin 97]

$$T^{\lambda}(u) = Q_{\emptyset}(u-K) \cdot \frac{\det\left(x_j^{1-k+\lambda_k} Q_j(u-k+1+\lambda_k)\right)_{1\leq j,k\leq K}}{\Delta(x_1,\cdots,x_K)\prod_{k=1}^{K} Q_{\emptyset}(u-k+\lambda_k)}$$

where $x_1, \cdots x_K$ are the eigenvalues of g (they are distinct) and $\Delta(x_1,\cdots,x_K) = \det\left(x_j^{1-k}\right)_{1\leq j,k\leq K}$

Diagonalization of the Hamiltonian

For GL(K) (and GL(K|M)), the above determinant expression allows to recover

- the Bethe equation
- the spectrum

assuming that the Q-operators are polynomial functions of u

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Explicit expression of Q-operators

[Kazakov SL Tsuboi 12, arXiv:1010.4022]

Differential expression of the Q-operators

$$Q_I(u) = \lim_{\substack{\forall i\in\bar{I}\\ t_i\to 1/x_i}} B_I \cdot \left[\bigotimes_{i=1}^{L}\left(u_i + |\bar{I}| + \hat{D}\right)\Pi_I\right]$$

$$\text{where } B_I \equiv \frac{\prod_{i\in\bar{I}}(1-g\ t_i)^{\otimes L}}{\Pi_I}, \qquad \Pi_I \equiv \frac{1}{\prod_{i\in\bar{I}}\det(1-g\ t_i)}$$

- Explicit proof of the determinant expression of T and the existence of the Bäcklund flow (at the level of operators)
- Explicit proof of the polynomiality of Q-operators

⇝ gives one derivation of the spin chain's spectrum

- Differential expression understood as a manifestation of classical integrability [Alexandrov Kazakov SL Tsuboi 11, arXiv:1112.3310]

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Other constructions

Many other constructions exist for several integrable models in the literature, and the analyticity properties often play a key role.

For the present spin chains, see also

[Bazhanov Łukowski Meneghelli Staudacher Tsuboi 08-12]

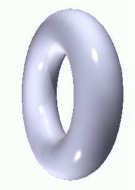
Integrable systems and AdS/CFT duality

[arXiv:1206.4061]

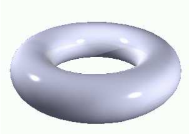
Finite Size effects

Field theories with a finite interaction range are usually integrable only when the space period is large enough.

$\rightsquigarrow$Finite size effects ?



short space-period
infinite time periodicity $R \rightarrow \infty$
Path integral $Z \sim e^{-RE_0(L)}$



Long operators
finite time-periodicity
$\Rightarrow$ finite temperature
Bethe equation, bound states

“free Energy” : $f(L) = E_0(L)$

TBA equations

⇝ Equations of the form

$$Y_{a,s}(u)=\sum_{a',s'} K_{a,s}^{(a',s')}\star\log\left(1+Y_{a',s'}(u)^{\pm 1}\right)$$

[Zamolodchykov 90]

- $Y_{a,s}(u)$ is a function of $a, s \in \mathbb{Z}$ and u in $\mathbb{R}$

SU(N)×SU(N)
principal chiral field

AdS/CFT

[Gromov Kazakov Kozak Vieira 09]

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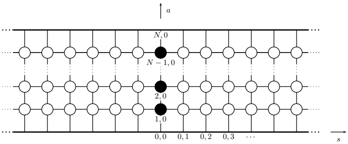

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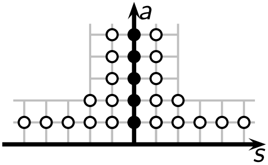

[Gromov Kazakov Kozak Vieira 09]

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Different forms of the finite-size spectral problem.

TBA approach

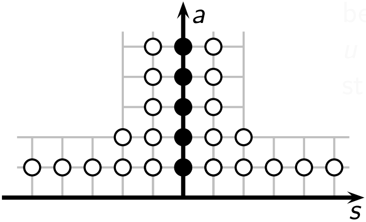
- infinite set of integral equations
- analyticity of Y-functions ← analyticity of the convolution kernels

⇒

Y-system equation

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

+ Analyticity



⇔

Hirota equation

T-system

Gauge

- Finite parameterization

+ ??

zeroes and poles, behavior at $u \to \infty$, analyticity strips

Y-system and Hirota equation

Y-system Equation

The TBA integral equation imply the universal ‘local’ relation

$$Y^+_{a,s}\,Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

where $Y^\pm_{a,s} = Y_{a,s}(u \pm \frac{i}{2})$

- change of variable $Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$

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Gauge freedom

Y-functions and Hirota equation are invariant under gauge transformations $T_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[-a+s]} g_4^{[-a-s]} T_{a,s}$

$f^{[\pm k]} = f(u \pm ki/2)$

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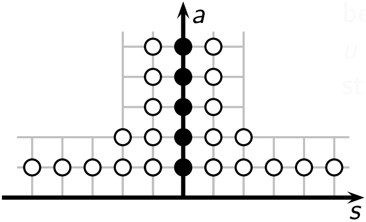

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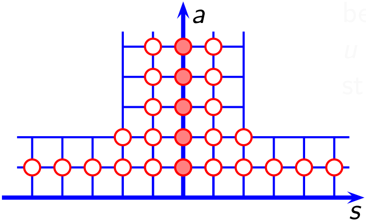
T-system / Gauge

- Finite parameterization

+ ??

zeroes and poles, behavior at $u \to \infty$, analyticity strips



Different forms of the finite-size spectral problem.

TBA approach

- infinite set of integral equations ⇒
- analyticity of Y-functions

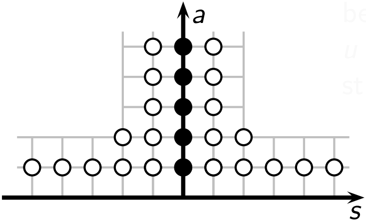
← analyticity of the convolution kernels

Y-system equation

$$\mathrm{Y}^+\,\mathrm{Y}^- = \frac{(1+\mathrm{Y}_{s+1})(1+\mathrm{Y}_{s-1})}{(1+1/\mathrm{Y}_{a+1})(1+1/\mathrm{Y}_{a-1})}$$

⇔

+ Analyticity



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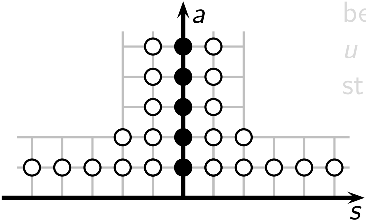

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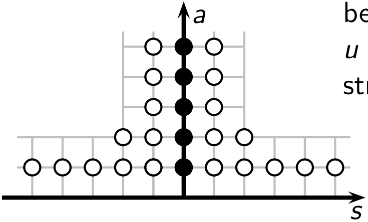

$\Leftrightarrow$

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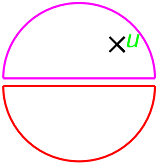
Riemann-Hilbert Problem

General statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{F(v)-G(v)}{v-u} \mathrm{d}v = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if Q is analytic on the upper-half-plane, and $Q \xrightarrow[|u|\to\infty]{} P(u)$,

$$Q(u) = P(u) + \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{\rho(v)}{v-u} \mathrm{d}v$$

where $\rho = Q(u) - P(u) + \bar{Q}(u) - \bar{P}(u)$

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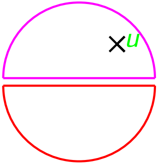

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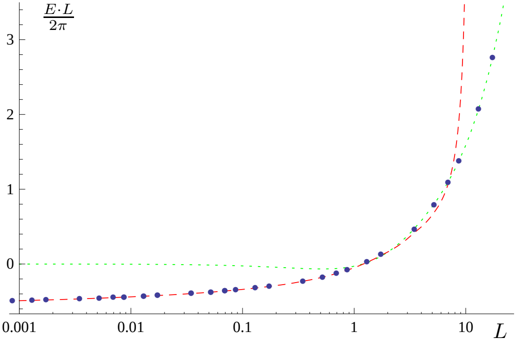
Principal chiral field

[Kazakov SL 10, 1007.1770]

For the principal chiral field, the Q-functions have a known polynomial behavior at $u \to \infty$, and are analytic in the upper-half-plane.

$\Rightarrow$ parameterization in terms of $N-1$ densities
$Q_i(u) = P_i(u) + \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{\rho_i(v)}{v-u}\mathrm{d}v$



Symmetries $\leftarrow$ Classical limit

[Gromov Kazakov SL Volin 11, arXiv:1110.0562]

In the classical limit, $T_{a,s}(u) = \chi_{a,s}\left(\Omega(u)\right)$ where $\Omega \in U(2,2|4)$.

characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  $\mathrm{sdet} = 1$ & invariance under a $\mathbb{Z}_4$ transformation.
  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C}$ (or $\{\lambda_i\} = \{1/\lambda_i\}$ for Ω's eigenvalues)

[Bena Polchinksi Roiban]

≪Quantum case≫ (ie finite-size, outside the classical limit)

$T_{1,s} = -\hat{T}_{1,-s}$ $\hat{T}$: analytic continuation from $s > 0$

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“quantum $\mathbb{Z}_4$” symmetry

(ie outside the classical limit) [Gromov Kazakov SL Volin 11, arXiv:1110.0562]

statement

$$T_{1,s} = -\hat{T}_{1,-s},$$

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$$\hat{T}_{1,0} = 0 \;\Rightarrow\; \quad q = -\bar{q}$$

$$= \qquad\qquad =$$

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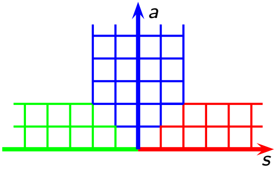

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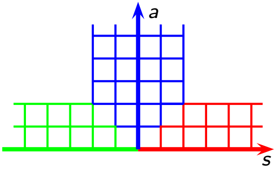

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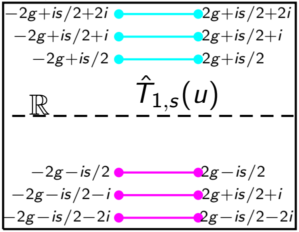

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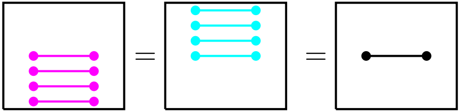

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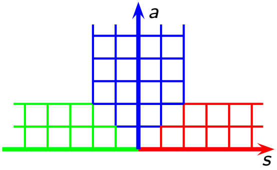

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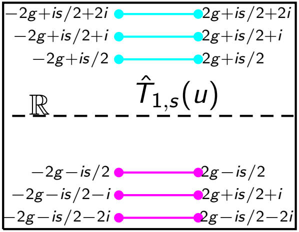

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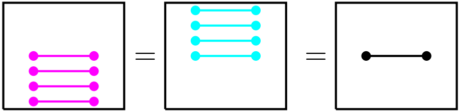

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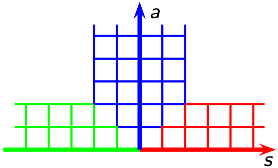

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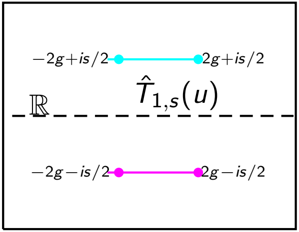

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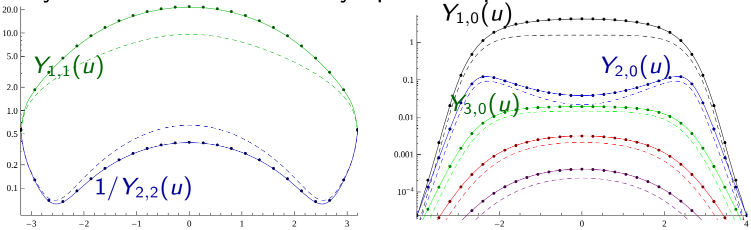
FiNLIE for AdS/CFT

[Gromov Kazakov SL Volin 11, arXiv:1110.0562]

These analyticity properties allow to derive a finite set of non-linear integral equations

- Proved to be equivalent to previous Y-system
- In particular these Y-system results allow to obtain non-trivial expansion coefficients for SYM or Strings.

Numerical Y-functions for Konishi state ($g = 1.6$): Dots are obtained from FiNLIE and lines from standard Y-system iterations. The asymptotic expression is dashed.



Conclusion

- A “general” procedure to simplify Y-systems, provided we understand
  - the analyticity strips
  - the pole structure and behavior at $u \to \infty$
  - the symmetries
- to be generalized
  - currently restricted to simple symmetric states
  - $\left\{\begin{array}{l}\text{numeric efficiency}\\ \text{best FiNLIE formulation}\end{array}\right.$ are to be studied
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  - BFKL
- Understanding of AdS/CFT
  - strong coupling construction of T (? $T = \langle \text{trace } \Omega \rangle$)
  - weak coupling interpretation of T
  - ⇝ proving the Y-system for AdS/CFT ?

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Thank you !

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