Non-linear Integral equations for AdS/CFT spectrum.

Sébastien Leurent
LPT-ENS (Paris)

[arXiv:1110.0562] N. Gromov, V. Kazakov, SL & D. Volin

[arXiv:1007.1770] V. Kazakov & SL
[arXiv:1010.2720] N. Gromov, V.Kazakov, SL & Z.Tsuboi
[arXiv:1010.4022] V. Kazakov, SL & Z.Tsuboi

ETH-Zürich 16 Novembre 2011

1. Integrability and Bethe equations
   - Coordinate Bethe Ansatz
   - Thermodynamic Bethe Ansatz
   - Y-system for the spectrum of AdS/CFT
   - Hirota equation
2. Solving Hirota ⇝ Q-functions
   - Wronskian solution of Hirota equation
   - Q-Q relations
3. FiNLIE ⇝ symmetries & analyticity
   - New symmetries
   - a Riemann-Hilbert Problem
   - FiNLIE

Outline

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p} = 1$
- For two particles, the Bethe Ansatz is

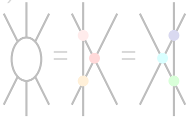
$$\psi(x_1, x_2) = \begin{cases} e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) \times e^{i(p_2 x_1 + p_1 x_2)} & \text{if } x_1 \lessapprox x_2 \\ S(p_1, p_2) \times e^{i(p_1 x_1 + p_2 x_2)} + e^{i(p_1 x_1 + p_2 x_2)} & \text{if } x_1 \gtrapprox x_2 \end{cases}$$

periodic iff $\Psi(x_1, x_2) = \Psi(x_1 + L, x_2)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1, p_2) = 1 \\ e^{ip_2 L} = S(p_1, p_2) \end{cases}$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma, \sigma') e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$



- S is fixed by symmetries

$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$

$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p} = 1$
- For two particles, the Bethe Ansatz is

$$\psi(x_1, x_2) = \begin{cases} e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) \times e^{i(p_2 x_1 + p_1 x_2)} & \text{if } x_1 \lessapprox x_2 \\ S(p_1, p_2) \times e^{i(p_1 x_1 + p_2 x_2)} + e^{i(p_1 x_1 + p_2 x_2)} & \text{if } x_1 \gtrapprox x_2 \end{cases}$$

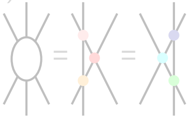
periodic iff $\Psi(x_1, x_2) = \Psi(x_1 + L, x_2)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1, p_2) = 1 \\ e^{ip_2 L} = S(p_1, p_2) \end{cases}$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma, \sigma') e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$

- S is fixed by symmetries



$$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$$

$$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p} = 1$
- For two particles, the Bethe Ansatz is

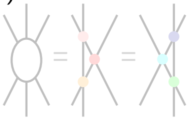
$$\psi(x_1,x_2) = \begin{cases} e^{i(p_1x_1+p_2x_2)} + S(p_1,p_2)\times e^{i(p_2x_1+p_1x_2)} & \text{if } x_1 \lessapprox x_2 \\ S(p_1,p_2)\times e^{i(p_1x_1+p_2x_2)} + e^{i(p_1x_1+p_2x_2)} & \text{if } x_1 \gtrapprox x_2 \end{cases}$$

periodic iff $\Psi(x_1,x_2) = \Psi(x_1+L,x_2)$, ie

$$\begin{cases} e^{ip_1L}\times S(p_1,p_2) = 1 \\ e^{ip_2L} = S(p_1,p_2) \end{cases}$$

- For more particles, $\psi(x_1,x_2,\cdots) \propto \sum_\sigma C(\sigma,\sigma')e^{i\sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i,\ e^{iLp_i} = \prod_{j\neq i} S(p_j,p_i)$



- S is fixed by symmetries

$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$

$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p} = 1$
- For two particles, the Bethe Ansatz is

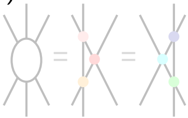
$$\psi(x_1, x_2) = \begin{cases} e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) \times e^{i(p_2 x_1 + p_1 x_2)} & \text{if } x_1 \lessapprox x_2 \\ S(p_1, p_2) \times e^{i(p_1 x_1 + p_2 x_2)} + e^{i(p_1 x_1 + p_2 x_2)} & \text{if } x_1 \gtrapprox x_2 \end{cases}$$

periodic iff $\Psi(x_1, x_2) = \Psi(x_1 + L, x_2)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1, p_2) = 1 \Leftrightarrow e^{ip_1 L} = S(p_2, p_1) \\ e^{ip_2 L} = S(p_1, p_2) \end{cases}$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma, \sigma') e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$



- S is fixed by symmetries

$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$

$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p}=1$
- For two particles, the Bethe Ansatz is

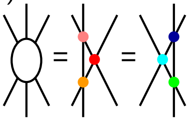
$$\psi(x_1,x_2)=\begin{cases} e^{i(p_1x_1+p_2x_2)}+S(p_1,p_2)\times e^{i(p_2x_1+p_1x_2)} & \text{if } x_1\lessapprox x_2\\ S(p_1,p_2)\times e^{i(p_1x_1+p_2x_2)}+e^{i(p_1x_1+p_2x_2)} & \text{if } x_1\gtrapprox x_2\end{cases}$$

periodic iff $\Psi(x_1,x_2)=\Psi(x_1+L,x_2)$, ie

$$\begin{cases} e^{ip_1L}\times S(p_1,p_2)=1\Leftrightarrow e^{ip_1L}=S(p_2,p_1)\\ e^{ip_2L}=S(p_1,p_2)\end{cases}$$

- For more particles, $\psi(x_1,x_2,\cdots)\propto\sum_\sigma C(\sigma,\sigma')e^{i\sum p_ix_{\sigma(i)}}$ in each domain $x_{\sigma'(1)}\lessapprox x_{\sigma'(2)}\lessapprox\cdots\lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i}=\prod_{j\neq i}S(p_j,p_i)$



- S is fixed by symmetries

$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$

$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$

Bethe Ansatz
Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p} = 1$
- For two particles, the Bethe Ansatz is

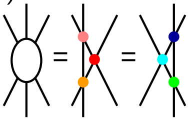
Main conditions

- many conserved charges
- unidimensional space (eg spin chain)
- $L \gg$ interaction range

$$e^{ip_2 L} = S(p_1, p_2)$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma, \sigma')e^{i\sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j\neq i} S(p_j, p_i)$



- S is fixed by symmetries

$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$

$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{i\,L\,p} = 1$
- For two particles, the Bethe Ansatz is

Main conditions

- many conserved charges
- unidimensional space (eg spin chain)
- $L \gg$ interaction range

Examples

This ansatz describes

- several 2-dimensional field-theories such as the Principal Chiral Model
- Spin chains under some condition on the form the Hamiltonian.

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j\neq i} S_{j,i}$
- $E = \sum_i E_i$

For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic “box” is big

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j \neq i} S_{j,i}$
- $E = \sum_i E_i$

For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic “box” is big

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j \neq i} S_{j,i}$
- $E = \sum_i E_i$

For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

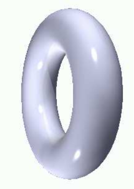
- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic “box” is big

Outline

“Thermodynamic Bethe Ansatz” for finite-size vacuum energy

“Double Wick rotation”



Periodic space (size L), infinite time-period $R \to \infty$:
Path integral

$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

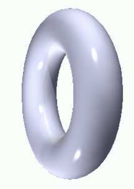
Periodic space of size $R \gg 1$ and time period L:

$$\Rightarrow \text{free energy} \quad f(L) = E_0(L)$$

To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

“Thermodynamic Bethe Ansatz” for finite-size vacuum energy

“Double Wick rotation”



Periodic space (size L),
infinite time-period $R \to \infty$:
Path integral

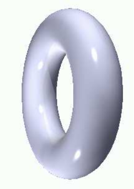
$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

Periodic space of size $R \gg 1$ and
time period L:

$$\Rightarrow \text{free energy} \quad f(L) = E_0(L)$$

To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

"Thermodynamic Bethe Ansatz" for finite-size vacuum energy
"Double Wick rotation"



Periodic space (size L), infinite time-period $R \to \infty$:
Path integral

$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

Periodic space of size $R \gg 1$ and time period L:

$$\Rightarrow \text{free energy } \; f(L) = E_0(L)$$

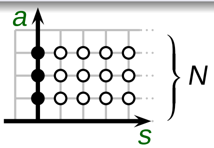
To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

Y-systems

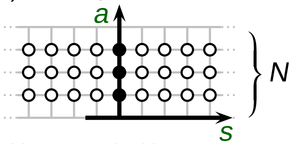
up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

- for $SU(N)$ Gross-Neveu,



- for $SU(N) \times SU(N)$ Principal Chiral Model,



- $Y_{a,s}{}^{+} Y_{a,s}{}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$$

- + analyticity condition

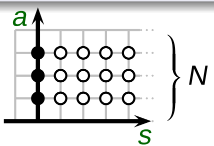
Integral form : TBA equation

Y-systems

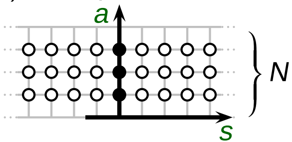
up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

- for $SU(N)$ Gross-Neveu,



- for $SU(N) \times SU(N)$ Principal Chiral Model,



- $Y_{a,s}{}^{+}\, Y_{a,s}{}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$$

- + analyticity condition

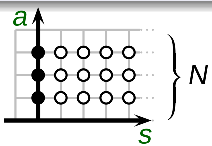
Integral form : TBA equation

Y-systems

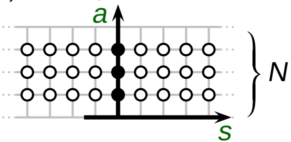
up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

- for $SU(N)$ Gross-Neveu,



- for $SU(N) \times SU(N)$ Principal Chiral Model,



- $Y_{a,s}{}^{+}\, Y_{a,s}{}^{-} = \dfrac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\dfrac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$$

- + analyticity condition

Integral form : TBA equation

Outline

AdS/CFT spectral problem

AdS/CFT correspondence

Conjectured duality between
4-dimensional ($\mathcal{N} = 4$) Super-Yang-Mills theory and
type IIB string theory on $AdS_5 \times S^5$ background

- weak-strong duality

- Integrability $\leftrightsquigarrow$ Spin-chain mapping

[Beisert Eden Staudacher 07]

$$\mathrm{trace}(ZXZZ \cdots XZ) \leftrightsquigarrow |{\downarrow}{\uparrow}{\downarrow}{\downarrow} \cdots {\uparrow}{\downarrow}\rangle$$

AdS/CFT spectral problem

AdS/CFT correspondence

Conjectured duality between
4-dimensional ($\mathcal{N}=4$) Super-Yang-Mills theory and
type IIB string theory on $AdS_5 \times S^5$ background

- weak-strong duality

- Integrability $\leftrightsquigarrow$ Spin-chain mapping

[Beisert Eden Staudacher 07]

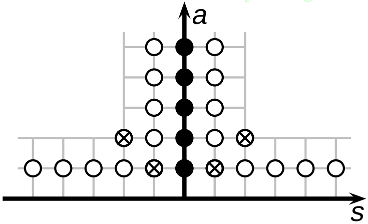
$$\mathrm{trace}(ZXZZ\cdots XZ) \leftrightsquigarrow |{\downarrow}{\uparrow}{\downarrow}{\downarrow}\cdots{\uparrow}{\downarrow}\rangle$$

Resolution of *AdS/CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)



$\Rightarrow$

Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

$\Leftrightarrow$

+ Analyticity
[Cavaglia Fioravanti Tateo 09]

Hirota equation

T-system — Gauge

More details

$$T^+ T^- = T_{a+1} T_{a-1} + T_{s+1} T_{s-1}$$

- Finite parameterization
[Gromov Kazakov S.L. Tsuboi 10]

+ ??

$$Y^{\pm}_{a,s} = Y_{a,s}(u \pm i/2)$$

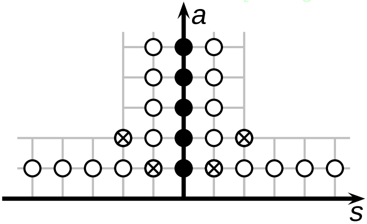
Resolution of *AdS/CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)

$\Rightarrow$



Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

$\Leftrightarrow$

+ Analyticity [Cavaglia Fioravanti Tateo 09]

Hirota equation

T-system — Gauge

More details

$$T^+ T^- = T_{a+1} T_{a-1} + T_{s+1} T_{s-1}$$

- Finite parameterization [Gromov Kazakov S.L. Tsuboi 10]

+ ??

$$Y^{\pm}_{a,s} = Y_{a,s}(u \pm i/2)$$

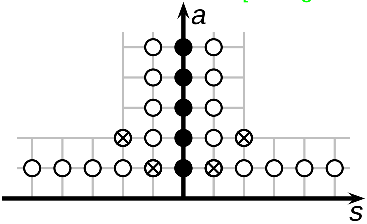
Resolution of *AdS*/*CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)

$\Longleftrightarrow$



Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})} \Leftrightarrow$$

+ Analyticity
[Cavaglia Fioravanti Tateo 09]

Hirota equation

T-system / Gauge

More details

$$T^+ T^- = T_{a+1} T_{a-1} + T_{s+1} T_{s-1}$$

- Finite parameterization

[Gromov Kazakov S.L. Tsuboi 10]

+ ??

$$Y^\pm_{a,s} = Y_{a,s}(u \pm i/2)$$

Outline

Resolution of AdS/CFT Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)

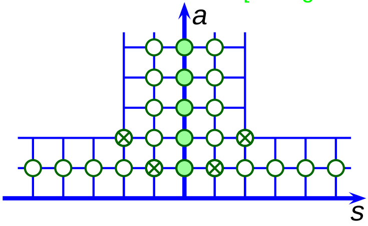
$\Longleftrightarrow$

Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})} \Leftrightarrow$$

+ Analyticity
[Cavaglia Fioravanti Tateo 09]



Hirota equation

T-system / Gauge

More details

$$T^+ T^- = T_{a+1} T_{a-1} + T_{s+1} T_{s-1}$$

- Finite parameterization
[Gromov Kazakov S.L. Tsuboi 10]

+ ??

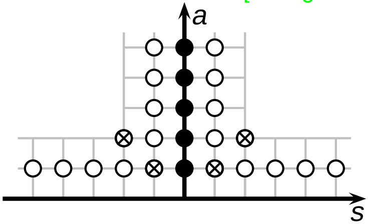
$$Y^{\pm}_{a,s} = Y_{a,s}(u \pm i/2)$$

Resolution of *AdS*/*CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)



$\Longleftrightarrow$

Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})} \Leftrightarrow$$

+ Analyticity
[Cavaglia Fioravanti Tateo 09]

Hirota equation

T-system / Gauge

More details

$$T^+T^- = T_{a+1}T_{a-1} + T_{s+1}T_{s-1}$$

- Finite parameterization

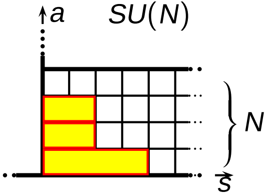
[Gromov Kazakov S.L. Tsuboi 10]

+ ??

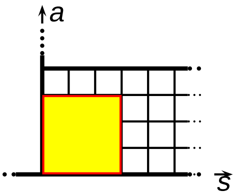
$$Y^{\pm}_{a,s} = Y_{a,s}(u \pm i/2)$$

Hirota equation and characters of the symmetry group

- mapping : young-tableau ↔ representation of the symmetry group

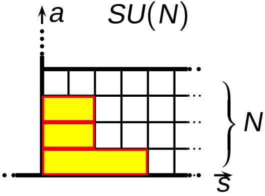


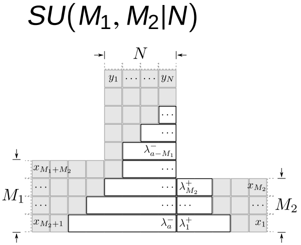
- Lattice node ↔ “rectangular” representations



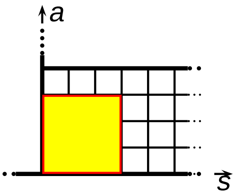
Hirota equation and characters of the symmetry group

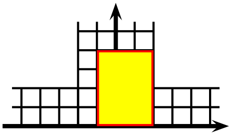
- mapping : young-tableau ↔ representation of the symmetry group





- Lattice node ↔ “rectangular” representations





Hirota equation and characters of the symmetry group

- mapping : young tableau ↔ representation of the

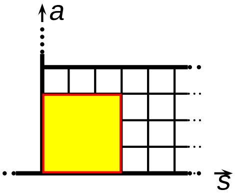
Characters

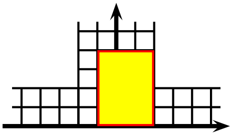
The characters associated to rectangular representations satisfy $\chi_{a,s}{}^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

The Hirota equation $T_{a,s}{}^{+} T_{a,s}{}^{-} = T_{a,s+1} T_{a,s-1} + T_{a+1,s} T_{a-1,s}$ is generalisation of this relation.

[Benichou 11]

- Lattice node ↔ “rectangular” representations





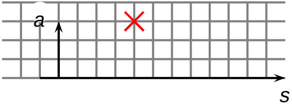
Outline

Q-functions solve Hirota equation

Hirota equation is solved by determinants of Q-functions :
eg. for $SU(4)$,

$$T_{3,s} = \begin{vmatrix} q_1^{[+s+2]} & q_2^{[+s+2]} & q_3^{[+s+2]} & q_4^{[+s+2]} \\ q_1^{[+s]} & q_2^{[+s]} & q_3^{[+s]} & q_4^{[+s]} \\ q_1^{[+s-2]} & q_2^{[+s-2]} & q_3^{[+s-2]} & q_4^{[+s-2]} \\ p_1^{[-s]} & p_2^{[-s]} & p_3^{[-s]} & p_4^{[-s]} \end{vmatrix} \quad \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} 3 \\ \left.\vphantom{a}\right\} 4-3 \end{matrix}$$

- $q_i^{[+k]} = q_i(u + k\frac{i}{2})$



[Baxter 72], [Pasquier Gaudin 92], [Bazhanov Lykyanov Zamolodchikov 96], [Derkachov, 99], [Bytsko Teschner 06],
[Bazhanov Frassek Lukowski Meneghelli Staudacher 10]

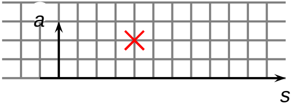
[Kazakov S.L. Tsuboi 10]

Q-functions solve Hirota equation

Hirota equation is solved by determinants of Q-functions : eg. for $SU(4)$,

$$T_{2,s} = \begin{vmatrix} q_1^{[+s+1]} & q_2^{[+s+1]} & q_3^{[+s+1]} & q_4^{[+s+1]} \\ q_1^{[+s-1]} & q_2^{[+s-1]} & q_3^{[+s-1]} & q_4^{[+s-1]} \\ p_1^{[-s+1]} & p_2^{[-s+1]} & p_3^{[-s+1]} & p_4^{[-s+1]} \\ p_1^{[-s-1]} & p_2^{[-s-1]} & p_3^{[-s-1]} & p_4^{[-s-1]} \end{vmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} 2 \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} 4-2 \end{matrix}$$

- $q_i^{[+k]} = q_i(u + k\frac{i}{2})$



[Baxter 72], [Pasquier Gaudin 92], [Bazhanov Lykyanov Zamolodchikov 96], [Derkachov, 99], [Bytsko Teschner 06], [Bazhanov Frassek Lukowski Meneghelli Staudacher 10]

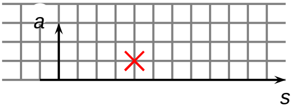
[Kazakov S.L. Tsuboi 10]

Q-functions solve Hirota equation

Hirota equation is solved by determinants of Q-functions :
eg. for $SU(4)$,

$$T_{1,s} = \begin{vmatrix} q_1^{[+s]} & q_2^{[+s]} & q_3^{[+s]} & q_4^{[+s]} \\ p_1^{[-s+2]} & p_2^{[-s+2]} & p_3^{[-s+2]} & p_4^{[-s+2]} \\ p_1^{[-s]} & p_2^{[-s]} & p_3^{[-s]} & p_4^{[-s]} \\ p_1^{[-s-2]} & p_2^{[-s-2]} & p_3^{[-s-2]} & p_4^{[-s-2]} \end{vmatrix} \begin{matrix} \}\ 1 \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}\ 4-1 \end{matrix}$$

- $q_i^{[+k]} = q_i(u + k\frac{i}{2})$



[Baxter 72], [Pasquier Gaudin 92], [Bazhanov Lykyanov Zamolodchikov 96], [Derkachov, 99], [Bytsko Teschner 06],
[Bazhanov Frassek Lukowski Meneghelli Staudacher 10]

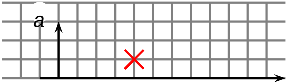
[Kazakov S.L. Tsuboi 10]

Q-functions solve Hirota equation

Hirota equation is solved by determinants of Q-functions : eg. for $SU(4)$,

$$T_{1,s} = \begin{vmatrix} q_1^{[+s]} & q_2^{[+s]} & q_3^{[+s]} & q_4^{[+s]} \\ p_1^{[-s+2]} & p_2^{[-s+2]} & p_3^{[-s+2]} & p_4^{[-s+2]} \\ p_1^{[-s]} & p_2^{[-s]} & p_3^{[-s]} & p_4^{[-s]} \\ p_1^{[-s-2]} & p_2^{[-s-2]} & p_3^{[-s-2]} & p_4^{[-s-2]} \end{vmatrix} \quad \begin{matrix} \} 1 \\ \\ \} 4-1 \\ \\ \end{matrix}$$

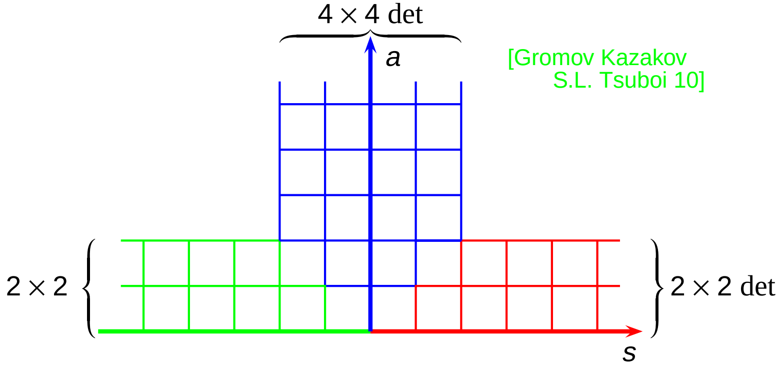
- $q_i^{[+k]} = q_i(u + k\frac{i}{2})$



Finiteness

q-functions are the building blocks of any Hirota solution. They allow to parameterize the whole Y-system in terms of a finite number of Q-functions.

Wronskian parameterization of AdS/CFT T-functions.



[Gromov Kazakov S.L. Tsuboi 10]

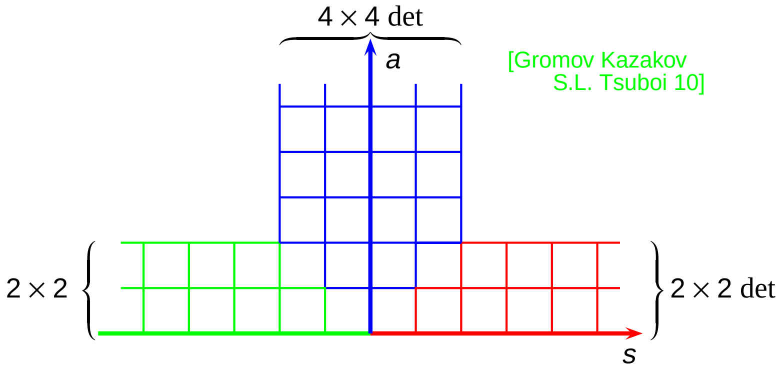
- eg $\mathcal{T}_{1,s}\big|_{s\geq 1} = \begin{vmatrix} q_1^{[+s]} & q_2^{[+s]} \\ p_1^{[-s]} & p_2^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & -\bar{Q}^{[-s]} \end{vmatrix}$

  up to a gauge transformation

  under reality assumption

Skip QQ-relations ⇝ FiNLIE

Wronskian parameterization of AdS/CFT T-functions.



[Gromov Kazakov S.L. Tsuboi 10]

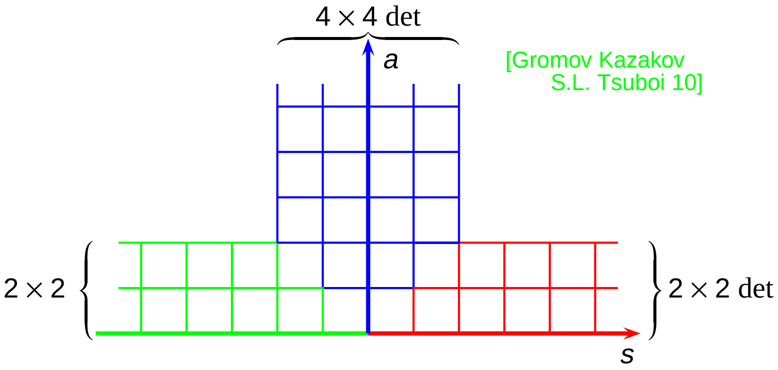
- eg $\mathcal{T}_{1,s}\big|_{s\geq 1} = \begin{vmatrix} q_1^{[+s]} & q_2^{[+s]} \\ p_1^{[-s]} & p_2^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & -\bar{Q}^{[-s]} \end{vmatrix}$

  up to a gauge transformation

  under reality assumption

Skip QQ-relations ⇝ FiNLIE

Wronskian parameterization of AdS/CFT T-functions.



[Gromov Kazakov S.L. Tsuboi 10]

- eg $\mathcal{T}_{1,s}\big|_{s\geq 1} = \begin{vmatrix} q_1^{[+s]} & q_2^{[+s]} \\ p_1^{[-s]} & p_2^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & -\bar{Q}^{[-s]} \end{vmatrix}$
  up to a gauge transformation
  under reality assumption

Skip QQ-relations ⇝ FiNLIE

Outline

q-functions for upper band

- right band : $T_{1,s} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix}$.

  $\forall s, T_{1,s} \in \mathbb{R} \quad \Rightarrow \quad P = -\bar{Q}$

- upper band :

$$T_{a,1} = \begin{vmatrix} q_1^{[+a+2]} & q_2^{[+a+2]} & q_3^{[+a+2]} & q_4^{[+a+2]} \\ q_1^{[+a]} & q_2^{[+a]} & q_3^{[+a]} & q_4^{[+a]} \\ q_1^{[+a-2]} & q_2^{[+a-2]} & q_3^{[+a-2]} & q_4^{[+a-2]} \\ p_1^{[-a]} & p_2^{[-a]} & p_3^{[-a]} & p_4^{[-a]} \end{vmatrix} \in \mathbb{R} \quad \Rightarrow \quad ?$$

$$\rightsquigarrow \bar{p}_1 = \begin{vmatrix} q_1^{[+2]} & q_2^{[+2]} & q_3^{[+2]} \\ q_1 & q_2 & q_3 \\ q_1^{[-2]} & q_2^{[-2]} & q_3^{[-2]} \end{vmatrix} \equiv q_{123}$$

- one defines this way 2^4 q-functions for the upper band, only 4 of which are independent.

q-functions for upper band

- right band : $T_{1,s} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix}$.

  $\forall s, T_{1,s} \in \mathbb{R} \quad \Rightarrow \quad P = -\bar{Q}$

- upper band :

$$T_{a,1} = \begin{vmatrix} q_1^{[+a+2]} & q_2^{[+a+2]} & q_3^{[+a+2]} & q_4^{[+a+2]} \\ q_1^{[+a]} & q_2^{[+a]} & q_3^{[+a]} & q_4^{[+a]} \\ q_1^{[+a-2]} & q_2^{[+a-2]} & q_3^{[+a-2]} & q_4^{[+a-2]} \\ p_1^{[-a]} & p_2^{[-a]} & p_3^{[-a]} & p_4^{[-a]} \end{vmatrix} \in \mathbb{R} \quad \Rightarrow \quad ?$$

$$\rightsquigarrow \bar{p}_1 = \begin{vmatrix} q_1^{[+2]} & q_2^{[+2]} & q_3^{[+2]} \\ q_1 & q_2 & q_3 \\ q_1^{[-2]} & q_2^{[-2]} & q_3^{[-2]} \end{vmatrix} \equiv q_{123}$$

- one defines this way 2^4 q-functions for the upper band, only 4 of which are independent.

q-functions for upper band

- right band : $T_{1,s} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix}$.

  $\forall s, T_{1,s} \in \mathbb{R} \quad \Rightarrow \quad P = -\bar{Q}$
- upper band :

  $$T_{a,1} = \begin{vmatrix} q_1^{[+a+2]} & q_2^{[+a+2]} & q_3^{[+a+2]} & q_4^{[+a+2]} \\ q_1^{[+a]} & q_2^{[+a]} & q_3^{[+a]} & q_4^{[+a]} \\ q_1^{[+a-2]} & q_2^{[+a-2]} & q_3^{[+a-2]} & q_4^{[+a-2]} \\ p_1^{[-a]} & p_2^{[-a]} & p_3^{[-a]} & p_4^{[-a]} \end{vmatrix} \in \mathbb{R} \quad \Rightarrow \quad ?$$

  $$\rightsquigarrow \bar{p}_1 = \begin{vmatrix} q_1^{[+2]} & q_2^{[+2]} & q_3^{[+2]} \\ q_1 & q_2 & q_3 \\ q_1^{[-2]} & q_2^{[-2]} & q_3^{[-2]} \end{vmatrix} \equiv q_{123}$$
- one defines this way 2^4 q-functions for the upper band, only 4 of which are independent.

q-functions for upper band

- right band : $T_{1,s} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix}$.

  $\forall s, T_{1,s} \in \mathbb{R} \quad \Rightarrow \quad P = -\bar{Q}$

- upper band :

$$T_{a,1} = \begin{vmatrix} q_1^{[+a+2]} & q_2^{[+a+2]} & q_3^{[+a+2]} & q_4^{[+a+2]} \\ q_1^{[+a]} & q_2^{[+a]} & q_3^{[+a]} & q_4^{[+a]} \\ q_1^{[+a-2]} & q_2^{[+a-2]} & q_3^{[+a-2]} & q_4^{[+a-2]} \\ p_1^{[-a]} & p_2^{[-a]} & p_3^{[-a]} & p_4^{[-a]} \end{vmatrix} \in \mathbb{R} \quad \Rightarrow \quad ?$$

$$\rightsquigarrow \bar{p}_1 = \begin{vmatrix} q_1^{[+2]} & q_2^{[+2]} & q_3^{[+2]} \\ q_1 & q_2 & q_3 \\ q_1^{[-2]} & q_2^{[-2]} & q_3^{[-2]} \end{vmatrix} \equiv q_{123}$$

- one defines this way 2^4 q-functions for the upper band, only 4 of which are independent.

q-functions for upper band

- right band : $T_{1,s} = \begin{vmatrix} 1 & Q^{[+s]} \\ 1 & P^{[-s]} \end{vmatrix}$.

  $\forall s, T_{1,s} \in \mathbb{R} \quad \Rightarrow \quad P = -\bar{Q}$

- upper band :

$$T_{a,1} = \begin{vmatrix} q_1^{[+a+2]} & q_2^{[+a+2]} & q_3^{[+a+2]} & q_4^{[+a+2]} \\ q_1^{[+a]} & q_2^{[+a]} & q_3^{[+a]} & q_4^{[+a]} \\ q_1^{[+a-2]} & q_2^{[+a-2]} & q_3^{[+a-2]} & q_4^{[+a-2]} \\ p_1^{[-a]} & p_2^{[-a]} & p_3^{[-a]} & p_4^{[-a]} \end{vmatrix} \in \mathbb{R} \quad \Rightarrow \quad ?$$

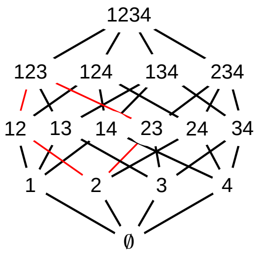
$$\rightsquigarrow \bar{p}_1 = \begin{vmatrix} q_1^{[+2]} & q_2^{[+2]} & q_3^{[+2]} \\ q_1 & q_2 & q_3 \\ q_1^{[-2]} & q_2^{[-2]} & q_3^{[-2]} \end{vmatrix} \equiv q_{123}$$

- one defines this way 2^4 q-functions for the upper band, only 4 of which are independent.

q-q relations

The definition (eg $\begin{vmatrix} q_i^{[+2]} & q_j^{[+2]} & q_k^{[+2]} \\ q_i & q_j & q_k \\ q_i^{[-2]} & q_j^{[-2]} & q_k^{[-2]} \end{vmatrix} = q_{ijk}$) implies as set of relations such as : $q_{ijk}\, q_i = q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^-$

there is one such relation at every facet of the hypercube

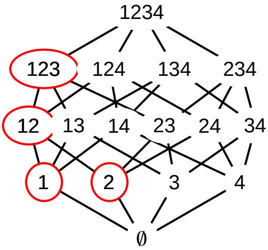


- Choice of basis

$\rightsquigarrow$ reality, analyticity strip, L/R symmetry etc. get very natural in terms of these q-functions

q-q relations

The definition (eg $\begin{vmatrix} q_i^{[+2]} & q_j^{[+2]} & q_k^{[+2]} \\ q_i & q_j & q_k \\ q_i^{[-2]} & q_j^{[-2]} & q_k^{[-2]} \end{vmatrix} = q_{ijk}$) implies as set of relations such as : $q_{ijk} q_i = q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^-$
there is one such relation at every facet of the hypercube

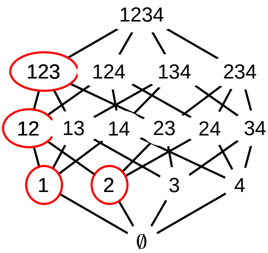


- Choice of basis

⇝ reality, analyticity strip, L/R symmetry etc. get very natural in terms of these q-functions

q-q relations

The definition (eg $\begin{vmatrix} q_i^{[+2]} & q_j^{[+2]} & q_k^{[+2]} \\ q_i & q_j & q_k \\ q_i^{[-2]} & q_j^{[-2]} & q_k^{[-2]} \end{vmatrix} = q_{ijk}$) implies as set of relations such as : $q_{ijk} q_i = q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^-$
there is one such relation at every facet of the hypercube



- Choice of basis

$\rightsquigarrow$ reality, analyticity strip, L/R symmetry etc. get very natural in terms of these q-functions

Outline

Symmetries ⇝ Classical limit

In the classical limit, $T_{a,s}(u) = \chi_{a,s}(\Omega(u))$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T \hat{C} \qquad (\text{or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues})$$

[Bena Polchinksi Roiban]

≪Quantum case≫ (ie finite-size, outside the classical limit)

$$T_{1,s} = -\hat{T}_{1,-s}$$

Symmetries $\rightsquigarrow$ Classical limit

In the classical limit, $T_{a,s}(u) = \chi_{a,s}(\Omega(u))$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T \hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

≪Quantum case≫ (ie finite-size, outside the classical limit)

$$T_{1,s} = -\hat{T}_{1,-s}$$

Symmetries $\rightsquigarrow$ Classical limit

In the classical limit, $T_{a,s}(u) = \chi_{a,s}(\Omega(u))$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T \hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

≪Quantum case≫ (ie finite-size, outside the classical limit)

$$T_{1,s} = -\hat{T}_{1,-s}$$

Symmetries $\rightsquigarrow$ Classical limit

In the classical limit, $T_{a,s}(u) = \chi_{a,s}(\Omega(u))$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T \hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

≪Quantum case≫ (ie finite-size, outside the classical limit)

$$T_{1,s} = -\hat{T}_{1,-s}$$

≪quantum $\mathbb{Z}_4$≫ symmetry
(ie outside the classical limit)

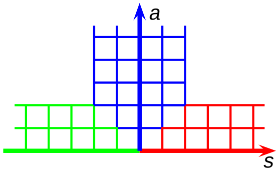
statement

$$\mathcal{T}_{1,s} = -\hat{\mathcal{T}}_{1,-s},$$

where $\hat{\mathcal{T}}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

$\hat{\mathcal{T}}_{1,0} = 0 \Rightarrow \quad Q = -\bar{Q}$

$Q(u) = -iu + \frac{1}{2i\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$



$$\mathcal{T}_{1,s}\big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$

≪quantum $\mathbb{Z}_4$≫ symmetry

(ie outside the classical limit)

statement

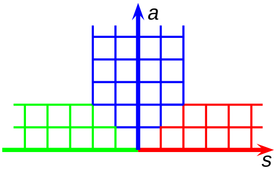
$$\mathcal{T}_{1,s} = -\hat{\mathcal{T}}_{1,-s},$$

where $\hat{\mathcal{T}}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

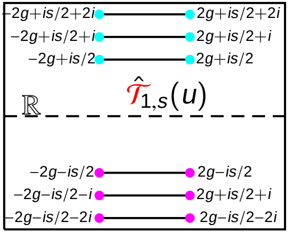
$$\hat{\mathcal{T}}_{1,0} = 0 \Rightarrow \quad Q = -\bar{Q}$$

$$= \qquad\qquad =$$

$$Q(u) = -iu + \frac{1}{2i\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$$



$$\mathcal{T}_{1,s}\Big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$



≪quantum $\mathbb{Z}_4$≫ symmetry

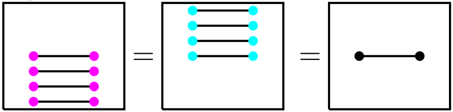
(ie outside the classical limit)

statement

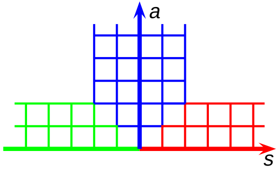
$$\mathcal{T}_{1,s} = -\hat{\mathcal{T}}_{1,-s},$$

where $\hat{\mathcal{T}}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

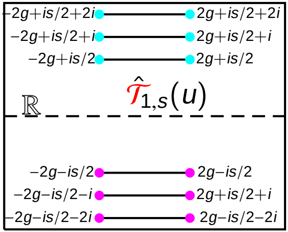
$$\hat{\mathcal{T}}_{1,0} = 0 \;\Rightarrow\; Q = -\bar{Q}$$



$$Q(u) = -iu + \frac{1}{2I\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$$



$$\mathcal{T}_{1,s}\big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$



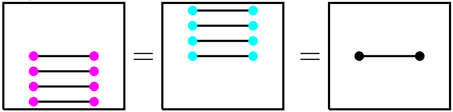
$\ll$quantum $\mathbb{Z}_4\gg$ symmetry
(ie outside the classical limit)

statement

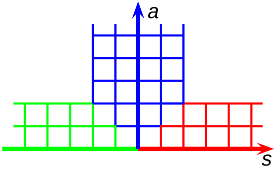
$$\mathcal{T}_{1,s} = -\hat{\mathcal{T}}_{1,-s},$$

where $\hat{\mathcal{T}}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

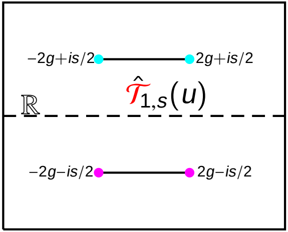
$$\hat{\mathcal{T}}_{1,0} = 0 \Rightarrow \quad Q = -\bar{Q}$$



$$Q(u) = -iu + \frac{1}{2I\pi}\int_{-2g}^{2g} \frac{\rho(v)}{v-u}$$



$$\mathcal{T}_{1,s}\big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$



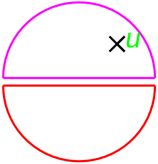
Outline

Riemann-Hilbert Problem

General statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{F(v)-G(v)}{v-u} = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if $Q = -\bar{Q}$ is analytic except on $[-2g, 2g]$ and $Q \xrightarrow[|u|\to\infty]{} -iu$,

$$Q(u) = -iu + \frac{1}{2I\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$$

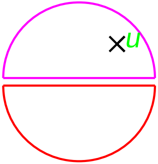
where $\rho = Q^{[+0]} + \bar{Q}^{[-0]}$

Riemann-Hilbert Problem

General statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{F(v)-G(v)}{v-u} = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if $Q = -\bar{Q}$ is analytic except on $[-2g, 2g]$ and $Q \xrightarrow[|u|\to\infty]{} -iu$,

$$Q(u) = -iu + \frac{1}{2I\pi}\int_{-2g}^{2g} \frac{\rho(v)}{v-u}$$

where $\rho = Q^{[+0]} + \bar{Q}^{[-0]}$

Riemann-Hilbert Problem

General statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{F(v)-G(v)}{v-u} = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$

FiNLIE-equations

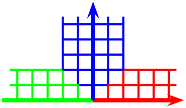
Appropriate choices of F and G allow to derive non-trivial integral equations from analyticity constraints.

These equations can be shown to be equivalent to the TBA-equations.

Outline

AdS/CFT FiNLIE

$$
\begin{aligned}
T_{a,+1} &= q_1^{[+a]}\bar{q}_2^{[-a]} + q_2^{[+a]}\bar{q}_1^{[-a]} + q_3^{[+a]}\bar{q}_4^{[-a]} + q_4^{[+a]}\bar{q}_3^{[-a]}\,,\\
T_{a,0} &= q_{12}^{[+a]}\bar{q}_{12}^{[-a]} + q_{34}^{[+a]}\bar{q}_{34}^{[-a]} - q_{14}^{[+a]}\bar{q}_{14}^{[-a]}\\
&\quad - q_{23}^{[+a]}\bar{q}_{23}^{[-a]} - q_{13}^{[+a]}\bar{q}_{24}^{[-a]} - q_{24}^{[+a]}\bar{q}_{13}^{[-a]}\,,\\
q_\emptyset q_{ij} &= q_i^+ q_j^- - q_j^+ q_i^-\,,\\
q_{ijk} q_i &= q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^-\,.
\end{aligned}
$$

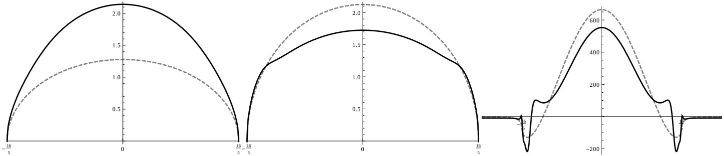


$$
Y_{1,1} = -\sqrt{\frac{R^{(+)}}{R^{(-)}}\frac{B^{(-)}}{R^{(+)}}\frac{\mathcal{T}_{1,2}}{T_{2,1}}\left(\frac{T_{1,0}}{Q^+Q^-}\right)^{1+\check{*}\mathcal{Z}}\left(\frac{Q^2}{T_{0,0}}\right)^{*\frac{1}{2}(\mathcal{Z}_1+\mathcal{K}_1)}\left(\frac{T_{1,1}}{\mathcal{T}_{1,1}}\right)^{*\mathcal{K}_1}}\,.
$$

$$
U^2 = \frac{\Lambda^2 T_{00}^-}{\hat{x}^{L-2} Y_{1,1} Y_{2,2} T_{1,0}}\left(\frac{Y_{1,1}Y_{2,2}-1}{\rho/\mathcal{F}^+}\right)^{\hat{*}2\mathcal{Z}}\left(\frac{T_{2,1}\mathcal{T}_{1,1}^-}{\hat{T}_{1,1}^-\mathcal{T}_{1,2}Y_{2,2}}\right)^{*2\Psi}
$$

Numeric implementation of FiNLIE

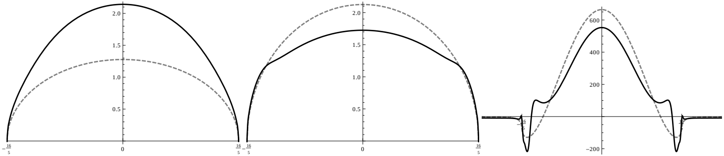
Numerical densities obtained for Konishi state by our FiNLIE algorithm: These three densities (black curves) describe the finite size Konishi state, and are compared to their asymptotic expression dashed gray curve



- Checked to reproduce previous Y-system
- In particular these Y-system results allow to obtain non-trivial expansion coefficients for SYM or Strings.

Numeric implementation of FiNLIE

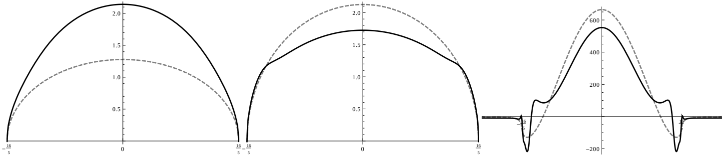
Numerical densities obtained for Konishi state by our FiNLIE algorithm: These three densities (black curves) describe the finite size Konishi state, and are compared to their asymptotic expression dashed gray curve



- Checked to reproduce previous Y-system
- In particular these Y-system results allow to obtain non-trivial expansion coefficients for SYM or Strings.

Numeric implementation of FiNLIE

Numerical densities obtained for Konishi state by our FiNLIE algorithm: These three densities (black curves) describe the finite size Konishi state, and are compared to their asymptotic expression dashed gray curve



- Checked to reproduce previous Y-system
- In particular these Y-system results allow to obtain non-trivial expansion coefficients for SYM or Strings.

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs
  - $\partial \log T_{0,0} \xrightarrow{u\to\infty} \frac{2E}{u}$
  - Exact Bethe equations arise as absence of poles of T-functions

- to be continued
  - currently restricted to symmetric sl_2 "sector" states
  - $\left\{ \begin{array}{l} \text{numeric efficiency} \\ \text{best FiNLIE formulation} \end{array} \right.$ are to be studied
  - application to other Y-systems ?
  - BFKL
  - strong coupling construction of T (? $T = \langle \mathrm{trace}\ \Omega \rangle$)
  - weak coupling interpretation of T

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs
  - $\partial \log T_{0,0} \xrightarrow{u\to\infty} \frac{2E}{u}$
  - Exact Bethe equations arise as absence of poles of T-functions
- to be continued
  - currently restricted to symmetric sl_2 "sector" states
  - $\left\{ \begin{array}{l} \text{numeric efficiency} \\ \text{best FiNLIE formulation} \end{array} \right.$ are to be studied
  - application to other Y-systems ?
  - BFKL
  - strong coupling construction of T (? $T = \langle \text{trace } \Omega \rangle$)
  - weak coupling interpretation of T

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs
  - $\partial \log T_{0,0} \xrightarrow{u \to \infty} \frac{2E}{u}$
  - Exact Bethe equations arise as absence of poles of T-functions
- to be continued
  - currently restricted to symmetric sl_2 "sector" states
  - $\left\{ \begin{array}{l} \text{numeric efficiency} \\ \text{best FiNLIE formulation} \end{array} \right.$ are to be studied
  - application to other Y-systems ?
  - BFKL
  - strong coupling construction of T (? $T = \langle \mathrm{trace}\ \Omega \rangle$)
  - weak coupling interpretation of T

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs
  - $\partial \log T_{0,0} \xrightarrow{u\to\infty} \frac{2E}{u}$

Really

Thank you !

- to be continued
  - currently restricted to symmetric sl_2 "sector" states
  - $\left\{ \begin{array}{l} \text{numeric efficiency} \\ \text{best FiNLIE formulation} \end{array} \right.$ are to be studied
  - application to other Y-systems ?
  - BFKL
  - strong coupling construction of T (? $T = \langle \text{trace } \Omega \rangle$)
  - weak coupling interpretation of T

Thermodynamic Bethe Ansatz

- Equations of the form

$$Y_{a,s}(u) = -L\ E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right)$$
$$+\langle \mathit{Source\ Terms}\rangle$$

- Vacuum energy

$$E_0 = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ du$$

Back to the presentation

- Extra assumption : Excited states obey the same equations.
Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles

- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

$\Rightarrow$ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

Thermodynamic Bethe Ansatz

- Equations of the form
$$Y_{a,s}(u) = -L\ E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right)$$
$$+\langle \textit{Source Terms}\rangle$$

- Vacuum energy
$$E = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$$

- Extra assumption : Excited states obey the same equations.
  Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles

Back to the presentation

- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

⇒ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

Thermodynamic Bethe Ansatz

- Equations of the form

$$Y_{a,s}(u) = -L\ E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right) + \langle Source\ Terms \rangle$$

- Vacuum energy

$$E = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$$

- Extra assumption : Excited states obey the same equations.
  Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles

- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

$\Rightarrow$ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

Back to the presentation

Y-system and Hirota equation

Y-system Equation

$$Y^+_{a,s}\, Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

where $Y^\pm_{a,s} = Y_{a,s}(u \pm \frac{i}{2})$

- change of variable $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$

Hirota equation

$$T^+_{a,s} T^-_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$

Back to the presentation

Gauge freedom

Y-functions and Hirota equation are invariant under gauge transformations $T_{a,s} \rightarrow g_1^{[a+s]} g_2^{[a-s]} g_3^{[-a+s]} g_4^{[-a-s]} T_{a,s}$

$f^{[\pm k]} \equiv f(u \pm ki/2)$

Y-system and Hirota equation

Y-system Equation

$$Y_{a,s}^{+}\,Y_{a,s}^{-}=\frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

where $Y_{a,s}^{\pm}=Y_{a,s}(u\pm\frac{i}{2})$

- change of variable $Y_{a,s}=\frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$

Hirota equation

$$T_{a,s}^{+}T_{a,s}^{-}=T_{a+1,s}T_{a-1,s}+T_{a,s+1}T_{a,s-1}$$

Gauge freedom

Y-functions and Hirota equation are invariant under gauge transformations $T_{a,s}\to g_1^{[a+s]}g_2^{[a-s]}g_3^{[-a+s]}g_4^{[-a-s]}T_{a,s}$

$f^{[\pm k]}\equiv f(u\pm ki/2)$

Back to the presentation