Hirota Equation, Bethe Ansatz, and Baxter's Q-operators

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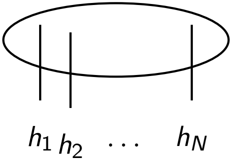
[arXiv:1010.4022]   V. Kazakov, SL, Z.Tsuboi

Saclay, January 24, 2011

Outline

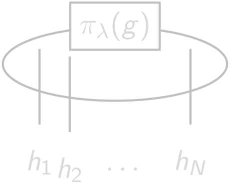
1. Spin chains and co-derivative
   - T-operators for $GL(K|M)$ spin chain
   - Co-derivatives
   - "Master Identity"
2. Explicit operatorial Bäcklund flow
   - Bäcklund Flow
   - Explicit nested T and Q-operators
   - Hints of proof
3. Diagonalization of T-operators
   - Bethe Equations
   - Wronskian formulae
4. Sigma-models and Q-functions
   - $SU(N)$ Principal Chiral Field [1007.1770]
   - AdS_5/CFT_4 Y-system [1010.2720]

Heisenberg $GL(2)$ Spin Chain



- Hilbert space $\mathcal{H} = \bigotimes_i h_i = \left(\mathbb{C}^2\right)^{\otimes N}$
- $T(u) =$ $\mathrm{trace}\,\underbrace{(u + 2P) \otimes (u + 2P) \otimes \cdots}_{N \text{ times}}$ (permutation)
- $H = \sum_i \vec{S_n} \cdot \vec{S_{n+1}} = \left.\frac{\mathrm{d}\log T}{\mathrm{d}u}\right|_{u\to 0}$
- $[T(u), T(v)] = 0$
- Solved by simultaneous diagonalization of all $T(u)$'s

$GL(K|M)$



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- $R_n(u) = (u - \theta_n) + 2\,\mathcal{P}_{n,\lambda}$
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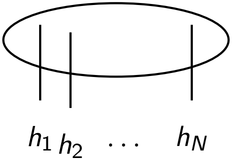

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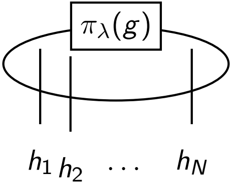

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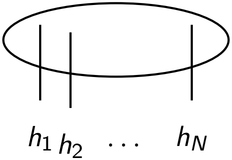

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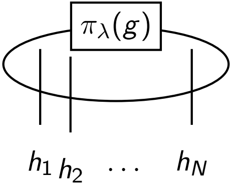

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Expression in terms of co-derivative

- $\hat{D} \otimes f(g) = \frac{\partial}{\partial \phi} \otimes f(e^{\phi} g)\Big|_{\phi=0}$ $\qquad \phi \in GL(K)$
- If $f(g)$ acts on $\mathcal{H}$, then $\hat{D} \otimes f$ acts on $\tilde{\mathcal{H}} = \mathbb{C}^K \otimes \mathcal{H}$
- $\hat{D} \otimes g = \mathcal{P}\,(1 \otimes g)$ and Leibniz rule :
  $\hat{D} \otimes (f \cdot \tilde{f}) = [\mathbb{I} \otimes f] \cdot \left[\hat{D} \otimes \tilde{f}\right] + \left[\hat{D} \otimes f\right] \cdot \left[\mathbb{I} \otimes \tilde{f}\right]$
  $\rightsquigarrow$ compute any $\hat{D} \otimes f(g)$
- $\hat{D} \otimes \pi_\lambda(g) = \sum_{\alpha,\beta} e_{\beta\alpha} \otimes \pi_\lambda(e_{\alpha\beta})$
  hence
  $R_N(u) \otimes R_{N-1}(u) \otimes \cdots \otimes R_1(u) \pi_\lambda(g) = \bigotimes_{i=1}^{N} (u - \theta_i + 2\hat{D}) \pi_\lambda(g)$
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Expression in terms of co-derivative

[arxiv:0711.2470 ; V.Kazakov & P.Vieira]

from these expressions, explicit proof that

- $[\![T^{\{\lambda\}}(u), T^{\{\mu\}}(v)]\!] = 0$
- for rectangular representations, (ie. rectangular young diagram)

$$T^{(a,s)}(u+1)\,T^{(a,s)}(u-1) = T^{(a+1,s)}(u+1)\,T^{(a-1,s)}(u-1) + T^{(a,s+1)}(u-1)\,T^{(a,s-1)}(u+1).$$

α,β

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and $T^{\{\lambda\}}(u) = \bigotimes_{i=1}^{N}(u-\theta_i+2\hat{D})\chi_\lambda(g)$

Construction of other commuting charges

if for all $g, h \in GL(K|M)$, $f(g) = f(h^{-1}gh)$ and $\tilde{f}(g) = \tilde{f}(h^{-1}gh)$,
then $[\![\bigotimes_{i=1}^{N}(u-\theta_i+2\hat{D})f(g), \bigotimes_{i=1}^{N}(v-\theta_i+2\hat{D})\tilde{f}(g)]\!] = 0$

Remark : linear combinations of T-operators.

"Master Identity"

when $\Pi = \prod_j w(t_j)$,

$$(t-z)\left[\bigotimes(u_i+2+2\hat{D})\ w(z)w(t)\Pi\right]\cdot\left[\bigotimes(u_i+2\hat{D})\ \Pi\right]$$
$$= t\left[\bigotimes(u_i+2\hat{D})\ w(z)\Pi\right]\cdot\left[\bigotimes(u_i+2+2\hat{D})\ w(t)\Pi\right]$$
$$- z\left[\bigotimes(u_i+2+2\hat{D})\ w(z)\Pi\right]\cdot\left[\bigotimes(u_i+2\hat{D})\ w(t)\Pi\right]$$

where $w(z) = \det\frac{1}{1-zg} = \sum_{s=0}^{\infty} z^s\ \chi_s(g)$

Combinatorial Identity

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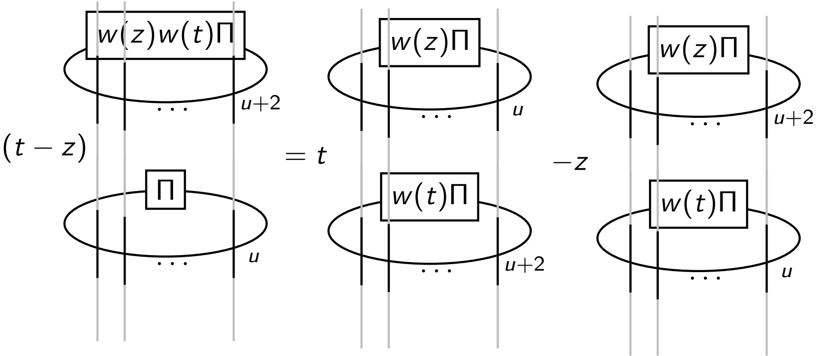
$$



Example of consequence of Master identity : an Hirota relation

If $\Pi = 1$, the coefficient of $t^s \cdot z^{s'}$ is

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Hirota relation

If $s' = s+1$, $\chi_{s'}\chi_{s-1} - \chi_{s'-1}\chi_s = -\chi^{(2,s)}$, and we get

$$\begin{aligned}T^{(2,s)}(u+2)\cdot T^{(0,s)}(u) = -\ T^{(1,s+1)}(u)\cdot T^{(1,s-1)}(u+2) \\ + T^{(1,s)}(u+2)\cdot T^{(1,s)}(u)\end{aligned}$$

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Outline

Bäcklund Transformations : linear system

Bäcklund Transformations

if $T^{(a,s)}(u)$ is a solution of Hirota equation and

$$T^{(a+1,s)}(u)F^{(a,s)}(u) - T^{(a,s)}(u)F^{(a+1,s)}(u) = x_j T^{(a+1,s-1)}(u+2)F^{(a,s+1)}(u-2),$$

$$T^{(a,s+1)}(u)F^{(a,s)}(u) - T^{(a,s)}(u)F^{(a,s+1)}(u) = x_j T^{(a+1,s)}(u+2)F^{(a-1,s+1)}(u-2).$$

Then $F^{(a,s)}(u)$ is a solution of Hirota equation.

Moreover, if $T^{(a,s)}(u) = 0, \forall a > K$, one can choose $F^{(a,s)}(u) = 0, \forall a > K-1$.

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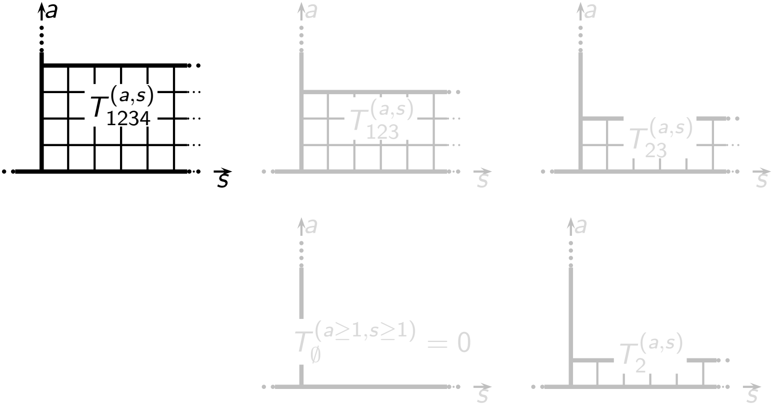
$GL(4)$ Bäcklund flow and lattices' boundaries



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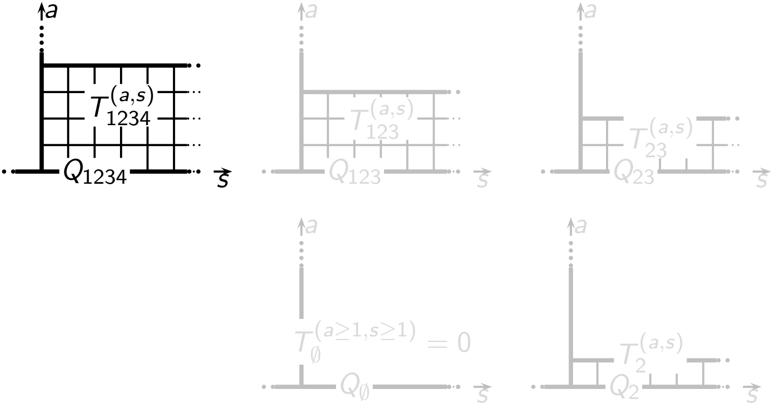

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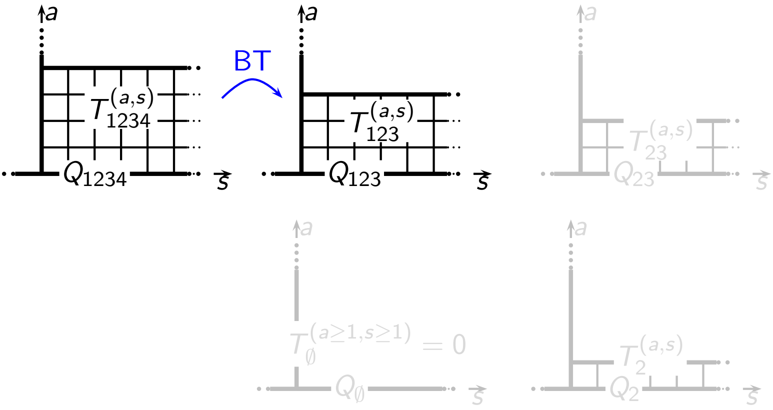

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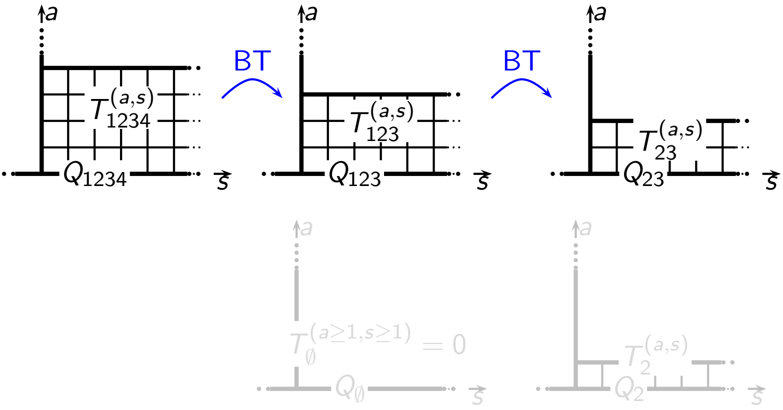

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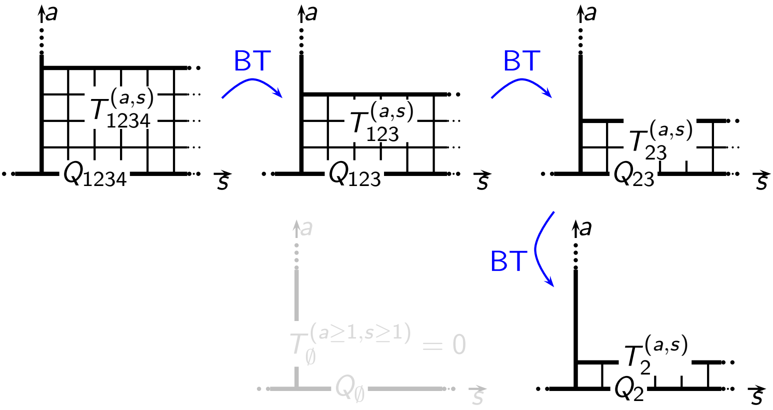

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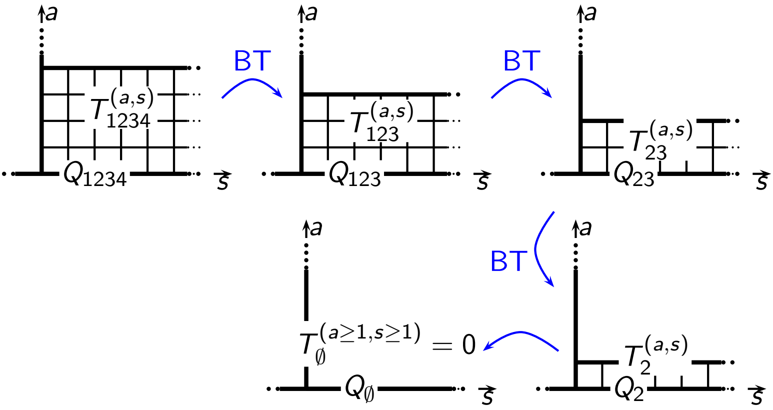

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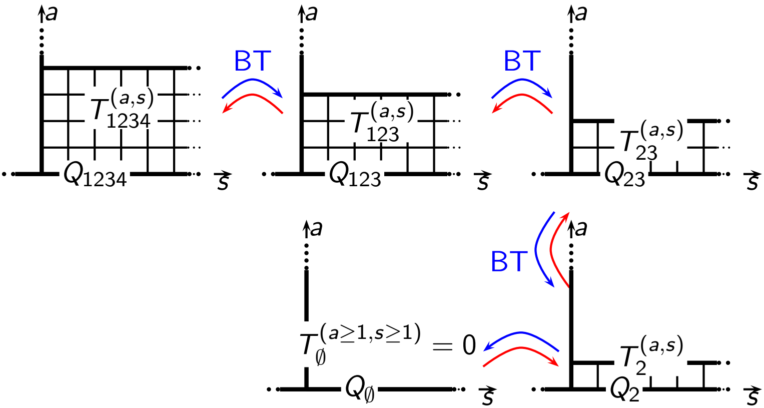

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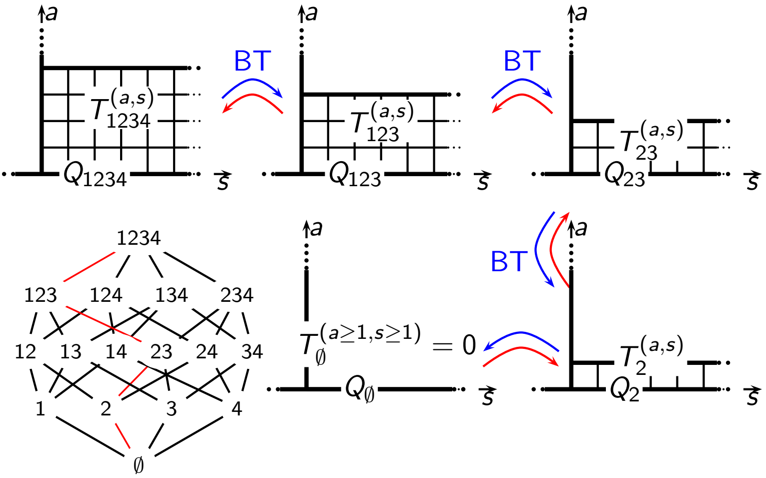

Explicit Bäcklund flow

Explicit solution of this linear system

$$T_I^{\{\lambda\}}(u) = \lim_{\substack{t_j \to \frac{1}{x_j} \\ j \in \bar{I}}} B_{\bar{I}} \cdot \left[\bigotimes_{i=1}^{N} (u_i + 2\hat{D} + 2|\bar{I}|)\ \chi_\lambda(g_I) \Pi_{\bar{I}} \right],$$

- $[\![T_I^{\{\lambda\}}(u), T_J^{\{\mu\}}(v)]\!] = 0$
- $T_I^s(u) Q_{I,j}(u) = T_{I,j}^s(u) Q_I(u) - x_j T_{I,j}^{s-1}(u+2) Q_I(u-2).$
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$$Q_I = T_I^{(a,0)} \qquad g_{\{j_1, j_2, \cdots, j_k\}} = \mathrm{diag}(x_{j_1}, x_{j_2}, \cdots, x_{j_k})$$

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Example of derivation : QQ-relations $\Leftarrow$ Master identity

Choose $\Pi = \Pi_{\overline{Iij}}$, $t = t_j$ and $z = t_i$ in the "master identity" :

$$(t_j - t_i)\left[\bigotimes(u_i + 2 + 2\hat{D})\ w(t_i)w(t_j)\Pi_{\overline{Iij}}\right] \cdot \left[\bigotimes(u_i + 2\hat{D})\ \Pi_{\overline{Iij}}\right]$$
$$= t_j\left[\bigotimes(u_i + 2\hat{D})\ w(t_i)\Pi_{\overline{Iij}}\right] \cdot \left[\bigotimes(u_i + 2 + 2\hat{D})\ w(t_j)\Pi_{\overline{Iij}}\right]$$
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Toward "Master identity" : proof of a first identity [Kazakov, Vieira]

$$\hat{D}\;\; w(x) = \frac{gx}{1-gx} w(x)$$

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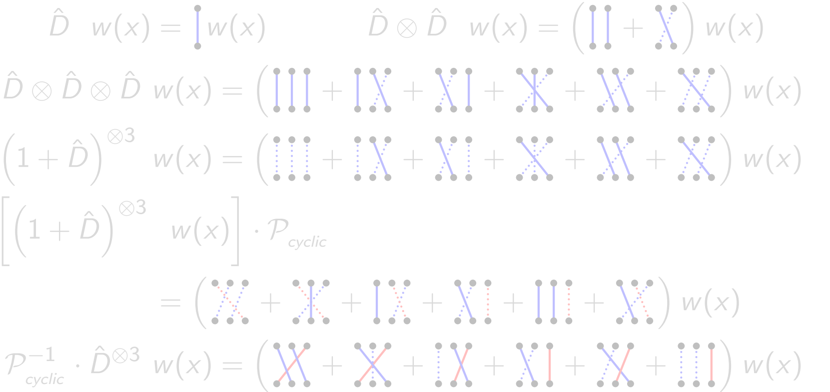

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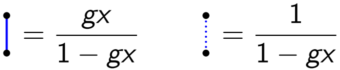

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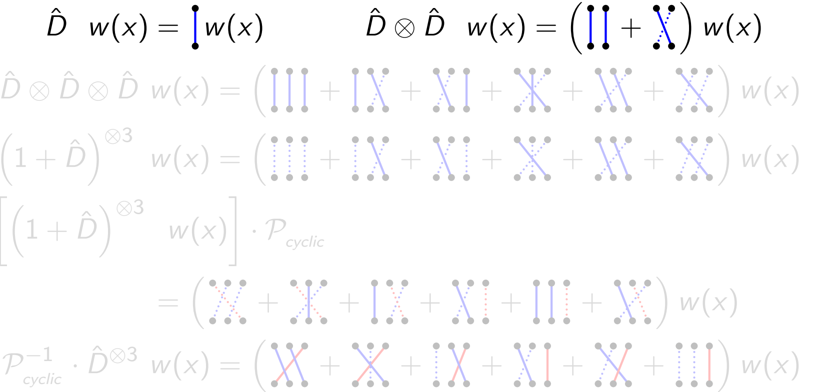

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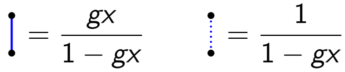

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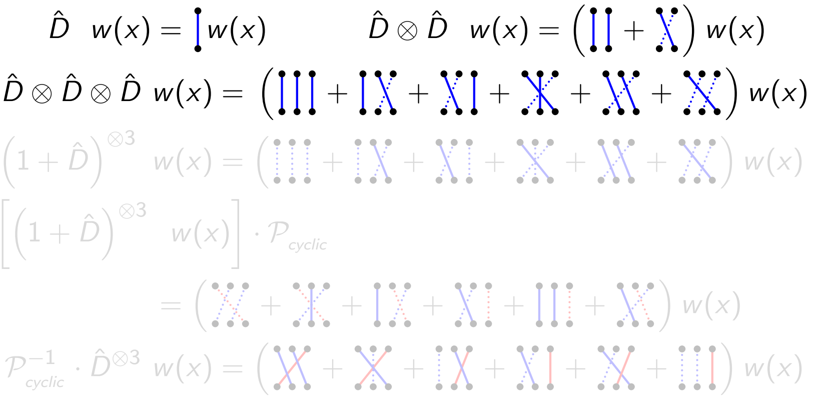

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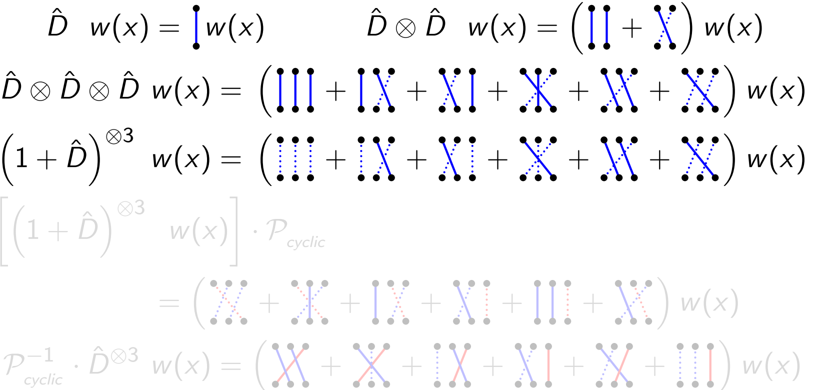

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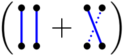
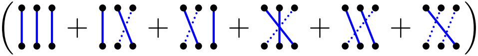
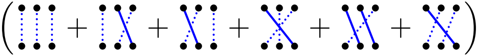
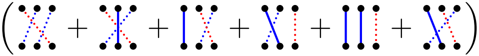
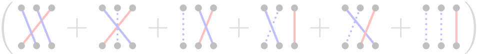
$\hat{D}\;\; w(x) =$  $w(x)$ $\qquad \hat{D}\otimes\hat{D}\;\; w(x) =$  $w(x)$

$\hat{D}\otimes\hat{D}\otimes\hat{D}\; w(x) =$  $w(x)$

$\left(1+\hat{D}\right)^{\otimes 3}\; w(x) =$  $w(x)$

$\left[\left(1+\hat{D}\right)^{\otimes 3}\;\; w(x)\right]\cdot\mathcal{P}_{cyclic}$

$=$  $w(x)$

$\mathcal{P}^{-1}_{cyclic}\cdot\hat{D}^{\otimes 3}\; w(x) =$  $w(x)$

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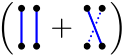
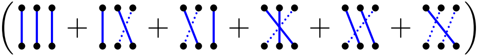
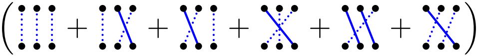
$\hat{D}\; w(x) =$  $w(x)$ $\qquad \hat{D}\otimes\hat{D}\; w(x) =$  $w(x)$

$\hat{D}\otimes\hat{D}\otimes\hat{D}\; w(x) =$  $w(x)$

$\left(1+\hat{D}\right)^{\otimes 3}\; w(x) =$  $w(x)$

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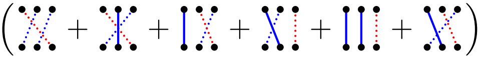
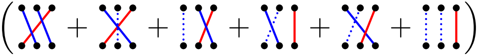
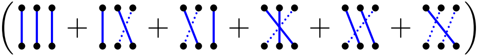
$=$  $w(x)$

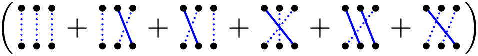
$\mathcal{P}_{cyclic}^{-1} \cdot \hat{D}^{\otimes 3}\; w(x) =$  $w(x)$

Toward "Master identity" : proof of a first identity [Kazakov, Vieira]

Identity [KV 0711.2470 ; V.Kazakov & P.Vieira]

$$\left[(1+\hat{D})^{\otimes N} w(z_1)\right]\cdot\left[\hat{D}^{\otimes N} w(z_2)\right] = \frac{z_2}{z_1}\left[\hat{D}^{\otimes N} w(z_1)\right]\cdot\left[(1+\hat{D})^{\otimes N} w(z_2)\right]$$

$\hat{D}\otimes\hat{D}\otimes\hat{D}\ w(x) =$  $w(x)$

$\left(1+\hat{D}\right)^{\otimes 3}\ w(x) =$  $w(x)$

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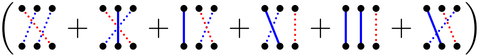
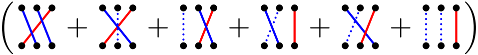
$=$  $w(x)$

$\mathcal{P}_{cyclic}^{-1}\cdot\hat{D}^{\otimes 3}\ w(x) =$  $w(x)$

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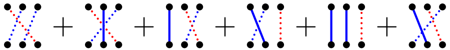
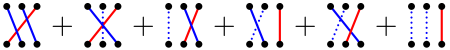
$$\left[\hat{D}^{\otimes N} w(z_1)\det(g)\right] \cdot \left[\hat{D}^{\otimes N} w(z_2)\right] = \frac{z_2}{z_1}\left[\hat{D}^{\otimes N} w(z_1)\right] \cdot \left[\hat{D}^{\otimes N} w(z_2)\det(g)\right]$$

cyclic

$= \Big($  $\Big)\, w(x)$

$\mathcal{P}^{-1}_{cyclic} \cdot \hat{D}^{\otimes 3}\ w(x) = \Big($  $\Big)\, w(x)$

Toward "Master identity" : Fundamental property for bilinear equations

Hypothesis

If $(A_j(g))_{1\leq j\leq k}$ and $(B_j(g))_{1\leq j\leq k}$ are class functions of $g \in GL(K|M)$, such that for all $N \in \mathbb{Z}_{\geq 0}$ and for all g

$$\sum_j \left[\hat{D}^{\otimes N} A_j(g)\right] \cdot \left[\hat{D}^{\otimes N} B_j(g)\right] = 0$$

then for any set $(t_m)_{m\leq P} \in \mathbb{C}^n$, for all $N \in \mathbb{Z}_{\geq 0}$ and $g \in GL(K)$:

$$\sum_j \left[\bigotimes_{i=1}^{N} (u_i + \hat{D})\ A_j(g)\Pi(g)\right] \cdot \left[\bigotimes_{i=1}^{N} (u_i + \hat{D})\ B_j(g)\Pi(g)\right] = 0$$

$$\text{where } \Pi(g) = \prod_{m=1}^{P} w(t_m)$$

Toward "Master identity" : Fundamental property for bilinear equations

Hypothesis

If $(A_j(g))_{1\leq j\leq k}$ and $(B_j(g))_{1\leq j\leq k}$ are class functions of $g\in GL(K|M)$, such that for all $N\in\mathbb{Z}_{\geq 0}$ and for all g

$$\sum_j \left[\hat{D}^{\otimes N} A_j(g)\right]\cdot\left[\hat{D}^{\otimes N} B_j(g)\right]=0$$

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$$\sum_j \left[\bigotimes_{i=1}^{N}(u_i+\hat{D})\ A_j(g)\Pi(g)\right]\cdot\left[\bigotimes_{i=1}^{N}(u_i+\hat{D})\ B_j(g)\Pi(g)\right]=0$$

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Generalization of the identity of [KV 0711.2470] $\Rightarrow$ "Master identity"

we have already, $\forall N \geq 1$,

$$z\left[\hat{D}^{\otimes N} w(z)\det(g)\right]\cdot\left[\hat{D}^{\otimes N} w(t)\right] - t\left[\hat{D}^{\otimes N} w(z)\right]\cdot\left[\hat{D}^{\otimes N} w(t)\det(g)\right] = 0$$

3-terms consequence

$$(t-z)\left[\bigotimes(u_i+2+2\hat{D})\ w(z)w(t)\Pi\right]\cdot\left[\bigotimes(u_i+2\hat{D})\ \Pi\right] = t\left[\bigotimes(u_i+2\hat{D})\ w(z)\Pi\right]\cdot\left[\bigotimes(u_i+2+2\hat{D})\ w(t)\Pi\right] - z\left[\bigotimes(u_i+2+2\hat{D})\ w(z)\Pi\right]\cdot\left[\bigotimes(u_i+2\hat{D})\ w(t)\Pi\right]$$

Generalization of the identity of [KV 0711.2470] $\Rightarrow$ "Master identity"

we have already, $\forall N \geq \not{1}^{\,0}$,

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$$= (z-t)\left[\hat{D}^{\otimes N} w(z)w(t)\det(g)\right] \cdot \left[\hat{D}^{\otimes N}\; 1\right]$$

3-terms consequence

$$(t-z)\left[\bigotimes(u_i+2+2\hat{D})\; w(z)w(t)\Pi\right] \cdot \left[\bigotimes(u_i+2\hat{D})\; \Pi\right]$$
$$= t\left[\bigotimes(u_i+2\hat{D})\; w(z)\Pi\right] \cdot \left[\bigotimes(u_i+2+2\hat{D})\; w(t)\Pi\right]$$
$$- z\left[\bigotimes(u_i+2+2\hat{D})\; w(z)\Pi\right] \cdot \left[\bigotimes(u_i+2\hat{D})\; w(t)\Pi\right]$$

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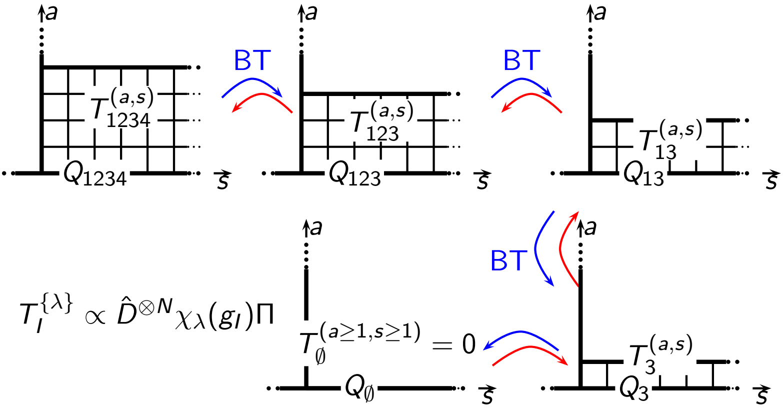
$$=(z-t)\left[\hat{D}^{\otimes N} w(z)w(t)\det(g)\right]\cdot\left[\hat{D}^{\otimes N}\;\; 1\right]$$

3-terms consequence

$$\begin{aligned}(t-z)&\left[\bigotimes(u_i+2+2\hat{D})\;\; w(z)w(t)\Pi\right]\cdot\left[\bigotimes(u_i+2\hat{D})\;\;\Pi\right]\\ &= t\left[\bigotimes(u_i+2\hat{D})\;\; w(z)\Pi\right]\cdot\left[\bigotimes(u_i+2+2\hat{D})\;\; w(t)\Pi\right]\\ &\quad - z\left[\bigotimes(u_i+2+2\hat{D})\;\; w(z)\Pi\right]\cdot\left[\bigotimes(u_i+2\hat{D})\;\; w(t)\Pi\right]\end{aligned}$$

Outline

$GL(4)$ Bäcklund flow



$$T_I^{\{\lambda\}} \propto \hat{D}^{\otimes N} \chi_\lambda(g_I)\Pi$$

Series representation for T-operators

TQ-rewriting

the TQ relation can be re-written as

$$\frac{Q_{I,j}(u)}{Q_I(u)}\mathfrak{W}_I(u,z) = \left(1 - x_j \frac{Q_I(u-2)}{Q_I(u)} z e^{2\partial_u}\right)\mathfrak{W}_{I,j}(u,z)$$

$$\text{where } e^{2\partial_u} f(u) = f(u+2) \qquad \mathfrak{W}_I(u,z) \equiv \sum_{s=0}^{\infty} z^s T_I^s(u)$$

Hence for any "nesting path" $I_K \supset I_{K-1} \supset \cdots \supset I_0 = \emptyset$, $I_k = \{i_1, i_2, \cdots, i_k\}$,

$$\mathfrak{W}_{I_k}(u,z) = \mathcal{O}_k \frac{Q_{I_k}(u)}{Q_{I_{k-1}}(u)} \mathcal{O}_{k-1} \frac{Q_{I_{k-1}}(u)}{Q_{I_{k-2}}(u)} \cdots \mathcal{O}_1 \frac{Q_{I_1}(u)}{Q_{I_0}(u)} Q_{I_0}(u)$$

$$\text{where } \mathcal{O}_k = \left(1 - x_{i_k} \frac{Q_{I_{k-1}}(u-2)}{Q_{I_{k-1}}(u)} z e^{2\partial_u}\right)^{-1}$$

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Series representation for T-operators

TQ-rewriting

the TQ relation can be re-written as

$$Q_{I,j}(u)\ldots$$

example

$$T^1_{I_K}(u) = Q_{I_K}(u) \sum_{m=1}^{K} x_{i_m} \frac{Q_{I_m}(u+2)}{Q_{I_m}(u)} \frac{Q_{I_{m-1}}(u-2)}{Q_{I_{m-1}}(u)}$$

$I_k = \{i_1, i_2, \cdots, i_k\}$,

$$\mathfrak{W}_{I_k}(u,z) = \mathcal{O}_k \frac{Q_{I_k}(u)}{Q_{I_{k-1}}(u)} \mathcal{O}_{k-1} \frac{Q_{I_{k-1}}(u)}{Q_{I_{k-2}}(u)} \cdots \mathcal{O}_1 \frac{Q_{I_1}(u)}{Q_{I_0}(u)} Q_{I_0}(u)$$

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Bethe Equations

At the level of operators, the QQ-relations imply

$$Q_{I,i}(u) \mid x_i Q_I(u-2) Q_{I,i,j}(u) Q_{I,i}(u+2) + x_j Q_I(u) Q_{I,i,j}(u+2) Q_{I,i}(u-2).$$

On a given eigen-state,

$$Q_I(u) = c_I \prod_{k=1}^{K_I} (u - u_k^{(I)}),$$

$$-1 = \frac{x_i}{x_j} \frac{Q_I(u_k^{(I,i)} - 2) Q_{I,i}(u_k^{(I,i)} + 2) Q_{I,i,j}(u_k^{(I,i)})}{Q_I(u_k^{(I,i)}) Q_{I,i}(u_k^{(I,i)} - 2) Q_{I,i,j}(u_k^{(I,i)} + 2)}$$

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Wronskian Formulae

from QQ-relations

$$Q_I(u) = \frac{\det\left(x_j^{|I|-1-k} Q_j(u-2k)\right)_{\substack{j\in I\\ 0\le k\le |I|-1}}}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\le k\le |I|-1}}}.$$

From TQ-relations

$$T_I^s(u) = \frac{\begin{vmatrix} \left(x_k^{|I|-1+s} Q_j(u+2s)\right)_{j\in I} \\ \left(x_j^{|I|-1-k} Q_j(u-2k)\right)_{\substack{j\in I\\ 1\le k\le |I|-1}} \end{vmatrix}}{Q_\emptyset(0)^{|I|-1}\det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\le k\le |I|-1}}}.$$

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Wronskian Formulae

from QQ-relations

Wronskian determinant

$$T_I^{(a,s)}(u) = \frac{\det\left(x_j^{|I|-1-k+s\Theta} Q_j(u-2k+2s\Theta)\right)_{\substack{j\in I\\ 0\leq k\leq |I|-1}}}{Q_\emptyset(0)^{|I|-1} \det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\leq k\leq |I|-1}}}$$

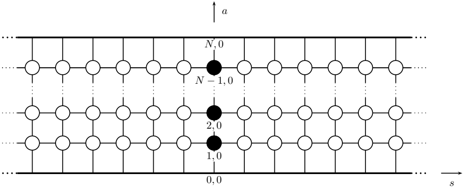
$$\Theta = \begin{cases} 1 \text{ if } k < a \\ 0 \text{ if } k \geq a \end{cases}$$

$$T_I^s(u) = \frac{\cdots}{Q_\emptyset(0)^{|I|-1} \det\left(x_j^{|I|-1-k}\right)_{\substack{j\in I\\ 0\leq k\leq |I|-1}}}.$$

Outline

$SU(N)$ principal chiral field [1007.1770 ; V. Kazakov & SL]

- The finite size PCF is described by a Y-system



- The corresponding T-system is solved by

$$T_{a,s} = \left| \begin{matrix} \left(\overline{q_j}^{[s+a+1+\frac{N}{2}-2k]}\right)_{1\leq j\leq N, 1\leq k\leq a} \\ \left(q_j^{[-s+a+1+\frac{N}{2}-2k]}\right)_{1\leq j\leq N, a<k\leq N} \end{matrix} \right|$$

- There exists "analyticity strips", consistent with $q_i(u) = \langle\text{polynomial}\rangle_i + \langle\text{resolvant}\rangle_i$

$\rightsquigarrow$ FiNLIE

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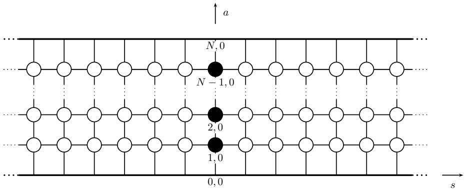

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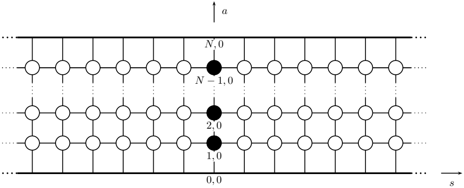

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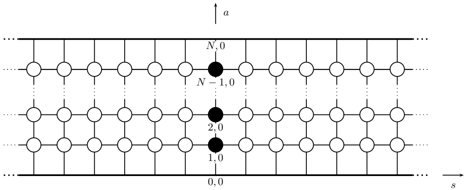

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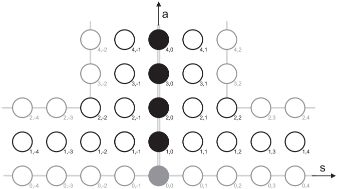
AdS_5/CFT_4 Y-system [1010.2720 ; N. Gromov, V.Kazakov, SL & Z.Tsuboi]



- The corresponding T-system is solved by

$T_{1,s}|_{s\geq 1} = Q_1^{[s]} Q_{\bar{1}}^{[-s]} - Q_2^{[s]} Q_{\bar{2}}^{[-s]}$,

$T_{2,s}|_{s\geq 2} = Q_{12}^{[s]} Q_{\overline{12}}^{[-s]}$, $T_{a,+2}|_{a\geq 2} = Q_{12}^{[a]} Q_{\overline{12}}^{[-a]}$,

$T_{a,+1}|_{a\geq 1} =$

$(-1)^{a+1}\left(Q_{12\hat{1}}^{[a]} Q_{\overline{12\hat{1}}}^{[-a]} - Q_{12\hat{2}}^{[a]} Q_{\overline{12\hat{2}}}^{[-a]} + Q_{12\hat{3}}^{[a]} Q_{\overline{12\hat{3}}}^{[-a]} - Q_{12\hat{4}}^{[a]} Q_{\overline{12\hat{4}}}^{[-a]}\right)$

$T_{a,0}|_{a\geq 0} = Q_{12\hat{1}\hat{2}}^{[a]} Q_{43\hat{4}\hat{3}}^{[-a]} - Q_{12\hat{1}\hat{3}}^{[a]} Q_{43\hat{4}\hat{2}}^{[-a]} + Q_{12\hat{1}\hat{4}}^{[a]} Q_{43\hat{3}\hat{2}}^{[-a]} +$
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...

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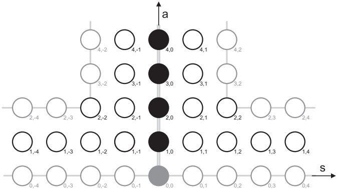

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$$T_{a,0}\big|_{a\geq 0} = Q_{12\hat 1\hat 2}^{[a]}Q_{43\hat 4\hat 3}^{[-a]} - Q_{12\hat 1\hat 3}^{[a]}Q_{43\hat 4\hat 2}^{[-a]} + Q_{12\hat 1\hat 4}^{[a]}Q_{43\hat 3\hat 2}^{[-a]} +$$

$$Q_{12\hat 2\hat 3}^{[a]}Q_{43\hat 4\hat 1}^{[-a]} - Q_{12\hat 2\hat 4}^{[a]}Q_{43\hat 3\hat 1}^{[-a]} + Q_{12\hat 3\hat 4}^{[a]}Q_{43\hat 2\hat 1}^{[-a]},$$

$\ldots$

Outlook

- The whole integrability of rational spin chains is encoded in one "master identity"
  - $\rightsquigarrow$ Does this identity have some deeper meaning and origin that combinatorics ?
- How much can this construction be generalized ?
  - $\rightsquigarrow$ Non-compact representations in auxiliary space
  - $\rightsquigarrow$ Other representations in quantum space
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