Integrability and Y-system for AdS/CFT correspondence

Sébastien Leurent
LPT-ENS (Paris)

[arXiv:1110.0562] N. Gromov, V. Kazakov, SL & D. Volin

[arXiv:1007.1770] V. Kazakov & SL
[arXiv:1010.2720] N. Gromov, V.Kazakov, SL & Z.Tsuboi
[arXiv:1010.4022] V. Kazakov, SL & Z.Tsuboi

SCSGC Conference, London 03 Novembre 2011

Outline

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{iLp} = 1$
- For two particles, the Bethe Ansatz is

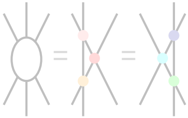
$$\psi(x,y) = \begin{cases} e^{i(p_1 x + p_2 y)} + S(p_1,p_2) \times e^{i(p_2 x + p_1 y)} & \text{if } x \lessapprox y \\ S(p_1,p_2) \times e^{i(p_1 x + p_2 y)} + e^{i(p_1 x + p_2 y)} & \text{if } x \gtrapprox y \end{cases}$$

periodic iff $\Psi(x,y) = \Psi(x+L,y)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1,p_2) = 1 \\ e^{ip_2 L} = S(p_1,p_2) \end{cases}$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma) e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$



- S is fixed by symmetries

$x < y < z \rightsquigarrow y < x < z \rightsquigarrow y < z < x \rightsquigarrow z < y < x$

$x < y < z \rightsquigarrow x < z < y \rightsquigarrow z < x < y \rightsquigarrow z < y < x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{iLp} = 1$
- For two particles, the Bethe Ansatz is

$$\psi(x,y) = \begin{cases} e^{i(p_1 x + p_2 y)} + S(p_1,p_2) \times e^{i(p_2 x + p_1 y)} & \text{if } x \lessapprox y \\ S(p_1,p_2) \times e^{i(p_1 x + p_2 y)} + e^{i(p_1 x + p_2 y)} & \text{if } x \gtrapprox y \end{cases}$$

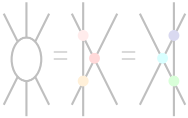
periodic iff $\Psi(x,y) = \Psi(x+L,y)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1,p_2) = 1 \\ e^{ip_2 L} = S(p_1,p_2) \end{cases}$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma) e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\leadsto$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$

- S is fixed by symmetries



$x < y < z \leadsto y < x < z \leadsto y < z < x \leadsto z < y < x$

$x < y < z \leadsto x < z < y \leadsto z < x < y \leadsto z < y < x$

Bethe Ansatz

Quantization condition in a periodic box of size L

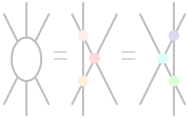
- For one particle, the wave function is periodic iff $e^{iLp}=1$
- For two particles, the Bethe Ansatz is

$$\psi(x,y)=\begin{cases} e^{i(p_1x+p_2y)}+S(p_1,p_2)\times e^{i(p_2x+p_1y)} & \text{if } x \lessapprox y \\ S(p_1,p_2)\times e^{i(p_1x+p_2y)}+e^{i(p_1x+p_2y)} & \text{if } x \gtrapprox y \end{cases}$$

periodic iff $\Psi(x,y)=\Psi(x+L,y)$, ie

$$\begin{cases} e^{ip_1L}\times S(p_1,p_2)=1 \\ e^{ip_2L}=S(p_1,p_2) \end{cases}$$

- For more particles, $\psi(x_1,x_2,\cdots)\propto\sum_\sigma C(\sigma)e^{i\sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)}\lessapprox x_{\sigma'(2)}\lessapprox\cdots\lessapprox x_{\sigma'(n)}$.



$\rightsquigarrow$ $\forall i, e^{iLp_i}=\prod_{j\neq i}S(p_j,p_i)$

- S is fixed by symmetries

$x<y<z \rightsquigarrow y<x<z \rightsquigarrow y<z<x \rightsquigarrow z<y<x$

$x<y<z \rightsquigarrow x<z<y \rightsquigarrow z<x<y \rightsquigarrow z<y<x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{iLp} = 1$
- For two particles, the Bethe Ansatz is

$$\psi(x,y) = \begin{cases} e^{i(p_1 x + p_2 y)} + S(p_1,p_2) \times e^{i(p_2 x + p_1 y)} & \text{if } x \lessapprox y \\ S(p_1,p_2) \times e^{i(p_1 x + p_2 y)} + e^{i(p_1 x + p_2 y)} & \text{if } x \gtrapprox y \end{cases}$$

periodic iff $\Psi(x,y) = \Psi(x+L,y)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1,p_2) = 1 \Leftrightarrow e^{ip_1 L} = S(p_2,p_1) \\ e^{ip_2 L} = S(p_1,p_2) \end{cases}$$

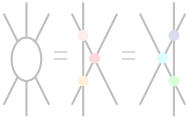
- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma) e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$

- S is fixed by symmetries

$x < y < z \rightsquigarrow y < x < z \rightsquigarrow y < z < x \rightsquigarrow z < y < x$

$x < y < z \rightsquigarrow x < z < y \rightsquigarrow z < x < y \rightsquigarrow z < y < x$



Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{iLp} = 1$

- For two particles, the Bethe Ansatz is

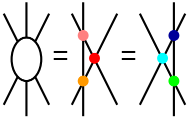
$$\psi(x,y) = \begin{cases} e^{i(p_1 x + p_2 y)} + S(p_1,p_2) \times e^{i(p_2 x + p_1 y)} & \text{if } x \lnapprox y \\ S(p_1,p_2) \times e^{i(p_1 x + p_2 y)} + e^{i(p_1 x + p_2 y)} & \text{if } x \gnapprox y \end{cases}$$

periodic iff $\Psi(x,y) = \Psi(x+L,y)$, ie

$$\begin{cases} e^{ip_1 L} \times S(p_1,p_2) = 1 \Leftrightarrow e^{ip_1 L} = S(p_2,p_1) \\ e^{ip_2 L} = S(p_1,p_2) \end{cases}$$

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma \mathcal{C}(\sigma) e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lnapprox x_{\sigma'(2)} \lnapprox \cdots \lnapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$



- S is fixed by symmetries

$x < y < z \rightsquigarrow y < x < z \rightsquigarrow y < z < x \rightsquigarrow z < y < x$

$x < y < z \rightsquigarrow x < z < y \rightsquigarrow z < x < y \rightsquigarrow z < y < x$

Bethe Ansatz

Quantization condition in a periodic box of size L

- For one particle, the wave function is periodic iff $e^{iLp} = 1$
- For two particles, the Bethe Ansatz is

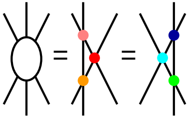
$$\Big(e^{i(p_1 x + p_2 y)} + S(p_1, p_2) \times e^{i(p_2 x + p_1 y)} \quad \text{if } x<y$$

Main conditions

- many conserved charges
- unidimensional space (eg spin chain)
- $L \gg$ interaction range

- For more particles, $\psi(x_1, x_2, \cdots) \propto \sum_\sigma C(\sigma) e^{i \sum p_i x_{\sigma(i)}}$ in each domain $x_{\sigma'(1)} \lessapprox x_{\sigma'(2)} \lessapprox \cdots \lessapprox x_{\sigma'(n)}$.

$\rightsquigarrow$ $\forall i, e^{iLp_i} = \prod_{j \neq i} S(p_j, p_i)$



- S is fixed by symmetries

$x < y < z \rightsquigarrow y < x < z \rightsquigarrow y < z < x \rightsquigarrow z < y < x$

$x < y < z \rightsquigarrow x < z < y \rightsquigarrow z < x < y \rightsquigarrow z < y < x$

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j \neq i} S_{j,i}$
- $E = \sum_i E_i$

For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic box is big

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j\neq i} S_{j,i}$
- $E = \sum_i E_i$

For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic box is big

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j \neq i} S_{j,i}$
- $E = \sum_i E_i$

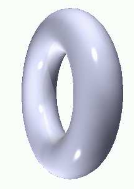
For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic box is big

“Thermodynamic Bethe Ansatz” for finite-size vacuum energy

“Double Wick rotation”



Periodic space (size L),
infinite time-period $R \to \infty$:
Path integral

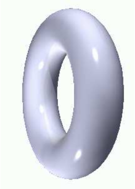
$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

Periodic space of size $R \gg 1$ and
time period L:

$$\Rightarrow \text{free energy} \quad f(L) = E_0(L)$$

To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

“Thermodynamic Bethe Ansatz” for finite-size vacuum energy
“Double Wick rotation”



Periodic space (size L),
infinite time-period $R \to \infty$:
Path integral

$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

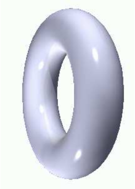
Periodic space of size $R \gg 1$ and time period L:

$$\Rightarrow \text{free energy} \;\; f(L) = E_0(L)$$

To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

“Thermodynamic Bethe Ansatz” for finite-size vacuum energy

“Double Wick rotation”



Periodic space (size L),
infinite time-period $R \to \infty$:
Path integral

$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

Periodic space of size $R \gg 1$ and
time period L:

$$\Rightarrow \text{free energy} \quad f(L) = E_0(L)$$

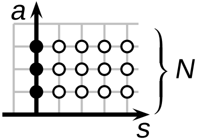
To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

Y-systems

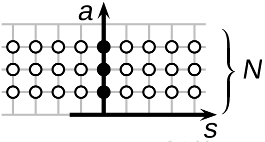
up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

- for $SU(N)$ Gross-Neveu,



- for $SU(N)\times SU(N)$ Principal Chiral Model,



- $Y^+_{a,s}\,Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

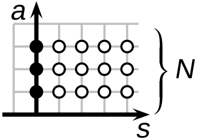
$$Y^\pm_{a,s} \equiv Y_{a,s}(u \pm i/2)$$

Y-systems

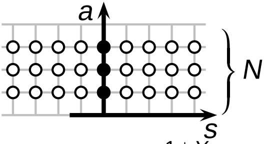
up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

- for $SU(N)$ Gross-Neveu,



- for $SU(N)\times SU(N)$ Principal Chiral Model,



- $Y^+_{a,s}\, Y^-_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ “Y-system equation”

$$Y^\pm_{a,s} \equiv Y_{a,s}(u \pm i/2)$$

Y-systems

up to a change of variables

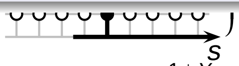
$Y_{(a,s)}(u)$= density of particles of type (a, s) and rapidity $u \in \mathbb{C}$.

"TBA-equation"

Derivation of Y-system from "TBA-equations" of the form

$$Y_{a,s}(u) = \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right) + \langle Source\ Terms \rangle$$

- "TBA equations" contain slightly more information than "Y-system equations" about analyticity conditions.



- $Y^{+}_{a,s}\, Y^{-}_{a,s} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y^{\pm}_{a,s} \equiv Y_{a,s}(u \pm i/2)$$

Y-systems

up to a change of variables

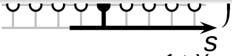
$Y_{(a,s)}(u)$= density of particles of type (a, s) and rapidity $u \in \mathbb{C}$.

"TBA-equation"

Derivation of Y-system from "TBA-equations" of the form

$$Y_{a,s}(u) = \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right) + \langle Source\ Terms \rangle$$

- "TBA equations" contain slightly more information than "Y-system equations" about analyticity conditions.



- $Y_{a,s}^+ Y_{a,s}^- = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}} \frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y_{a,s}^{\pm} \equiv Y_{a,s}(u \pm i/2)$$

AdS/CFT spectral problem

AdS/CFT correspondence

Conjectured duality between
4-dimensional ($\mathcal{N} = 4$) Super-Yang-Mills theory and
type IIB string theory on $AdS_5 \times S^5$ background

- weak-strong duality

- Integrability $\leftrightsquigarrow$ Spin-chain mapping

[Beisert Eden Staudacher 07]

$$\mathrm{trace}(ZDZZ \cdots DZ) \leftrightsquigarrow |{\downarrow}{\uparrow}{\downarrow}{\downarrow} \cdots {\uparrow}{\downarrow}\rangle$$

AdS/CFT spectral problem

AdS/CFT correspondence

Conjectured duality between
4-dimensional ($\mathcal{N} = 4$) Super-Yang-Mills theory and
type IIB string theory on $AdS_5 \times S^5$ background

- weak-strong duality

- Integrability $\leftrightsquigarrow$ Spin-chain mapping

[Beisert Eden Staudacher 07]

$$\mathrm{trace}\big(ZDZZ \cdots DZ\big) \leftrightsquigarrow |{\downarrow}{\uparrow}{\downarrow}{\downarrow} \cdots {\uparrow}{\downarrow}\rangle$$

Outline

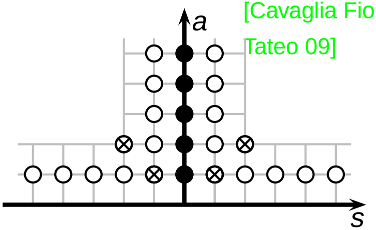
Resolution of *AdS/CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)

⟺



Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

+ Analyticity

[Cavaglia Fioravanti Tateo 09]

⟺

Hirota equation

T-system — Gauge

More details

$$T^+ T^- = T_{a+1}T_{a-1} + T_{s+1}T_{s-1}$$

- Finite parameterization

[Gromov Kazakov S.L. Tsuboi 10]

+ ??

$$Y^{\pm}_{a,s} = Y_{a,s}(u \pm i/2)$$

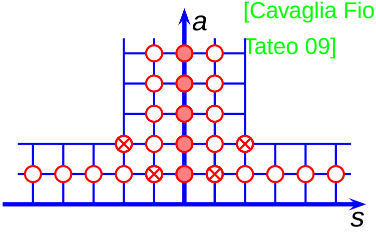
Resolution of *AdS/CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)

$\Longleftrightarrow$



Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

+ Analyticity [Cavaglia Fioravanti Tateo 09]

$\Leftrightarrow$

Hirota equation

T-system / Gauge

More details

$$T^+ T^- = T_{a+1} T_{a-1} + T_{s+1} T_{s-1}$$

- Finite parameterization [Gromov Kazakov S.L. Tsuboi 10]

+ ??

$$Y^{\pm}_{a,s} = Y_{a,s}(u \pm i/2)$$

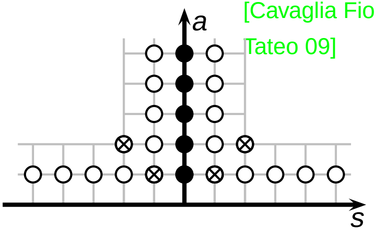
Resolution of *AdS*/*CFT* Spectral problem.

TBA approach

[Gromov Kazakov Kozak Vieira 09]
[Bombardelli Fioravanti Tateo 09]
[Autyunov Frolov 09]

- infinite set of NLIEs
- complicated kernels (zhukovski cuts)

$\Longleftrightarrow$



Y-system equation

[Gromov Kazakov Vieira 09]

$$Y^+ Y^- = \frac{(1+Y_{s+1})(1+Y_{s-1})}{(1+1/Y_{a+1})(1+1/Y_{a-1})}$$

+ Analyticity

[Cavaglia Fioravanti Tateo 09]

$\Leftrightarrow$

Hirota equation

T-system / Gauge

More details

$$T^+ T^- = T_{a+1}T_{a-1} + T_{s+1}T_{s-1}$$

- Finite parameterization

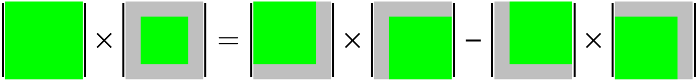
[Gromov Kazakov S.L. Tsuboi 10]

+ ??

$$Y^\pm_{a,s} = Y_{a,s}(u \pm i/2)$$

Outline

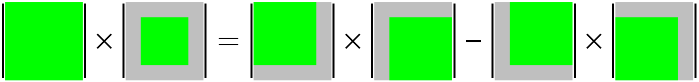
Bilinear relation on determinants



- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times 1 = a \cdot d - b \cdot c$
- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \times e = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \end{vmatrix} \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- $T_{a,s-1} T_{a,s+1} = T_{a,s}^{+} T_{a,s}^{-} - T_{a+1,s} T_{a-1,s}$

$\rightsquigarrow$ parameterization of the infinite number of T-functions in terms of a finite number of functions (the coefficients in the determinant).

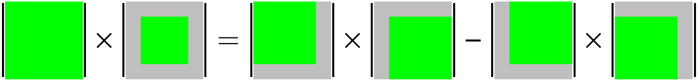
Bilinear relation on determinants



- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times 1 = a \cdot d - b \cdot c$
- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \times e = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \end{vmatrix} \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- $T_{a,s-1} T_{a,s+1} = T^+_{a,s} T^-_{a,s} - T_{a+1,s} T_{a-1,s}$

$\rightsquigarrow$ parameterization of the infinite number of T-functions in terms of a finite number of functions (the coefficients in the determinant).

Bilinear relation on determinants



- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times 1 = a \cdot d - b \cdot c$
- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \times e = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \end{vmatrix} \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- $T_{a,s-1} T_{a,s+1} = T^{+}_{a,s} T^{-}_{a,s} - T_{a+1,s} T_{a-1,s}$

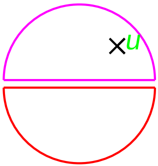
$\rightsquigarrow$ parameterization of the infinite number of T-functions in terms of a finite number of functions (the coefficients in the determinant).

Riemann-Hilbert Problem

Statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{F(v)-G(v)}{v-u} = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if Q is analytic on the upper half plane and decreases quickly enough, then $Q(u) = \frac{1}{2I\pi}\int_{-\infty}^{\infty} \frac{\rho(v)}{v-u}$

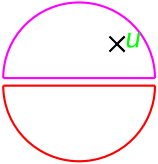
where $\rho = 2\,\mathrm{Re}\left(Q^{[+0]}\right)$

Riemann-Hilbert Problem

Statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then
$$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{F(v)-G(v)}{v-u} = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if Q is analytic on the upper half plane and decreases quickly enough, then $Q(u) = \frac{1}{2I\pi}\int_{-\infty}^{\infty} \frac{\rho(v)}{v-u}$

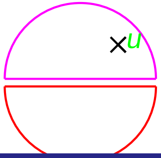
where $\rho = 2\,\mathrm{Re}\left(Q^{[+0]}\right)$

Riemann-Hilbert Problem

Statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{F(v)-G(v)}{v-u} = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if Q is analytic on the upper half plane and decreases quickly enough, then $Q(u) = \frac{1}{2I\pi}\int_{-\infty}^{\infty} \frac{\rho(v)}{v-u}$

FiNLIE-equations

Appropriate choices of F and G allow to derive non-trivial integral equations from analyticity constraints.

Symmetries ⇝ Classical limit

In the classical limit, $g \to \infty$, and $T_{a,s} \to T_{a,s}(u/g)$.
$\Rightarrow$ shifts by $\pm\frac{i}{2}$ in Hirota equation can be neglected.
$\Rightarrow T_{a,s}(u) = \chi_{a,s}\left(\Omega(u)\right)$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C} \qquad (\text{or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues})$$

[Bena Polchinksi Roiban]

Quantum case

$$T_{1,s} = -\hat{T}_{1,-s}$$

Symmetries $\rightsquigarrow$ Classical limit

In the classical limit, $g \to \infty$, and $T_{a,s} \to T_{a,s}(u/g)$.
$\Rightarrow$ shifts by $\pm\frac{i}{2}$ in Hirota equation can be neglected.
$\Rightarrow T_{a,s}(u) = \chi_{a,s}(\Omega(u))$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

Quantum case

$$T_{1,s} = -\hat{T}_{1,-s}$$

Symmetries $\leftsquigarrow$ Classical limit

In the classical limit, $g \to \infty$, and $T_{a,s} \to T_{a,s}(u/g)$.
$\Rightarrow$ shifts by $\pm\frac{i}{2}$ in Hirota equation can be neglected.
$\Rightarrow T_{a,s}(u) = \chi_{a,s}\left(\Omega(u)\right)$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

Quantum case

$$T_{1,s} = -\hat{T}_{1,-s}$$

Symmetries $\leftsquigarrow$ Classical limit

In the classical limit, $g \to \infty$, and $T_{a,s} \to T_{a,s}(u/g)$.
$\Rightarrow$ shifts by $\pm\frac{i}{2}$ in Hirota equation can be neglected.
$\Rightarrow T_{a,s}(u) = \chi_{a,s}\left(\Omega(u)\right)$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\det = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

Quantum case

$$T_{1,s} = -\hat{T}_{1,-s}$$

≪quantum $\mathbb{Z}_4$≫ symmetry

statement

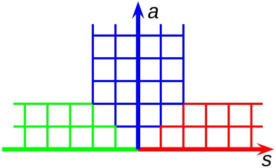
$$T_{1,s} = -\hat{T}_{1,-s},$$

where $\hat{T}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

$$\hat{T}_{1,0} = 0 \Rightarrow \quad Q = -\bar{Q}$$

$=$ $=$

$$Q(u) = -iu + \frac{1}{2i\pi} \int_{-2g}^{2g} \frac{\rho(v)}{v-u}$$



$$T_{1,s}\big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$

≪quantum $\mathbb{Z}_4$≫ symmetry

statement

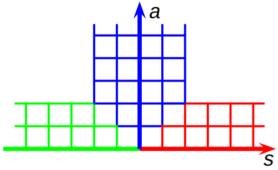
$$T_{1,s} = -\hat{T}_{1,-s},$$

where $\hat{T}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

$$\hat{T}_{1,0} = 0 \Rightarrow \quad Q = -\bar{Q}$$

$$= \qquad\qquad =$$

$$Q(u) = -iu + \frac{1}{2i\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$$



$$T_{1,s}\Big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$

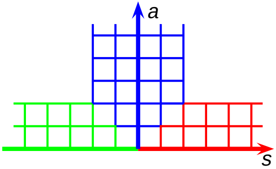
≪quantum $\mathbb{Z}_4$≫ symmetry

statement

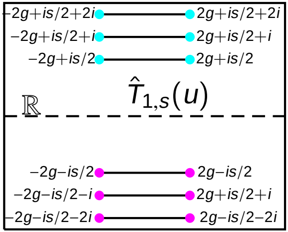
$$T_{1,s} = -\hat{T}_{1,-s},$$

where $\hat{T}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

$\hat{T}_{1,0} = 0 \Rightarrow \quad Q = -\bar{Q}$



$$T_{1,s}\Big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$



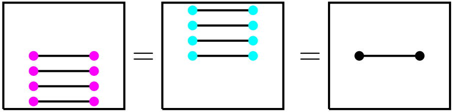
≪quantum $\mathbb{Z}_4$≫ symmetry

statement

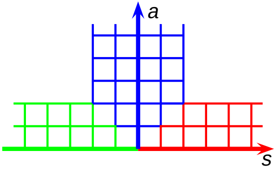
$$T_{1,s} = -\hat{T}_{1,-s},$$

where $\hat{T}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

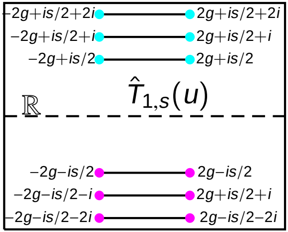
$$\hat{T}_{1,0} = 0 \;\Rightarrow\; Q = -\bar{Q}$$



$$Q(u) = -iu + \frac{1}{2l\pi}\int_{-2g}^{2g} \frac{\rho(v)}{v-u}$$



$$T_{1,s}\big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$



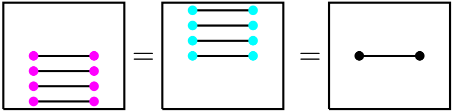
≪quantum $\mathbb{Z}_4$≫ symmetry

statement

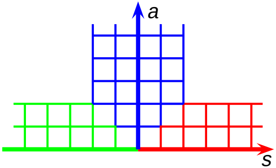
$$T_{1,s} = -\hat{T}_{1,-s},$$

where $\hat{T}_{1,s} = -\bar{Q}^{[-s]} - Q^{[+s]}$ in a Riemann sheet where Zhukovski cuts are on $[-2g, 2g]$ up to a shift

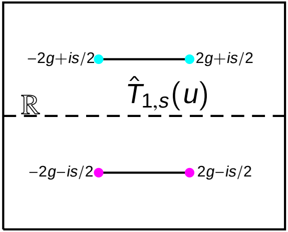
$$\hat{T}_{1,0} = 0 \;\Rightarrow\; Q = -\bar{Q}$$



$$Q(u) = -iu + \frac{1}{2I\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$$



$$T_{1,s}\big|_{s\geq 1} = -\bar{Q}^{[-s]} - Q^{[+s]}$$



Outline

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs

… the physical meaning of which still has to be better understood

  - The TBA approach only gives meaning to Y-functions
  - analytic properties, symmetries

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs

· · · the physical meaning of which still has to be better understood
  - The TBA approach only gives meaning to Y-functions
  - analytic properties, symmetries

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs

⋯ the physical meaning of which still has to be better understood

  - The TBA approach only gives meaning to Y-functions
  - analytic properties, symmetries

Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs

··· the physical meaning of which still has to be better understood
  - The TBA approach only gives meaning to Y-functions
  - analytic properties, symmetries

Thank you !

Y-system and Hirota equation

Y-system Equation

$$Y_{a,s}^{+}\, Y_{a,s}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$$

[Gromov Kazakov Vieira 09] where $Y_{a,s}^{\pm} = Y_{a,s}(u \pm \frac{i}{2})$

- change of variable $Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$

Hirota equation

$$T_{a,s}^{+}\,T_{a,s}^{-} = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s-1}$$

Gauge freedom

Y-functions and Hirota equation are invariant under gauge transformations $T_{a,s} \to g_1^{[a+s]}\, g_2^{[a-s]}\, g_3^{[-a+s]}\, g_4^{[-a-s]}\, T_{a,s}$

$f^{[\pm k]} \equiv f(u \pm ki/2)$

Back to the presentation